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# Multiplicative spectrum of ultrametric Banach algebras of continuous functions

by Alain Escassut and Nicolas Mainetti

**Abstract** Let  $K$  be an ultrametric complete field and let  $E$  be an ultrametric space. Let  $A$  be the Banach  $K$ -algebra of bounded continuous functions from  $E$  to  $K$  and let  $B$  be the Banach  $K$ -algebra of bounded uniformly continuous functions from  $E$  to  $K$ . Maximal ideals and continuous multiplicative semi-norms on  $A$  (resp. on  $B$ ) are studied by defining relations of stickiness and contiguousness on ultrafilters that are equivalence relations. So, the maximal spectrum of  $A$  (resp. of  $B$ ) is in bijection with the set of equivalence classes with respect to stickiness (resp. to contiguousness). Every prime ideal of  $A$  or  $B$  is included in a unique maximal ideal and every prime closed ideal of  $A$  (resp. of  $B$ ) is a maximal ideal, hence every continuous multiplicative semi-norm on  $A$  (resp. on  $B$ ) has a kernel that is a maximal ideal. If  $K$  is locally compact, every maximal ideal of  $A$ , (resp. of  $B$ ) is of codimension 1. Every maximal ideal of  $A$  or  $B$  is the kernel of a unique continuous multiplicative semi-norm and every continuous multiplicative semi-norm is defined as the limit along an ultrafilter on  $E$ . Consequently, on  $A$  as on  $B$  the set of continuous multiplicative semi-norms defined by points of  $E$  is dense in the whole set of all continuous multiplicative semi-norms. Ultrafilters show bijections between the set of continuous multiplicative semi-norms of  $A$ ,  $Max(A)$  and the Banaschewski compactification of  $E$  which is homeomorphic to the topological space of continuous multiplicative semi-norms. The Shilov boundary of  $A$  (resp.  $B$ ) is equal to the whole set of continuous multiplicative semi-norms.

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## Introduction and preliminaries:

### Definitions and notation:

Let  $K$  be a field complete with respect to an ultrametric absolute value  $|\cdot|$ . It is well known that the set of maximal ideals is not sufficient to describe spectral properties of an ultrametric Banach algebra: we have to consider the set of continuous multiplicative semi-norms [6], [7], [9], [10], [11], ]. Many studies were made on continuous multiplicative semi-norms on algebras of analytic functions, analytic elements and their applications to holomorphic functional calculus [3], [5], [6]. Here we mean to study continuous multiplicative semi-norms on Banach algebras of continuous functions. We will consider two main cases: Banach algebras of bounded continuous functions and Banach algebras of bounded uniformly continuous functions (with an application to Banach algebras of bounded functions).

**Definitions and notation:** Let  $E$  denote a metric space whose distance  $\delta$  is ultrametric, let  $A$  be the Banach  $K$ -algebra of bounded continuous functions from  $E$  to  $K$  and let  $B$  be the Banach  $K$ -algebra of bounded uniformly continuous functions from  $E$  to  $K$ .

We will call *clopen* any closed open subset of  $E$ . Let  $H$  be a subset of  $E$  different from  $E$  and  $\emptyset$ . We will call *codiameter of  $H$*  the number  $\delta(H, E \setminus H)$  and we will denote it by  $\text{codiam}(H)$ . The subset  $H$  will be said to be *uniformly open* if  $\text{codiam}(H) > 0$ . Given  $a \in E$  and  $r > 0$  we will denote by  $d(a, r^-)$  the ball  $\{x \in E \mid \delta(a, x) < r\}$ . Let  $\mathcal{F}$  be a filter on  $E$ . Given a function  $f$  from  $E$  to  $K$  admitting a limit along  $\mathcal{F}$ , we will denote by  $\lim_{\mathcal{F}} f(x)$  that limit.

Given a set  $F$ , we shall denote by  $U(F)$  the set of ultrafilters on  $F$ . Now, let  $X$  be a topological space. Given  $\mathcal{F} \in U(X)$ , we will denote by  $\overline{\mathcal{F}}$  the filter generated by the closures of elements of  $\mathcal{F}$ . Two ultrafilters  $\mathcal{F}, \mathcal{G}$  on  $F$  will be said to be *sticked* if  $\overline{\mathcal{F}}$  and  $\overline{\mathcal{G}}$  are secant. We will denote by  $(\mathcal{S})$  the relation defined on  $U(X)$  as  $\mathcal{U}(\mathcal{S})\mathcal{V}$  if  $\mathcal{U}$  and  $\mathcal{V}$  are stucked.

Next, two filters  $\mathcal{F}, \mathcal{G}$  on  $E$  will be said to be *contiguous* if for every  $H \in \mathcal{F}, L \in \mathcal{G}$ , we have  $\delta(H, L) = 0$ . We shall denote by  $(\mathcal{T})$  the relation defined on  $U(E)$  as  $\mathcal{U}(\mathcal{T})\mathcal{V}$  if  $\mathcal{U}$  and  $\mathcal{V}$  are contiguous.

An ultrafilter  $\mathcal{U}$  on the set  $E$  is said to be *principal* if there exists  $a \in E$  such that  $\mathcal{U} = \{H \subset E \mid a \in H\}$ .

A closed open set of  $E$  is called a *clopen*.

**Remark 1:** Let  $H \subset E$  be different from  $\emptyset$  and from  $E$ . Then  $\text{codiam}(H) = \text{codiam}(E \setminus H)$ .

**Remark 2:** Two stucked filters on  $E$  are contiguous.

**Remark 3:** A uniformly open subset of  $E$  is open and closed.

**Remark 4:** Let  $\mathcal{U}, \mathcal{V}$  be contiguous ultrafilters on  $E$  and assume  $\mathcal{U}$  is convergent. Then  $\mathcal{V}$  is convergent and has the same limit as  $\mathcal{U}$ . Moreover  $\mathcal{U}$  and  $\mathcal{V}$  are stucked.

**Remark 5:** We can construct contiguous ultrafilters that are not stucked. Let  $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$  be sequences in  $K$  such that

- (1)  $|a_n| < |a_{n+1}|$ ,
- (2)  $\lim_{n \rightarrow +\infty} a_n - b_n = 0$ ,
- (3)  $|a_n - b_n| \geq \frac{1}{n}$ .

Let  $\mathcal{U}$  be a filter thinner than the sequence  $(a_n)$ . We will define a filter  $\tilde{\mathcal{U}}$  thinner than the sequence  $(b_n)$  completing the example. By definition,  $\mathcal{U}$  admits a basis made of images of subsequences  $(a_{\sigma(n)})_{n \in \mathbb{N}}$  of the sequence  $(a_n)$ . Let  $\mathcal{Z}$  be the family of images of such sequences, making a basis of  $\mathcal{U}$ .

Given such a subsequence  $(a_{\sigma(m)})_{m \in \mathbb{N}}$ , set  $Q = \{a_{\sigma(m)} \mid m \in \mathbb{N}\}$  and set  $\tilde{Q} = \{b_{\sigma(m)} \mid m \in \mathbb{N}\}$ . Let  $\tilde{\mathcal{U}}$  be the filter admitting for basis the family  $\{\tilde{Q}, Q \in \mathcal{Z}\}$ .

Then we can check that  $\mathcal{U}$  is an ultrafilter if and only if so is  $\tilde{\mathcal{U}}$ . Indeed, suppose  $\mathcal{U}$  is an ultrafilter and suppose  $\tilde{\mathcal{U}}$  is not. Let  $\mathcal{V}$  be an ultrafilter strictly thinner than  $\tilde{\mathcal{U}}$ .

Let  $X \in \mathcal{V} \setminus \tilde{\mathcal{U}}$ . Since  $\mathcal{V}$  is thinner than  $\tilde{\mathcal{U}}$ , we may assume that  $X$  is the image of a subsequence  $(b_{\tau(m)})_{m \in \mathbb{N}}$  of the sequence  $(b_n)_{n \in \mathbb{N}}$  where this image  $\{b_{\tau(m)} \mid m \in \mathbb{N}\}$  is strictly included in the image  $\{b_{\sigma(m)} \mid m \in \mathbb{N}\}$  of a subsequence  $(b_{\sigma(m)})_{m \in \mathbb{N}}$ . But then, the set  $\{a_{\tau(m)} \mid m \in \mathbb{N}\}$  is strictly included in  $\{a_{\sigma(m)} \mid m \in \mathbb{N}\}$  which belongs to  $\mathcal{U}$ . But  $\{a_{\tau(m)} \mid m \in \mathbb{N}\}$  doesn't belong to  $\mathcal{U}$  because if it belonged to  $\mathcal{U}$ , then  $\{b_{\tau(m)} \mid m \in \mathbb{N}\}$  would belong to  $\tilde{\mathcal{U}}$ , which is excluded by hypothesis. But then, the filter generated by  $\mathcal{U}$  and the set  $\{a_{\tau(m)} \mid m \in \mathbb{N}\}$  is strictly thinner than  $\mathcal{U}$ , a contradiction since  $\mathcal{U}$  is an ultrafilter.

Thus we have proved that if  $\mathcal{U}$  is an ultrafilter so is  $\tilde{\mathcal{U}}$ . The converse is obvious.

Now, by (2),  $\mathcal{U}$  and  $\tilde{\mathcal{U}}$  are contiguous. Next, by (1) and (3), both  $Q, \tilde{Q}$  are closed sets such that  $Q \cap \tilde{Q} = \emptyset$  which shows that  $\mathcal{U}$  and  $\tilde{\mathcal{U}}$  are not stucked.

**Remark 6:** Relation  $(\mathcal{S})$  is not the equality between ultrafilters, even when the ultrafilters are not convergent. In [13], Labib Haddad introduced the following equivalence relation  $(\mathcal{L})$  on ultrafilters. Given two ultrafilters  $\mathcal{U}, \mathcal{V}$  we write  $\mathcal{U}(\mathcal{L})\mathcal{V}$  if there exists an ultrafilter  $\mathcal{W}$  such that every closed set  $H$  lying in  $\mathcal{W}$  also lies in  $\mathcal{U}$  and similarly, every closed set  $H$  lying in  $\mathcal{W}$  also lies in  $\mathcal{V}$ . So, Relation  $(\mathcal{L})$  is clearly thinner than Relation  $(\mathcal{S})$ . However, it is shown that two ultrafilters  $\mathcal{U}, \mathcal{V}$  satisfying  $\mathcal{U}(\mathcal{L})\mathcal{V}$  may be distinct without converging.

