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Pierre Bérard, Philippe Castillon

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# SPECTRAL POSITIVITY AND RIEMANNIAN COVERINGS

PIERRE BÉRARD AND PHILIPPE CASTILLON

ABSTRACT. Let  $(M, g)$  be a complete non-compact Riemannian manifold. We consider operators of the form  $\Delta_g + V$ , where  $\Delta_g$  is the non-negative Laplacian associated with the metric  $g$ , and  $V$  a locally integrable function. Let  $\rho : (\widehat{M}, \widehat{g}) \rightarrow (M, g)$  be a Riemannian covering, with Laplacian  $\Delta_{\widehat{g}}$  and potential  $\widehat{V} = V \circ \rho$ . If the operator  $\Delta + V$  is non-negative on  $(M, g)$ , then the operator  $\Delta_{\widehat{g}} + \widehat{V}$  is non-negative on  $(\widehat{M}, \widehat{g})$ . In this note, we show that the converse statement is true provided that  $\pi_1(\widehat{M})$  is a co-amenable subgroup of  $\pi_1(M)$ .

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## 1. INTRODUCTION

Let  $(M^n, g)$  be a complete, connected Riemannian manifold. Denote by  $\Delta$  the non-negative Laplacian, by  $\mu$  the measure associated with the metric  $g$ , and consider the Schrödinger operator  $J = \Delta + V$ , where  $V$  is a locally integrable function on  $M$ . Such operators appear naturally when one studies minimal (or constant mean curvature) immersions  $M^n \looparrowright N^{n+1}$ . Such hypersurfaces are critical points of a volume functional whose second derivative is given by the quadratic form associated with the Jacobi (stability) operator  $J = \Delta - \text{Ric}_N(\nu, \nu) - |A|^2$ , where  $\nu$  is a unit normal vector field along  $M$  and  $|A|$  the norm of the second fundamental form of the immersion. Stable hypersurfaces are those for which the Jacobi operator is non-negative, *i.e.* those critical points of the volume functional which are local minima up to second order.

When studying stable minimal hypersurfaces, a natural question is: “What conclusions on the Riemannian manifold  $(M, g)$  can one draw from the fact that the operator  $\Delta + V$  is non-negative in the sense of quadratic forms?” *i.e.* from the fact that the associated quadratic form is non-negative on Lipschitz functions with compact support in  $M$  (or equivalently on  $C^1$ -functions with compact support),

$$(\star) \quad 0 \leq \int_M (|df|^2 + Vf^2) d\mu \quad \forall f \in \text{Lip}_0(M).$$

When investigating the above question, it is often useful to pass to the universal cover of  $M$ . In this paper, we study the behaviour of non-negativity under Riemannian covering. More precisely, let  $\rho : (\widehat{M}, \widehat{g}) \rightarrow (M, g)$  be a

Riemannian covering, let  $V$  be a locally integrable function on  $M$ , and let  $\widehat{V} = V \circ \rho$ . It follows from [6] that  $\Delta_g + V \geq 0$  on  $(M, g)$  (in the sense of quadratic forms) implies that  $\Delta_{\widehat{g}} + \widehat{V} \geq 0$  on  $(\widehat{M}, \widehat{g})$ . It is a natural question to investigate the converse statement. In [9], Proposition 2.5, the authors prove that the converse statement is true provided that the inverse image  $\rho^{-1}(\Omega)$  of any relatively compact subdomain  $\Omega \subset M$  has sub-exponential volume growth.

Note that this statement is not true in general and needs some additional hypotheses on the covering, as the following simple example shows. Consider a group  $\Gamma$  acting on the hyperbolic plane with a compact quotient  $\mathbb{H}^2/\Gamma$ . It is well known that the operator  $\Delta - \frac{1}{4}$  is non-negative on  $\mathbb{H}^2$ , but it is not the case on  $\mathbb{H}^2/\Gamma$  which is a compact surface.

