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# Limiting laws of supercritical branching random walks

Julien Barral <sup>\*</sup>, Rémi Rhodes <sup>†</sup>, Vincent Vargas <sup>‡§</sup>

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## Abstract

In this note, we make explicit the law of the renormalized supercritical branching random walk, giving credit to a conjecture formulated in [4] for a continuous analog of the branching random walk. In particular, we express the law of the corresponding limiting renormalized Gibbs measures, confirming, in this discrete model, conjectures formulated by physicists about the Poisson-Dirichlet nature of the jumps in the limit.

## 1. Introduction and notations

We consider a branching random walk on the real line  $\mathbb{R}$ . Initially, a particle sits at the origin. Its children form the first generation; their displacements from the origin correspond to a point process on the line. These children have children of their own (who form the second generation), and behave relative to their respective positions like independent copies of the same point process  $L$ . And so on.

We write  $|u| = n$  if an individual  $u$  is in the  $n$ -th generation, and denote its position  $V(u)$  (and  $V(e) = 0$  for the initial ancestor). We assume throughout the paper that, for some  $\delta > 0$  and  $\delta_- > 0$ ,

$$\forall t > 0, \quad \mathbb{E} \left[ \left( \sum_{|u|=1} e^{-tV(u)} \right)^{1+\delta} \right] + \mathbb{E} \left[ \left( \sum_{|u|=1} 1 \right)^{1+\delta} \right] + \mathbb{E} \left[ \sum_{|u|=1} e^{\delta_- V(u)} \right] < +\infty. \quad (1)$$

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where  $\mathbb{E}$  denotes expectation with respect to  $\mathbb{P}$ , the law of the branching random walk. We define the (logarithmic) moment generating function

$$\forall t \geq 0, \quad \psi(t) = \ln \mathbb{E} \left[ \sum_{|u|=1} e^{-tV(u)} \right] \in ]-\infty; +\infty[.$$

From (1), we have  $\psi(t) < +\infty$  for  $t \in [-\delta_-, +\infty[$ . Following Biggins and Kyprianou [5], we assume

$$\psi(0) > 0, \quad \psi(1) = \psi'(1) = 0. \quad (2)$$

The number of particles in each generation forms a Galton-Watson tree. Therefore the condition  $\psi(0) > 0$  ensures that this Galton-Watson tree is super-critical. We introduce the martingale

$$W_n = \sum_{|u|=n} e^{-V(u)}, \quad n = 0, 1, 2, \dots \left( \sum_{\emptyset} := 0 \right). \quad (3)$$

Since this martingale is positive, it converges almost surely. When  $\psi'(1) < 0$ , it is proved in [7] that the martingale  $(W_n)_n$  converges almost surely to a non-degenerate limit which is strictly positive on the set of non-extinction. We finally assume that the distribution of  $L$  is non-lattice and

$$\mathbb{E} \left[ \sum_{|u|=1} V(u)^2 e^{-V(u)} \right] + \mathbb{E} \left[ X(\ln_+(X + \tilde{X}))^3 \right] < +\infty \quad (4)$$

where

$$X = \sum_{|u|=1} e^{-V(u)}, \quad \tilde{X} = \sum_{|u|=1} V(u)_+ e^{-V(u)}. \quad (5)$$

We introduce the low temperature Gibbs measure ( $\beta > 1$ ):

$$W_{n,\beta} = \sum_{|u|=n} e^{-\beta V(u)} \quad (6)$$

and the derivative martingale:

$$Z_n = \sum_{|u|=n} V(u) e^{-V(u)} \quad (7)$$

It is well known that the martingale  $(Z_n)_n$  converges almost surely towards a random variable  $Z_\infty$ , which is strictly positive conditionally to the survival of the system (see [6, 1]).

We claim:

**Theorem 1.** *The random variables  $(n^{\frac{3}{2}\beta} W_{n,\beta})_n$  converge in law towards a non trivial random variable  $W_{\infty,\beta}$ , which possesses moments of order  $r$  for  $0 \leq r < \frac{1}{\beta}$ . The law of  $W_{\infty,\beta}$  is, up to a multiplicative constant  $c$ , that of a stable subordinator  $T_\beta$  of index  $1/\beta$  taken at an independent time  $Z_\infty$ :*

$$W_{\infty,\beta} \stackrel{law}{=} cT_\beta(Z_\infty).$$

## 2. The case of Random multiplicative cascades

We point out that Theorem 1 has direct applications to random multiplicative cascades since random multiplicative cascades can be seen as branching random walks where the number of children for each particle has been set to two. Random multiplicative cascades were introduced by B. Mandelbrot in [14] as a toy model for energy dissipation in a turbulent flow. More precisely, let  $\mathcal{I}_n$  denote the set of dyadic subintervals of  $[0, 1]$  of length  $2^{-n}$ , namely

$$\mathcal{I}_n = \left\{ [k2^{-n}, (k+1)2^{-n}[; n \in \mathbb{N}, k \in \{0, 1, \dots, 2^n - 1\} \right\}.$$

Each interval in  $\mathcal{I}_n$  has precisely two subintervals in  $\mathcal{I}_{n+1}$ , its left half and its right half. We set  $\mathcal{I} = \bigcup_{n \in \mathbb{N}} \mathcal{I}_n$ .