## Basic results:

First, we will show that Relation  $(\mathcal{S})$  may be also defined in terms of clopens.

**Theorem 1:** *Two ultrafilters  $\mathcal{U}, \mathcal{V}$  are stucked if and only if for any two clopens  $H \in \mathcal{U}, L \in \mathcal{V}$ , we have  $H \cap L \neq \emptyset$ .*

Theorem 2 may be viewed as a particular version of a theorem due to Urysohn, although it is not a direct consequence of a theorem of Urysohn because Urysohn's theorem only concerns functions with values in  $[0, 1]$ .

**Theorem 2:** *Let  $\mathcal{U}, \mathcal{V}$  be two ultrafilters on  $E$  that are not stucked. There exist  $H \in \mathcal{U}, L \in \mathcal{V}$  and  $f \in A$  such that  $f(x) = 1 \forall x \in H, f(x) = 0 \forall x \in L$ .*

**Theorem 3:** *Let  $\mathcal{U}, \mathcal{V}$  be two ultrafilters on  $E$  that are not contiguous. There exist  $H \in \mathcal{U}, L \in \mathcal{V}$  and  $f \in B$  such that  $f(x) = 1 \forall x \in H, f(x) = 0 \forall x \in L$ .*

**Notation:** Let  $T$  be a  $K$ -algebra of bounded functions from  $E$  to  $K$ .

Given a filter  $\mathcal{F}$  on  $E$ , we will denote by  $\mathcal{I}(\mathcal{F}, T)$  the ideal of the  $f \in T$  such that  $\lim_{\mathcal{F}} f(x) = 0$ . We will denote by  $\mathcal{I}^*(\mathcal{F}, T)$  the ideal of the  $f \in T$  such that there exists a subset  $L \in \mathcal{F}$  such that  $f(x) = 0 \forall x \in L$ .

Given  $a \in E$  we will denote by  $\mathcal{I}(a, T)$  the ideal of the  $f \in T$  such that  $f(a) = 0$ .

We will denote by  $Max(T)$  the set of maximal ideals and by  $Max_E(T)$  the set of maximal ideals of the form  $\mathcal{I}(a, T)$ ,  $a \in E$ .

Proposition A is easy:

**Proposition A:** *Let  $T$  be a  $K$ -algebra of bounded functions from  $E$  to  $K$ . Given an ultrafilter  $\mathcal{U}$  on  $E$ ,  $\mathcal{I}(\mathcal{U}, T)$ ,  $\mathcal{I}^*(\mathcal{U}, T)$  are prime ideals.*

**Theorem 4:** *Let  $\mathcal{U}$ ,  $\mathcal{V}$  be two ultrafilters on  $E$ . Then  $\mathcal{I}(\mathcal{U}, A) = \mathcal{I}(\mathcal{V}, A)$  if and only if  $\mathcal{U}$  and  $\mathcal{V}$  are sticked. Further,  $\mathcal{I}(\mathcal{U}, B) = \mathcal{I}(\mathcal{V}, B)$  if and only if  $\mathcal{U}$  and  $\mathcal{V}$  are contiguous.*

**Corollary 4.1:** *Both Relations  $(\mathcal{S})$ ,  $(\mathcal{T})$  are equivalence relations on  $U(E)$ .*

**Remark 7 :** Relations  $(\mathcal{S})$ ,  $(\mathcal{T})$  are not transitive when applying to the set of all filters on  $E$ . However, given a topological space  $X$  satisfying the normality axiom, (i.e. any two closed disjoint subsets  $H$ ,  $L$  admit disjoint open neighborhoods), then  $(\mathcal{S})$  is transitive for ultrafilters and therefore is an equivalence relation on  $U(X)$ . Similarly, given a metric space  $X$ , then  $(\mathcal{T})$  is transitive for ultrafilters and therefore is an equivalence relation on  $U(X)$  [13].

**Notation:** We will denote by  $Y_{(\mathcal{S})}(E)$  the set of equivalence classes on  $U(E)$  with respect to Relation  $(\mathcal{S})$  and by  $Y_{(\mathcal{T})}(E)$  the set of equivalence classes on  $U(E)$  with respect to Relation  $(\mathcal{T})$ .

Let  $f \in A$  and let  $\epsilon$  be  $> 0$ . We set  $D(f, \epsilon) = \{x \in E \mid |f(x)| \leq \epsilon\}$ .

We will need the following Proposition that is immediate:

**Proposition B:** *Let  $\mathcal{M}$  be a maximal ideal of  $A$  (resp. of  $B$ ) of the form  $\mathcal{I}(\mathcal{U}, A)$  (resp.  $\mathcal{I}(\mathcal{U}, B)$ ). If  $\mathcal{U}$  converges in  $E$  then  $\mathcal{M}$  is of codimension 1.*

## Main Theorems:

Theorem 5 looks like certain Bezout-Corona statements [10], [14]:

**Theorem 5:** *Let  $f_1, \dots, f_q \in A$  (resp.  $f_1, \dots, f_q \in B$ ) satisfy*

$$\inf_{x \in E} (\max_{1 \leq j \leq q} |f_j(x)|) > 0.$$

*Then there exists  $g_1, \dots, g_q \in A$  (resp.  $g_1, \dots, g_q \in B$ ) such that*

$$\sum_{j=1}^q f_j(x)g_j(x) = 1 \quad \forall x \in E.$$

**Corollary 5.1:** *Let  $I$  be an ideal of  $A$  (resp. of  $B$ ) different from  $A$  (resp. from  $B$ ). The family  $D(f, \epsilon)$ ,  $f \in I$ ,  $\epsilon > 0$ , generates a filter  $\mathcal{F}_{I,A}$  (resp.  $\mathcal{F}_{I,B}$ ) on  $E$  such that  $I \subset \mathcal{I}(\mathcal{F}_{I,A}, A)$  (resp.  $I \subset \mathcal{I}(\mathcal{F}_{I,B}, B)$ ).*

By Proposition B, we now have Corollary 5.2:

**Corollary 5.2:** *Let  $\mathcal{M}$  be a maximal ideal of  $A$ . There exists an ultrafilter  $\mathcal{U}$  on  $E$  such that  $\mathcal{M} = \mathcal{I}(\mathcal{U}, A)$ . Moreover, if  $\mathcal{U}$  converges in  $E$ , then  $\mathcal{M}$  is of codimension 1.*

**Corollary 5.3:** *Suppose  $E$  is complete. Let  $\mathcal{M}$  be a maximal ideal of  $A$  and let  $\mathcal{U}$  be an ultrafilter on  $E$  such that  $\mathcal{M} = \mathcal{I}(\mathcal{U}, A)$ . If  $\mathcal{U}$  is a Cauchy ultrafilter, then  $\mathcal{M}$  is of codimension 1.*

**Corollary 5.4:** *For every maximal ideal  $\mathcal{M}$  of  $A$ , there exists a unique  $\mathcal{H} \in Y_{(\mathcal{S})}(E)$  such that  $\mathcal{M} = \mathcal{I}(\mathcal{U}, A) \forall \mathcal{U} \in \mathcal{H}$ .*

*Moreover, the mapping  $\Phi$  that associates to each  $\mathcal{M} \in \text{Max}(A)$  the unique  $\mathcal{H} \in Y_{(\mathcal{S})}(E)$  such that  $\mathcal{M} = \mathcal{I}(\mathcal{U}, A) \forall \mathcal{U} \in \mathcal{H}$ , is a bijection from  $\text{Max}(A)$  onto  $Y_{(\mathcal{S})}(E)$ .*

In the particular case when we consider the discrete topology on  $E$ , we have Corollary 5.5:

**Corollary 5.5:** *For every maximal ideal  $\mathcal{M}$  of the Banach  $K$ -algebra  $T$  of all bounded functions on  $E$ , there exists a unique ultrafilter  $\mathcal{U}$  on  $E$  such that  $\mathcal{M} = \mathcal{I}(\mathcal{U}, T)$ .*

*Moreover, the mapping  $\Phi$  that associates to each  $\mathcal{M} \in \text{Max}(T)$  the unique  $\mathcal{U}$  such that  $\mathcal{M} = \mathcal{I}(\mathcal{U}, T) \forall \mathcal{U} \in \mathcal{H}$ , is a bijection from  $\text{Max}(T)$  onto  $U(E)$ .*

**Theorem 6:** *Let  $\mathcal{M}$  be a maximal ideal of  $B$ . There exists an ultrafilter  $\mathcal{U}$  on  $E$  such that  $\mathcal{M} = \mathcal{I}(\mathcal{U}, B)$ . Moreover, if  $\mathcal{U}$  is a cauchy ultrafilter, then  $\mathcal{M}$  is of codimension 1.*

**Corollary 6.1:** *For every maximal ideal  $\mathcal{M}$  of  $B$  there exists a unique  $\mathcal{H} \in Y_{(\mathcal{T})}(E)$  such that  $\mathcal{M} = \mathcal{I}(\mathcal{U}, B) \forall \mathcal{U} \in \mathcal{H}$ .*

*Moreover, the mapping  $\Psi$  that associates to each  $\mathcal{M} \in \text{Max}(B)$  the unique  $\mathcal{H} \in Y_{(\mathcal{T})}(E)$  such that  $\mathcal{M} = \mathcal{I}(\mathcal{U}, B) \forall \mathcal{U} \in \mathcal{H}$ , is a bijection from  $\text{Max}(B)$  onto  $Y_{(\mathcal{T})}(E)$ .*

**Theorem 7:** *Let  $K$  be a locally compact field. Then every maximal ideal of  $A$  (resp.  $B$ ) is of codimension 1.*

The Banaschewski compactification of  $E$  is directly linked to  $\text{Max}(A)$ .

**Definition and notation:** Let  $\mathcal{B}(E)$  be the boolean ring of clopens provided with the classical laws  $\Delta$  (the symmetrical difference taking place of the addition) and  $\cap$  (taking place of the multiplication).

