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SPECTRAL POSITIVITY AND RIEMANNIAN COVERINGS

PIERRE BÉRARD AND PHILIPPE CASTILLON

ABSTRACT. Let (M, g) be a complete non-compact Riemannian manifold. We consider operators of the form $\Delta_g + V$, where Δ_g is the non-negative Laplacian associated with the metric g , and V a locally integrable function. Let $\rho : (\widehat{M}, \widehat{g}) \rightarrow (M, g)$ be a Riemannian covering, with Laplacian $\Delta_{\widehat{g}}$ and potential $\widehat{V} = V \circ \rho$. If the operator $\Delta + V$ is non-negative on (M, g) , then the operator $\Delta_{\widehat{g}} + \widehat{V}$ is non-negative on $(\widehat{M}, \widehat{g})$. In this note, we show that the converse statement is true provided that $\pi_1(\widehat{M})$ is a co-amenable subgroup of $\pi_1(M)$.

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1. INTRODUCTION

Let (M^n, g) be a complete, connected Riemannian manifold. Denote by Δ the non-negative Laplacian, by μ the measure associated with the metric g , and consider the Schrödinger operator $J = \Delta + V$, where V is a locally integrable function on M . Such operators appear naturally when one studies minimal (or constant mean curvature) immersions $M^n \looparrowright N^{n+1}$. Such hypersurfaces are critical points of a volume functional whose second derivative is given by the quadratic form associated with the Jacobi (stability) operator $J = \Delta - \text{Ric}_N(\nu, \nu) - |A|^2$, where ν is a unit normal vector field along M and $|A|$ the norm of the second fundamental form of the immersion. Stable hypersurfaces are those for which the Jacobi operator is non-negative, *i.e.* those critical points of the volume functional which are local minima up to second order.

When studying stable minimal hypersurfaces, a natural question is: “What conclusions on the Riemannian manifold (M, g) can one draw from the fact that the operator $\Delta + V$ is non-negative in the sense of quadratic forms?” *i.e.* from the fact that the associated quadratic form is non-negative on Lipschitz functions with compact support in M (or equivalently on C^1 -functions with compact support),

$$(\star) \quad 0 \leq \int_M (|df|^2 + Vf^2) d\mu \quad \forall f \in \text{Lip}_0(M).$$

When investigating the above question, it is often useful to pass to the universal cover of M . In this paper, we study the behaviour of non-negativity under Riemannian covering. More precisely, let $\rho : (\widehat{M}, \widehat{g}) \rightarrow (M, g)$ be a Riemannian covering, let V be a locally integrable function on M , and let

$\widehat{V} = V \circ \rho$. It follows from [6] that $\Delta_g + V \geq 0$ on (M, g) (in the sense of quadratic forms) implies that $\Delta_{\widehat{g}} + \widehat{V} \geq 0$ on $(\widehat{M}, \widehat{g})$. It is a natural question to investigate the converse statement. In [9], Proposition 2.5, the authors prove that the converse statement is true provided that the inverse image $\rho^{-1}(\Omega)$ of any relatively compact subdomain $\Omega \subset M$ has sub-exponential volume growth.

Note that this statement is not true in general and needs some additional hypotheses on the covering, as the following simple example shows. Consider a group Γ acting on the hyperbolic plane with a compact quotient \mathbb{H}^2/Γ . It is well known that the operator $\Delta - \frac{1}{4}$ is non-negative on \mathbb{H}^2 , but it is not the case on \mathbb{H}^2/Γ which is a compact surface.

In [2, 3], R. Brooks worked on a closely related problem. Indeed, he investigated the behaviour of the infimum of the spectrum of the Laplacian under normal Riemannian coverings. A key assumption in his work is the amenability of the covering group. In this paper, we prove the following result which is very much inspired by [2, 3].

Theorem 1.1. *Let (M, g) be a complete Riemannian manifold of dimension $n \geq 2$. Let $\rho : (\widehat{M}, \widehat{g}) \rightarrow (M, g)$ be a Riemannian covering. Let V be a locally integrable function on (M, g) , and let $\widehat{V} = V \circ \rho$. Assume the $\pi_1(\widehat{M})$ is a co-amenable sub-group of $\pi_1(M)$. Then, the operator $\Delta_g + V$ is non-negative on (M, g) if and only if the operator $\Delta_{\widehat{g}} + \widehat{V}$ is non-negative on $(\widehat{M}, \widehat{g})$.*

As a particular case, when $\rho : (\widehat{M}, \widehat{g}) \rightarrow (M, g)$ is a normal covering, with covering group G , the assumption on the fundamental groups of M and \widehat{M} is equivalent to the amenability of the group G .

