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# A REMARK ON THE ANISOTROPIC OUTER MINKOWSKI CONTENT

ANTONIN CHAMBOLLE, STEFANO LISINI, AND LUCA LUSSARDI

ABSTRACT. We study an anisotropic version of the outer Minkowski content of a closed set in  $\mathbb{R}^n$ . In particular, we show that it exists on the same class of sets for which the classical outer Minkowski content coincides with the Hausdorff measure, and we give its explicit form.

Keywords: outer Minkowski content, finite perimeter sets

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## 1. INTRODUCTION

As it is well known, the classical *Minkowski content* of a closed set  $S \subset \mathbb{R}^n$  is defined by

$$(1.1) \quad \mathcal{M}(S) := \lim_{\varepsilon \rightarrow 0^+} \frac{|\{x \in \mathbb{R}^n : \text{dist}(x, S) \leq \varepsilon\}|}{2\varepsilon}$$

whenever the limit in (1.1) exists and is finite; here  $|\cdot|$  denotes the Lebesgue measure in  $\mathbb{R}^n$ . The quantity  $\mathcal{M}$  measures the area of “ $(n-1)$ -dimensional sets”, and it is an alternative to the more classical Hausdorff measure  $\mathcal{H}^{n-1}$ . With the role of surface measure, the Minkowski content turns out to be important in many problems arising from real applications: for instance  $\mathcal{M}$  is related to evolution problems for closed sets [1, 10, 13].

Clearly, it poses as natural problems its existence and comparison with  $\mathcal{H}^{n-1}$ . Let us mention some known results in this direction. In [9, p. 275] the author proves that  $\mathcal{M}(S)$  exists and coincides with  $\mathcal{H}^{n-1}(S)$  whenever  $S$  is compact and  $(n-1)$ -*rectifiable*, i.e.  $S = f(K)$  for some  $K \subset \mathbb{R}^{n-1}$  compact and  $f: \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$  Lipschitz. A generalization of this result is contained in [4, p. 110]. Here, the authors consider *countable  $\mathcal{H}^{n-1}$ -rectifiable* compact sets in  $\mathbb{R}^n$ , i.e. sets which can be covered by a countable family of sets  $S_j$ , with  $j \in \mathbb{N}$ , such that  $S_0$  is  $\mathcal{H}^{n-1}$ -negligible and  $S_j$  is a  $(n-1)$ -dimensional surface in  $\mathbb{R}^n$  of class  $C^1$ , for any  $j > 0$ . In this case,  $\mathcal{M}(S)$  exists and coincides with  $\mathcal{H}^{n-1}(S)$  if a further density assumption on  $S$  holds: more precisely there must exist  $\gamma > 0$  and  $\eta$  a probability measure on  $\mathbb{R}^n$  satisfying  $\eta(B(x, r)) \geq \gamma r^n$ , for each  $r \in (0, 1)$  and for each  $x \in S$ , where  $B(x, r)$  is the open ball centered in  $x$  of radius  $r$ . Counterexamples [4, p. 109] show that the countable rectifiability is indeed not sufficient to ensure the existence of  $\mathcal{M}$ .

More recently [2], motivated by problems in stochastic geometry, a generalization of the Minkowski content has been introduced, the so-called *outer Minkowski content*  $\mathcal{SM}$ , which is

defined by

$$(1.2) \quad \mathcal{SM}(E) := \lim_{\varepsilon \rightarrow 0^+} \frac{|\{x \in \mathbb{R}^n : \text{dist}(x, E) \leq \varepsilon\} \setminus E|}{\varepsilon}, \quad E \subset \mathbb{R}^n \text{ compact.}$$

In [2] the authors investigate general conditions ensuring the existence of  $\mathcal{SM}$ : in particular, they prove that if  $E$  is a set with finite perimeter and  $\mathcal{M}(\partial E)$  exists and coincides with the perimeter of  $E$ , then also  $\mathcal{SM}(E)$  exists and coincides with the perimeter of  $E$  (in  $\Omega$ ).

Now, notice that the quantity which appears in the argument of the limit in (1.2) can be rewritten as (provided the set  $E$  is “nice” enough)

$$\frac{1}{\varepsilon} (|E + \varepsilon B(0, 1)| - |E|).$$

We consider in this short note a variant of this content, which is an anisotropic outer Minkowski content. The idea is to study the limit, as  $\varepsilon \rightarrow 0^+$ , of

$$(1.3) \quad \frac{1}{\varepsilon} (|E + \varepsilon C| - |E|),$$

where  $C \subset \mathbb{R}^n$  is a closed convex body. It is standard that if  $E$  is convex, then  $|E + \varepsilon C|$  is a polynomial in  $\varepsilon$  (of degree  $n$ ) whose coefficient of the first degree term (see also Remark 3.6 below) is precisely the anisotropic perimeter

$$(1.4) \quad \int_{\partial E} h_C(\nu_E) d\mathcal{H}^{n-1},$$

where  $h_C$  is the support function of  $C$ , defined by  $h_C(\nu) = \sup_{x \in C} x \cdot \nu$ , and  $\nu_E$  the outer normal to  $\partial E$ , see [11] for details. The convergence of (1.3) to (1.4) follows for convex sets  $E$  and can be easily extended to (very) smooth sets.

We show here two (expected) results: first, as a functional defined on sets, (1.3)  $\Gamma$ -converges to the natural limit (1.4) as  $\varepsilon \rightarrow 0$ .

Second, we show in Theorem 3.4 that given any set for which the (classical) outer Minkowski content equals the perimeter, then the limit of (1.3), as  $\varepsilon \rightarrow 0^+$ , coincides with (1.4).

The proof of Theorem 3.4 is quite technical, because we wanted to work under the only assumption of the convergence of the classical content. We show that this convergence implies that the boundary is flat enough in a relatively uniform way, so that the convergence of (1.3) holds. It would be easy to adapt our proof to get, in Theorem 3.4, an “if and only if”, under the additional assumption, though, that  $C$  is elliptic and regular in some appropriate sense, we leave this to the reader.

Eventually, we also deduce a  $\Gamma$ -convergence result (see [5, 7] for details) for functionals of the type

$$(1.5) \quad \frac{1}{\varepsilon} \int (\text{ess sup}_{x \in \varepsilon C} u - u(x)) dx$$

which coincides with (1.3) on characteristic functions of sets. The limit is (quite obviously) given by  $\int_{\Omega} h_C(-Du)$  (where the minus signs accounts for the fact that the outer normal was

appearing in (1.4), and not the inner normal which corresponds more naturally to the gradient of the characteristic function  $\chi_E$ )

As a simple corollary, one, if wants, is able to approximate functionals of the type

$$\int_{\partial E} \phi(\nu_E) d\mathcal{H}^{n-1},$$

for  $\phi$  a positively one-homogenous convex function  $\phi : \mathbb{R}^n \rightarrow [0, +\infty)$  (and positive away from 0). Indeed, it suffices to choose the convex body

$$C := \{x \in \mathbb{R}^n : x \cdot \nu \leq \phi(\nu) \forall \nu\}$$

and apply our results.

The paper is organized as follows: in section 3 we define the setting and we state the results, then in section 4 we prove the  $\Gamma$ -convergence result for (1.5), and then the pointwise convergence result for (1.3).

## 2. NOTATION AND PRELIMINARIES

**2.1. Notation.** Let  $n \geq 1$  be integer. Given a measurable set  $A \subset \mathbb{R}^n$ , we will denote by  $|A|$  its Lebesgue measure. If  $k \in \{0, \dots, n\}$ , the  $k$ -dimensional Hausdorff measure of  $S \subset \mathbb{R}^n$  will be denoted by  $\mathcal{H}^k(S)$ . We will use the notation  $x \cdot y$  for the standard scalar product in  $\mathbb{R}^n$  between  $x$  and  $y$ ,  $B(x, r)$  for the open ball of radius  $r$  centered in  $x$ . Finally, here convergence in  $L^1_{\text{loc}}(\Omega)$  means convergence in  $L^1(B \cap \Omega)$  for any ball  $B$ .

We say that a sequence of sets  $E_j \subset \mathbb{R}^n$  converges to  $E \subset \mathbb{R}^n$  in  $L^1_{\text{loc}}(\Omega)$  as  $j \rightarrow +\infty$ , if  $\chi_{E_j}$  converges to  $\chi_E$  in  $L^1_{\text{loc}}(\Omega)$  as  $j \rightarrow +\infty$ , where  $\chi_S$  denotes the characteristic function of the set  $S$ .

**2.2. Geometric measure theory.** In this paragraph we recall with some basic notions of geometric measure theory we will need; for an exhaustive treatment of the subject we refer the reader to [12].

Let  $n \geq 1$  be integer and let  $k \in \mathbb{N}$  with  $k \leq n$ . We say that  $S \subset \mathbb{R}^n$  is  $\mathcal{H}^k$ -rectifiable if  $S$  can be covered by a countable family of sets  $S_j$ , with  $j \in \mathbb{N}$ , such that  $S_0$  is  $\mathcal{H}^k$ -negligible and  $S_j$  is a  $k$ -dimensional surface in  $\mathbb{R}^n$  of class  $C^1$ , for any  $j > 0$ .

