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Multiplicative spectrum of ultrametric Banach algebras of continuous functions

by Alain Escassut and Nicolas Mainetti

Abstract Let K be an ultrametric complete field and let E be an ultrametric space. Let A be the Banach K -algebra of bounded continuous functions from E to K and let B be the Banach K -algebra of bounded uniformly continuous functions from E to K . Maximal ideals and continuous multiplicative semi-norms on A (resp. on B) are studied by defining relations of stickness and contiguousness on ultrafilters that are equivalence relations. So, the maximal spectrum of A (resp. of B) is in bijection with the set of equivalence classes with respect to stickness (resp. to contiguousness). Every prime ideal of A or B is included in a unique maximal ideal and every prime closed ideal of A (resp. of B) is a maximal ideal, hence every continuous multiplicative semi-norm on A (resp. on B) has a kernel that is a maximal ideal. If K is locally compact, every maximal ideal of A , (resp. of B) is of codimension 1. Every maximal ideal of A or B is the kernel of a unique continuous multiplicative semi-norm and every continuous multiplicative semi-norm is defined as the limit along an ultrafilter on E . Consequently, on A as on B the set of continuous multiplicative semi-norms defined by points of E is dense in the whole set of all continuous multiplicative semi-norms. Ultrafilters show bijections between the set of continuous multiplicative semi-norms of A , $Max(A)$ and the Banaschewski compactification of E which is homeomorphic to the topological space of continuous multiplicative semi-norms. The Shilov boundary of A (resp. B) is equal to the whole set of continuous multiplicative semi-norms.

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Introduction and preliminaries:

Definitions and notation:

Let K be a field complete with respect to an ultrametric absolute value $|\cdot|$. It is well known that the set of maximal ideals is not sufficient to describe spectral properties of an ultrametric Banach algebra: we have to consider the set of continuous multiplicative semi-norms [6], [7], [9], [10], [11],]. Many studies were made on continuous multiplicative semi-norms on algebras of analytic functions, analytic elements and their applications to holomorphic functional calculus [3], [5], [6]. Here we mean to study continuous multiplicative semi-norms on Banach algebras of continuous functions. We will consider two main cases: Banach algebras of bounded continuous functions and Banach algebras of bounded uniformly continuous functions (with an application to Banach algebras of bounded functions).

Definitions and notation: Let E denote a metric space whose distance δ is ultrametric, let A be the Banach K -algebra of bounded continuous functions from E to K and let B be the Banach K -algebra of bounded uniformly continuous functions from E to K .

We will call *clopen* any closed open subset of E . Let H be a subset of E different from E and \emptyset . We will call *codiameter of H* the number $\delta(H, E \setminus H)$ and we will denote it by $\text{codiam}(H)$. The subset H will be said to be *uniformly open* if $\text{codiam}(H) > 0$. Given $a \in E$ and $r > 0$ we will denote by $d(a, r^-)$ the ball $\{x \in E \mid \delta(a, x) < r\}$. Let \mathcal{F} be a filter on E . Given a function f from E to K admitting a limit along \mathcal{F} , we will denote by $\lim_{\mathcal{F}} f(x)$ that limit.

Given a set F , we shall denote by $U(F)$ the set of ultrafilters on F . Now, let X be a topological space. Given $\mathcal{F} \in U(X)$, we will denote by $\overline{\mathcal{F}}$ the filter generated by the closures of elements of \mathcal{F} . Two ultrafilters \mathcal{F}, \mathcal{G} on F will be said to be *sticked* if $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ are secant. We will denote by (\mathcal{S}) the relation defined on $U(X)$ as $\mathcal{U}(\mathcal{S})\mathcal{V}$ if \mathcal{U} and \mathcal{V} are stucked.

Next, two filters \mathcal{F}, \mathcal{G} on E will be said to be *contiguous* if for every $H \in \mathcal{F}, L \in \mathcal{G}$, we have $\delta(H, L) = 0$. We shall denote by (\mathcal{T}) the relation defined on $U(E)$ as $\mathcal{U}(\mathcal{T})\mathcal{V}$ if \mathcal{U} and \mathcal{V} are contiguous.