The relationship between [9], Proposition 2.5 and Theorem 1.1 is that a finitely generated group with sub-exponential growth is amenable. However, there exist amenable groups with exponential volume growth, so that our hypothesis is weaker than the one in [9].

For a complete Riemannian manifold $(M, g; V)$, equipped with a locally integrable function V , consider the set

$$I(M, g; V) = \{a \in \mathbb{R} \mid \Delta + aV \geq 0\}.$$

This Riemannian invariant is a closed interval which contains 0, see [4].

For any Riemannian covering $\rho : (\widehat{M}, \widehat{g}) \rightarrow (M, g)$, we have $I(M, g; V) \subset I(\widehat{M}, \widehat{g}; \widehat{V})$. Theorem 1.1 tells us that if $\pi_1(\widehat{M})$ is co-amenable in $\pi_1(M)$, then $I(M, g; V) = I(\widehat{M}, \widehat{g}; \widehat{V})$: the invariant $I(M, g; V)$ does not distinguish two manifolds which differ by a covering with amenable action. In particular, we have the following corollary.

Corollary 1.2. *Let (M, g) be either \mathbb{R}^2 with is a \mathbb{Z}^2 -invariant metric, or $S^1 \times \mathbb{R}$ with a \mathbb{Z} -invariant metric. Let K denote the Gaussian curvature of the metric g . Then, $I(M, g; K) = \{0\}$ or $I(M, g; K) = \mathbb{R}$. Furthermore, $I(M, g; K) = \mathbb{R}$ if and only if $K \equiv 0$.*

Remark. The Jacobi operator of an isometric minimal surface $M^2 \looparrowright \mathbb{R}^3$ into Euclidean 3-space is $\Delta + 2K$, where K is the Gaussian curvature of M .

More generally ([6], Section 3), the Jacobi operator of a minimal immersion $M \looparrowright \widehat{M}^3$ into a 3-manifold with scalar curvature \widehat{S} , can be written as $\Delta + K - (\widehat{S} + \frac{1}{2}|A|^2)$. When \widehat{S} is non-negative, the stability of the surface implies that $\Delta + K$ is non-negative. As a consequence, the study of stable minimal surfaces led to a general spectral problem on Riemannian surfaces, namely investigating the consequences of the non-negativity of operators of the form $\Delta + aK$, where K is the curvature of the surface, and $a \in \mathbb{R}$ is a real parameter. As a matter of fact, the papers [6, 4, 5, 1] derive topological properties of M from assumptions on $I(M, g; K)$ (where K is the Gaussian curvature of the metric g , in this 2-dimensional framework).

The paper is organized as follows. In Section 2, we recall basic facts on amenable groups and amenable group actions. For amenable groups, we refer to [8], and to [2], Section 1. For amenable group actions, we refer to [10, 7]. Section 3 contains the proofs.

The authors would like to thank Ivan Babenko for introducing them to amenable group actions.

2. AMENABLE GROUP ACTIONS

2.1. Definitions. Let X be a countable set, and let G be a discrete group acting on X .

Definition 2.1. A mean on X is a bounded linear functional $\mu : \ell^\infty(X) \rightarrow \mathbb{R}$ such that,

$$\mu(1) = 1 \quad \text{and} \quad \mu(f) \geq 0, \quad \text{whenever } f \geq 0.$$

Equivalently, a mean on X is a map on subsets of X , $\mu : \mathcal{P}(X) \rightarrow [0, 1]$, such that,

$$\begin{cases} \mu(X) = 1, & \text{and} \\ \mu(A \cup B) = \mu(A) + \mu(B) & \text{whenever } A \cap B = \emptyset, \end{cases}$$

i.e. μ is a finitely additive probability on X .

Definition 2.2. A G -action on X is amenable if there exists a G -invariant mean on X , i.e. if there exists a mean μ on X such that,

$$\mu(\gamma \cdot f) = \mu(f), \quad \forall \gamma \in G, \quad \forall f \in \ell^\infty(X),$$

where $(\gamma \cdot f)(x) = f(\gamma^{-1}x)$. The group G is amenable if its action on itself by left multiplications is amenable.