In [2, 3], R. Brooks worked on a closely related problem. Indeed, he investigated the behaviour of the infimum of the spectrum of the Laplacian under normal Riemannian coverings. A key assumption in his work is the amenability of the covering group. In this paper, we prove the following result which is very much inspired by [2, 3].

**Theorem 1.1.** *Let  $(M, g)$  be a complete Riemannian manifold of dimension  $n \geq 2$ . Let  $\rho : (\widehat{M}, \widehat{g}) \rightarrow (M, g)$  be a Riemannian covering. Let  $V$  be a locally integrable function on  $(M, g)$ , and let  $\widehat{V} = V \circ \rho$ . Assume the  $\pi_1(\widehat{M})$  is a co-amenable sub-group of  $\pi_1(M)$ . Then, the operator  $\Delta_g + V$  is non-negative on  $(M, g)$  if and only if the operator  $\Delta_{\widehat{g}} + \widehat{V}$  is non-negative on  $(\widehat{M}, \widehat{g})$ .*

As a particular case, when  $\rho : (\widehat{M}, \widehat{g}) \rightarrow (M, g)$  is a normal covering, with covering group  $G$ , the assumption on the fundamental groups of  $M$  and  $\widehat{M}$  is equivalent to the amenability of the group  $G$ .

The relationship between [9], Proposition 2.5 and Theorem 1.1 is that a finitely generated group with sub-exponential growth is amenable. However, there exist amenable groups with exponential volume growth, so that our hypothesis is weaker than the one in [9].

For a complete Riemannian manifold  $(M, g; V)$ , equipped with a locally integrable function  $V$ , consider the set

$$I(M, g; V) = \{a \in \mathbb{R} \mid \Delta + aV \geq 0\}.$$

This Riemannian invariant is a closed interval which contains 0, see [4].

For any Riemannian covering  $\rho : (\widehat{M}, \widehat{g}) \rightarrow (M, g)$ , we have  $I(M, g; V) \subset I(\widehat{M}, \widehat{g}; \widehat{V})$ . Theorem 1.1 tells us that if  $\pi_1(\widehat{M})$  is co-amenable in  $\pi_1(M)$ , then  $I(M, g; V) = I(\widehat{M}, \widehat{g}; \widehat{V})$ : the invariant  $I(M, g; V)$  does not distinguish two manifolds which differ by a covering with amenable action. In particular, we have the following corollary.

**Corollary 1.2.** *Let  $(M, g)$  be either  $\mathbb{R}^2$  with is a  $\mathbb{Z}^2$ -invariant metric, or  $S^1 \times \mathbb{R}$  with a  $\mathbb{Z}$ -invariant metric. Let  $K$  denote the Gaussian curvature of the metric  $g$ . Then,  $I(M, g; K) = \{0\}$  or  $I(M, g; K) = \mathbb{R}$ . Furthermore,  $I(M, g; K) = \mathbb{R}$  if and only if  $K \equiv 0$ .*

**Remark.** The Jacobi operator of an isometric minimal surface  $M^2 \looparrowright \mathbb{R}^3$  into Euclidean 3-space is  $\Delta + 2K$ , where  $K$  is the Gaussian curvature of  $M$ . More generally ([6], Section 3), the Jacobi operator of a minimal immersion  $M \looparrowright \widehat{M}^3$  into a 3-manifold with scalar curvature  $\widehat{S}$ , can be written as  $\Delta + K - (\widehat{S} + \frac{1}{2}|A|^2)$ . When  $\widehat{S}$  is non-negative, the stability of the surface implies that  $\Delta + K$  is non-negative. As a consequence, the study of stable minimal surfaces led to a general spectral problem on Riemannian surfaces, namely investigating the consequences of the non-negativity of operators of the form  $\Delta + aK$ , where  $K$  is the curvature of the surface, and  $a \in \mathbb{R}$  is a real parameter. As a matter of fact, the papers [6, 4, 5, 1] derive topological properties of  $M$  from assumptions on  $I(M, g; K)$  (where  $K$  is the Gaussian curvature of the metric  $g$ , in this 2-dimensional framework).