We will denote by  $\Sigma(E)$  the set of homomorphisms from  $\mathcal{B}(E)$  to  $\mathbb{F}_2$ :  $\Sigma(E)$  is also called *the Stone space of  $\mathcal{B}(E)$*  and is provided with the topology of simple convergence, while  $\mathbb{F}_2$  is provided with the discrete topology, so  $\Sigma(E)$  is compact in the compact space  $\mathbb{F}_2^{\mathcal{B}(E)}$ .

Given  $a \in E$ , we denote by  $\zeta_a$  the ring homomorphism from  $\mathcal{B}(E)$  to  $\mathbb{F}_2$  defined as  $\zeta_a(O) = 1 \forall O \in \mathcal{B}(E)$  such that  $a \in O$  and  $\zeta_a(O) = 0 \forall O \in \mathcal{B}(E)$  such that  $a \notin O$ .

We will denote by  $\Sigma_E(E)$  the set of the  $\zeta_a$ ,  $a \in E$ .

We know the following proposition [15]:

**Proposition C :** *There is a natural bijection between  $\Sigma(E)$  and  $Max(A)$ . Moreover,  $\Sigma(E)$  is compact and  $\Sigma_E(E)$  is dense in  $\Sigma(E)$ .*

Here we can describe more precisely this bijection thanks to the following theorem:

**Theorem 8 :** *Let  $\mathcal{U}, \mathcal{V}$  be sticked ultrafilters on  $E$ . Then a clopen belongs to  $\mathcal{U}$  if and only if it belongs to  $\mathcal{V}$ .*

**Corollary 8.1** *For every maximal ideal  $\mathcal{M}$  of  $A$  and  $\mathcal{U}, \mathcal{V} \in \Phi(\mathcal{M})$ , then a clopen belongs to  $\mathcal{U}$  if and only if it belongs to  $\mathcal{V}$ .*

**Notation:** Given  $\mathcal{M} \in Max(A)$  we will denote by  $\Xi(\mathcal{M})$  the mapping from  $\mathcal{B}(E)$  to  $\mathbb{F}_2$  defined as  $\Xi(\mathcal{M})(O) = 1$  whenever any  $\mathcal{U} \in \Phi(\mathcal{M})$  is secant with  $O$  and  $\Xi(\mathcal{M})(O) = 0$  whenever any  $\mathcal{U} \in \Phi(\mathcal{M})$  is not secant with  $O$ .

**Theorem 9:** *Given  $\mathcal{M} \in Max(A)$ ,  $\Xi(\mathcal{M})$  is a ring homomorphism from  $\mathcal{B}(E)$  onto  $\mathbb{F}_2$ .*

**Theorem 10:**  *$\Xi$  is a bijection from  $Max(A)$  onto  $\Sigma(E)$ . Given  $a \in E$  then  $\Xi(\mathcal{I}(a, A))$  is  $\zeta_a$  defined above. The restriction of  $\Xi$  to  $Max_E(A)$  is a bijection from  $Max_E(A)$  onto  $\Sigma_E(E)$  and  $\Sigma_E(E)$  is dense in  $\Sigma(E)$ .*

**Definition:**  $\Sigma(E)$  is called *the Banaschewski compactification of  $E$* .

We will now examine prime closed ideals of  $A$  and  $B$ .

**Theorem 11:** *Let  $\mathcal{U}$  be an ultrafilter on  $E$  and let  $\mathcal{P}$  be a prime ideal included in  $\mathcal{I}(\mathcal{U}, A)$  (resp.  $\mathcal{I}(\mathcal{U}, B)$ ). Let  $L \in \mathcal{U}$  be a clopen (resp. let  $L \in \mathcal{U}$  be uniformly open) and let  $H = E \setminus L$ . Let  $u$  be the function defined on  $E$  by  $u(x) = 1 \forall x \in H$ ,  $u(x) = 0 \forall x \in L$ . Then  $u$  belongs to  $\mathcal{P}$ .*

**Corollary 11.1:** *Let  $\mathcal{U}$  be an ultrafilter on  $E$  and let  $\mathcal{I}^{**}(\mathcal{U}, A)$  the ideal of the  $f \in A$  such that there exists a clopen  $H \in \mathcal{U}$  such that  $f(x) = 0 \forall x \in H$ . Then  $\mathcal{I}^{**}(\mathcal{U}, A)$  is included in every prime ideal  $\mathcal{P} \subset \mathcal{I}(\mathcal{U}, A)$ .*

**Corollary 11.2:** *Let  $\mathcal{U}$  be an ultrafilter on  $E$  and let  $T$  be the Banach  $K$ -algebra of all bounded functions on  $E$ . Then  $\mathcal{I}^*(\mathcal{U}, T)$  is the smallest prime ideal among all prime ideals  $\mathcal{P} \subset \mathcal{I}(\mathcal{U}, T)$ .*

**Corollary 11.3:** *Let  $\mathcal{U}$  be an ultrafilter on  $E$  and let  $\mathcal{I}^{***}(\mathcal{U}, B)$  the ideal of the  $f \in B$  such that there exists a uniformly open subset  $H \in \mathcal{U}$  such that  $f(x) = 0 \forall x \in H$ . Then  $\mathcal{I}^{***}(\mathcal{U}, B)$  is included in every prime ideal  $\mathcal{P} \subset \mathcal{I}(\mathcal{U}, B)$ .*

**Theorem 12:** *The closure of a prime ideal of  $A$  (resp. of  $B$ ) is a maximal ideal.*

**Corollary 12.1:** *Let  $\mathcal{P}$  be a prime ideal of  $A$  (resp. of  $B$ ). There exists a unique maximal ideal  $\mathcal{M}$  of  $A$  (resp. of  $B$ ) containing  $\mathcal{P}$ .*

**Corollary 12.2:** *Every prime closed ideal of  $A$  (resp. of  $B$ ) is a maximal ideal.*

Now, since the kernel of a continuous multiplicative semi-norm is a closed prime ideal, we will show Corollary 12.3:

**Notation and definition:** Let  $T$  be a normed commutative  $K$ -algebra with unity. We denote by  $Mult(T, \|\cdot\|)$  the set of multiplicative semi-norms of  $T$  provided with the topology of simple convergence. Given  $\phi \in Mult(T, \|\cdot\|)$ , the set of the  $x \in T$  such that  $\phi(x) = 0$  is a closed prime ideal and is called *the kernel of  $\phi$* . It is denoted by  $Ker(\phi)$ .

We denote by  $Mult_m(T, \|\cdot\|)$  the set of multiplicative semi-norms of  $T$  whose kernel is a maximal ideal and by  $Mult_1(T, \|\cdot\|)$  the set of multiplicative semi-norms of  $T$  whose kernel is a maximal ideal of codimension 1.

Suppose now  $T$  is a  $K$ -algebra of bounded functions from  $E$  to  $K$  normed by the norm of uniform convergence on  $E$ . Let  $a \in E$ . The mapping  $\varphi_a$  from  $T$  to  $\mathbb{R}$  defined by  $\varphi_a(f) = |f(a)|$  belongs to  $Mult(T, \|\cdot\|)$ .

Let  $\mathcal{U}$  be an ultrafilter on  $E$ . By Urysohn's Theorem, given  $f \in T$ , the mapping from  $E$  to  $\mathbb{R}$  that sends  $x$  to  $|f(x)|$  admits a limit along  $\mathcal{U}$ . We set  $\varphi_{\mathcal{U}}(f) = \lim_{\mathcal{U}} |f(x)|$ .

Propositions D, E below are immediate and well known:

**Proposition D:** *Let  $T = A$  or  $B$  and let  $a \in E$ . Then  $\mathcal{I}(a, T)$  is a maximal ideal of  $T$  of codimension 1 and  $\varphi_a$  belongs to  $Mult_1(T, \|\cdot\|)$ .*

**Notation:** Let  $T = A$  or  $B$ . We denote by  $Mult_E(T, \|\cdot\|)$  the set of multiplicative semi-norms of  $T$  of the form  $\varphi_a$ ,  $a \in E$ . Consequently, by definition,  $Mult_E(T, \|\cdot\|)$  is a subset of  $Mult_1(T, \|\cdot\|)$ .

Proposition E is immediate:

**Proposition E :** *Let  $T = A$  or  $B$  and let  $a \in E$  and let  $\mathcal{U}$  be an ultrafilter on  $E$ . Then  $\varphi_{\mathcal{U}}$  belongs to the closure of  $Mult_E(T, \|\cdot\|)$ .*

Now, Corollaries 12.3 is an immediate consequence of Theorem 12 and Propositions D, E:

**Corollary 12.3 :**  *$Mult(A, \|\cdot\|) = Mult_m(A, \|\cdot\|)$ ,  $Mult(B, \|\cdot\|) = Mult_m(B, \|\cdot\|)$ . Further, if  $K$  is locally compact then  $Mult(A, \|\cdot\|) = Mult_1(A, \|\cdot\|)$ ,  $Mult(B, \|\cdot\|) = Mult_1(B, \|\cdot\|)$ .*



**Remark 8:** Suppose  $K$  is locally compact and  $E$  is a disk in an algebraically closed complete ultrametric field. There do exist ultrafilters that do not converge. Let  $\mathcal{U}$  be such an ultrafilter. Then  $\varphi_{\mathcal{U}}$  belongs to  $Mult_1(B, \|\cdot\|)$  but does not belong to  $Mult_E(B, \|\cdot\|)$ .

**Remark 9:** In  $\mathcal{H} \in Y_{(\mathcal{S})}(E)$ , the various ultrafilters  $\mathcal{U} \in \mathcal{H} \in Y_{(\mathcal{S})}(E)$  define various prime ideals of  $A$ . It is not clear whether these ideals are minimal among the set of prime ideals of  $A$ . Similarly, in  $\mathcal{H} \in Y_{(\mathcal{T})}(E)$  the various ultrafilters  $\mathcal{U} \in \mathcal{H} \in Y_{(\mathcal{T})}(E)$  define various prime ideals of  $B$  and it is not clear whether these ideals are minimal among the set of prime ideals of  $B$ .

**Remark 10:** The ideal  $\mathcal{I}^{**}(\mathcal{U}, A)$  is not a prime ideal of  $A$ , as the following example shows.