Let  $E \subset \mathbb{R}^n$  be a measurable set and  $\Omega \subset \mathbb{R}^n$  be an open domain. We say that  $E$  has *finite perimeter in  $\Omega$*  if the distributional derivative of  $\chi_E$  is a real Radon measure on  $\Omega$ ; we will denote  $\text{Per}(E; \Omega) := |D\chi_E|(\Omega)$ . The *upper* and *lower  $n$ -dimensional densities of  $E$  at  $x$*  are respectively defined by

$$\Theta_n^*(E, x) := \limsup_{r \searrow 0} \frac{|E \cap B(x, r)|}{\alpha_n r^n}, \quad \Theta_{*n}(E, x) := \liminf_{r \searrow 0} \frac{|E \cap B(x, r)|}{\alpha_n r^n},$$

where  $\alpha_n$  is the volume of the  $(n - 1)$ -dimensional unit disc (the diameter of the unit ball). If  $\Theta_n^*(E, x) = \Theta_{*n}(E, x)$  their common value is denoted by  $\Theta_n(E, x)$ . For every  $t \in [0, 1]$  we

define

$$(2.1) \quad E^t := \{x \in \mathbb{R}^n : \Theta_n(E, x) = t\}.$$

The *essential boundary* of  $E$  is the set  $\partial_* E := \mathbb{R}^n \setminus (E^0 \cup E^1)$ . It turns out that if  $E$  has finite perimeter in  $\Omega$ , then  $\mathcal{H}^{n-1}(\partial_* E \setminus E^{1/2}) = 0$ , and  $\text{Per}(E; \Omega) = \mathcal{H}^{n-1}(\partial_* E \cap \Omega)$ .

Moreover, one can define a subset of  $E^{1/2}$  as the set of points  $x$  where there exists a unit vector  $\nu_E(x)$  such that:

$$\frac{E - x}{\rho} \rightarrow \{y \in \mathbb{R}^n : y \cdot \nu_E(x) \leq 0\}, \quad \text{as } \rho \rightarrow 0 \text{ in } L^1_{\text{loc}}(\mathbb{R}^n),$$

and which is referred to as the *outer normal to  $E$  at  $x$* . The set where  $\nu_E(x)$  exists is called the *reduced boundary* and is denoted by  $\partial^* E$ . One can show that  $\mathcal{H}^{n-1}(\partial_* E \setminus \partial^* E) = 0$ , moreover, one has the decomposition  $D\chi_E = (-\nu_E)\mathcal{H}^{n-1} \llcorner \partial^* E$ .

### 3. STATEMENT OF THE RESULTS

Let us assume that  $C$  is a closed convex body, that is, bounded and with 0 in its interior. We denote its support function by  $h_C(\nu) = \sup_{x \in C} x \cdot \nu$ , and its polar function is  $h_C^\circ(x) := \sup_{h_C(\nu) \leq 1} x \cdot \nu$ . It is well known, then, that both  $h_C$  and  $h_C^\circ$  are convex, one-homogeneous and Lipschitz functions, moreover  $C = \{h_C^\circ \leq 1\}$ .

By assumptions, there also exists  $a, b$  with  $0 < a < b$  such that  $B(0, a) \subseteq C \subseteq B(0, b)$ , in particular, we have for all  $\nu, x \in \mathbb{R}^n$

$$(3.1) \quad a|\nu| \leq h_C(\nu) \leq b|\nu|, \quad \frac{1}{b}|x| \leq h_C^\circ(x) \leq \frac{1}{a}|x|.$$

Let  $\Omega \subset \mathbb{R}^n$  be an open domain. Given a Lebesgue measurable set  $E \subset \Omega$ , we introduce the outer  $\varepsilon, C$ -Minkowski content,

$$(3.2) \quad \mathcal{SM}_{\varepsilon, C}^0(E; \Omega) := \frac{1}{\varepsilon} (|\Omega \cap (E + \varepsilon C)| - |E|).$$

Actually, this definition is not very practical, since it can change drastically with Lebesgue-negligible changes of the set  $E$ . For this reason, we introduce the functional, defined for a measurable function  $u$ :

$$(3.3) \quad F_{\varepsilon, C}(u; \Omega) := \frac{1}{\varepsilon} \int_{\Omega} \left( \text{ess sup}_{\Omega \cap (x - \varepsilon C)} u - u(x) \right) dx$$

which takes values in  $[0, +\infty]$ . Notice that one can check easily (using Fatou's lemma) that  $F_{\varepsilon, C}$  is l.s.c. in  $L^1_{\text{loc}}(\Omega)$ . We then define

$$(3.4) \quad \mathcal{SM}_{\varepsilon, C}(E; \Omega) := F_{\varepsilon, C}(\chi_E; \Omega).$$

It is also easy to check that the definition coincides with  $\mathcal{SM}_{\varepsilon, C}^0$  on smooth sets, and in general for a measurable set  $E$  we have

$$\mathcal{SM}_{\varepsilon, C}(E; \Omega) = \min_{|E' \Delta E|=0} \mathcal{SM}_{\varepsilon, C}^0(E'; \Omega) = \mathcal{SM}_{\varepsilon, C}^0(E^1; \Omega) = \mathcal{SM}_{\varepsilon, C}^0(\Omega \setminus E^0; \Omega)$$

where  $E^1$  (resp.,  $E^0$ ) is the set of points of Lebesgue density 1 (resp., 0) in  $\Omega$  (see (2.1)), and  $\Delta$  denotes the symmetric difference. Eventually, one can check easily that  $F_\varepsilon$  satisfies a *generalized coarea formula* [14, 6]: for any function  $u \in L^1_{\text{loc}}(\Omega)$ ,

$$(3.5) \quad F_{\varepsilon,C}(u; \Omega) = \int_{-\infty}^{\infty} \mathcal{SM}_{\varepsilon,C}(\{u > s\}; \Omega) ds.$$

Moreover, one can show that for any measurable sets,  $E$  and  $F$ ,

$$\mathcal{SM}_{\varepsilon,C}(E \cup F; \Omega) + \mathcal{SM}_{\varepsilon,C}(E \cap F; \Omega) \leq \mathcal{SM}_{\varepsilon,C}(E; \Omega) + \mathcal{SM}_{\varepsilon,C}(F; \Omega)$$

from which it follows that  $F_\varepsilon$  is convex on  $L^1_{\text{loc}}(\Omega)$ , see [6] for details.

Before to state our results we say that a family of functionals  $G_\varepsilon$  defined on the Lebesgue measurable subsets of  $\mathbb{R}^n$   $\Gamma$ -converges to  $G$  in  $L^1_{\text{loc}}(\Omega)$  as  $\varepsilon \rightarrow 0$  if

- for any Lebesgue measurable set  $E$ , and for any family of Lebesgue measurable sets  $E_\varepsilon \rightarrow E$  in  $L^1_{\text{loc}}(\Omega)$  as  $\varepsilon \rightarrow 0$ , we have

$$G(E) \leq \liminf_{\varepsilon \rightarrow 0} G_\varepsilon(E_\varepsilon),$$

- for any Lebesgue measurable set  $E$ , there exist a family of Lebesgue measurable sets  $E_\varepsilon$  such that  $E_\varepsilon \rightarrow E$  in  $L^1_{\text{loc}}(\Omega)$  as  $\varepsilon \rightarrow 0$  and

$$\limsup_{\varepsilon \rightarrow 0} G_\varepsilon(E_\varepsilon) \leq G(E).$$

In the same way, we say that a family of functionals  $F_\varepsilon$  defined on  $L^1_{\text{loc}}(\Omega)$   $\Gamma$ -converges to  $F$  in  $L^1_{\text{loc}}(\Omega)$  as  $\varepsilon \rightarrow 0$  if

- for any  $u \in L^1_{\text{loc}}(\Omega)$ , and for any family of elements of  $L^1_{\text{loc}}(\Omega)$  such that  $u_\varepsilon \rightarrow u$  in  $L^1_{\text{loc}}(\Omega)$  as  $\varepsilon \rightarrow 0$ , we have

$$F(u) \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon),$$

- for any  $u \in L^1_{\text{loc}}(\Omega)$ , there exist a family of elements of  $L^1_{\text{loc}}(\Omega)$   $u_\varepsilon$  such that  $u_\varepsilon \rightarrow u$  in  $L^1_{\text{loc}}(\Omega)$  as  $\varepsilon \rightarrow 0$  and

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \leq F(u).$$

We will show the following result:

**Theorem 3.1.** *As  $\varepsilon \rightarrow 0$ ,  $\mathcal{SM}_{\varepsilon,C}$  and  $\mathcal{SM}_{\varepsilon,C}^0$   $\Gamma$ -converge to*

$$(3.6) \quad \text{Per}_{h_C}(E) := \begin{cases} \int_{\partial^* E} h_C(\nu_E(x)) d\mathcal{H}^{n-1}(x) & \text{if } E \text{ has finite perimeter,} \\ +\infty & \text{else} \end{cases}$$

in  $L^1_{\text{loc}}(\Omega)$ , where here, we let  $\nu_E(x)$  be the outer<sup>1</sup> normal to  $\partial^* E$  at  $x$ . Moreover, if  $\{E_\varepsilon\}_{\varepsilon > 0}$  are sets with locally finite measure and  $\sup_{\varepsilon > 0} \mathcal{SM}_{\varepsilon,C}(E_\varepsilon; \Omega) < \infty$ , then, up to subsequences,  $E_\varepsilon$  converge to some set  $E$  in  $L^1_{\text{loc}}(\Omega)$ .