Any finite group G is amenable. It suffices to take $\mu(f) = \frac{1}{\sharp(G)} \sum_{\gamma \in G} f(\gamma)$, where $\sharp(G)$ is the number of elements of G . The group \mathbb{Z} is amenable. To see this, consider the means $\mu_n, n \in \mathbb{N}$, defined by $\mu_n(f) = \frac{1}{2n+1} \sum_{k=-n}^n f(k)$ and take their weak- $*$ limit. More generally, the group \mathbb{Z}^n is amenable for all n . The basic example of a non-amenable group is the free group on two generators \mathbb{F}_2 , and it can be shown that any group containing a subgroup isomorphic to \mathbb{F}_2 is not amenable.

Remarks.

(i) If the group G is amenable, any action of G is amenable. Indeed, let ν be an invariant mean on G , and let X be a set endowed with a G -action.

Choose $x_0 \in X$. Given $f \in \ell^\infty(X)$, define $\hat{f} \in \ell^\infty(G)$ by $\hat{f}(\gamma) = f(\gamma \cdot x_0)$. Then, the map $\mu : \ell^\infty(X) \rightarrow \mathbb{R}$ defined by $\mu(f) = \nu(\hat{f})$ is an invariant mean on X (we take the mean on a G -orbit).

(ii) Note that the converse statement is not true. Indeed, van Douwen proved that any finitely generated non-abelian free group admits a faithful transitive amenable action ([7], Introduction). In other words, there exist amenable actions by non-amenable groups.

Definition 2.3. *A subgroup $H < G$ is co-amenable if the G -action on G/H is amenable.*

Remark. Note that a normal subgroup $H \triangleleft G$ is co-amenable if and only if G/H is an amenable group.

When the subgroup H is not normal, one can consider left cosets $[\sigma]_L = \sigma H \in G/H$, and right cosets $[\sigma]_R = H\sigma \in H \backslash G$. The group G acts on both sets. The G -action on G/H is defined by $\gamma \cdot [\sigma]_L = [\gamma\sigma]_L$; the G -action on $H \backslash G$ is defined by $\gamma \cdot [\sigma]_R = [\sigma\gamma^{-1}]_R$. There is a natural bijective map between the coset spaces,

$$\begin{cases} F : G/H \rightarrow H \backslash G, \\ F : [\sigma]_L \mapsto [\sigma^{-1}]_R, \end{cases}$$

and this map is equivariant for the actions of G . It follows ([7], Lemma 2.1) that,

Proposition 2.4. *The G -action on G/H is amenable if and only if the G -action on $H \backslash G$ is amenable. It follows that a subgroup $H < G$ is amenable if any (and hence both) of the actions of G on the coset spaces is amenable.*

Proof. Since the map F is equivariant, the push-forward of a G -invariant mean on G/H provides an invariant mean on $H \backslash G$ and conversely. \square

2.2. Følner's condition. Følner's characterization of amenable groups ([8], Section 3.6) has an analog for amenable actions.

Theorem 2.5 (Følner's condition). *A G -action on X is amenable if and only if,*

$$\begin{cases} \forall \eta \in (0, 1), \quad \forall \gamma_1, \dots, \gamma_n \in G, \quad \exists E \subset X \text{ such that,} \\ \#(E) < \infty \text{ and } \eta \#(E) \leq \#(E \cap \gamma_i \cdot E), \quad \forall 1 \leq i \leq n. \end{cases}$$

Equivalently, a G -action on X is amenable if and only if,

$$\begin{cases} \forall \epsilon > 0, \quad \forall \gamma_1, \dots, \gamma_n \in G, \quad \exists E \subset X \text{ such that,} \\ \#(E) < \infty \text{ and } \#(E \triangle \gamma_i \cdot E) \leq \epsilon \#(E), \quad \forall 1 \leq i \leq n. \end{cases}$$

Here $\#(E)$ denotes the number of elements of E , and $A \triangle B$ the symmetric difference of two sets A, B .

Proof. See [10], Section 4, Theorems 4.4 and 4.9. \square

Remarks.

(1) When G is countable, the second condition is equivalent to saying that there exists a sequence $\{E_k\}_{k \in \mathbb{N}}$ of finite subsets of X such that,

$$\lim_{k \rightarrow \infty} \frac{\#(E_k \triangle \gamma \cdot E_k)}{\#(E_k)} = 0, \quad \forall \gamma \in G.$$

This is the formulation in [10].