Suppose  $E$  is the disk  $d(0, 1)$  in the field  $K$  and let  $(a_n)$  be a sequence of limit 0 such that  $|a_n| < |a_{n+1}|$ . Let  $\mathcal{U}$  be an ultrafilter of limit 0. Let  $r_n = |a_n|$ ,  $n \in \mathbb{N}$ , let  $H = \bigcup_{n=0}^{\infty} d(a_n, r_n^-)$  and let  $H' = E \setminus H$ . Let  $f(x) = x \forall x \in H$ ,  $f(x) = 0 \forall x \in H'$  and let  $g(x) = x - f(x)$ . Then  $f(x)g(x) = 0 \forall x \in E$ . However, neither  $f$  nor  $g$  is identically zero on any clopen belonging to  $\mathcal{U}$  because such a clopen must contain the origin that is on the boundary of both  $H$  and  $H'$ .

Similarly,  $\mathcal{I}^{***}(\mathcal{U}, B)$  is not a prime ideal of  $B$ .

**Theorem 13:**  $\Phi \circ \Xi^{-1}$  is a homeomorphism from  $\Sigma(E)$  onto  $Mult(A, \|\cdot\|)$ .

**Corollary 13.1:** The topology of  $Mult(A, \|\cdot\|)$  and this of the Banaschewski compactification induce the same topology on  $Max(A)$ .

**Corollary 13.2:** The topology of  $Mult(A, \|\cdot\|)$  does not depend on the field  $K$ .

**Remark 11:** Let  $F$  be a set compact for two topologies, admitting a subset  $E$  dense for both topologies. In general, we may not conclude that the two topologies are identical, as shows the following example.

Let  $F = [0, 1]$  be provided with the topology  $\mathcal{N}$  induced by this of  $\mathbb{R}$  and let  $E = ]0, 1[$ . Now, let  $\mathcal{Q}$  be the topology on  $F$  defined as follows:

For  $a \in E$ , a neighborhood of  $a$  is a subset of  $F$  containing an open interval included in  $E$ .

A neighborhood of 0 is a subset of  $F$  containing a subset of the form  $\{0\} \cup ]1 - \epsilon, 1[$ .

A neighborhood of 1 is a subset of  $F$  containing a subset of the form  $\{1\} \cup ]0, \epsilon[$ .

So we have defined  $\mathcal{Q}$  a topology on  $F$ . Of course,  $\mathcal{Q}$  is different from  $\mathcal{N}$ . Then  $E$  is obviously dense in  $F$  for  $\mathcal{Q}$ . Next, we can check that  $F$  is compact for  $\mathcal{Q}$ .

**Theorem 14:** Let  $T = A$  or  $B$ . The topology induced on  $E$  by this of  $Mult_E(T, \|\cdot\|)$  is equivalent to the metric topology defined by  $\delta$ .

Theorem 15 was proved in [15] for the algebra  $B$ . We can find it again for  $A$  and  $B$  in a different way.

**Theorem 15:** Let  $T = A$  or  $B$  and let  $\mathcal{M}$  be a maximal ideal of  $T$ . Let  $T'$  be the field  $\frac{T}{\mathcal{M}}$ . Let  $\theta$  be the canonical surjection from  $T$  onto  $T'$ . Given any ultrafilter  $\mathcal{U}$  such that  $\mathcal{I}(\mathcal{U}, T) = \mathcal{M}$ , the quotient norm  $\| \cdot \|'$  on  $T'$  is defined by  $\|\theta(f)\|' = \lim_{\mathcal{U}} |f(s)|$  and hence is multiplicative.

**Definition:** Recall that given a commutative Banach  $K$ -algebra with unity  $T$ , every maximal ideal of  $T$  is the kernel of at least one continuous multiplicative semi-norm [4].  $T$  is said to be *multibjective* if every maximal ideal is the kernel of only one continuous multiplicative semi-norm.

**Remark 12:** There exist ultrametric Banach  $K$ -algebras that are not multibjective [2], [4], [5].

**Theorem 16:**  $A, B$  are multibjective.

**Corollary 16.1:** The  $K$ -algebra of all bounded functions from a set  $X$  to  $K$  is multibjective.

By Corollaries 5.2, 5.5 and Theorem 16, we have Corollary 16.2:

**Corollary 16.2:** For every  $\phi \in \text{Mult}(A, \| \cdot \|)$ , there exists an ultrafilter  $\mathcal{U}$  on  $E$  such that  $\phi(f) = \lim_{\mathcal{U}} |f(x)| \forall f \in A$ .

Moreover, the mapping  $\tilde{\Phi}$  that associates to each  $\phi \in \text{Mult}(A, \| \cdot \|)$  the unique  $\mathcal{H} \in Y_{(\mathcal{S})}(E)$  such that  $\phi(f) = \lim_{\mathcal{U}} |f(x)| \forall f \in A, \forall \mathcal{U} \in \mathcal{H}$ , is a bijection from  $\text{Mult}(A, \| \cdot \|)$  onto  $Y_{(\mathcal{S})}(E)$ .

**Corollary 16.3:** Let  $T$  be the Banach algebra of all bounded functions from  $E$  to  $K$ . For every  $\phi \in \text{Mult}(T, \| \cdot \|)$  there exists a unique ultrafilter  $\mathcal{U}$  on  $F$  such that  $\phi(f) = \lim_{\mathcal{U}} |f(x)| \forall f \in T$ . The mapping  $\tilde{\Phi}$  that associates to each  $\phi \in \text{Mult}(T, \| \cdot \|)$  the unique ultrafilter  $\mathcal{U}$  such that  $\phi(f) = \lim_{\mathcal{U}} |f(x)| \forall f \in T$ , is a bijection from  $\text{Mult}(T, \| \cdot \|)$  onto the set of ultrafilters on  $F$ .

**Corollary 16.4:** For every  $\phi \in \text{Mult}(B, \| \cdot \|)$  there exists a unique  $\mathcal{H} \in Y_{(\mathcal{T})}(E)$  such that  $\phi(f) = \lim_{\mathcal{U}} |f(x)| \forall f \in B, \forall \mathcal{U} \in \mathcal{H}$ .

Moreover, the mapping  $\tilde{\Psi}$  that associates to each  $\phi \in \text{Mult}(B, \| \cdot \|)$  the unique  $\mathcal{H} \in Y_{(\mathcal{T})}(E)$  such that  $\phi(f) = \lim_{\mathcal{U}} |f(x)| \forall f \in B, \forall \mathcal{U} \in \mathcal{H}$ , is a bijection from  $\text{Mult}(B, \| \cdot \|)$  onto  $Y_{(\mathcal{T})}(E)$ .

**Remark 13:** Consider two ultrafilters  $\mathcal{U}, \mathcal{V}$  which are contiguous but not sticked. They define the same maximal ideal and the same multiplicative semi-norm on  $B$ , but not on  $A$ . This means that for every bounded uniformly continuous fonction  $f$  from  $E$  to  $K$ , we have

$\lim_{\mathcal{U}} |f(x)| = \lim_{\mathcal{V}} |f(x)|$ . But there exist bounded continuous functions  $g$  from  $E$  to  $K$  such that  $\lim_{\mathcal{U}} |g(x)| \neq \lim_{\mathcal{V}} |g(x)|$ . Actually, by Theorem 1, we can find a bounded continuous function  $u$  such that  $\lim_{\mathcal{U}} |u(x)| = 1$ ,  $\lim_{\mathcal{V}} |u(x)| = 0$ .

Now, by Propositions D and E, we have Corollary 16.5:

**Corollary 16.5:**  *$Mult_E(A, \|\cdot\|)$  is dense in  $Mult(A, \|\cdot\|)$ ,  $Mult_E(B, \|\cdot\|)$  is dense in  $Mult(B, \|\cdot\|)$ .*

**Remark 14:** In [8], it is showed that in the algebra of bounded analytic functions in the open unit disk of a complete ultrametric algebraically closed field, any maximal ideal which is not defined by a point of the open unit disk is of infinite codimension. Here, we may ask whether the same holds. In the general case no answer is obvious. We can only answer a particular case:

**Theorem 17:** *Suppose  $K$  is algebraically closed. Let  $\mathcal{U}$  be an ultrafilter on  $K$  and let  $P \in K[x]$ ,  $P \neq 0$  satisfy  $\lim_{\mathcal{U}} P(x) = 0$ . Then  $\mathcal{U}$  is a principal ultrafilter.*

As a consequence, we have Theorem 18:

**Theorem 18:** *Suppose  $K$  is algebraically closed. Let  $F$  be a closed bounded subset of  $K$  with infinitely many points and let  $\mathcal{M}$  be a maximal ideal of  $A$  (resp.  $B$ ) which is not principal. Then  $\mathcal{M}$  is of infinite codimension.*

**Remark 15:** Suppose  $K$  is algebraically closed and let  $E = K$ . Then the algebras  $A$ ,  $B$  contain no polynomial. In such a case, it is not clear whether maximal ideals not defined by points of  $K$  are of infinite codimension.

**Remark 15:** Concerning uniformly continuous functions, it has been shown that two ultrafilters that are not contiguous define two distinct continuous multiplicative semi-norms.

Now, concerning bounded analytic functions inside the disk  $F = \{x \in K \mid |x| < 1\}$ , in [9], it was shown that the same property holds for a large set of ultrafilters on  $F$ . However, the question remains whether it holds for all ultrafilters on  $F$ .

Let us recall some results on the Shilov boundary of an ultrametric normed algebra:

**Proposition F [5], [6]:** *Let  $T$  be a normed  $K$ -algebra whose norm is  $\|\cdot\|$ . For each  $x \in T$ , let  $\|x\|_{si} = \lim_{n \rightarrow \infty} \|x^n\|^{\frac{1}{n}}$ . Then  $\|\cdot\|_{si}$  is power multiplicative semi-norm on  $T$ .*

**Definitions:** Let  $T$  be a normed  $K$ -algebra whose norm is  $\|\cdot\|$ . We call *spectral semi-norm of  $T$*  the semi-norm defined by Proposition F.

We call *Shilov boundary of  $T$*  a closed subset  $S$  of  $\text{Mult}(T, \|\cdot\|)$  that is minimum with respect to inclusion, such that, for every  $x \in T$ , there exists  $\phi \in S$  such that  $\phi(x) = \|x\|_{si}$ .