<sup>1</sup>Observe that with this classical but not so natural choice, we have  $\text{Per}_{h_C}(E) = \int_{\Omega} h_C(-DX_E)$ .

In particular, we deduce from [6]:

**Corollary 3.2.** *As  $\varepsilon \rightarrow 0$ ,  $F_{\varepsilon,C}$   $\Gamma$ -converges to*

$$TV_{-C}(u; \Omega) := \begin{cases} \int_{\Omega} h_C(-Du) & \text{if } u \in BV(\Omega), \\ +\infty & \text{else} \end{cases}$$

in  $L^1_{\text{loc}}(\Omega)$ , where  $h_C(-Du)$  stands for  $h_C(-\frac{dDu}{|Du|})|Du|$  [8]. Moreover, if  $\{u_\varepsilon\}_{\varepsilon>0}$  are functions in  $L^1_{\text{loc}}(\Omega)$  with  $\sup_\varepsilon F_{\varepsilon,C}(u_\varepsilon; \Omega) < \infty$ , then  $\{u_\varepsilon\}_{\varepsilon>0}$  is precompact in  $L^1_{\text{loc}}(\Omega)$ .

For any measurable set  $E$  we can also consider

$$\mathcal{M}_{\varepsilon,C}(E; \Omega) := (\mathcal{SM}_{\varepsilon,C}(E; \Omega) + \mathcal{SM}_{\varepsilon,C}(\Omega \setminus E; \Omega))/2.$$

From Theorem 3.1 the following Corollary follows easily:

**Corollary 3.3.** *As  $\varepsilon \rightarrow 0$ ,  $\mathcal{M}_{\varepsilon,C}$   $\Gamma$ -converges to  $(\text{Per}_{h_C}(E) + \text{Per}_{h_C}(\Omega \setminus E))/2$  in  $L^1_{\text{loc}}(\Omega)$ .*

Concerning the pointwise convergence of  $\mathcal{SM}_{\varepsilon,C}^0$ , we also have the following interesting result, from which the  $\Gamma$ -lim sup inequality in Theorem 3.1 follows in a straightforward way.

**Theorem 3.4.** *Assume that the set  $E$  is a finite-perimeter set such that*

$$(3.7) \quad \lim_{\varepsilon \rightarrow 0} \mathcal{SM}_{\varepsilon,B(0,1)}^0(E; \Omega) = \text{Per}(E; \Omega).$$

Then,

$$(3.8) \quad \lim_{\varepsilon \rightarrow 0} \mathcal{SM}_{\varepsilon,C}^0(E; \Omega) = \text{Per}_{h_C}(E; \Omega).$$

**Remark 3.5.** The sets which satisfy (3.7) are studied in [2]. A sufficient condition is that the Minkowski content of the reduced boundary coincides with its  $(n-1)$ -dimensional measure, that is,

$$\lim_{\varepsilon \rightarrow 0} \frac{|\{x \in \Omega : \text{dist}(x, \partial^* E) \leq \varepsilon\}|}{2\varepsilon} = \text{Per}(E; \Omega).$$

The proof is quite elementary (see [2, Thm 13]): in that case, we can introduce, for  $x \in \Omega$ , the signed distance function

$$d_E(x) := \text{dist}(x, E) - \text{dist}(x, \Omega \setminus E)$$

and we have that (thanks to the co-area formula)

$$\begin{aligned} \frac{1}{2\varepsilon} |\{x \in \Omega : \text{dist}(x, \partial^* E) \leq \varepsilon\}| &\geq \frac{1}{2\varepsilon} |\{x \in \Omega : |d_E(x)| \leq \varepsilon\}| \\ &= \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \text{Per}(\{d_E < s\}; \Omega) ds = \frac{1}{2} \int_{-1}^1 \text{Per}(\{d_E < \varepsilon s\}; \Omega) ds. \end{aligned}$$

In particular, since

$$\text{Per}(E; \Omega) \leq \liminf_{\varepsilon \rightarrow 0} \text{Per}(\{d_E < \varepsilon s\}; \Omega)$$

for all  $s$ , by Fatou's lemma, we find that also

$$(3.9) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} |\{x \in \Omega : |d_E(x)| \leq \varepsilon\}| = \text{Per}(E; \Omega),$$

while (for the same reasons)

$$(3.10) \quad \begin{aligned} \text{Per}(E; \Omega) &\leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} |\{x \in \Omega : 0 < d_E(x) \leq \varepsilon\}|, \\ \text{Per}(E; \Omega) &\leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} |\{x \in \Omega : 0 < -d_E(x) \leq \varepsilon\}|. \end{aligned}$$

It follows from (3.9) and (3.10) that the inequality in (3.10) must in fact be an equality, and the  $\liminf$  a  $\lim$ . In particular, we deduce (3.7).

**Remark 3.6.** In case  $E$  is a convex body, then it is well known that (see [11])

$$|E + \varepsilon C| = |E| + \varepsilon \text{Per}_{h_C}(E; \Omega) + O(\varepsilon^2).$$

Also, if  $\partial E$  is compact and rectifiable (that is, included in the image of a Lipschitz map from  $\mathbb{R}^{n-1}$  to  $\mathbb{R}^n$ ), and  $\mathcal{H}^{n-1}(\partial E \setminus \partial^* E) = 0$ , then the Minkowski content coincides with the perimeter, see [9, Thm 3.2.39 p. 275]. It is easy to build examples, though, where this is not true and still, (3.8) holds, see again [2].

As before, for any measurable set  $E \subset \Omega$  we let

$$\mathcal{M}_{\varepsilon, C}^0(E; \Omega) := (\mathcal{SM}_{\varepsilon, C}^0(E; \Omega) + \mathcal{SM}_{\varepsilon, C}^0(\Omega \setminus E; \Omega))/2.$$

Then the following pointwise convergence result holds.

**Theorem 3.7.** *Assume that the set  $E$  is a finite-perimeter set such that*

$$\lim_{\varepsilon \rightarrow 0} \mathcal{M}_{\varepsilon, B(0,1)}^0(E; \Omega) = \text{Per}(E; \Omega).$$

*Then,*

$$\lim_{\varepsilon \rightarrow 0} \mathcal{M}_{\varepsilon, C}^0(E; \Omega) = (\text{Per}_{h_C}(E; \Omega) + \text{Per}_{h_C}(\Omega \setminus E; \Omega))/2.$$

*In particular, we get*

$$\lim_{\varepsilon \rightarrow 0} \mathcal{M}_{\varepsilon, C}^0(E; \Omega) = \int_{\partial^* E} \frac{h_C(\nu_E(x)) + h_C(-\nu_E(x))}{2} d\mathcal{H}^{n-1}(x).$$

#### 4. PROOF OF THE RESULTS

**4.1. Proof of Theorem 3.1.** First, the compactness statement is proved as follows. Notice that for any  $E$  measurable we can rewrite  $F_{\varepsilon, C}(\chi_E; \Omega)$  as

$$(4.1) \quad F_{\varepsilon, C}(\chi_E; \Omega) := \frac{1}{\varepsilon} \int_{\Omega} \text{ess sup}_{\Omega \cap (x - \varepsilon C)} |\chi_E - \chi_E(x)| dx.$$

By assumption the family  $\{\chi_{E_\varepsilon}\}_{\varepsilon > 0}$  satisfies  $F_{\varepsilon, C}(\chi_{E_\varepsilon}; \Omega) \leq c$  for some  $c > 0$ . Let  $\sigma > 0$  and let

$$\Omega_\sigma = \{x \in \Omega \cap B(0, 1/\sigma) : d(x, \partial\Omega) > \sigma\};$$



of course we have  $\Omega_\sigma \subset\subset \Omega$ . Since  $B(0, a) \subseteq C$ , there exists  $r > 0$  such that for any  $\xi \in \mathbb{R}^n$  with  $|\xi| \leq 1$  and any  $\varepsilon < \varepsilon_0$ , for  $\varepsilon_0$  sufficiently small, it holds, using the form (4.1) for  $F_{\varepsilon, C}$ ,

$$\int_{\Omega_\sigma} \frac{|\chi_{E_\varepsilon}(x + r\varepsilon\xi) - \chi_{E_\varepsilon}(x)|}{\varepsilon|\xi|} dx \leq F_{\varepsilon, C}(\chi_{E_\varepsilon}; \Omega) \leq c$$

and thus

$$\sup_{\varepsilon < \varepsilon_0, |\xi|=1} \int_{\Omega_\sigma} \frac{|\chi_{E_\varepsilon}(x + r\varepsilon\xi) - \chi_{E_\varepsilon}(x)|}{\varepsilon|\xi|} dx \leq c.$$

Since obviously  $\|\chi_{E_\varepsilon}\|_{L^1(\Omega_\sigma)}$  is bounded, we are in position to apply Riesz-Frechet-Kolmogorov compactness Theorem in  $L^1(\Omega_\sigma)$ , and then we find a positive infinitesimal sequence  $\varepsilon_j$  such that  $E_{\varepsilon_j}$  converges to some measurable set  $E$  in  $L^1(\Omega_\sigma)$ . Now it is easy to conclude applying a diagonal argument.