The paper is organized as follows. In Section 2, we recall basic facts on amenable groups and amenable group actions. For amenable groups, we refer to [8], and to [2], Section 1. For amenable group actions, we refer to [10, 7]. Section 3 contains the proofs.

The authors would like to thank Ivan Babenko for introducing them to amenable group actions.

## 2. AMENABLE GROUP ACTIONS

**2.1. Definitions.** Let  $X$  be a countable set, and let  $G$  be a discrete group acting on  $X$ .

**Definition 2.1.** A mean on  $X$  is a bounded linear functional  $\mu : \ell^\infty(X) \rightarrow \mathbb{R}$  such that,

$$\mu(1) = 1 \quad \text{and} \quad \mu(f) \geq 0, \quad \text{whenever } f \geq 0.$$

Equivalently, a mean on  $X$  is a map on subsets of  $X$ ,  $\mu : \mathcal{P}(X) \rightarrow [0, 1]$ , such that,

$$\begin{cases} \mu(X) = 1, & \text{and} \\ \mu(A \cup B) = \mu(A) + \mu(B) & \text{whenever } A \cap B = \emptyset, \end{cases}$$

i.e.  $\mu$  is a finitely additive probability on  $X$ .

**Definition 2.2.** A  $G$ -action on  $X$  is amenable if there exists a  $G$ -invariant mean on  $X$ , i.e. if there exists a mean  $\mu$  on  $X$  such that,

$$\mu(\gamma \cdot f) = \mu(f), \quad \forall \gamma \in G, \quad \forall f \in \ell^\infty(X),$$

where  $(\gamma \cdot f)(x) = f(\gamma^{-1}x)$ . The group  $G$  is amenable if its action on itself by left multiplications is amenable.

Any finite group  $G$  is amenable. It suffices to take  $\mu(f) = \frac{1}{\sharp(G)} \sum_{\gamma \in G} f(\gamma)$ , where  $\sharp(G)$  is the number of elements of  $G$ . The group  $\mathbb{Z}$  is amenable. To see this, consider the means  $\mu_n, n \in \mathbb{N}$ , defined by  $\mu_n(f) = \frac{1}{2n+1} \sum_{k=-n}^n f(k)$  and take their weak- $*$  limit. More generally, the group  $\mathbb{Z}^n$  is amenable for all  $n$ . The basic example of a non-amenable group is the free group on two generators  $\mathbb{F}_2$ , and it can be shown that any group containing a subgroup isomorphic to  $\mathbb{F}_2$  is not amenable.

**Remarks.**

(i) If the group  $G$  is amenable, then any action of  $G$  is amenable. Indeed, let  $\nu$  be an invariant mean on  $G$ , and let  $X$  be a set endowed with a  $G$ -action. Choose  $x_0 \in X$ . Given  $f \in \ell^\infty(X)$ , define  $\hat{f} \in \ell^\infty(G)$  by  $\hat{f}(\gamma) = f(\gamma \cdot x_0)$ . Then, the map  $\mu : \ell^\infty(X) \rightarrow \mathbb{R}$  defined by  $\mu(f) = \nu(\hat{f})$  is an invariant mean on  $X$  (we take the mean on a  $G$ -orbit).

(ii) Note that the converse statement is not true: there exist amenable actions by non-amenable groups. Indeed, van Douwen proved that any finitely generated non-abelian free group admits a faithful transitive amenable action ([7], Introduction).

**Definition 2.3.** *A subgroup  $H < G$  is co-amenable if the  $G$ -action on  $G/H$  is amenable.*

**Remark.** Note that a normal subgroup  $H \triangleleft G$  is co-amenable if and only if  $G/H$  is an amenable group.