**Proposition G** [5], [7] : *Every normed  $K$ -algebra admits a Shilov boundary.*

**Theorem 19:** *The Shilov boundary  $S$  of  $A$  (resp.  $B$ ) is equal to  $\text{Mult}(A, \|\cdot\|)$  (resp.  $\text{Mult}(B, \|\cdot\|)$ ).*

## The Proofs:

Lemma 1 is classical due to the ultrametric distance of  $E$ :

**Lemma 1:** *For every  $r > 0$ ,  $E$  admits a partition of the form  $(d(a_i, r^-))_{i \in I}$ .*

**Definition and notation:** A function  $f$  from  $E$  to  $K$  will be said to be *uniformly locally constant* if there exists  $r > 0$  such that for every  $a \in E$ ,  $f(x)$  is constant in  $d(a, r^-)$ .

**Lemma 2:** *The set of bounded uniformly locally constant functions from  $E$  to  $K$  is a  $K$ -subalgebra of  $B$  and is dense in  $B$*

**Proof:** It is obvious that  $S$  is a  $K$ -algebra and is included in  $B$ . We will check that  $S$  is dense in  $B$ . Let  $f \in B$  and let  $\epsilon$  be  $> 0$ . There exists  $r > 0$  such that  $|f(x) - f(y)| \leq \epsilon$  for all  $x, y \in E$  such that  $\delta(x, y) \leq \epsilon$ . Now, by Lemma 1,  $E$  admits a partition of the form  $(d(a_i, r^-))_{i \in I}$ . Let  $h$  be the function defined by  $h(x) = f(a_i) \forall x \in d(a_i, r^-)$ . Clearly,  $\|f - h\| \leq \epsilon$ .

**Remark 16:** Lemma 2 suggests that in our general study, we can't find an interesting complete subalgebra of  $B$ .

**Notation:** We will denote by  $|\cdot|_\infty$  the Archimedean absolute value of  $\mathbb{R}$ .

**Lemma 3:** *Let  $m, M \in \mathbb{R}_+^*$  and let  $f \in A$ . Then the sets  $H = \{x \in E \mid ||f(x)| - m|_\infty \geq M\}$ ,  $L = \{x \in E \mid ||f(x)| - m|_\infty \leq M\}$ , are clopen. Moreover, if  $f \in B$ , then  $H, L$  are uniformly open.*

**Lemma 4:** *Let  $H$  be a clopen. Then the characteristic function  $u$  of  $H$  belongs to  $A$ . Moreover, if  $H$  is uniformly open, then  $u$  belongs to  $B$ .*

Given a bounded function in the set  $E$ ,  $|f(x)|$  obviously takes values in a compact of  $\mathbb{R}$ , therefore the following Lemma 5 comes from Urysohn's Theorem [1].

**Lemma 5:** *Let  $\mathcal{U}$  be an ultrafilter on  $E$ . Let  $f$  be a bounded function from  $E$  to  $K$ . The function  $|f|$  from  $E$  to  $\mathbb{R}_+$  defined as  $|f|(x) = |f(x)|$  admits a limit along  $\mathcal{U}$ . Moreover, if  $K$  is locally compact, then  $f(x)$  admits a limit along  $\mathcal{U}$ .*

**Lemma 6:** *Let  $\mathcal{U}, \mathcal{V}$  be sticked (resp. contiguous) ultrafilters on  $E$  and let  $f \in A$  (resp. let  $f \in B$ ). Then  $\lim_{\mathcal{U}} |f(x)| = \lim_{\mathcal{V}} |f(x)|$ .*

Lemma 7 is immediate:

**Lemma 7:** *Let  $f \in B$  and let  $\tilde{E}$  be the completion of  $E$ . Then  $f$  has continuation to a function  $\tilde{f}$  uniformly continuous on  $\tilde{E}$ .*

**Proof of Theorem 1:** If  $\mathcal{U}, \mathcal{V}$  are sticked, then by definition, given a clopen  $H \in \mathcal{U}$  and a clopen  $L \in \mathcal{V}$  we have  $H \cap L \neq \emptyset$ .

Now, suppose that two ultrafilters  $\mathcal{U}, \mathcal{V}$  are not sticked. We can find closed subsets  $F \in \mathcal{U}, G \in \mathcal{V}$  of  $E$  such that  $F \cap G = \emptyset$ . For each  $x \in F$ , let  $r(x)$  be the distance from  $x$  to  $G$  and let  $H = \bigcup_{x \in F} d(x, (\frac{1}{2}r(x))^-)$ . So,  $H$  is open. Suppose  $H$  is not closed and let  $(c_n)_{n \in \mathbb{N}}$  be a sequence of  $H$  converging to a point  $c \in E \setminus H$ . Since the distance is ultrametric, each point  $c_n$  belongs to a ball  $d(a_n, r(a_n))^-$  with  $a_n \in F$ . Suppose the sequence  $(r(a_n))_{n \in \mathbb{N}}$  does not tend to 0. There exists a subsequence  $(r(a_{q(m)}))_{m \in \mathbb{N}}$  and  $s > 0$  such that  $r(a_{q(m)}) \geq s \forall m \in \mathbb{N}$  and consequently,  $c$  belongs to one of the balls  $d(a_{q(m)}, (\frac{1}{2}r(a_{q(m)}))^-)$ , a contradiction. Thus, the sequence  $(r(a_n))_{n \in \mathbb{N}}$  must tend to 0. But since  $a_n$  belongs to  $F$  and since  $F$  is closed, clearly  $c$  lies in  $F$ , a contradiction. Thus,  $H$  is a clopen. By Construction,  $H$  belongs to  $\mathcal{U}$  and satisfies  $H \cap G = \emptyset$ , hence  $H$  does not belong to  $\mathcal{V}$ . Now, let  $L = E \setminus H$ . Then  $L$  also is a clopen that does not belong to  $\mathcal{U}$ . But since  $\mathcal{V}$  is an ultrafilter that is not secant with  $H$ , it is secant with  $L$  and hence  $L$  belongs to  $\mathcal{V}$ , which ends the proof.

**Proof of Theorem 2:** Since  $\mathcal{U}, \mathcal{V}$  are not sticked, by Theorem 1 we can find clopens  $H \in \mathcal{U}, L \in \mathcal{V}$  of  $E$  such that  $H \cap L = \emptyset$ . Then the set  $H' = E \setminus H$  also is a clopen. Let  $u$  be the characteristic function of  $H$ . Since  $H$  and  $H'$  are open,  $u$  is continuous, which ends the proof.

**Proof of Theorem 3:** Since  $\mathcal{U}$  and  $\mathcal{V}$  are not contiguous, there exist  $H \in \mathcal{U}, L \in \mathcal{V}$  such that  $\delta(H, L) = \mu > 0$ . Let  $H' = \{x \in E \mid \delta(x, H) \leq \frac{\mu}{2}\}$ . Then  $H'$  is a clopen containing  $H$  and by ultrametricity, we can check that  $\delta(H', L) \geq \mu$ . Let  $u$  be the function defined in  $E$  by  $u(x) = 1 \forall x \in H'$  and  $f(x) = 0 \forall x \in E \setminus H'$ . Since  $u$  is constant in any ball of diameter  $\frac{\mu}{2}$ ,  $u$  belongs to  $B$ .

**Proof of Proposition A:** It is obvious and well known that  $\mathcal{I}(\mathcal{U}, T)$  is prime. Let us check that so is  $\mathcal{I}^*(\mathcal{U}, T)$ . Suppose  $\mathcal{I}^*(\mathcal{U}, T)$  is not prime. There exists  $f, g \notin \mathcal{I}^*(\mathcal{U}, T)$  such that  $fg \in \mathcal{I}^*(\mathcal{U}, T)$ . Thus, there exists  $L \in \mathcal{U}$  such that  $f(x)g(x) = 0 \forall x \in L$ , but neither  $f$  nor  $g$  are identically zero on  $L$ . Let  $F$  be the set of the  $x \in L$  such that  $f(x) = 0$  and let  $G$  be the set of the  $x \in L$  such that  $g(x) = 0$ . Then  $F \cup G = L$ , hence  $\mathcal{U}$  is secant at least with one of the two sets  $F$  and  $G$ . Suppose it is secant with  $F$ . The intersection of  $\mathcal{U}$  with  $F$  is a filter thinner than  $\mathcal{U}$ , hence it is  $\mathcal{U}$ . Thus,  $f$  is identically zero on a set  $F \in \mathcal{U}$ , a contradiction. And similarly if it is secant with  $G$ .

**Proof of Theorem 4:** First, if  $\mathcal{U}$  and  $\mathcal{V}$  are not sticked, by Theorem 2 we have  $\mathcal{I}(\mathcal{F}, A) \neq \mathcal{I}(\mathcal{G}, A)$ .

Now, suppose that  $\mathcal{U}, \mathcal{V}$  are sticked and let  $f \in \mathcal{I}(\mathcal{U}, A)$ . Let  $\epsilon$  be  $> 0$  and let  $H \in \mathcal{U}$  be such that  $|f(x)| \leq \epsilon \forall x \in H$ . Since  $|f|$  has a limit  $l$  along  $\mathcal{V}$ , we can find  $L \in \mathcal{V}$  such that  $|f(x) - l| \leq \epsilon \forall x \in L$ . Since  $f$  is continuous, it satisfies  $|f(x)| \leq \epsilon \forall x \in \overline{H}$  and  $|f(x) - l| \leq \epsilon \forall x \in \overline{L}$ . But since  $\mathcal{U}, \mathcal{V}$  are sticked, there exists  $a \in \overline{H} \cap \overline{L}$ , hence  $l \leq 2\epsilon$ . And since  $\epsilon$  is arbitrary, then  $l = 0$  and hence  $f \in \mathcal{I}(\mathcal{V}, A)$ . Thus,  $\mathcal{I}(\mathcal{U}, A) \subset \mathcal{I}(\mathcal{V}, A)$ . And symmetrically, we have  $\mathcal{I}(\mathcal{V}, A) \subset \mathcal{I}(\mathcal{U}, A)$ , hence the two ideals are equal.