To extend this compactness result to Corollary 3.2, one can consider for each  $\delta > 0$  and function  $u_\varepsilon$  a function

$$u_\varepsilon^\delta = \sum_{k \in \mathbb{Z}} s_k \chi_{\{s_{k+1} \geq u_\varepsilon > s_k\}},$$

where  $s_k \in (k\delta, (k+1)\delta)$  is a level appropriately chosen so that

$$\mathcal{SM}_{\varepsilon, C}(\{u_\varepsilon > s_k\}; \Omega) \leq (1 + \sup_{\varepsilon > 0} F_{\varepsilon, C}(u_\varepsilon; \Omega))/\delta.$$

Then, the previous compactness result (and a diagonal argument) shows that there exists a positive infinitesimal sequence  $\varepsilon_k$  such that  $u_{\varepsilon_k}^{1/n}$  converges to some  $u^{1/n}$  in  $L^1_{\text{loc}}(\Omega)$ , for all  $n \geq 1$ . Since  $\|u_{\varepsilon_k}^{1/m} - u_{\varepsilon_k}^{1/n}\|_\infty \leq 2/\min\{m, n\}$  and  $\|u^{1/m} - u^{1/n}\|_\infty \leq 2/\min\{m, n\}$  for all  $m, n, k$ , we easily deduce that (up to a subsequence), there exists  $u$  such that  $u_{\varepsilon_k} \rightarrow u$  in  $L^1_{\text{loc}}(\Omega)$ .

In order to prove the  $\Gamma$ -convergence, we must show that for any  $E$ ,

- if  $E_\varepsilon \rightarrow E$  in  $L^1_{\text{loc}}(\Omega)$ , then

$$(4.2) \quad \text{Per}_{h_C}(E; \Omega) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{SM}_{\varepsilon, C}(E_\varepsilon; \Omega),$$

- and that there exists  $E_\varepsilon \rightarrow E$  with

$$(4.3) \quad \limsup_{\varepsilon \rightarrow 0} \mathcal{SM}_{\varepsilon, C}(E_\varepsilon; \Omega) \leq \text{Per}_{h_C}(E; \Omega).$$

As it is standard that one can approximate any set  $E$  with finite perimeter by means of (almost) smooth sets such that  $\text{Per}(E_k; \Omega) \rightarrow \text{Per}(E; \Omega)$  (for instance, minimizers of  $\text{Per}(F; \Omega) + k|E \Delta E'|$ ) then (4.3) will follow, using a diagonal argument and Remark 3.5, from Theorem 3.4 (which we will prove later on).

Hence, we focus on the proof of (4.2). Let us introduce the anisotropic (essential) distance function to a set  $E$ :

$$\text{dist}_C(x, E) := \text{ess inf}_{y \in E} h_C^\circ(x - y).$$

(Equivalently, this is the  $h_C^\circ$ -distance to the set  $E^1$  of points where the Lebesgue density of  $E$  is 1, or to the complement of  $E^0$ .) Then,  $\text{dist}_C(x, E) < \varepsilon$  if and only if there exists a set of positive measure in  $E$  of points  $y$  with  $h_C^\circ(x - y) < \varepsilon$ , or, in other words, such that  $x - y \in \varepsilon C$ , which is equivalent to say that  $x \in E^1 + \varepsilon C$ . In particular, it follows that

$$(E^1 + \varepsilon C \setminus E^1) \cap \Omega = \{x \in \Omega \setminus E^1 : \text{dist}_C(x, E) < \varepsilon\}.$$

On the other hand, if one lets  $d(x) := \text{dist}_C(x, E)$ , it is standard that  $d$  is Lipschitz and that  $h_C(\nabla d) = 1$  a.e. in  $\{d > 0\}$ , and 0 a.e. in  $\{d = 0\} \supset E^1$ . The proof follows the same lines as in [3]. First, for any  $x, y \in \Omega$ , if  $\delta > 0$ , one can find a set with positive measure in  $E$  of point  $y'$  with  $d(y) \leq h_C^\circ(y - y') \leq d(y) + \delta$ . Then, for these points,

$$(4.4) \quad d(x) - d(y) \leq h_C^\circ(x - y') - h_C^\circ(y - y') + \delta \leq h_C^\circ(x - y) + \delta$$

and sending  $\delta$  to zero and using (3.1), it follows that  $d$  is Lipschitz. Moreover,  $\nabla d = 0$  a.e. in  $\{d = 0\}$ . Now, from (4.4) we also see that  $d(x + tz) - d(x) \leq th_C^\circ(z)$  for all  $z$ ; therefore, if  $d$  is differentiable at  $x$  it follows that  $\nabla d(x) \cdot z \leq 1$  for all  $z \in C$ , hence  $h_C(\nabla d(x)) \leq 1$ .

We show the reverse inequality for points  $x$  where  $d(x) > 0$ : for such a point, there exists  $y \in \overline{E^1}$  with  $d(x) = h_C^\circ(x - y)$ . For each  $x' \in (y, x]$  (which means that  $x' \neq y$  and  $x'$  lies on the line segment with extreme points  $y$  and  $x$ ), one has  $d(x') = h_C^\circ(x' - y) > 0$ , otherwise there would exist  $y'$  with  $h_C^\circ(x' - y') < h_C^\circ(x' - y)$ , but then, it would follow that

$$h_C^\circ(x - y') \leq h_C^\circ(x - x') + h_C^\circ(x' - y') < h_C^\circ(x - x') + h_C^\circ(x' - y) = h_C^\circ(x - y)$$

since  $x' \in (y, x]$ , a contradiction. It follows that for  $z = x - y$ ,  $t \in (0, 1)$ ,

$$d(x - tz) = h_C^\circ(x - tz - y) = (1 - t)d(x),$$

and if in addition  $x$  is a point of differentiability, it follows that

$$-\nabla d(x) \cdot z = -d(x) = -h_C^\circ(z).$$

But since  $h_C(\nabla d(x)) \leq 1$  and  $z/h_C^\circ(z) \in C$ , it follows that  $h_C(\nabla d(x)) = 1$ . If moreover  $h_C^\circ$  is differentiable as well in  $x - y$ , we find in addition that  $\nabla d(x) = \nabla h_C^\circ(x - y)$ . If  $h_C$  is differentiable in  $\nabla d(x)$ , we find that  $y = x - d(x)\nabla h_C(\nabla d(x))$  and in particular, in that case, the projection  $y$  must be unique. For a general convex set  $C$  this might not be the case, even at points of differentiability.

Now, let  $\{E_\varepsilon\}_{\varepsilon > 0}$  be a family of sets, with  $|(E_\varepsilon \Delta E) \cap \Omega| \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , and assume that  $\liminf_{\varepsilon \rightarrow 0} \mathcal{SM}_{\varepsilon, C}(E_\varepsilon; \Omega) < +\infty$ . We consider a subsequence  $E_k := E_{\varepsilon_k}$  such that this lim inf is in fact a limit. We have

$$((E_k^1 + \varepsilon_k C) \setminus E_k^1) \cap \Omega \supseteq \{x \in \Omega : 0 < \text{dist}_C(x, E_k) < \varepsilon_k\}$$

(the difference being the possible set of points  $x \notin E_k^1$  with  $\text{dist}_C(x, E_k) = 0$ ). It follows, letting  $d_k(x) := \max\{\text{dist}_C(x, E_k)/\varepsilon_k, 1\}$ ,

$$\mathcal{SM}_{\varepsilon_k, C}(E_k; \Omega) \geq \frac{1}{\varepsilon_k} \int_{\{0 < \text{dist}_C(\cdot, E_k) < \varepsilon_k\}} h_C(\nabla \text{dist}_C(x, E_k)) dx = \int_{\Omega} h_C(\nabla d_k(x)) dx.$$

In particular,  $(d_k)_{k \geq 1}$  have equibounded total variation: we may assume that a subsequence (not relabelled) converges to some limit  $d$ , with values in  $[0, 1]$ , in  $L^1_{\text{loc}}(\Omega)$ . (And, in fact, we may even assume that the convergence is pointwise, out of a negligible set.)

