Primitives of p-adic meromorphic functions

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Dedicated to the memory of Nicole De Grande-De Kimpe

Abstract. Let K be an algebraically closed field of characteristic 0, complete with respect to an ultrametric absolute value. We show that a meromorphic function in K or in an open disk admits primitives if and only if all its residues are null.

1. Introduction and results

Notation and definitions: Let K be an algebraically closed field of characteristic 0, complete with respect to an ultrametric absolute value $| \ . \ |$. Given $\alpha \in K$ and $r \in \mathbb{R}_+^*$, we denote by $d(\alpha, r)$ the disk $\{x \in K \mid |x - \alpha| \leq r\}$ and by $d(\alpha, r^-)$ the disk $\{x \in K \mid |x - \alpha| < r\}$, by $\mathcal{A}(K)$ the K-algebra of analytic functions in K (i.e. the set of power series with an infinite radius of convergence) and by $\mathcal{M}(K)$ the field of meromorphic functions in K [2],[4].

In the same way, given $\alpha \in K, \ r > 0$ we denote by $\mathcal{A}(d(\alpha, r^-))$ the K-algebra of analytic functions in $d(\alpha, r^-)$ (i.e. the set of power series with an infinite radius of convergence $\geq r$) and by $\mathcal{M}(d(\alpha, r^-))$ the field of fractions of $\mathcal{A}(d(\alpha, r^-))$.

Here we mean to characterize meromorphic functions in K or inside an open disk that admit primitives. In complex analysis, a meromorphic function having all its residues equal to 0 admits primitives. On our field K, such a conclusion seems obvious but apparently has not been published in any paper. Actually, the proof requires the use of analytic elements [1], [3].

Notation and definitions: Let $f \in \mathcal{M}(d(a, R^-))$ have a pole of order q at a: f then admits an expansion in a Laurent series inside a disk $d(anr^-)$ of the form $f(x) = \sum_{n=-q}^{\infty} a_n (x - a)^n$. As usual, a_{-1} is called *residue of f at a* and will be denoted by $\mathrm{res}_a(f)$.

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Main Theorem : *Let $f \in \mathcal{M}(K)$ (resp. $f \in \mathcal{M}(d(a, R^-))$). Then f admits primitives in $\mathcal{M}(K)$ (resp. in $\mathcal{M}(d(a, R^-))$) if and only if f has no residue different from zero.*

2. The proofs

Let D be a subset of K. If D is bounded of diameter R we denote by $\widetilde{D}$ the disk $d(a, R)$ for any $a \in D$. If D is not bounded we put $\widetilde{D} = K$.
Given a point $a \in K$ we put $\delta(a, D) = \inf\{|x - a| \; |x \in D\}$. Then $\delta(a, D)$ is named *the distance of a to D*.

Recall that given any subset D of K, $\widetilde{D} \setminus \overline{D}$ *admits a unique partition of the form $(T_i)_{i\in I}$, whereas each T_i is a disk of the form $d(a_i, r_i^-)$ with $r_i = \delta(a_i, D)$.* Such disks $d(a_i, r_i^-)$ are called *the holes* of D [1].

Given $a \in K$ and r', $r'' \in \mathbb{R}_+^*$ with $r' < r''$, we denote by $\Gamma(a, r', r'')$ the annulus $\{x \in K \mid r' < |x - a| < r''\}$.

A set D is said to be *infraconnected* if for every $a \in D$, the mapping I_a from D to $\mathbb{R}_+$ defined by $I_a(x) = |x - a|$ has an image whose closure in $\mathbb{R}_+$ is an interval. In other words, a set D is not infraconnected if and only if there exist a and $b \in D$ and an annulus $\Gamma(a, r_1, r_2)$ with $0 < r_1 < r_2 < |a-b|$ such that $\Gamma(a, r_1, r_2) \cap D = \emptyset$.

Given a subset E of K, we denote by $R(E)$ the algebra of rational functions with no pole in E and the functions that are uniform limits of a sequence of $R(E)$ are called *analytic elements on E*. We denote by $H(E)$ the set of analytic elements on E. Moreover, we denote by $R_b(E)$ the sub-K-algebra of $R(E)$ consisting of the $h \in R(E)$ that are bounded in E. So, $R_b(E)$ admits the norm of uniform convergence as a K-algebra norm and we denote by $H_b(E)$ its completion that consists of all $h \in H(E)$ that are bounded in E. The algebra $H_b(E)$ is then a K-Banach algebra. Moreover, when E is unbounded, we denote by $H_0(E)$ the set of $f \in H(E)$ such that $\displaystyle\lim_{|x|\to+\infty, x\in E} f(x) = 0$. Particularly, when E is of the form $K \setminus d(a, R^-)$, then $H_0(E)$ is K-Banach algebra with respect to the norm of uniform convergence on E denoted by $\| \, . \, \|_E$.