(2) As a consequence of this criterion, one can show ([2], Proposition 1) that groups with sub-exponential growth are amenable. This fact relates our Theorem 1.1 to Proposition 2.5 in [9].

For finitely generated groups, Følner's property has the following consequence. Let $\alpha_1, \dots, \alpha_n$ be a symmetric system of generators for G . Define the Cayley graph of the G -action on X as follows,

- the vertices of the graph are the elements of X ,
- $[x, y]$ is an edge of the graph if and only if there exists some $i \in \{1, \dots, n\}$, such that $x = \alpha_i \cdot y$.

For a finite set $E \subset X$, define the *boundary* ∂E of E (in the Cayley graph) as,

$$\partial E = \{x \in E \mid \exists i \in \{1, \dots, n\}, \text{ such that } \alpha_i \cdot x \notin E\}.$$

Proposition 2.6. *If the G -action on X is amenable, then*

$$\forall \epsilon > 0, \exists E \subset X, \text{ finite, such that } \frac{\#\partial E}{\#(E)} \leq \epsilon.$$

Proof. Let E be a finite subset of X . Then

$$\partial E = \cup_{i=1}^n \{x \in E \mid \alpha_i \cdot x \notin E\}.$$

Then,

$$\begin{aligned} \#\partial E &\leq \sum_{i=1}^n \#\{x \in E \mid \alpha_i \cdot x \notin E\} \\ &\leq \sum_{i=1}^n \left(\#(E) - \#\{x \in E \mid \alpha_i \cdot x \in E\} \right) \\ &\leq \sum_{i=1}^n \left(\#(E) - \#(E \cap \alpha_i^{-1} \cdot E) \right). \end{aligned}$$

Choose $\epsilon > 0$ and $\eta \in (0, 1)$ such that $n(1 - \eta) \leq \epsilon$. Apply Følner's theorem: there exists a finite set $E \subset X$ such that $\eta \#(E) \leq \#(E \cap \alpha_i^{-1} \cdot E)$ for $i = 1, \dots, n$. Using the above inequalities, we obtain for this subset,

$$\#\partial E \leq \sum_{i=1}^n (\#(E) - \eta \#(E)) = n(1 - \eta) \#(E) \leq \epsilon \#(E).$$

This proves the proposition. \square

Remark. A geometric interpretation of the previous proposition is that there is no linear isoperimetric inequality on the Cayley graph of an amenable action by a finitely generated group.

3. PROOFS

As we already mentioned in the Introduction, it follows from [6] that for any Riemannian covering $\rho : (\widehat{M}, \widehat{g}) \rightarrow (M, g)$, and any locally integrable function V on M , with $\widehat{V} = V \circ \rho$, $\Delta_g + V \geq 0$ on M implies that $\Delta_{\widehat{g}} + \widehat{V} \geq 0$ on \widehat{M} . We shall therefore concentrate on the converse statement. We begin the proof by considering normal coverings and then explain how to handle the general case.

3.1. Normal coverings. We first assume that $\pi_1(\widehat{M})$ is a normal subgroup of $\pi_1(M)$. Therefore, $\rho : (\widehat{M}, \widehat{g}) \rightarrow (M, g)$ is a normal Riemannian covering, with amenable covering group $G = \pi_1(M)/\pi_1(\widehat{M})$. In view of the introduction to this section, we only need to prove that $\Delta_{\widehat{g}} + \widehat{V} \geq 0$ implies that $\Delta_g + V \geq 0$.

Let f be a function in $C_0^1(M)$. We want to prove that $0 \leq \int_M |df|^2 + Vf^2$. The general idea is as follows. Lift f to \widehat{f} on \widehat{M} . This function is not compactly supported, but it behaves like f on fundamental domains. Multiply \widehat{f} by a cut-off function ξ which is 1 inside some Ω to be chosen later on. By assumption, $\int_{\widehat{M}} |d(\xi\widehat{f})|^2 + \widehat{V}\xi^2\widehat{f}^2$ is non-negative. We will conclude using Proposition 2.6 which tells us that the effect of the cut-off function is negligible.

◊ Fix some $\epsilon > 0$. Let $F \subset \widehat{M}$ be a fundamental domain for the action of the covering group G , and let β_1, \dots, β_n be the elements of G such that $\beta_i \cdot F \cap F \neq \emptyset$. These elements generate G . As G is amenable, there exists a finite set $E \subset G$ such that $\sharp(\partial E) \leq \epsilon \sharp(E)$ (Proposition 2.6).