By assumption,  $|\{0 < d_k < 1\}| \leq c\varepsilon_k$ , in particular we deduce that  $d \in \{0, 1\}$  a.e. in  $\Omega$ . Observe also that ( $X^c$  denoting here the complement of  $X$  in  $\Omega$ , that is  $\Omega \setminus X$ ), if  $B$  is a ball in  $\mathbb{R}^n$ ,

$$\begin{aligned} |B \cap \{d_k < \varepsilon_k\} \Delta E| &= |B \cap \{d_k < \varepsilon_k\} \cap E^c| + |B \cap \{d_k < \varepsilon_k\}^c \cap E| \\ &= |B \cap \{d_k < \varepsilon_k\} \cap E_k^1 \cap E^c| + |B \cap \{d_k < \varepsilon_k\} \cap (E_k^1)^c \cap E^c| \\ &\quad + |B \cap \{d_k < \varepsilon_k\}^c \cap E_k^1 \cap E| + |B \cap \{d_k < \varepsilon_k\}^c \cap (E_k^1)^c \cap E| \\ &\leq |B \cap (E_k \Delta E)| + |\{d_k < \varepsilon_k\} \setminus E_k^1|. \end{aligned}$$

As  $|B \cap (E_k \Delta E)| \rightarrow 0$  (by assumption), while  $|\{d_k < \varepsilon_k\} \setminus E_k^1| \leq c\varepsilon_k$ , we deduce that  $\{d_k < \varepsilon_k\} \rightarrow E$  in  $L^1_{\text{loc}}(\Omega)$ . It follows that  $\{d = 1\} = \Omega \setminus E$ , hence  $d_k \rightarrow 1 - \chi_E$  as  $k \rightarrow \infty$ . Thanks to Reshetniak's lower semicontinuity Theorem, we deduce that

$$\int_{\Omega} h_C(-D\chi_E) \leq \liminf_{k \rightarrow \infty} \int_{\Omega} h_C(\nabla d_k(x)) dx \leq \lim_{k \rightarrow \infty} \mathcal{SM}_{\varepsilon_k, C}(E_k; \Omega).$$

Since  $\int_{\Omega} h_C(-D\chi_E) = \text{Per}_{h_C}(E)$ , (4.2) follows.

As already mentioned, the proof of (4.3) will follow from Theorem (3.4), which is given in the next Section.

**4.2. Proof of Theorem 3.4.** Now, we consider a set  $E \subset \Omega$  such that (3.7) holds. We will identify  $E$  with the set of points where its Lebesgue density is 1, moreover, a necessary condition for (3.7) is that  $\overline{E} = \bigcap_{\varepsilon > 0} E + B(0, \varepsilon)$  coincides with  $E$  up to a negligible set, in other words,  $|\overline{E} \setminus E| = 0$ .

A first remark is that, clearly, using (3.1),

$$a\mathcal{SM}_{a\varepsilon, B(0,1)}(E; \Omega) \leq \mathcal{SM}_{\varepsilon, C}(E; \Omega) \leq b\mathcal{SM}_{b\varepsilon, B(0,1)}(E; \Omega)$$

hence any limit of  $\mathcal{SM}_{\varepsilon, C}(E; \Omega)$  is in between  $a\text{Per}(E; \Omega)$  and  $b\text{Per}(E; \Omega)$ . In particular, we can introduce the measures

$$\mu_{\varepsilon} := \frac{1}{\varepsilon} (\chi_{E+\varepsilon C} - \chi_E)$$

which are equibounded. Then, up to a subsequence, we have  $\mu_{\varepsilon_k} \xrightarrow{*} \mu$  as measures in  $\Omega$ , with  $a\mathcal{H}^{n-1} \llcorner \partial^* E \leq \mu \leq b\mathcal{H}^{n-1} \llcorner \partial^* E$ . We introduce the Besicovitch derivative  $g(x) \in [a, b]$  of the measure  $\mu$  w.r.  $\mathcal{H}^{n-1} \llcorner \partial^* E$ , defined by

$$g(x) = \lim_{\rho \rightarrow 0} \frac{\mu(B(x, \rho))}{\mathcal{H}^{n-1}(\partial^* E \cap B(x, \rho))}$$

and which is defined  $\mathcal{H}^{n-1}$ -a.e. on  $\Omega \cap \partial^* E$ . Moreover, since  $\partial^* E$  is rectifiable, it is also given by

$$(4.5) \quad g(x) = \lim_{\rho \rightarrow 0} \frac{\mu(B(x, \rho))}{\alpha_n \rho^{n-1}}.$$

(We recall  $\alpha_n$  is the diameter of the unit disc.)

Theorem 3.4 will follow if we can show that  $g(x) = h_C(\nu_E(x))$  for  $\mathcal{H}^{n-1}$ -a.e.  $x \in \partial^* E$ . We fix from now on a point  $\bar{x} \in \partial^* E$  where (4.5) exists. Observe that from (4.2), it follows that  $g(x) \geq h_C(\nu_E(x))$   $\mathcal{H}^{n-1}$ -a.e. in  $\partial^* E$ , so that we need to show that  $g(\bar{x}) \leq h_C(\nu_E(\bar{x}))$ . We recall that

$$\lim_{\rho \rightarrow 0} \frac{\mathcal{H}^{n-1}(\partial^* E \cap B(\bar{x}, \rho))}{\alpha_n \rho^{n-1}} = 1$$

and

$$\lim_{\rho \rightarrow 0} \int_{B(0,1)} \left| \chi_{\frac{E-\bar{x}}{\rho}} - \chi_{\{y: y \cdot \nu_E(\bar{x}) \leq 0\}} \right| dy = 0$$

hold. We denote  $\nu = \nu_E(\bar{x})$  and without loss of generality we will assume that it is the direction of the last coordinate  $x_n$ . We will use the notation  $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$  to distinguish between the component  $x' \perp \nu$  and  $x_n$  (along  $\nu$ ) of a point  $x \in \mathbb{R}^n$ .

We also introduce the measures

$$\lambda_\varepsilon := \frac{1}{\varepsilon} (\chi_{E+\varepsilon B(0,b)} - \chi_E) \geq \mu_\varepsilon,$$

the main assumption of Theorem 3.4 ensures that these measures converge weakly-\* to  $\lambda = b \mathcal{H}^{n-1} \llcorner \partial^* E$  as  $\varepsilon \rightarrow 0$ . Now we use a classical procedure: since for a.e.  $\rho > 0$ ,

$$\mu_{\varepsilon_k}(B(x, \rho)) \rightarrow \mu(B(x, \rho)) \quad \text{and} \quad \lambda_{\varepsilon_k}(B(x, \rho)) \rightarrow \lambda(B(x, \rho)),$$

we can build a sequence  $(\rho_k)_{k \in \mathbb{N}}$  with  $\varepsilon'_k = \varepsilon_k / \rho_k \rightarrow 0$  such that

$$\lim_{k \rightarrow \infty} \frac{\mu_{\varepsilon_k}(B(\bar{x}, \rho_k))}{\alpha_n \rho_k^{n-1}} = g(\bar{x})$$

and

$$\lim_{k \rightarrow \infty} \frac{\lambda_{\varepsilon_k}(B(\bar{x}, \rho_k))}{\alpha_n \rho_k^{n-1}} = b.$$

In fact, the rest of the proof would be relatively easy if we could ensure that  $\varepsilon_k \sim \rho_k$  as  $k \rightarrow \infty$ , using then a blow-up argument. However, this is not clear in general, and we need to consider the situation where  $\varepsilon_k = o(\rho_k)$ , hence  $\varepsilon'_k \rightarrow 0$ . As is usual, we do a blow-up using the change of variables  $x = \bar{x} + \rho_k y$ . We let  $E_k = (E - \bar{x}) / \rho_k$ , and we observe that

$$(4.6) \quad \lim_{k \rightarrow \infty} \frac{1}{\alpha_n \varepsilon'_k} \int_{B(0,1)} (\chi_{E_k + \varepsilon'_k C} - \chi_{E_k}) dy = g(\bar{x}),$$

$$(4.7) \quad \lim_{k \rightarrow \infty} \frac{1}{\alpha_n \varepsilon'_k} \int_{B(0,1)} (\chi_{E_k + \varepsilon'_k B(0,b)} - \chi_{E_k}) dy = b,$$

$$(4.8) \quad \lim_{k \rightarrow \infty} \int_{B(0,1)} |\chi_{E_k} - \chi_{\{y: y \cdot \nu \leq 0\}}| dy = 0.$$

Moreover, for any  $\beta > 0$  (small), one can check easily that if we replace in (4.7) and (4.8)  $B(0, 1)$  with  $B(0, 1 - \beta)$ , or even with  $C(0, 1 - \beta) := \{(y', y_n) \in B(0, 1) : |y'| \leq 1 - \beta\}$ , (4.8) still holds and the right-hand side in (4.7) is replaced with  $b(1 - \beta)^{n-1}$ . Indeed, it follows from (4.2) (with  $C = B(0, b)$ ) and (4.8) that for any open set  $A \subseteq B(0, 1)$ ,

$$b\mathcal{H}^{n-1}(A \cap \{y \cdot \nu = 0\}) \leq \liminf_{k \rightarrow \infty} \frac{1}{\varepsilon'_k} \int_A (\chi_{E_k + \varepsilon'_k B(0, b)} - \chi_{E_k}) dy.$$