When the subgroup  $H$  is not normal, one can consider left cosets  $[\sigma]_L = \sigma H \in G/H$ , and right cosets  $[\sigma]_R = H\sigma \in H \backslash G$ . The group  $G$  acts on both sets. The  $G$ -action on  $G/H$  is defined by  $\gamma \cdot [\sigma]_L = [\gamma\sigma]_L$ ; the  $G$ -action on  $H \backslash G$  is defined by  $\gamma \cdot [\sigma]_R = [\sigma\gamma^{-1}]_R$ . There is a natural bijective map between the coset spaces,

$$\begin{cases} F : G/H \rightarrow H \backslash G, \\ F : [\sigma]_L \mapsto [\sigma^{-1}]_R, \end{cases}$$

and this map is equivariant for the actions of  $G$ . It follows ([7], Lemma 2.1) that,

**Proposition 2.4.** *The  $G$ -action on  $G/H$  is amenable if and only if the  $G$ -action on  $H \backslash G$  is amenable. It follows that a subgroup  $H < G$  is co-amenable if any (and hence both) of the actions of  $G$  on the coset spaces is amenable.*

**Proof.** Since the map  $F$  is equivariant, the push-forward of a  $G$ -invariant mean on  $G/H$  provides an invariant mean on  $H \backslash G$  and conversely.  $\square$

**2.2. Følner's condition.** Følner's characterization of amenable groups ([8], Section 3.6) has an analog for amenable actions.

**Theorem 2.5** (Følner's condition). *A  $G$ -action on  $X$  is amenable if and only if,*

$$\begin{cases} \forall \eta \in (0, 1), \quad \forall \gamma_1, \dots, \gamma_n \in G, \quad \exists E \subset X \text{ such that,} \\ \#(E) < \infty \text{ and } \eta \#(E) \leq \#(E \cap \gamma_i \cdot E), \quad \forall 1 \leq i \leq n. \end{cases}$$

*Equivalently, a  $G$ -action on  $X$  is amenable if and only if,*

$$\begin{cases} \forall \epsilon > 0, \quad \forall \gamma_1, \dots, \gamma_n \in G, \quad \exists E \subset X \text{ such that,} \\ \#(E) < \infty \text{ and } \#(E \Delta \gamma_i \cdot E) \leq \epsilon \#(E), \quad \forall 1 \leq i \leq n. \end{cases}$$

*Here  $\#(E)$  denotes the number of elements of  $E$ , and  $A \Delta B$  the symmetric difference of two sets  $A, B$ .*

**Proof.** See [10], Section 4, Theorems 4.4 and 4.9.  $\square$

**Remarks.**

(1) When  $G$  is countable, the second condition is equivalent to saying that there exists a sequence  $\{E_k\}_{k \in \mathbb{N}}$  of finite subsets of  $X$  such that,

$$\lim_{k \rightarrow \infty} \frac{\#(E_k \triangle \gamma \cdot E_k)}{\#(E_k)} = 0, \quad \forall \gamma \in G.$$

This is the formulation in [10].

(2) As a consequence of this criterion, one can show ([2], Proposition 1) that groups with sub-exponential growth are amenable. This fact relates our Theorem 1.1 to Proposition 2.5 in [9].

For finitely generated groups, Følner's property has the following consequence. Let  $\alpha_1, \dots, \alpha_n$  be a symmetric system of generators for  $G$ . Define the Cayley graph of the  $G$ -action on  $X$  as follows,

- the vertices of the graph are the elements of  $X$ ,
- $[x, y]$  is an edge of the graph if and only if there exists some  $i \in \{1, \dots, n\}$ , such that  $x = \alpha_i \cdot y$ .