Let  $W$  be some positive random variable with mean  $1/2$  and let  $(W_I)_{I \in \mathcal{I}}$  be an independent collection of random variables, each of which has the law of  $W$ . We now define a sequence of random measures on  $[0, 1]$ . Let  $dx$  denote the Lebesgue measure on  $[0, 1]$ . We define

$$\mu_n(dx) = 2^n w_n(x) dx, \quad \text{where } w_n(x) = \prod_{j=0}^{n-1} W_{I_j(x)}$$

where  $I_j(x)$  denotes the interval  $I \in \mathcal{I}_j$  that contains  $x$ . We recall the following theorem of [11] first conjectured in the seminal work [14]:

**Theorem 2** (Kahane, Peyriere). *The weak limit  $\mu(dx) = \lim_{n \rightarrow \infty} \mu_n(dx)$  exists almost surely. Moreover,  $\mathbb{E}[W \ln W] < 0$  if and only if  $\mu$  is almost surely non trivial.*

We are interested in the situation where  $\mathbb{E}[W \ln W] > 0$  in which case the limiting measure vanishes. More precisely, as suggested in [14], we want to find a renormalization sequence  $(\lambda_n)_n$  such that  $\nu_n(dx) = \lambda_n \mu_n(dx)$  converges. We define:

$$\psi(t) = \ln \mathbb{E} \left[ \sum_{I \in \mathcal{I}_1} W_I^t \right].$$

We assume that  $\psi$  is well defined for  $t \in [-\delta_-, +\infty[$  with  $\delta_- > 0$ . We assume that  $\psi(1) = 0$  and  $\psi'(1) = 0$ . We define for  $\beta > 0$ :

$$w_{n,\beta}(x) = \prod_{j=0}^{n-1} W_{I_j(x)}^\beta \quad \text{and} \quad \mu_{n,\beta}(dx) = 2^n w_{n,\beta}(x) dx.$$

When  $0 < \beta < 1$ , the situation boils down to the non-degenerate (classical) situation. For  $\beta = 1$ , this corresponds to the boundary case (i.e.  $\mathbb{E}[W \ln W] = 0$ ) which

was studied in [2]. It is well known (cf. [6] for instance) that the derivative measure  $-\frac{d\mu_{n,\beta}}{d\beta}|_{\beta=1}(dx)$  converges almost surely to a random measure  $Z_\infty(dx)$  which is positive on every interval.

As previously, we are especially interested in the low temperature case  $\beta > 1$ . From Theorem 1, we deduce the following:

**Corollary 3.** *The random measures  $(n^{\frac{3}{2}\beta}\mu_{n,\beta}(dx))_n$  weakly converge in law towards a random measure  $\mu_{\infty,\beta}(dx)$  defined on the Borelian subsets of  $[0, 1]$ . The law of the measure  $\mu_{\infty,\beta}(dx)$  is given by the derivative (in the sense of distributions) of the function  $t \rightarrow cT_\beta(Z_\infty[0, t])$  where  $c > 0$  is some positive constant and  $T_\beta$  is a stable subordinator of index  $1/\beta$  independent from  $Z_\infty(dx)$ .*

We will denote this limiting law  $cT_\beta(Z_\infty(dx))$ . We stress that this implies the following result on renormalized Gibbs measures:

**Corollary 4.** *We have the following convergence in law (for the weak convergence of measures):*

$$\frac{\mu_{n,\beta}(dx)}{\mu_{n,\beta}([0, 1])} \xrightarrow{n \rightarrow \infty} \frac{T_\beta(Z_\infty(dx))}{T_\beta(Z_\infty[0, 1])}.$$

### 3. The continuous case: Gaussian multiplicative Chaos

In this section, we discuss the continuous analogue of the Random multiplicative cascades with lognormal weights: Gaussian multiplicative Chaos. One can work with other weights than lognormal but we stick to this case for the sake of simplicity. Gaussian multiplicative Chaos was introduced by J.P. Kahane in [10]. Gaussian multiplicative Chaos and lognormal Random multiplicative cascades bear striking similarities; in particular, one can introduce a continuous analogue to the discrete star equation (cf. [3, 16]) which plays a key role in the identification of the limit in the above theorems. In some sense, there is a correspondance between both models in the sense that roughly all theorems in the discrete case have continuous analogues. In the paper [4] (section 6 perspectives), the authors conjectured an analogue to corollary 4 for Gaussian multiplicative Chaos. In fact, theorem 1 leads to an even stronger version of the conjecture in [4].