Lemma 1: *Let $f \in H(d(0, R))$ and let $r \in]0, R[$. Then f admits primitives in $H(d(0, r))$.*

Proof: Since f lies in $H(d(0, R))$, $f(x)$ has expansion of the form $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ with $\displaystyle\lim_{n\to\infty} |a_n| R^n = 0$. Particularly, the radius of convergence

$$\rho = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$$

is such that $\rho \geq R$. But by Theorem 1.5.4 [2] the primitive $F(x) = \displaystyle\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ also has a radius of convergence equal to S. Consequently, for all $r < R$, by Proposition 13.3 [1], F belongs to $H(d(0, r))$.

In the proof of the Main Theorem, we need Krasner's Mittag-Leffler Theorem. Here for simplicity, we will only state it when the set D is closed and bounded.

Theorem A : (Krasner-Mittag-Leffler Theorem) [1], [3] *Let D be a closed and bounded infraconnected subset of K and let $f \in H(D)$. There exists a unique sequence of holes $(T_n)_{n\in \mathbb{N}^*}$ of D and a unique sequence $(f_n)_{n\in\mathbb{N}}$ in $H(D)$ such that $f_0 \in H(\widetilde{D})$, $f_n \in H_0(K \setminus T_n)$ $(n > 0)$, $\lim_{n\to\infty} f_n = 0$ and*

(1) $\quad f = \sum_{n=0}^{\infty} f_n$ *and* $\|f\|_D = \sup_{n\in\mathbb{N}} \|f_n\|_D$.

Moreover for every hole $T_n = d(a_n, r_n^-)$, we have

(2) $\quad \|f_n\|_D = \|f_n\|_{K\setminus T_n} \leq \|f\|_D$.

If $\widetilde{D} = d(a, r)$ we have

(3) $\quad \|f_0\|_D = \|f_0\|_{\widetilde{D}} \leq \|f\|_D$.

Let us recall Theorems B and C [1].

Theorem B (Theorem 19.5 [1]): *Let E be an open set in K such that $\overline{E}$ also is open. Then E is infraconnected if and only if for every $f \in H(E)$ such that $f'(x) = 0$ whenever $x \in E$, we have $f = ct$.*

Theorem C (Theorem 25.5 [1]): *Let $(a_j)_{j\in\mathbb{N}}$ be a sequence in $d(0, r^-)$ such that $|a_n| \leq |a_{n+1}|$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} |a_n| = r$. Let $(q_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}^*$ and let $B \in]1, +\infty[$. There exists $f \in A(d(0, r^-))$ satisfying*

i) $f(0) = 1$

ii) $\|f\|_{d(0,|a_n|)} \leq B \prod_{j=0}^{n} |\frac{a_n}{a_j}|^{q_j}$ *whenever* $n \in \mathbb{N}$

iii) *for each* $n \in \mathbb{N}$, a_n *is a zero of* f *of order* $z_n \geq q_n$.

Proof of the Main Theorem: Let α be a pole of f. According to the Laurent series of f at α, if f admits primitives then f has no residue different from zero at α because $\dfrac{1}{x-\alpha}$ has no primitive in $H(d(\alpha, r))$ (whenever $r > 0$).

Now suppose that for every pole α of f we have $res_\alpha(f) = 0$. Without loss of generality, we may assume that f admits infinitely many poles and has no pole at the origin. Moreover, when f is a meromorphic function in $d(a, R^-)$, we may assume that $a = 0$.

Let $(a_n)_{n\in\mathbb{N}}$ be the sequence of poles of f in K (resp. in $d(0, R^-)$), with $|a_n| \leq |a_{n+1}|$, $\forall n \in \mathbb{N}$, each pole a_n being of order q_n and let $\psi(x) = \prod_{n=0}^{\infty} (1 - \dfrac{x}{a_n})^{q_n - 1}$. For each $S > 0$ (resp. for each $S \in]0, R[$), we denote by $t(S)$ the greatest integer n such that $|a_n| \leq S$. And for each $r \in]0, S[$, we set $D(r, S) = d(0, S^-) \setminus (\bigcup_{j=0}^{t(S)} d(a_j, r^-))$. According to results on analytic elements [1], [2], f obviously defines an element of $H(D(r, S))$.