Suppose now that  $\mathcal{U}, \mathcal{V}$  are not contiguous. By Theorem 3, there exist  $H \in \mathcal{U}, L \in \mathcal{V}$  and  $f \in B$  such that  $f(x) = 1 \forall x \in H, f(x) = 0 \forall x \in L$ . Consequently,  $f$  belongs to  $\mathcal{I}(\mathcal{U}, B)$  but does not belong to  $\mathcal{I}(\mathcal{V}, B)$ . Thus,  $\mathcal{I}(\mathcal{F}, B) \neq \mathcal{I}(\mathcal{G}, B)$ .

Finally, suppose that  $\mathcal{U}, \mathcal{V}$  are contiguous. Let  $f \in \mathcal{I}(\mathcal{U}, B)$ . Let  $l = \lim_{\mathcal{V}} |f(x)|$ , suppose  $l > 0$  and let  $L \in \mathcal{V}$  be such that  $||f(x)| - l|_{\infty} \leq \frac{l}{3} \forall x \in L$ , hence  $|f(x)| \geq \frac{2l}{3} \forall x \in L$ . Let  $H \in \mathcal{U}$  be such that  $|f(x)| \leq \frac{l}{3} \forall x \in H$ . Since  $f \in B$ , there exists  $\rho > 0$  such that  $\delta(x, y) \leq \rho$  implies  $|f(x) - f(y)| \leq \frac{l}{4}$ . And since  $f$  is uniformly continuous, there exist  $a \in H, b \in L$  such that  $\delta(a, b) \leq \rho$ , hence  $|f(a) - f(b)| \leq \frac{l}{4}$ , a contradiction because  $|f(a)| \leq \frac{l}{3}$  and  $|f(b)| \geq \frac{2l}{3}$ .

**Proof of Proposition B:** Suppose that  $\mathcal{U}$  converges to a point  $a$ . Given  $f \in A$  (resp.  $f \in B$ ), we have  $\lim_{\mathcal{U}} f(x) = f(a)$ . So, the mapping  $\theta$  from  $A$  (resp.  $B$ ) to  $K$  defined as  $\theta(f) = f(a)$  admits  $\mathcal{M}$  for kernel and hence  $\mathcal{M}$  is of codimension 1.

**Proof of Theorem 5:** Let  $M = \inf_{x \in E} (\max_{1 \leq j \leq q} |f_j(x)|)$ . Let  $E_j = \{x \in E \mid |f_j(x)| \geq M\}$ ,  $j = 1, \dots, q$  and let  $F_j = \bigcup_{m=1}^j E_m$ ,  $j = 1, \dots, q$ . Let  $g_1(x) = \frac{1}{f_1(x)} \forall x \in E_1$  and  $g_1(x) = 0 \forall x \in E \setminus E_1$ . Since  $|f_1(x)| \geq M \forall x \in E_1$ ,  $|g_1(x)|$  is clearly bounded.

Suppose first  $f_1, \dots, f_q \in A$ . Since  $E$  is ultrametric, each  $E_j$  is obviously a clopen and so is each  $F_j$ . And since  $f_1$  is continuous  $g_1$  is continuous, hence belongs to  $A$ .

Now, suppose that  $f_1, \dots, f_q \in B$ . By Lemma 3,  $E_1$  has a strictly positive codiameter  $\rho$  and so does  $E \setminus E_1$ . Then  $g_1$  is obviously uniformly continuous in  $E \setminus E_1$ . And, since  $|f_1(x)| \geq M \forall x \in E_1$ ,  $g_1$  is uniformly continuous in  $E_1$ . Hence it is uniformly continuous in  $E$ . Thus  $g_1$  belongs to  $B$ .

Suppose now we have constructed  $g_1, \dots, g_k \in A$  (resp.  $g_1, \dots, g_k \in B$ ) satisfying  $\sum_{j=1}^k f_j g_j(x) = 1 \forall x \in F_k$  and  $\sum_{j=1}^k f_j g_j(x) = 0 \forall x \in E \setminus F_k$ . Let  $g_{k+1}$  be defined on  $E$  by  $g_{k+1}(x) = \frac{1}{f_{k+1}(x)} \forall x \in F_{k+1} \setminus F_k$  and  $g_{k+1}(x) = 0 \forall x \in E \setminus (F_{k+1} \setminus F_k)$ . Then  $g_{k+1}$  is bounded.

Now we can check that  $\sum_{j=1}^{k+1} f_j g_j(x) = 1 \forall x \in F_{k+1}$  and  $\sum_{j=1}^k f_j g_j(x) = 0 \forall x \in E \setminus F_{k+1}$ . So, by an immediate recurrence, we can get bounded functions  $g_1, \dots, g_q$  such that

$$\sum_{j=1}^q f_j g_j(x) = 1 \quad \forall x \in E.$$

Now suppose that  $f_1, \dots, f_q \in A$ . Since  $F_k$  and  $F_{k+1}$  are clopens, so is  $E \setminus (F_{k+1} \setminus F_k)$  and consequently,  $g_{k+1}$  is continuous. Similarly as for  $g_1$ , since  $|f_{k+1}(x)| \geq M \quad \forall x \in E_{k+1}$ ,  $|g_{k+1}(x)|$  is clearly bounded, hence belongs to  $A$ . And similarly, if  $g_1, \dots, g_k \in B$ ,  $g_{k+1}$  belongs to  $B$  for the same reason as  $g_1$  above. So, by induction, we can get  $g_1, \dots, g_q \in B$  such that  $\sum_{j=1}^q f_j g_j(x) = 1 \quad \forall x \in E$ .

**Proof of Theorem 6:** By Theorem 5, there exists an ultrafilter  $\mathcal{U}$  on  $E$  such that  $\mathcal{M} = \mathcal{I}(\mathcal{U}, B)$ . Now, suppose that  $\mathcal{M}$  is a Cauchy ultrafilter. Since the functions of  $B$  are uniformly continuous, by Lemma 7 they have continuation to the completion  $\tilde{E}$  of  $E$  and  $\mathcal{U}$  defines an ultrafilter that converges in  $\tilde{E}$  to a point  $a$ . Given  $f \in B$ , let  $\tilde{f}$  be the continuation of  $f$  in  $\tilde{E}$ : we have  $\lim_{\mathcal{U}} f(x) = \tilde{f}(a)$ . So by Proposition B  $\mathcal{M}$  is of codimension 1.

**Proof of Theorem 7:** Let  $\mathcal{M}$  be a maximal ideal of  $A$  (resp.  $B$ ). By Corollary 4.2 there exists an ultrafilter  $\mathcal{U}$  such that  $\mathcal{M} = \mathcal{I}(\mathcal{U}, A)$  (resp.  $\mathcal{M} = \mathcal{I}(\mathcal{U}, B)$ ). Let  $f \in A$  (resp.  $f \in B$ ). By Lemma 2, the function  $f$  has a limit  $\chi(f)$  along  $\mathcal{U}$ . Thus, the mapping  $\chi$  from  $A$  (resp.  $B$ ) to  $K$  is a  $K$ -algebra homomorphism and therefore  $\mathcal{M}$  is of codimension 1.

**Proof of Theorem 8:** Let  $\mathcal{U}, \mathcal{V}$  be stucked ultrafilters and let  $O$  be a clopen that belongs to  $\mathcal{U}$ . Suppose it does not belong to  $\mathcal{V}$ . Then  $\mathcal{V}$  is secant with  $E \setminus O$ . But since  $\mathcal{V}$  is an ultrafilter,  $E \setminus O$  belongs to  $\mathcal{V}$ . But  $E \setminus O$  is a clopen, hence it has a non-empty intersection with  $O$  (because  $\mathcal{U}$  and  $\mathcal{V}$  are stucked), a contradiction. Thus,  $O$  belongs to  $\mathcal{V}$ , which proves Theorem 8.

**Lemma 8 :** *Let  $O$  be a clopen and let  $\mathcal{U}$  be an ultrafilter that is not secant with  $O$ . There exists a clopen  $L$  that belongs to  $\mathcal{U}$  and satisfies  $L \cap O = \emptyset$ .*

**Proof :** Let  $H$  be a clopen that belongs to  $\mathcal{U}$  and let  $L = H \setminus O$ . Since  $\mathcal{U}$  is not secant with  $O$ , it is secant with  $L$ . But since both  $H, O$  are clopen, so is  $L$ . And by definition,  $L \cap O = \emptyset$ .

**Proof of Theorem 9:** Let  $\mathcal{U} \in \Phi(\mathcal{M})$ . Let  $O_1, O_2$  be clopen and set  $\theta = \Xi(\mathcal{M})$ . We first have to check that  $\theta(O_1 \Delta O_2) = \theta(O_1) + \theta(O_2)$  in  $\mathbb{F}_2$ . Let  $\mathcal{U} \in \Phi(\mathcal{M})$ .

If  $O_1$  belongs to  $\mathcal{U}$  and if  $O_2 \notin \mathcal{U}$ , the conclusion is immediate. Similarly, so is it whenever  $O_1 \notin \mathcal{U}$  and  $O_2 \in \mathcal{U}$ . Now, consider the case when  $O_1 \in \mathcal{U}$  and  $O_2 \in \mathcal{U}$ .

Then  $O_1 \cap O_2$  belongs to  $\mathcal{U}$ . But since  $\mathcal{U}$  is an ultrafilter, it cannot be secant with  $(O_1 \cup O_2) \setminus (O_1 \cap O_2)$ . Consequently,  $\theta(O_1 \Delta O_2) = 0$ . So we have checked that  $\theta(O_1 \Delta O_2) = \theta(O_1) + \theta(O_2)$ .

Concerning  $\theta(O_1 \cap O_2)$ , clearly,  $\theta(O_1 \cap O_2) = 1$  if and only if both  $O_1, O_2$  belong to  $\mathcal{U}$  i.e.  $\theta(O_1) = \theta(O_2) = 1$ , hence  $\theta(O_1 \cap O_2) = \theta(O_1)\theta(O_2)$ . This finishes proving that  $\theta$  is a ring homomorphism.