For a finite set  $E \subset X$ , define the *boundary*  $\partial E$  of  $E$  (in the Cayley graph) as,

$$\partial E = \{x \in E \mid \exists i \in \{1, \dots, n\}, \text{ such that } \alpha_i \cdot x \notin E\}.$$

**Proposition 2.6.** *If the  $G$ -action on  $X$  is amenable, then*

$$\forall \epsilon > 0, \exists E \subset X, \text{ finite, such that } \frac{\#(\partial E)}{\#(E)} \leq \epsilon.$$

**Proof.** Let  $E$  be a finite subset of  $X$ . Then

$$\partial E = \cup_{i=1}^n \{x \in E \mid \alpha_i \cdot x \notin E\}.$$

Then,

$$\begin{aligned} \#(\partial E) &\leq \sum_{i=1}^n \#\{x \in E \mid \alpha_i \cdot x \notin E\} \\ &\leq \sum_{i=1}^n \left( \#(E) - \#\{x \in E \mid \alpha_i \cdot x \in E\} \right) \\ &\leq \sum_{i=1}^n \left( \#(E) - \#(E \cap \alpha_i^{-1} \cdot E) \right). \end{aligned}$$

Choose  $\epsilon > 0$  and  $\eta \in (0, 1)$  such that  $n(1 - \eta) \leq \epsilon$ . Apply Følner's theorem: there exists a finite set  $E \subset X$  such that  $\eta \#(E) \leq \#(E \cap \alpha_i^{-1} \cdot E)$  for  $i = 1, \dots, n$ . Using the above inequalities, we obtain for this subset,

$$\#(\partial E) \leq \sum_{i=1}^n (\#(E) - \eta \#(E)) = n(1 - \eta) \#(E) \leq \epsilon \#(E).$$

This proves the proposition.  $\square$

**Remark.** A geometric interpretation of the previous proposition is that there is no linear isoperimetric inequality on the Cayley graph of an amenable action by a finitely generated group.

## 3. PROOFS

As we already mentioned in the Introduction, it follows from [6] that for any Riemannian covering  $\rho : (\widehat{M}, \widehat{g}) \rightarrow (M, g)$ , and any locally integrable function  $V$  on  $M$ , with  $\widehat{V} = V \circ \rho$ ,  $\Delta_g + V \geq 0$  on  $M$  implies that  $\Delta_{\widehat{g}} + \widehat{V} \geq 0$  on  $\widehat{M}$ . We shall therefore concentrate on the converse statement. We begin the proof by considering normal coverings and then explain how to handle the general case.

**3.1. Normal coverings.** We first assume that  $\pi_1(\widehat{M})$  is a normal subgroup of  $\pi_1(M)$ . Therefore,  $\rho : (\widehat{M}, \widehat{g}) \rightarrow (M, g)$  is a normal Riemannian covering, with amenable covering group  $G = \pi_1(M)/\pi_1(\widehat{M})$ . In view of the introduction to this section, we only need to prove that  $\Delta_{\widehat{g}} + \widehat{V} \geq 0$  implies that  $\Delta_g + V \geq 0$ .

Let  $f$  be a function in  $C_0^1(M)$ . We want to prove that  $0 \leq \int_M |df|^2 + Vf^2$ . The general idea is as follows. Lift  $f$  to  $\widehat{f}$  on  $\widehat{M}$ . This function is not compactly supported, but it behaves like  $f$  on fundamental domains. Multiply  $\widehat{f}$  by a cut-off function  $\xi$  which is 1 inside some  $\Omega$  to be chosen later on. By assumption,  $\int_{\widehat{M}} |d(\xi\widehat{f})|^2 + \widehat{V}\xi^2\widehat{f}^2$  is non-negative. We will conclude using Proposition 2.6 which tells us that the effect of the cut-off function is negligible.

◊ Fix some  $\epsilon > 0$ . Let  $F \subset \widehat{M}$  be a fundamental domain for the action of the covering group  $G$ , and let  $\beta_1, \dots, \beta_n$  be the elements of  $G$  such that  $\beta_i \cdot F \cap F \neq \emptyset$ . These elements generate  $G$ . As  $G$  is amenable, there exists a finite set  $E \subset G$  such that  $\sharp(\partial E) \leq \epsilon \sharp(E)$  (Proposition 2.6).