The above result on renormalized Gibbs measures, i.e. corollary 4, and the conjecture in [4] on the limit of a continuous renormalized Gibbs measure share a common feature which is proved or conjectured to hold for many supercritical Gibbs measures (also called the glassy phase in the physics litterature) : the limiting object is a purely atomic measure whose jumps are distributed according to a Poisson-Dirichlet distribution, i.e. the ordered jumps of a renormalized stable subordinator. This universal feature (proved for instance in the case of the REM, etc.) has been conjectured by physicists (see [8]).

## 4. Proofs

### 4.1 Proof of Theorem 1

It is proved in [13] that the sequence  $(n^{\frac{3}{2}\beta}W_{n,\beta})_n$  converges in law towards a random variable  $W_{\infty,\beta}$ . This random variable is non trivial since

$$\begin{aligned} n^{\frac{3}{2}\beta}W_{n,\beta} &\geq \left(n^{3/2} \max_{|u|} e^{-V(u)}\right)^\beta \\ &\geq \left(n^{3/2} e^{-\min_{|u|} V(u)}\right)^\beta \\ &= \left(e^{-\left(\min_{|u|} V(u) - \frac{3}{2} \ln n\right)}\right)^\beta \end{aligned}$$

It is proved in [1] that the last quantity  $\min_{|u|} V(u) - \frac{3}{2} \ln n$  converges in law as  $n \rightarrow \infty$  towards a non trivial limit. So  $W_{\infty,\beta}$  cannot be 0. Let us prove that the random variable  $W_{\infty,\beta}$  possesses moments of order  $r$  for  $r < \frac{1}{\beta}$ . From [13], for some positive constant  $c$ , we have  $\mathbb{P}(n^{\frac{3}{2}\beta}W_{n,\beta} > e^{\beta x}) \leq cxe^{-x}$  for all  $n$  large enough:

$$\begin{aligned} \mathbb{E}[(n^{\frac{3}{2}\beta}W_{n,\beta})^r] &= \int_0^\infty \mathbb{P}(n^{\frac{3}{2}\beta}W_{n,\beta} > u^{1/r}) du \\ &= \int_0^\infty \mathbb{P}(n^{\frac{3}{2}\beta}W_{n,\beta} > e^{\beta x}) e^{r\beta x} r\beta dx \\ &\leq c \int_0^\infty x e^{(r\beta-1)x} r\beta dx. \end{aligned}$$

This quantity is obviously finite when  $r < \frac{1}{\beta}$ . It just remains to identify the limiting law  $W_{\infty,\beta}$ . To that purpose, it is straightforward to check that  $W_{\infty,\beta}$  satisfies the star equation ([14]):

$$W_{\infty,\beta} \stackrel{\text{law}}{=} \sum_{|u|=1} e^{-\beta V(u)} W_{\infty,\beta}^{(u)},$$

where conditionally on  $(u, V(u), |u| = 1)$  the family  $(W_{\infty,\beta}^{(u)})_{|u|=1}$  are independent copies of  $W_{\infty,\beta}$ . This equation is carefully studied in [12]. Since  $t \mapsto \psi_\beta(t) = \ln \mathbb{E}[\sum_{|u|=1} e^{-\beta t V(u)}]$  vanishes for  $t = \frac{1}{\beta}$  with  $\psi'_\beta(\frac{1}{\beta}) = 0$ , the result follows from [12] (see also [9] for the case of homogenous trees).  $\square$

### 4.2 Convergence of the random measures of section 2

Notice that, for all continuous function  $f : [0, 1] \rightarrow \mathbb{R}$ ,  $|n^{3\beta/2}\mu_{n,\beta}(f)| \leq n^{3\beta/2}\mu_{n,\beta}([0, 1])\|f\|_\infty$ . Since  $(n^{3\beta/2}\mu_{n,\beta}([0, 1]))_n$  is tight, the sequence of random measures  $(n^{3\beta/2}\mu_{n,\beta}(dx))$  is tight for the weak convergence of measures. Uniqueness of the possible limit is established by Theorem 1.  $\square$

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