**Proof of Theorem 10:** Let us check that  $\Xi$  is injective. Suppose  $\mathcal{M}_1, \mathcal{M}_2$  are two distinct maximal ideals such that  $\Xi(\mathcal{M}_1) = \Xi(\mathcal{M}_2)$ . Let  $\mathcal{U}_1 \in \Phi(\mathcal{M}_1), \mathcal{U}_2 \in \Phi(\mathcal{M}_2)$ . Since  $\mathcal{U}_1, \mathcal{U}_2$  are not sticked, by Theorem 1 there exists a clopen  $O \in \mathcal{U}_1$  that does not belong to  $\mathcal{U}_2$ . Consequently,  $\Xi(\mathcal{M}_1)(O) = 1, \Xi(\mathcal{M}_2)(O) = 0$ , which proves that  $\Xi$  is injective.

Now, let us check that  $\Xi$  is surjective. Let  $\theta \in \Sigma(E)$ . The family of clopens  $O$  satisfying  $\theta(O) = 1$  clearly generates a filter  $\mathcal{F}$ . Let  $\mathcal{U}$  be an ultrafilter thinner than  $\mathcal{F}$  and let  $\mathcal{M} = \mathcal{I}(\mathcal{U}, A)$ . We will check that  $\theta = \Xi(\mathcal{M})$ . Let  $O$  be clopen that belongs to  $\mathcal{U}$ . Then  $\mathcal{F}$  is secant with  $O$ , hence  $\theta(O) = 1$ . Now, let  $\mathcal{V}$  be an ultrafilter sticked to  $\mathcal{U}$ . By Theorem 8, the clopens that belong to  $\mathcal{U}$  are the same as those which belong to  $\mathcal{V}$ . Consequently,  $\theta(O) = \Xi(\mathcal{M})(O)$  for every clopen which belongs to any  $\mathcal{V} \in \Phi(\mathcal{M})$ .

And now, let  $O$  be a clopen that does not belong to any  $\mathcal{V} \in \Phi(\mathcal{M})$ . Let us take again  $\mathcal{U} \in \Phi(\mathcal{M})$ . By Lemma 8, there exists a clopen  $L \in \mathcal{V}$  such that  $O \cap L = \emptyset$ . Then  $O \Delta L = O \cup L$  belongs to  $\mathcal{U}$ . Hence  $\theta(L \cup O) = 1 = \theta(L)$  and consequently,  $\theta(O) = 0$ , which finishes proving that  $\theta = \Xi(\mathcal{M})$ . So,  $\Xi$  is surjective.

$\Sigma(E)$  is compact because it is closed in  $\mathbb{F}_2^{\mathcal{B}(E)}$ . By definition, we have  $\Xi(\mathcal{I}(a, A)) = \zeta_a \forall a \in E$ . Let us check that  $\Sigma_E(E)$  is dense in  $\Sigma(E)$ . Let  $\theta = \Xi(\mathcal{M})$  ( $\mathcal{M} \in \text{Max}(A)$ ) and let  $O_1, \dots, O_q$  be clopens. We may assume that  $\theta(O_j) = 1 \forall j = 1, \dots, k$  and  $\theta(O_j) = 0 \forall j = k + 1, \dots, q$ . Let  $\mathcal{U} \in \Phi(\mathcal{M})$ . Then  $\mathcal{U}$  is secant with  $O_1, \dots, O_k$  and is not with  $O_{k+1}, \dots, O_q$ . Let  $a \in \bigcap_{j=1}^k O_j$ , hence clearly  $\theta(O_j) = \zeta_a(O_j) \forall j = 1, \dots, q$ . This finishes the proof of Theorem 10.

**Proof of Theorem 11:** We know that  $u \in A$  (resp.  $u \in B$ ). By construction,  $1 - u$  does not belong to  $\mathcal{I}(\mathcal{U}, A)$  (resp.  $\mathcal{I}(\mathcal{U}, B)$ ) because  $\lim_{\mathcal{U}} u(x) = 1$ . But  $u(1 - u) = 0$ , hence  $u$  belongs to  $\mathcal{P}$  because  $\mathcal{P}$  is prime.

**Proof of Theorem 12:** Let  $\mathcal{P}$  be a prime ideal of  $A$  included in a maximal ideal  $\mathcal{M} = \mathcal{I}(\mathcal{U}, A)$ , (resp.  $\mathcal{M} = \mathcal{I}(\mathcal{U}, B)$ ). Let  $f$  belong to  $\mathcal{I}(\mathcal{U}, A)$ , (resp.  $\mathcal{I}(\mathcal{U}, B)$ ). Let us take  $\epsilon > 0$  and find  $h \in \mathcal{P}$  such that  $\|f - h\| \leq \epsilon$ . By Lemma 3 we can find a clopen  $L \in \mathcal{U}$ , (resp. a uniformly open subset  $L \in \mathcal{U}$ ) such that  $|f(x)| \leq \epsilon \forall x \in L$ . Let  $u$  be the characteristic function of  $E \setminus L$ . By Theorem 11  $u$  belongs to  $\mathcal{P}$  and hence so does  $uf$ . We then check that  $\|f - uf\| \leq \epsilon$ . Thus,  $\mathcal{P}$  is dense in  $\mathcal{M}$ .

**Notation:** Let  $T = A$  or  $B$ . Given  $f_1, \dots, f_q \in T$ ,  $\epsilon > 0$ , on  $\text{Mult}_E(T, \|\cdot\|)$  we will denote by  $W(\varphi_a, f_1, \dots, f_q, \epsilon)$  the set of the  $\varphi_x$  such that  $\| |f_j(x)| - |f_j(a)| \|_{\infty} < \epsilon \forall j = 1, \dots, q$ .

**Definitions and notation:** Let  $T = A$  or  $B$ . Given  $\phi \in \text{Mult}(T, \|\cdot\|)$  and  $f_1, \dots, f_q \in T$ ,  $\epsilon > 0$ , we will denote by  $W(\phi, f_1, \dots, f_q, \epsilon)$  the set of the  $\psi$  such that  $\| |\phi(f_j)| - |\psi(f_j)| \|_{\infty} < \epsilon \forall j = 1, \dots, q$ . Such neighborhoods of  $\phi$  will be called *basic neighborhoods of  $\phi$* .

By definition of the topology of simple convergence on  $\text{Mult}(T, \|\cdot\|)$  we know that the set of basic neighborhoods of  $\phi$  makes a fundamental system of neighborhoods of  $\phi$ .



Similarly, given  $\zeta \in \Sigma(E)$  and clopens  $O_1, \dots, O_q$  we will denote by  $Z(\zeta, O_1, \dots, O_q)$  the set of the  $\xi$  such that  $\zeta(O_j) = \xi(O_j) \forall j = 1, \dots, q$ . Such neighborhoods of  $\zeta$  will be called *basic neighborhoods of  $\zeta$* .

Then by definition of the topology of simple convergence on  $\Sigma(E)$ , we know that the set of basic neighborhoods of  $\zeta$  makes a fundamental system of neighborhoods of  $\zeta$ .

Given  $\psi \in Mult(A, \|\cdot\|)$ , we set  $\bar{\psi} = \Xi \circ \Phi^{-1}(\psi)$ .

**Lemma 9:** quad Let  $\phi \in Mult(A, \|\cdot\|)$ , let  $O_1, \dots, O_q$  be clopens and let  $\epsilon \in ]0, 1[$ . Let  $D = \bigcap_{j=1}^q O_j$  and let  $u$  be the characteristic function of  $D$ . Then, given  $\psi \in Mult(A, \|\cdot\|)$ ,  $\psi$  belongs to  $W(\phi, u, \epsilon)$  if and only if  $\bar{\psi}$  belongs to  $Z(\bar{\phi}, O_1, \dots, O_q)$ .

**Proof:** Let  $\mathcal{U}$  be an ultrafilter such that  $Ker(\phi) = \mathcal{I}(\mathcal{U}, A)$ . Of course  $\phi(u)$  is equal to 0 or 1. Since  $\epsilon < 1$ , we can see that  $\psi(u) = 1$  if and only if  $\mathcal{U}$  is secant with  $D$  and  $\psi(u) = 0$  if and only if  $\mathcal{U}$  is not secant with  $D$ . But this holds if and only if  $\bar{\psi}(O_j) = 1 \forall j = 1, \dots, q$ , hence  $\bar{\psi}$  belongs to  $Z(\bar{\phi}, O_1, \dots, O_q)$ .

**Proof of Theorem 13:** Let  $\phi \in Mult(A, \|\cdot\|)$  and consider first a basic neighborhood of  $\bar{\phi}$ :  $Z(\bar{\phi}, O_1, \dots, O_q)$ . Since  $\bigcap_{j=1}^q O_j$  is a clopen, its characteristic function  $u$  belongs to  $A$ . Then by Lemma 9, given  $\epsilon \in ]0, 1[$ , a  $\psi \in Mult(A, \|\cdot\|)$  belongs to  $W(\phi, u, \epsilon)$  if and only if  $\bar{\psi}$  belongs to  $Z(\bar{\phi}, O_1, \dots, O_q)$ . Consequently, any basic neighborhood of  $\bar{\phi}$  is the image of a basic neighborhood of  $\phi$  by the bijection  $\Xi \circ \Phi^{-1}$ . Therefore, the topology of  $Mult(A, \|\cdot\|)$  is at least as thin as this of  $\Sigma(E)$ .

Now, conversely, consider a basic neighborhood of  $\phi$ :  $W(\phi, f_1, \dots, f_q, \epsilon)$  with  $\epsilon \in ]0, 1[$ . The set of the  $x \in E$  such that  $||f_j(x)| - \phi(f_j)|_\infty \leq \epsilon \forall j = 1, \dots, q$  is a clopen  $D$  hence its characteristic function  $u$  belongs to  $A$ . Now, let  $\zeta \in \Sigma(E)$ . It is of the form  $\Xi(\mathcal{M})$  where  $\mathcal{M}$  is a maximal ideal  $\mathcal{I}(\mathcal{U}, A)$  with  $\mathcal{U}$  an ultrafilter on  $E$ . And then, given any clopen  $O$ , we have  $\Xi(\mathcal{M})(O) = 1$  if and only if  $\mathcal{U}$  is secant with  $O$ . Thus,  $\zeta$  belongs to  $Z(\bar{\phi}, D)$  if and only if ultrafilters  $\mathcal{U}$  such that  $\Xi^{-1}(\zeta) = \mathcal{I}(\mathcal{U}, A)$  are secant with  $D$ . Now, suppose that  $\zeta$  belongs to  $Z(\bar{\phi}, D)$ . Of course,  $Ker(\Phi \circ \Xi^{-1}(\zeta))$  is equal to  $\mathcal{I}(\mathcal{U}, A)$ . Set  $\hat{\zeta} = \Phi \circ \Xi^{-1}(\zeta)$ . Since  $\mathcal{U}$  is secant with  $D$ , the inequality  $|\lim_{\mathcal{U}} |f_j(x)| - \phi(f_j)|_\infty \leq \epsilon$  holds for every  $j = 1, \dots, q$ . Therefore,  $\hat{\zeta}$  belongs to  $W(\phi, f_1, \dots, f_q, \epsilon)$ . This proves that  $\Xi \circ \Phi^{-1}(W(\phi, f_1, \dots, f_q, \epsilon))$  contains a neighborhood of  $\zeta$  in  $\Sigma(E)$  and finishes proving that the two topologies are equivalent.