◊ Lift  $\text{supp}(f)$  to  $F$ . As it is compact, there exists some  $\alpha > 0$  such that the  $\alpha$ -neighborhood of  $\partial(F \cup \cup_{i=1}^n \beta_i \cdot F)$  does not intersect the lift of  $\text{supp}(f)$ . Let  $\Omega = \cup_{\gamma \in E} \gamma \cdot F$  and consider the cut-off function  $\xi : \widehat{M} \rightarrow \mathbb{R}$  defined by

$$\xi(x) = \begin{cases} 0, & \text{if } x \notin \Omega, \\ \frac{1}{\alpha}d(x, \partial\Omega), & \text{if } x \in \Omega \text{ and } d(x, \partial\Omega) < \alpha, \\ 1, & \text{if } x \in \Omega \text{ and } d(x, \partial\Omega) \geq \alpha. \end{cases}$$

The function  $\xi$  satisfies the inequalities,

$$\begin{cases} 0 \leq \xi \leq 1 & \text{on } \Omega, \\ |d\xi| \leq \frac{1}{\alpha} & \text{on } \gamma \cdot F, \text{ if } \gamma \in \partial E, \\ \xi = 1 \text{ and } |d\xi| = 0 & \text{on } \gamma \cdot F, \text{ if } \gamma \in E \setminus \partial E. \end{cases}$$

◊ Call  $\widehat{f}$  the lift of  $f$  to  $\widehat{M}$ , and consider the test function  $\xi\widehat{f}$  on  $\widehat{M}$ . From the assumption  $\Delta_{\widehat{g}} + \widehat{V} \geq 0$ , we have that

$$(a) \quad 0 \leq \int_{\widehat{M}} |d(\xi\widehat{f})|^2 + \widehat{V}\xi^2\widehat{f}^2.$$

◊ Let  $c := \sharp(E)$  and  $b := \sharp(\partial E)$ . Then,  $b \leq \epsilon c$ . We have

$$|d(\widehat{f}\xi)|^2 \leq \widehat{f}^2|d\xi|^2 + 2|\widehat{f}\xi||d\widehat{f}||d\xi| + \xi^2|d\widehat{f}|^2,$$

and it follows from the above estimates on  $\xi$  that,

$$(b) \quad \int_{\widehat{M}} |d(\widehat{f}\xi)|^2 \leq \frac{b}{\alpha^2} \int_M f^2 + \frac{2b}{\alpha} \left( \int_M f^2 \right)^{1/2} \left( \int_M |df|^2 \right)^{1/2} + c \int_M |df|^2.$$

Consider the positive and negative parts of  $\widehat{V}$ ,  $\widehat{V} = \widehat{V}_+ - \widehat{V}_-$ . Then,

$$(c) \quad \begin{aligned} \int_{\widehat{M}} \widehat{V} \widehat{f}^2 \xi^2 &= \int_{\widehat{M}} \widehat{V}_+ \widehat{f}^2 \xi^2 - \int_{\widehat{M}} \widehat{V}_- \widehat{f}^2 \xi^2 \\ &\leq c \int_M V_+ f^2 - (c-b) \int_M V_- f^2 \\ &\leq c \int_M V f^2 + b \int_M V_- f^2. \end{aligned}$$

The inequalities (a)–(c) yield,

$$0 \leq \int_M |df|^2 + V f^2 + \frac{b}{c} \left[ \frac{1}{\alpha^2} \int_M f^2 + \frac{2}{\alpha} \left( \int_M |df|^2 \int_M f^2 \right)^{1/2} + \int_M V_- f^2 \right].$$

Since  $0 \leq b/c \leq \epsilon$ , letting  $\epsilon$  tend to zero, we find that  $0 \leq \int_M |df|^2 + V f^2$ .  $\square$

**3.2. General coverings.** We first make some preparation and then explain how to make the proof for normal coverings work in the general case.