Together with (4.7), we deduce that as soon as  $\mathcal{H}^{n-1}(\partial A \cap \{y \cdot \nu = 0\}) = 0$ ,

$$(4.9) \quad b\mathcal{H}^{n-1}(A \cap \{y \cdot \nu = 0\}) = \lim_{k \rightarrow \infty} \frac{1}{\varepsilon'_k} \int_A (\chi_{E_k + \varepsilon'_k B(0, b)} - \chi_{E_k}) dy.$$

We fix a (small) value of  $\beta > 0$ . Then, we choose a value  $\theta > 10b$  and we consider the points  $z \in \mathbb{Z}^{n-1}$  such that the hypersquares  $(\theta\varepsilon'_k(z + (0, 1)^{n-1})) \times \{0\}$  are contained in  $B(0, 1 - \beta)$ . There is a finite number  $N_k$  of such squares and we enumerate the corresponding points  $\{z_1^k, \dots, z_{N_k}^k\}$ . For  $i = 1, \dots, N_k$ , we let

$$C_i^k = [(\theta\varepsilon'_k(z_i^k + (0, 1)^{n-1})) \times \mathbb{R}] \cap B(0, 1), \quad C_i'^k = (\theta\varepsilon'_k(z_i^k + (0, 1)^{n-1})) \times \{0\}.$$

We then let

$$(4.10) \quad a_i^k = \int_{C_i^k} |\chi_{E_k} - \chi_{\{y_n \leq 0\}}| dy \leq 2(\theta\varepsilon'_k)^{n-1}$$

and  $\delta_k = \sum_{i=1}^{N_k} a_i^k$ : from (4.8) we know that  $\delta_k \rightarrow 0$  as  $k \rightarrow \infty$ . We then consider

$$Z_k = \{i = 1, \dots, N_k : a_i^k \leq \sqrt{\delta_k}(\varepsilon'_k)^{n-1}\},$$

$Z'_k = \{1, \dots, N_k\} \setminus Z_k$ . It follows that

$$(4.11) \quad \delta_k \geq \sqrt{\delta_k}(\varepsilon'_k)^{n-1} \#Z'_k \Rightarrow (\varepsilon'_k)^{n-1} \#Z'_k \leq \sqrt{\delta_k}$$

which gives a control on the “bad” surface, of the cylinders  $C_i^k$  where the integral  $a_i^k$  is “large”. For each  $i = 1, \dots, N_k$ , we have

$$(4.12) \quad \frac{1}{\varepsilon'_k} \int_{C_i^k} |\chi_{E_k + \varepsilon'_k B(0, b)} - \chi_{E_k}| dy = \frac{1}{\varepsilon'_k} \int_{C_i'^k} \int_{-\sqrt{1-y'^2}}^{\sqrt{1-y'^2}} |\chi_{E_k + \varepsilon'_k B(0, b)} - \chi_{E_k}| dy_n dy' \\ \geq b \left| \left\{ y' \in C_i'^k : |(\{y'\} \times \mathbb{R}) \cap (B(0, 1) \cap E_k)| > 0, |(\{y'\} \times \mathbb{R}) \cap (B(0, 1) \setminus E_k)| \geq \varepsilon'_k b, \right\} \right|$$

since clearly, each time a point  $(y', y_n) \in E_k$ , then  $(y', y_n + s) \in E_k + \varepsilon'_k B(0, b)$  for  $|s| \leq \varepsilon'_k b$ .

We denote by  $D_i^k$  the set in the right-hand side of (4.12). For  $y' \notin D_i^k$ ,

$$\int_{-\sqrt{1-y'^2}}^{\sqrt{1-y'^2}} |\chi_{E_k} - \chi_{\{y_n \leq 0\}}| dy \geq \sqrt{\beta} - \varepsilon'_k b \geq \frac{\sqrt{\beta}}{2}$$

as soon as  $\varepsilon'_k \leq \sqrt{\beta}/(2b)$ . It follows that  $|C_i'^k \setminus D_i^k| \leq 2a_i^k/\sqrt{\beta}$ , hence if  $i \in Z_k$ , so that  $a_i^k \leq \sqrt{\delta_k}(\varepsilon'_k)^{n-1} = \sqrt{\delta_k}|C_i'^k|/\theta^{n-1}$ , we get that (4.12) bounds

$$(4.13) \quad b|D_i^k| \geq b|C_i'^k| \left(1 - \frac{2\sqrt{\delta_k}}{\theta^{n-1}\sqrt{\beta}}\right) = b|C_i'^k| \left(1 - K\sqrt{\delta_k}\right).$$

To sum up, (4.12) and (4.13) show that for any  $i \in Z_k$ ,

$$\frac{1}{\varepsilon'_k} \int_{C_i^k} \left| \chi_{E_k + \varepsilon'_k B(0,b)} - \chi_{E_k} \right| dy \geq b |C_i^k| \left( 1 - K \sqrt{\delta_k} \right).$$

In particular, it follows that (using (4.11))

$$(4.14) \quad \liminf_{k \rightarrow \infty} \sum_{i \in Z_k} \frac{1}{\varepsilon'_k} \int_{C_i^k} \left| \chi_{E_k + \varepsilon'_k B(0,b)} - \chi_{E_k} \right| dy \\ \geq b \lim_{k \rightarrow \infty} \left( \left| \bigcup_{i=1}^{N_k} C_i^k \right| - (\theta \varepsilon'_k)^{n-1} \# Z'_k \right) \left( 1 - K \sqrt{\delta_k} \right) = \alpha_n b (1 - \beta)^{n-1}$$

and together with (4.9) (with  $A = C(0, 1 - \beta)$ ) we deduce that

$$\delta'_k := \sum_{i \in Z_k} \left( \frac{1}{\varepsilon'_k} \int_{C_i^k} \left| \chi_{E_k + \varepsilon'_k B(0,b)} - \chi_{E_k} \right| dy - b |C_i^k| \left( 1 - K \sqrt{\delta_k} \right) \right) \xrightarrow{k \rightarrow \infty} 0.$$

So now we introduce

$$\tilde{Z}_k = \left\{ i \in Z_k : \frac{1}{\varepsilon'_k} \int_{C_i^k} \left| \chi_{E_k + \varepsilon'_k B(0,b)} - \chi_{E_k} \right| dy - b |C_i^k| \left( 1 - K \sqrt{\delta_k} \right) \leq (\varepsilon'_k)^{n-1} \sqrt{\delta'_k} \right\},$$

and its complement  $\tilde{Z}'_k = Z_k \setminus \tilde{Z}_k$ . Then as before, one sees that

$$(4.15) \quad \delta'_k \geq \sqrt{\delta'_k} (\varepsilon'_k)^{n-1} \# \tilde{Z}'_k \Rightarrow (\varepsilon'_k)^{n-1} \# \tilde{Z}'_k \leq \sqrt{\delta'_k}.$$

We see at this point that (4.14) still holds if  $Z_k$  is replaced with  $\tilde{Z}_k$ , and  $Z'_k$  with  $Z'_k \cup \tilde{Z}'_k$ . Together with (4.9) (with again  $A = C(0, 1 - \beta)$ ) it follows that

$$\limsup_{k \rightarrow \infty} \frac{1}{\varepsilon'_k} \int_{C(0,1-\beta) \setminus \bigcup_{i \in \tilde{Z}_k} C_i^k} \left| \chi_{E_k + \varepsilon'_k B(0,b)} - \chi_{E_k} \right| dy = 0,$$

and as a consequence

$$(4.16) \quad \limsup_{k \rightarrow \infty} \frac{1}{\varepsilon'_k} \int_{C(0,1-\beta) \setminus \bigcup_{i \in \tilde{Z}_k} C_i^k} \left| \chi_{E_k + \varepsilon'_k C} - \chi_{E_k} \right| dy = 0.$$

We now need to estimate the quantity  $(1/\varepsilon'_k) \int_{C_i^k} \left| \chi_{E_k + \varepsilon'_k C} - \chi_{E_k} \right| dy$  for  $i \in \tilde{Z}_k$ , hence when

$$(4.17) \quad b |C_i^k| \left( 1 - K \sqrt{\delta_k} \right) \leq \frac{1}{\varepsilon'_k} \int_{C_i^k} \left| \chi_{E_k + \varepsilon'_k B(0,b)} - \chi_{E_k} \right| dy \leq b |C_i^k| \left( 1 - K \sqrt{\delta_k} + \frac{\sqrt{\delta'_k}}{b \theta^{n-1}} \right)$$

and

$$(4.18) \quad |C_i^k \setminus D_i^k| \leq 2 |C_i^k| \frac{\sqrt{\delta_k}}{\theta^{n-1} \sqrt{\beta}}.$$