Let us take $r < \min(|a_i - a_j|,\ 0 \leq i < j \leq t(S))$. In each hole $d(a_n, r^-)$ of $D(r, S)$, f has a unique pole that is a_n. It is of order q_n and the Mittag-Leffler term f_n of f in the hole $d(a_n, r^-)$ is of the form $\sum_{i=2}^{q_n} \dfrac{\lambda_{n,i}}{(x - a_n)^i}$ because, by hypothesis, the residue of f at a_n is null. On the other hand, as an element of $H(D(r, S))$, by Theorem A, the Mittag-Lefler decomposition of f on the infraconnected set $D(r,S)$ is of the form $\sum_{n=0}^{t(S)} f_n + g_S$ with $g_S \in H(d(0,S))$. Moreover, we notice that there exists $S' > S$ (resp $S' \in]S,R[$) such that f admits no pole $b \in \Gamma(0,S,S')$. Consequently, the holes of $D(r,S)$ are the same as these of $D(r,S')$ and therefore, the Mittag-Leffler decomposition of f on $D(r,S)$ also holds on $D(r,S')$. Hence g_S belongs to $H(d(0,S'))$. But then by Lemma 1, g_S admits primitives in $H(d(0,S))$.

Now, as an element of $H(D(r,S))$, each f_n admits a primitive F_n of the form

$$-\sum_{i=2}^{q_n} \frac{\lambda_{n,i}}{(i-1)(x-a_n)^{i-1}}.$$

And as an element of $H(D(r,S))$ again, f admits a primitive L_S satisfying $L_S(0) = 0$, equal to $\sum_{n=0}^{t(S))} F_n + G_S$ with $G_S \in H(d(0,S))$.

Now, fixing S we can take r arbitrary small: the Mittag-Leffler terms $f_1, ..., f_{t(S)}$ and G_S remain the same because if $r' < r < \min(|a_i - a_j|\ 0 \le i < j \le t(S))$, the Mittag-Leffler expansions on $D(r',S)$ holds on $D(r,S)$ and the two expansions are equal, due to the properties of the Mittag-Leffler expansions. Thus, the function L_S is defined in $d(0,S) \setminus \{a_1, ..., a_{t(S)}\}$ and satisfies $(L_S)'(x) = f(x),\ \forall x \in d(0,S) \setminus \{a_1, ..., a_{t(S)}\}$.

Consider first the case when f belongs to $\mathcal{M}(K)$. The function $\psi(x) = \prod_{n=0}^{\infty}(1 - \frac{x}{a_n})^{q_n - 1}$ is an entire function, hence an element of $H(d(0,S))$. Let $\phi_S(x) = L_S(x)\psi(x)$. Then for any $r > 0$, ϕ_S is an element of $H(D(r,S))$ meromorphic on each hole of $D(r,S)$ [1], (chap. 31). But actually, by construction, ϕ_S has no pole in any hole of $D(r,S)$. Consequently, ϕ belongs to $H(d(0,S))$.

Now, let $S' > S$. Similarly we can make a function $\phi_{S'} = L_{S'}\psi$. We will show that the restriction of $\phi_{S'}$ to $d(0,S)$ is ϕ_S. Indeed, by definition, both functions L_S and $L_{S'}$ are null at 0 and have a derivative equal to f in $D(r,S)$, for any $r > 0$. Consequently, L_S and $L_{S'}$ coincide in a disk and therefore so do ϕ_S and $\phi_{S'}$. Hence the equality $\phi_S(x) = \phi_{S'}(x)$ holds in all $d(0,S)$. Thus, we can define the function $\phi(x) = \phi_S(x),\ \forall x \in d(0,S)$. Since ϕ_S belongs to $H(d(0,S))$, ϕ belongs to $H(d(0,S))$ for all $S > 0$ and therefore is an entire function. Now, we can set $F(x) = \dfrac{\phi(x)}{\psi(x)}$ and then F belongs to $\mathcal{M}(K)$. On the other hand, since $(L_S)' = f$ and since $F(x) = L_S(x)$ in $d(0,S) \setminus \{a_1, ..., a_n, ...\}$, we have $F'(x) = f(x),\ \forall d(0,S) \setminus \{a_1, ..., a_n, ...\}$ and hence, by Theorem B we know that the equality $F'(x) = f(x)$ holds in each set $D(r,S)$, for all $r > 0$, hence in all $K \setminus \{a_1, ..., a_n, ...\}$.

Similarly, consider the case when f belongs to $\mathcal{M}(d(0,R^-))$. By Theorem C, we can find a function $\psi(x) \in \mathcal{A}(d(0,R^-))$ admitting each a_n as a zero of order $s_n \ge q_n - 1$. Then, as in the previous case, using the same notation, we can show that $L_S(x)\psi(x)$ lies in $H(d(0,S))$ for every $S < R$, because it has no pole in $d(0,S)$. Now, for every $S < S' < R$, ϕ_S is the restriction of $\phi_{S'}$ to $d(0,S)$. Let ϕ be defined as $\phi(x) = \phi_S(x)$ for all $x \in d(0,S),\ \forall S < R$. Then ϕ belongs to $\mathcal{A}(d(0,R^-))$ and hence the function $F(x) = \dfrac{\phi(x)}{\psi(x)}$ belongs to $\mathcal{M}(d(0, R^-))$. And similarly, we have $F' = f$, which ends the proof.

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