◊ Lift $\text{supp}(f)$ to F . As it is compact, there exists some $\alpha > 0$ such that the α -neighborhood of $\partial(F \cup \cup_{i=1}^n \beta_i \cdot F)$ does not intersect the lift of $\text{supp}(f)$. Let $\Omega = \cup_{\gamma \in E} \gamma \cdot F$ and consider the cut-off function $\xi : \widehat{M} \rightarrow \mathbb{R}$ defined by

$$\xi(x) = \begin{cases} 0, & \text{if } x \notin \Omega, \\ \frac{1}{\alpha}d(x, \partial\Omega), & \text{if } x \in \Omega \text{ and } d(x, \partial\Omega) < \alpha, \\ 1, & \text{if } x \in \Omega \text{ and } d(x, \partial\Omega) \geq \alpha. \end{cases}$$

The function ξ satisfies the inequalities,

$$\begin{cases} 0 \leq \xi \leq 1 & \text{on } \Omega, \\ |d\xi| \leq \frac{1}{\alpha} & \text{on } \gamma \cdot F, \text{ if } \gamma \in \partial E, \\ \xi = 1 \text{ and } |d\xi| = 0 & \text{on } \gamma \cdot F, \text{ if } \gamma \in E \setminus \partial E. \end{cases}$$

◊ Call \widehat{f} the lift of f to \widehat{M} , and consider the test function $\xi\widehat{f}$ on \widehat{M} . From the assumption $\Delta_{\widehat{g}} + \widehat{V} \geq 0$, we have that

$$(a) \quad 0 \leq \int_{\widehat{M}} |d(\xi\widehat{f})|^2 + \widehat{V}\widehat{f}^2\xi^2.$$

◊ Let $c := \sharp(E)$ and $b := \sharp(\partial E)$. Then, $b \leq \epsilon c$. We have

$$|d(\widehat{f}\xi)|^2 \leq \widehat{f}^2|d\xi|^2 + 2|\widehat{f}\xi||d\widehat{f}||d\xi| + \xi^2|d\widehat{f}|^2,$$

and it follows from the above estimates on ξ that,

$$(b) \quad \int_{\widehat{M}} |d(\widehat{f}\xi)|^2 \leq \frac{b}{\alpha^2} \int_M f^2 + \frac{2b}{\alpha} \left(\int_M f^2 \right)^{1/2} \left(\int_M |df|^2 \right)^{1/2} + c \int_M |df|^2.$$

Consider the positive and negative parts of \widehat{V} , $\widehat{V} = \widehat{V}_+ - \widehat{V}_-$. Then,

$$(c) \quad \begin{aligned} \int_{\widehat{M}} \widehat{V}\widehat{f}^2\xi^2 &= \int_{\widehat{M}} \widehat{V}_+\widehat{f}^2\xi^2 - \int_{\widehat{M}} \widehat{V}_-\widehat{f}^2\xi^2 \\ &\leq c \int_M V_+ f^2 - (c-b) \int_M V_- f^2 \\ &\leq c \int_M V f^2 + b \int_M V_- f^2. \end{aligned}$$

The inequalities (a)–(c) yield,

$$0 \leq \int_M |df|^2 + Vf^2 + \frac{b}{c} \left[\frac{1}{\alpha^2} \int_M f^2 + \frac{2}{\alpha} \left(\int_M |df|^2 \int_M f^2 \right)^{1/2} + \int_M V_- f^2 \right].$$

Since $0 \leq b/c \leq \epsilon$, letting ϵ tend to zero, we find that $0 \leq \int_M |df|^2 + Vf^2$. \square

3.2. General coverings. We first make some preparation and then explain how to make the proof for normal coverings work in the general case.

\diamond Let $\rho : (\widehat{M}, \widehat{g}) \rightarrow (M, g)$ be a Riemannian covering. Let $(\widetilde{M}, \widetilde{g})$ be the common universal covering for $(\widehat{M}, \widehat{g})$ and (M, g) . Let $H = \pi_1(\widetilde{M})$, *resp.* $G = \pi_1(M)$, denote the fundamental groups of \widetilde{M} and M respectively. We make the assumption that H is a co-amenable subgroup of G .