We choose  $k, i \in \tilde{Z}_k$  so that (4.17) and (4.18) hold, and consider the change of variable  $y = z_i^k + \varepsilon'_k \hat{y}$ . We let  $F = (E_k - z_i^k)/\varepsilon'_k$ ,  $Q = (C_i^k - z_i^k)/\varepsilon'_k \supset (0, \theta)^{n-1} \times (-\sqrt{\beta}/\varepsilon'_k, \sqrt{\beta}/\varepsilon'_k)$ ,  $D = (D_i^k - z_i^k)/\varepsilon'_k$ . We find that

$$(4.19) \quad \int_Q |\chi_{F+B(0,b)} - \chi_F| d\hat{y} \leq b\theta^{n-1} \left( 1 - K\sqrt{\delta_k} + \frac{\sqrt{\delta'_k}}{b\theta^{n-1}} \right),$$

$$(4.20) \quad \int_{(0,\theta)^{n-1} \times \left(-\frac{\sqrt{\beta}}{\varepsilon'_k}, \frac{\sqrt{\beta}}{\varepsilon'_k}\right)} |\chi_F - \chi_{\{\hat{y}_n \leq 0\}}| d\hat{y} \leq \theta^{n-1} \frac{\sqrt{\delta_k}}{\varepsilon'_k},$$

while (4.18) yields

$$(4.21) \quad |(0, \theta)^{n-1} \setminus D| \leq 2\sqrt{\frac{\delta_k}{\beta}}.$$

Let us set  $F_b = F + B(0, b) = \{\hat{y} \in \mathbb{R}^n : \text{dist}(\hat{y}, F) \leq b\}$ , and

$$F_s = \{\hat{y} \in F_b : \text{dist}(\hat{y}, \partial F_b) \geq b - s\}.$$

We observe that  $F \subset F_s$  for any  $s \in [0, b]$ . It is well known that if  $0 < s < b$ , the boundaries  $\partial F_s$  are  $C^{1,1}$ , with curvatures between  $-1/(b-s)$  and  $1/s$ . We want to show that for  $k$  large enough, these boundaries are essentially flat inside  $Q$ . Let now

$$\begin{aligned} D' &= \{\hat{y}' \in (0, \theta)^{n-1} : |(\{\hat{y}'\} \times \mathbb{R}) \cap Q \cap F| = 0\}, \\ D'' &= \{\hat{y}' \in (0, \theta)^{n-1} : (\{\hat{y}'\} \times \mathbb{R}) \cap Q \subset F_b\}, \\ D''' &= \{\hat{y}' \in (0, \theta)^{n-1} : |(\{\hat{y}'\} \times \mathbb{R}) \cap Q \cap (F_b \setminus F)| \geq 2b\}. \end{aligned}$$

The definition of  $D$  ensures that if  $\hat{y}' \in (0, \theta)^{n-1} \setminus D$ ,  $|(\{\hat{y}'\} \times \mathbb{R}) \cap Q \cap (F_b \setminus F)| \geq b$ . From (4.19) and (4.21), we have that

$$b\theta^{n-1} + \sqrt{\delta'_k} \geq b|((0, \theta)^{n-1} \setminus D) \setminus D'''| + 2b|D'''| \geq b\theta^{n-1} + b|D'''| - 2\sqrt{\frac{\delta_k}{\beta}},$$

hence

$$(4.22) \quad |D'''| \leq \frac{1}{b} \left( \sqrt{\delta'_k} + 2\sqrt{\frac{\delta_k}{\beta}} \right).$$

Now, we easily deduce from (4.20) that both  $|D'|$  and  $|D'' \setminus D'''|$  are bounded by a constant (multiple of  $\theta^{n-1}/\sqrt{\beta}$ ) times  $\sqrt{\delta_k}$ . It follows that there exists a constant  $K'$  (still depending on  $\theta, \beta$ ) such that

$$(4.23) \quad |D' \cup D''| \leq K' \left( \sqrt{\delta_k} + \sqrt{\delta'_k} \right).$$

Now, each time  $\hat{y}' \in (0, \theta)^{n-1} \setminus (D' \cup D'')$ , in  $(\{\hat{y}'\} \times \mathbb{R}) \cap Q$  one can find points which are in  $F$  (hence in  $F_s$  for all  $s \in [0, b]$ ), points which are not in  $F_b$ , and as a consequence there are also points in  $\partial F_s$ , for any  $s \in [0, b]$ . It follows that

$$(4.24) \quad \mathcal{H}^{n-1}(\partial F_s \cap Q) \geq \theta^{n-1} - K' \left( \sqrt{\delta_k} + \sqrt{\delta'_k} \right),$$

and as a consequence, also (from the coarea formula applied to the distance function to  $\partial F_b$ ),

$$(4.25) \quad |(F_b \setminus F_0) \cap Q| = \int_0^b \mathcal{H}^{n-1}(\partial F_s \cap Q) ds \geq b\theta^{n-1} - bK' \left( \sqrt{\delta_k} + \sqrt{\delta'_k} \right).$$

Inequalities (4.19) and (4.25) yield that (here the constant  $K'$  may vary from line to line, keeping the same kind of dependency in the parameters)

$$(4.26) \quad |(F_0 \setminus F) \cap Q| \leq bK' \left( \sqrt{\delta_k} + \sqrt{\delta'_k} \right).$$

Let us choose  $\eta > 0$ , small, and observe that (using (4.24) and (4.19))

$$\begin{aligned} & \int_{\eta}^{2\eta} \mathcal{H}^{n-1}(\partial F_s \cap Q) ds \\ & \leq |(F_b \setminus F_0) \cap Q| - (b - \eta) \left( \theta^{n-1} - K' \left( \sqrt{\delta_k} + \sqrt{\delta'_k} \right) \right) \leq \eta\theta^{n-1} + K' \left( \sqrt{\delta_k} + \sqrt{\delta'_k} \right) \end{aligned}$$

so that there exists  $\bar{s} \in [\eta, 2\eta]$  with

$$(4.27) \quad \mathcal{H}^{n-1}(\partial F_{\bar{s}} \cap Q) \leq \theta^{n-1} + \frac{K'}{\eta} \left( \sqrt{\delta_k} + \sqrt{\delta'_k} \right).$$

We recall that  $\partial F_{\bar{s}}$  has a  $(1/\eta)$ -Lipschitz normal, by construction. Eventually, we deduce that  $\partial F_{\bar{s}}$  must be almost flat if  $k$  is large enough: indeed, fix  $\zeta > 0$  and assume there is a point  $\hat{y} \in \partial F_{\bar{s}} \cap Q$  with  $|\nu_{\partial F_{\bar{s}}}(\hat{y}) \cdot \bar{\nu}| \leq 1 - \zeta$ . We let  $\bar{\nu} = \nu_{\partial F_{\bar{s}}}(\hat{y})$ . Then, for  $t$  small, we consider the ball  $B(\hat{y}, t\eta\zeta)$  (which we assume is in  $Q$ , and we let  $r = t\eta\zeta$ ). The regularity of  $\partial F_{\bar{s}}$  yields that, in that ball, it consists (at least) of a  $C^{1,1}$  graph which passes in between two spherical caps of radius  $\eta$ , which are tangent in  $\hat{y}$  and normal at that point to  $\bar{\nu}$ . We call  $\mathcal{S}$  the subset of  $B(\hat{y}, r)$  bounded by these two caps (see Figure 1). A simple calculation shows that the trace of these spherical caps on the sphere  $\partial B(\hat{y}, r)$  is given by the intersection of this sphere with the hyperplanes  $\{(y - \hat{y}) \cdot \bar{\nu} = \pm \frac{t\zeta}{2}r\}$  (hence,  $\mathcal{S} \subset B(\hat{y}, r) \cap \{|(y - \hat{y}) \cdot \bar{\nu}| < \frac{t\zeta}{2}r\}$ ). In particular, the surface  $\mathcal{H}^{n-1}(\partial F_{\bar{s}} \cap B(\hat{y}, r))$  can be estimated from below with the surface of the corresponding discs, that is,  $\alpha_n r^{n-1} \sqrt{1 - t^2 \zeta^2 / 4}^{n-1}$ .