**Proof of Theorem 14:** Let  $a \in E$ . The filter of neighborhoods of  $a$  admits for basis the family of balls  $d(a, r^-) = \{x \in E \mid \delta(x, a) \leq r\}$ ,  $r > 0$ . But we can check that such a ball is induced by a neighborhood of  $\varphi_a$  with respect to both topologies of  $Mult_E(A, \|\cdot\|)$  and  $Mult_E(B, \|\cdot\|)$ . Given  $\varphi_a \in Mult_E(T, \|\cdot\|)$ , we set  $W'(\varphi_a, f_1, \dots, f_q, \epsilon) = W(\varphi_a, f_1, \dots, f_q, \epsilon) \cap Mult_E(T, \|\cdot\|)$ . Let  $r \in ]0, 1[$ . By Lemma 4 there exists  $u \in B$  such that  $u(x) = 0 \forall x \in \mathcal{B}(a, r)$  and  $u(x) = 1 \forall x \in E \setminus d(a, r^-)$ .

Consequently,  $W'(\varphi_a, u, r)$  is the set of the  $\varphi_b$  such that  $|b - a| \leq r$ . This holds when we consider  $A$  as when we consider  $B$  and hence the topology of  $Mult_E(B, \|\cdot\|)$  as this of  $Mult_E(A, \|\cdot\|)$  is thinner or equal to the metric topology of  $E$ . Now, since each  $f_j$  is continuous, the set of the  $x \in E$  such that  $|f_j(x) - f_j(a)| \leq \epsilon \forall j = 1, \dots, q$  is open and hence contains a ball  $d(a, r^-)$  of  $E$ . Consequently, the topology of  $E$  is thinner or equal to this of  $Mult_E(T, \|\cdot\|)$ , which finishes proving that the topology induced on  $E$  by  $Mult(T, \|\cdot\|)$  coincides with the metric topology of  $E$ .

**Proof of Theorem 15:** Let  $t \in T'$  and let  $f \in T$  be such that  $\theta(f) = t$ . Let  $\mathcal{U}$  be an ultrafilter such that  $\mathcal{I}(\mathcal{U}, T) = \mathcal{M}$ . So,  $\|t\|' \geq \lim_{\mathcal{U}} |f(s)|$ . Conversely, let  $W \in \mathcal{U}$  be such that  $|f(x)| \leq \lim_{\mathcal{U}} |f(s)| + \epsilon \forall x \in W$ . There exists  $f_1, \dots, f_q \in \mathcal{M}$  and  $\epsilon > 0$  such that

$$\bigcap_{j=1}^q D(f_j, \epsilon) \subset W. \text{ Let } X = \bigcap_{j=1}^q D(f_j, \epsilon).$$

Suppose  $\mathcal{M}$  is a maximal ideal of  $T$ . There exists  $u \in T$  such that  $u(x) = 0 \forall x \in X$ ,  $u(x) = 1 \forall x \in F \setminus X$ . Then  $u(1 - u) = 0$ . But  $1 - u \notin \mathcal{M}$ . Hence,  $u$  belongs to  $\mathcal{M}$ . Then  $\theta(f - uf) = \theta(f) = t$ . But by construction,  $(f - uf)(x) = 0 \forall x \in F \setminus X$  and  $(f - uf)(x) = f(x) \forall x \in X$ . Consequently,  $\|f - uf\| \leq \lim_{\mathcal{U}} |f(s)| + \epsilon$  and therefore  $\|t\|' = \|\theta(f - uf)\|' \leq \lim_{\mathcal{U}} |f(s)| + \epsilon$ . This finishes proving the equality  $\|\theta(f)\|' = \lim_{\mathcal{U}} |f(s)|$ . Now, such a norm defined as  $\|\theta(f)\|' = \lim_{\mathcal{U}} |f(s)|$  is obviously multiplicative. The proof concerning  $A$  is exactly similar, the set  $X$  being then closed and open by Lemma 3 and the proof concerning  $B$  is also similar the set  $X$  being uniformly open.

**Proof of Theorem 16:** Let  $\mathcal{M}$  be a maximal ideal of  $A$  (resp. of  $B$ ) and let  $A'$  be the field  $\frac{A}{\mathcal{M}}$  (resp. let  $B'$  be the field  $\frac{B}{\mathcal{M}}$ ). By Theorem 15, the quotient norm of  $A'$  (resp. of  $B'$ ) is multiplicative. But then,  $A'$  (resp.  $B'$ ) admits only one continuous multiplicative semi-norm: its quotient  $K$ -algebra norm. Consequently,  $A$  (resp.  $B$ ) admits only one continuous multiplicative semi-norm whose kernel is  $\mathcal{M}$ , which proves that  $A$  (resp.  $B$ ) is multibjective.

**Proof of Theorem 17:** Let  $P(x) = \prod_{j=1}^q (x - a_j)$ . Let  $\mathcal{F}$  be the filter admitting for

basis the sets  $\Lambda(r) = \bigcup_{j=1}^q d(a_j, r)$ . Suppose first that  $\mathcal{U}$  is not secant with  $\mathcal{F}$ . There exists

$\rho > 0$  and  $H \in \mathcal{U}$  such that  $\Lambda(\rho) \cap H = \emptyset$ . Then  $|P(x)| \geq \rho^q \forall x \in H$ , a contradiction to the hypothesis  $\lim_{\mathcal{U}} P(x) = 0$ . Consequently,  $\mathcal{U}$  is secant with  $\mathcal{F}$ . Hence it is obviously secant with the filter of neighborhoods of one of the points  $a_j$  and therefore, it converges to this point.

**Proof of Theorem 18:** Indeed, by Theorem 17 the ideal  $\mathcal{M}$  contains no polynomials, hence  $\frac{A}{\mathcal{M}}$ , (resp.  $\frac{B}{\mathcal{M}}$ ) contains a subfield isomorphic to  $K(x)$ .

**Proof of Theorem 19:** Given  $\psi \in Mult(A, \|\cdot\|)$ ,  $f_1, \dots, f_q \in A$ ,  $\epsilon > 0$ , we set

$$V(\psi, f_1, \dots, f_q, \epsilon) = \{\phi \in Mult(A, \|\cdot\|) \mid |\phi(f_j) - \psi(f_j)|_\infty \leq \epsilon, j = 1, \dots, q\}.$$

By definition of the topology of simple convergence, the filter of neighborhoods of  $\psi$  admits for basis the family of sets

$$V(\psi, f_1, \dots, f_q, \epsilon), q \in \mathbb{N}^*, f_1, \dots, f_q \in A, \epsilon > 0.$$

Henceforth we take  $\epsilon \in ]0, \frac{1}{2}[$ .

Suppose that the Shilov boundary of  $A$  is not equal to  $Mult(A, \|\cdot\|)$  and let  $\psi \in Mult(A, \|\cdot\|) \setminus S$ . Since  $S$  is a closed subset of  $Mult(A, \|\cdot\|)$ , the set  $Mult(A, \|\cdot\|) \setminus S$  is an open subset of  $Mult(A, \|\cdot\|)$  and hence, there exist  $f_1, \dots, f_q$  such that  $V(\psi, f_1, \dots, f_q, \epsilon) \subset (Mult(A, \|\cdot\|) \setminus S)$ . Let  $L = \{x \in E \mid |\psi(f_j) - |f_j(x)||_\infty \leq \frac{\epsilon}{2}\}$ . By Lemma 3  $L$  is a clopen. Consequently, by Lemma 4 the characteristic function  $u$  of  $L$  belongs to  $A$  and obviously satisfies  $\psi(u) = 1$ . On the other hand, we have  $u(x) = 0 \forall x \notin L$ .

Now, there exists  $\theta \in S$  such that  $\theta(u) = \|u\| = 1$ . Consider the neighborhood  $V(\theta, f_1, \dots, f_q, u, \frac{\epsilon}{2})$ . Since  $Mult_1(A, \|\cdot\|)$  is dense in  $Mult(A, \|\cdot\|)$ , particularly, for every  $\varphi_a \in V(\theta, f_1, \dots, f_q, u, \frac{\epsilon}{2})$  we have  $||u(a)| - \theta(u)|_\infty \leq \frac{\epsilon}{2}$ , hence  $|u(a)| \geq 1 - \frac{\epsilon}{2} > 0$ . But since  $\theta \in S$ , we have  $V(\theta, f_1, \dots, f_q, \frac{\epsilon}{2}) \cap V(\psi, f_1, \dots, f_q, \frac{\epsilon}{2}) = \emptyset$  and so much the more:  $V(\theta, f_1, \dots, f_q, u, \frac{\epsilon}{2}) \cap V(\psi, f_1, \dots, f_q, u, \frac{\epsilon}{2}) = \emptyset$ . Let  $H = \{a \in E \mid \varphi_a \in V(\theta, f_1, \dots, f_q, u, \frac{\epsilon}{2})\}$ . Then  $H \cap L = \emptyset$ . But by definition of  $u$ , we have  $u(a) = 0 \forall a \in H$ , a contradiction.

The proof of the statement concerning  $B$  is similar. Following the same notation with  $B$  in place of  $A$ , we only have to remark that by lemma 3 here,  $L$  is a uniformly open subset of  $E$ . Consequently, by Lemma 4 the characteristic function  $u$  of  $L$  belongs to  $B$ . The end of the proof follows the same way as this for  $A$ .

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