\diamond Choose a reference point $x_0 \in \widetilde{M}$. Let \widehat{D} , *resp.* D , denote the closed Dirichlet fundamental domain for the action of H , *resp.* for the action of G , on \widetilde{M} , centered at x_0 . As $H < G$, we have $D \subset \widehat{D}$. Let β_1, \dots, β_n be the elements of G such that $\beta_i D \cap D \neq \emptyset$ in \widetilde{M} . These elements form a symmetric system of generators for G .

\diamond Let $X = Gx_0 \cap \widehat{D}$ denote the subset of points of the orbit of x_0 under the action of G which belong to \widehat{D} . For $x \in X$, let D_x denote the closed Dirichlet fundamental domain for the action of G on \widetilde{M} , centered at x . Then, $\widehat{D} = \cup_{x \in X} D_x$, and the interiors D_x° are pairwise disjoint.

\diamond Let $\widehat{\rho} : \widetilde{M} \rightarrow \widehat{M}$ be the universal covering of \widehat{M} . For $x \in X$, let $\Omega_x = \widehat{\rho}(D_x)$. Then, $\widehat{M} = \cup_{x \in X} \Omega_x$, and the interiors Ω_x° are pairwise disjoint. Since $\rho \circ \widehat{\rho}$ is the universal covering $\widetilde{M} \rightarrow M$, the map ρ is injective on each Ω_x° , and surjective from each Ω_x onto M .

\diamond At this point, we have tiled \widehat{M} by fundamental domains of the covering $\rho : \widehat{M} \rightarrow M$, indexed by the set X . We shall now define an amenable action of G on X . There is a natural map $\Phi : X \rightarrow H \backslash G$. Indeed, given $x \in X$, there is a unique $\gamma \in G$ such that $x = \gamma x_0$. Define $\Phi(x) = [\gamma]_R$. We claim that the map Φ is bijective. Let $\Psi : H \backslash G \rightarrow X$ be defined as follows. Take $H\gamma = [\gamma]_R \in H \backslash G$. Then the set $\{\sigma x_0 \mid \sigma \in [\gamma]_R\} = H(\gamma x_0)$ is the H -orbit of γx_0 . Since \widehat{D} is a fundamental domain for the action of H , there exists a unique $\sigma \in [\gamma]_R$ such that $\sigma x_0 \in \widehat{D}$. Let $\Psi([\gamma]_R) = \sigma x_0$. It is clear that $\Phi \circ \Psi = \text{Id}$ because $[\sigma]_R = [\gamma]_R$ in the above construction. On the other hand, $\Psi \circ \Phi = \text{Id}$ by unicity of σ . It follows that we can view the tiling of \widehat{M} by the sets Ω 's as indexed by $H \backslash G$, and we now use the G -action on $H \backslash G$.

\diamond We can now conclude the proof of Theorem 1.1. From the above construction, $\widehat{M} = \cup_{x \in H \backslash G} \Omega_x$, and we have the amenable G -action on $H \backslash G$.

\diamond By Proposition 2.6, given any $\epsilon > 0$, there exists a finite set $E \subset H \backslash G$, such that, $\sharp(\partial E) \leq \epsilon \sharp(E)$, where the boundary is defined with respect to the generators $\{\beta_1, \dots, \beta_n\}$ defined above.

\diamond We now define the set $\Omega = \cup_{x \in E} \Omega_x$ and we can reproduce the proof of the normal covering case. \square

3.3. Proof of Corollary 1.2. Since the groups \mathbb{Z} and \mathbb{Z}^2 are amenable [8], it suffices to look at the quotient, *i.e.* at the torus T^2 . Assume that $\Delta + aK \geq 0$ on T^2 , for some $a \neq 0$. As in [6], taking u to be the constant function $\mathbf{1}$, we find that $\int_{T^2} |d\mathbf{1}|^2 + aK\mathbf{1}^2 = 0$ because $\int_{T^2} K = 0$. Since

$\Delta + aK \geq 0$, the function $\mathbf{1}$ realizes the minimum of the Rayleigh quotient, so that it is an eigenfunction associated with the eigenvalue 0, and we have $(\Delta + aK)\mathbf{1} = 0$. This implies that $K \equiv 0$ since $a \neq 0$. As a consequence, if $I(g) \neq \{0\}$, then $K \equiv 0$ and hence $I(g) = \mathbb{R}$. If $K \not\equiv 0$, then $I(g) = \{0\}$. \square

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