Let us now estimate from below the surface  $\mathcal{H}^{n-1}(\partial F_{\bar{s}} \cap Q \setminus B(\hat{y}, r))$ . Since we know that given any  $\hat{y}' \in (0, \theta)^{n-1} \setminus (D' \cup D'')$ ,  $\partial F_{\bar{s}} \cap (\{\hat{y}'\} \times \mathbb{R}) \cap Q \neq \emptyset$ , it is enough to estimate from above the projection of  $\mathcal{S}$  onto  $(0, \theta)^{n-1}$ , which we denote by  $\Pi_{\bar{\nu}}(\mathcal{S})$ . This, in turn, is bounded by the projection of

$$B(\hat{y}, r) \cap \left\{ y : |(y - \hat{y}) \cdot \bar{\nu}| < \frac{t\zeta}{2}r \right\} = \left\{ \hat{y} + r(s\bar{\nu} + \xi) : |s| < \frac{t\zeta}{2}, |\xi| \leq \sqrt{1 - s^2}, \xi \cdot \bar{\nu} = 0 \right\}.$$



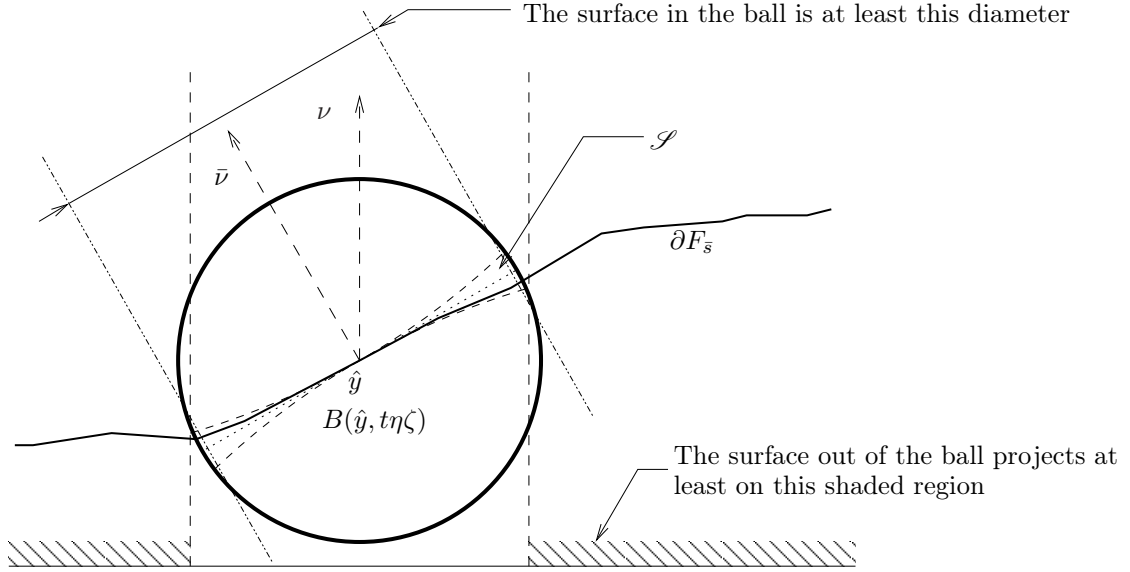


FIGURE 1. If the normal at  $\hat{y}$  to  $\partial F_{\bar{s}}$  is away from  $\nu$ , then its surface must exceed  $\theta^{n-1}$  by some quantity which is estimated.

Now, this projection is a subset of the vertical projection of the diameter of  $B(\hat{y}, r)$  perpendicular to  $\bar{\nu}$ , that is,  $\Delta = \{\hat{y} + r\xi : |\xi| \leq 1, \xi \cdot \bar{\nu} = 0\}$ , plus the disk  $\Pi_{\nu}(B(0, rt\zeta/2))$ . It follows (see the expansion of the volume of Minkowski sums of convex sets in [11], *cf* Remark 3.6) that

$$|\Pi_{\nu}(\mathcal{S})| \leq |\Pi_{\nu}(\Delta)| + \text{Per}(\Pi_{\nu}(\Delta)) \frac{rt\zeta}{2} + o(rt\zeta).$$

Here,  $\text{Per}(\Pi_{\nu}(\Delta))$  is the  $(n-2)$ -dimensional perimeter of  $\Pi_{\nu}(\Delta)$  in  $(0, \theta)^{n-1}$ , and a simple scaling argument shows that  $o(rt\zeta)$  is of the form  $r^{n-1}o(t\zeta)$ , where the latter “ $o$ ” depends only on the geometry of the vertical projection of the unit ball, that is, on  $\nu \cdot \bar{\nu}$  — and, in fact, would be largest for  $\nu = \bar{\nu}$ . Now, since  $\text{Per}(\Pi_{\nu}(\Delta)) \leq \mathcal{H}^{n-2}(\partial\Delta) = (n-1)\alpha_n r^{n-2}$ , we obtain that

$$|\Pi_{\nu}(\mathcal{S})| \leq (\nu \cdot \bar{\nu})\alpha_n r^{n-1} + 2(n-1)\alpha_n r^{n-2} \frac{rt\zeta}{2} \leq \alpha_n r^{n-1} (1 - (1 - (n-1)t)\zeta)$$

if  $t$  is small enough. It follows

(4.28)

$$\mathcal{H}^{n-1}(\partial F_{\bar{s}} \cap Q) \geq \theta^{n-1} - K'(\sqrt{\delta_k} + \sqrt{\delta'_k}) + \alpha_n r^{n-1} \left( \sqrt{1 - \frac{t^2 \zeta^2}{4}} - 1 + (1 - (n-1)t)\zeta \right).$$

For  $t = 0$ , the quantity between the right-hand side parentheses is  $\zeta > 0$ , and it decreases with  $t$ . It follows that one can find  $\bar{t} > 0$  (depending only on  $n$  and  $\zeta$ ) such that (4.28) reads

$$\mathcal{H}^{n-1}(\partial F_{\bar{s}} \cap Q) \geq \theta^{n-1} - K'(\sqrt{\delta_k} + \sqrt{\delta'_k}) + \alpha_n (\bar{t}\eta\zeta)^{n-1} \frac{\zeta}{2}.$$

Together with (4.27), it follows that if  $k$  is large enough (depending on  $K', \eta, \zeta$ ), we get a contradiction, and therefore  $\nu_{\partial F_{\bar{s}}}(\hat{y}) \cdot \nu \geq 1 - \zeta$  for any  $\hat{y} \in \partial F_{\bar{s}} \cap Q$  at distance at least  $\bar{t}\eta\zeta$  from  $\partial Q$ . (The normal cannot be reverted, nor can  $\hat{y}$  be close to the boundary  $\partial Q \cap \partial B(0, 1)$ , because of (4.20).)

We deduce that there exists a value  $\sigma$  such that

$$\partial F_{\bar{s}} \cap Q \subset \{\hat{y} \in Q : \sigma \leq \hat{y} \cdot \nu \leq \sigma + 2\zeta\theta\}$$

for  $k$  large enough, with moreover  $F \cap Q \subset F_{\bar{s}} \cap Q \subset \{\hat{y} \in Q : \hat{y} \cdot \nu \leq \sigma + 2\zeta\theta\}$ . In particular, it follows that  $(F \cap Q) + C \subset \{\hat{y} \cdot \nu \leq \phi(\nu) + \sigma + 2\zeta\theta\}$ . A consequence is that

$$\begin{aligned} ((F + C) \setminus F) \cap Q &\subseteq ((F_{\bar{s}} \setminus F) \cap Q) \cup ((F_b \setminus F) \cap Q) \setminus ((b, \theta - b)^{n-1} \times \mathbb{R}) \\ &\cup \{\hat{y} \in Q : \sigma \leq \hat{y} \cdot \nu \leq \sigma + 2\zeta\theta + \phi(\nu)\}. \end{aligned}$$

Now, using again (4.21) we can show that

$$|((F_b \setminus F) \cap Q) \setminus ((b, \theta - b)^{n-1} \times \mathbb{R})| \leq 2nb^2\theta^{n-2} + 2b\sqrt{\frac{\delta_k}{\beta}},$$

while, using (4.26) and (4.27),

$$|(F_{\bar{s}} \setminus F) \cap Q| \leq 2\eta\theta^{n-1} + (b+2)K' \left( \sqrt{\delta_k} + \sqrt{\delta'_k} \right).$$

We deduce that

$$|((F + C) \setminus F) \cap Q| \leq \theta^{n-1}(2\zeta\theta + \phi(\nu)) + 2\eta\theta^{n-1} + 2nb^2\theta^{n-2} + \mathcal{R}_k$$

where  $\mathcal{R}_k$  is a rest which goes to zero with  $\delta_k$  and  $\delta'_k$ . Returning to the original sets  $C_i^k$ , we find that if  $k, i \in \tilde{Z}_k$  and  $k$  is large enough,

$$(4.29) \quad \frac{1}{\varepsilon'_k} \int_{C_i^k} |\chi_{E_k + \varepsilon'_k C} - \chi_{E_k}| dy \leq |C_i^k| \left( \phi(\nu) + 2\zeta\theta + 2\eta + \frac{2nb^2}{\theta} + \frac{1}{\theta^{n-1}} \mathcal{R}_k \right).$$

Together with (4.16), (4.29) yields that

$$\limsup_{k \rightarrow 0} \frac{1}{\varepsilon'_k} \int_{C(0, 1-\beta)} |\chi_{E_k + \varepsilon'_k C} - \chi_{E_k}| dy \leq \alpha_n (1-\beta)^{n-1} \left( \phi(\nu) + 2\zeta\theta + 2\eta + \frac{2nb^2}{\theta} \right).$$

Sending first  $\zeta$ , then  $\eta$  to zero and eventually  $\theta$  to  $+\infty$ , and using (4.6) and (4.9), we deduce

$$g(\bar{x}) = \lim_{k \rightarrow 0} \frac{1}{\varepsilon'_k} \int_{B(0,1)} |\chi_{E_k + \varepsilon'_k C} - \chi_{E_k}| dy \leq b\alpha_n (1 - (1-\beta)^{n-1}) + \alpha_n (1-\beta)^{n-1} \phi(\nu),$$

and letting then  $\beta \rightarrow 0$  yields the desired inequality.

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