$\diamond$  Let  $\rho : (\widehat{M}, \widehat{g}) \rightarrow (M, g)$  be a Riemannian covering. Let  $(\widetilde{M}, \widetilde{g})$  be the common universal covering for  $(\widehat{M}, \widehat{g})$  and  $(M, g)$ . Let  $H = \pi_1(\widehat{M})$ , *resp.*  $G = \pi_1(M)$ , denote the fundamental groups of  $\widehat{M}$  and  $M$  respectively. We make the assumption that  $H$  is a co-amenable subgroup of  $G$ .

$\diamond$  Choose a reference point  $x_0 \in \widetilde{M}$ . Let  $\widehat{D}$ , *resp.*  $D$ , denote the closed Dirichlet fundamental domain for the action of  $H$ , *resp.* for the action of  $G$ , on  $\widetilde{M}$ , centered at  $x_0$ . As  $H < G$ , we have  $D \subset \widehat{D}$ . Let  $\beta_1, \dots, \beta_n$  be the elements of  $G$  such that  $\beta_i D \cap D \neq \emptyset$  in  $\widetilde{M}$ . These elements form a symmetric system of generators for  $G$ .

$\diamond$  Let  $X = Gx_0 \cap \widehat{D}$  denote the subset of points of the orbit of  $x_0$  under the action of  $G$  which belong to  $\widehat{D}$ . For  $x \in X$ , let  $D_x$  denote the closed Dirichlet fundamental domain for the action of  $G$  on  $\widetilde{M}$ , centered at  $x$ . Then,  $\widehat{D} = \cup_{x \in X} D_x$ , and the interiors  $D_x^\circ$  are pairwise disjoint.

$\diamond$  Let  $\widehat{\rho} : \widetilde{M} \rightarrow \widehat{M}$  be the universal covering of  $\widehat{M}$ . For  $x \in X$ , let  $\Omega_x = \widehat{\rho}(D_x)$ . Then,  $\widehat{M} = \cup_{x \in X} \Omega_x$ , and the interiors  $\Omega_x^\circ$  are pairwise disjoint. Since  $\rho \circ \widehat{\rho}$  is the universal covering  $\widetilde{M} \rightarrow M$ , the map  $\rho$  is injective on each  $\Omega_x^\circ$ , and surjective from each  $\Omega_x$  onto  $M$ .

$\diamond$  At this point, we have tiled  $\widehat{M}$  by fundamental domains of the covering  $\rho : \widehat{M} \rightarrow M$ , indexed by the set  $X$ . We shall now define an amenable action of  $G$  on  $X$ . There is a natural map  $\Phi : X \rightarrow H \backslash G$ . Indeed, given  $x \in X$ , there is a unique  $\gamma \in G$  such that  $x = \gamma x_0$ . Define  $\Phi(x) = [\gamma]_R$ . We claim that the map  $\Phi$  is bijective. Let  $\Psi : H \backslash G \rightarrow X$  be defined as follows. Take  $H\gamma = [\gamma]_R \in H \backslash G$ . Then the set  $\{\sigma x_0 \mid \sigma \in [\gamma]_R\} = H(\gamma x_0)$  is the  $H$ -orbit of  $\gamma x_0$ . Since  $\widehat{D}$  is a fundamental domain for the action of  $H$ , there exists a unique  $\sigma \in [\gamma]_R$  such that  $\sigma x_0 \in \widehat{D}$ . Let  $\Psi([\gamma]_R) = \sigma x_0$ . It is clear that  $\Phi \circ \Psi = \text{Id}$  because  $[\sigma]_R = [\gamma]_R$  in the above construction. On the other hand,  $\Psi \circ \Phi = \text{Id}$  by unicity of  $\sigma$ . It follows that we can view the tiling of  $\widehat{M}$  by the sets  $\Omega$ 's as indexed by  $H \backslash G$ , and we now use the  $G$ -action on  $H \backslash G$ .

$\diamond$  We can now conclude the proof of Theorem 1.1. From the above construction,  $\widehat{M} = \cup_{x \in H \backslash G} \Omega_x$ , and we have the amenable  $G$ -action on  $H \backslash G$ .



◊ By Proposition 2.6, given any  $\epsilon > 0$ , there exists a finite set  $E \subset H \backslash G$ , such that,  $\sharp(\partial E) \leq \epsilon \sharp(E)$ , where the boundary is defined with respect to the generators  $\{\beta_1, \dots, \beta_n\}$  defined above.

◊ We now define the set  $\Omega = \cup_{x \in E} \Omega_x$  and we can reproduce the proof of the normal covering case.  $\square$

**3.3. Proof of Corollary 1.2.** Since the groups  $\mathbb{Z}$  and  $\mathbb{Z}^2$  are amenable [8], it suffices to look at the quotient, *i.e.* at the torus  $T^2$ . Assume that  $\Delta + aK \geq 0$  on  $T^2$ , for some  $a \neq 0$ . As in [6], taking  $u$  to be the constant function  $\mathbf{1}$ , we find that  $\int_{T^2} |d\mathbf{1}|^2 + aK\mathbf{1}^2 = 0$  because  $\int_{T^2} K = 0$ . Since  $\Delta + aK \geq 0$ , the function  $\mathbf{1}$  realizes the minimum of the Rayleigh quotient, so that it is an eigenfunction associated with the eigenvalue 0, and we have  $(\Delta + aK)\mathbf{1} = 0$ . This implies that  $K \equiv 0$  since  $a \neq 0$ . As a consequence, if  $I(g) \neq \{0\}$ , then  $K \equiv 0$  and hence  $I(g) = \mathbb{R}$ . If  $K \neq 0$ , then  $I(g) = \{0\}$ .  $\square$

#### REFERENCES

- [1] Pierre Bérard and Philippe Castillon, *Inverse spectral positivity for surfaces*, arXiv:1111.5928 (2011).
- [2] Robert Brooks, *The fundamental group and the spectrum of the Laplacian*, Comment. Math. Helv. **56** (1981), 581-598.
- [3] Robert Brooks, *The bottom of the spectrum of a Riemannian covering*, Journal für die reine und angewandte Mathematik **357** (1985), 101-114.
- [4] Philippe Castillon, *An inverse spectral problem on surfaces*, Comment. Math. Helv. **81** (2006), 271-286.
- [5] José Espinar and Harold Rosenberg, *A Colding-Minicozzi inequality and its applications*, Trans. Amer. Math. Soc. **363** (2011), 2447-2465 [arXiv:0808.3011v5].
- [6] Doris Fischer-Colbrie and Richard Schoen, *The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature*, Comm. Pure Applied Math. **33** (1980), 199-211.
- [7] Yair Glasner and Nicolas Monod, *Amenable actions, free products and fixed point property*, Bull. London Math. Soc. **39** (2007), 138-150.
- [8] Frederick P. Greenleaf, *Invariant means on topological groups*. Van Nostrand Mathematical Studies  $\sharp$  16, 1969.
- [9] William H. Meeks, Joaquín Pérez and Antonio Ros, *Stable constant mean curvature surfaces*. In Handbook of Geometric Analysis, 1. Lizhen Ji, Peter Li, Richard Schoen and Leon Simon Editors. International Press, 2008.
- [10] Joseph M. Rosenblatt, *A generalization of Følner's condition*, Math. Scand. **33** (1973), 153-170.

Pierre Bérard  
 Université Grenoble 1  
 Institut Fourier (UJF-CNRS)  
 B.P. 74  
 38402 Saint Martin d'Hères Cedex  
 France  
 pierre.berard@ujf-grenoble.fr

Philippe Castillon  
 Université Montpellier II  
 Dépt des sciences mathématiques CC 51  
 I3M (UMR 5149)  
 34095 Montpellier Cedex 5  
 France  
 philippe.castillon@univ-montp2.fr