An ultrafilter \mathcal{U} on the set E is said to be *principal* if there exists $a \in E$ such that $\mathcal{U} = \{H \subset E \mid a \in H\}$.

A closed open set of E is called a *clopen*.

Remark 1: Let $H \subset E$ be different from \emptyset and from E . Then $\text{codiam}(H) = \text{codiam}(E \setminus H)$.

Remark 2: Two stucked filters on E are contiguous.

Remark 3: A uniformly open subset of E is open and closed.

Remark 4: Let \mathcal{U}, \mathcal{V} be contiguous ultrafilters on E and assume \mathcal{U} is convergent. Then \mathcal{V} is convergent and has the same limit as \mathcal{U} . Moreover \mathcal{U} and \mathcal{V} are stucked.

Remark 5: We can construct contiguous ultrafilters that are not stucked. Let $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$ be sequences in K such that

- (1) $|a_n| < |a_{n+1}|$,
- (2) $\lim_{n \rightarrow +\infty} a_n - b_n = 0$,
- (3) $|a_n - b_n| \geq \frac{1}{n}$.

Let \mathcal{U} be a filter thinner than the sequence (a_n) . We will define a filter $\tilde{\mathcal{U}}$ thinner than the sequence (b_n) completing the example. By definition, \mathcal{U} admits a basis made of images of subsequences $(a_{\sigma(n)})_{n \in \mathbb{N}}$ of the sequence (a_n) . Let \mathcal{Z} be the family of images of such sequences, making a basis of \mathcal{U} .

Given such a subsequence $(a_{\sigma(m)})_{m \in \mathbb{N}}$, set $Q = \{a_{\sigma(m)} \mid m \in \mathbb{N}\}$ and set $\tilde{Q} = \{b_{\sigma(m)} \mid m \in \mathbb{N}\}$. Let $\tilde{\mathcal{U}}$ be the filter admitting for basis the family $\{\tilde{Q}, Q \in \mathcal{Z}\}$.

Then we can check that \mathcal{U} is an ultrafilter if and only if so is $\tilde{\mathcal{U}}$. Indeed, suppose \mathcal{U} is an ultrafilter and suppose $\tilde{\mathcal{U}}$ is not. Let \mathcal{V} be an ultrafilter strictly thinner than $\tilde{\mathcal{U}}$.

Let $X \in \mathcal{V} \setminus \tilde{\mathcal{U}}$. Since \mathcal{V} is thinner than $\tilde{\mathcal{U}}$, we may assume that X is the image of a subsequence $(b_{\tau(m)})_{m \in \mathbb{N}}$ of the sequence $(b_n)_{n \in \mathbb{N}}$ where this image $\{b_{\tau(m)} \mid m \in \mathbb{N}\}$ is strictly included in the image $\{b_{\sigma(m)} \mid m \in \mathbb{N}\}$ of a subsequence $(b_{\sigma(m)})_{m \in \mathbb{N}}$. But then, the set $\{a_{\tau(m)} \mid m \in \mathbb{N}\}$ is strictly included in $\{a_{\sigma(m)} \mid m \in \mathbb{N}\}$ which belongs to \mathcal{U} . But $\{a_{\tau(m)} \mid m \in \mathbb{N}\}$ doesn't belong to \mathcal{U} because if it belonged to \mathcal{U} , then $\{b_{\tau(m)} \mid m \in \mathbb{N}\}$ would belong to $\tilde{\mathcal{U}}$, which is excluded by hypothesis. But then, the filter generated by \mathcal{U} and the set $\{a_{\tau(m)} \mid m \in \mathbb{N}\}$ is strictly thinner than \mathcal{U} , a contradiction since \mathcal{U} is an ultrafilter.

Thus we have proved that if \mathcal{U} is an ultrafilter so is $\tilde{\mathcal{U}}$. The converse is obvious.

Now, by (2), \mathcal{U} and $\tilde{\mathcal{U}}$ are contiguous. Next, by (1) and (3), both Q, \tilde{Q} are closed sets such that $Q \cap \tilde{Q} = \emptyset$ which shows that \mathcal{U} and $\tilde{\mathcal{U}}$ are not stucked.

Remark 6: Relation (\mathcal{S}) is not the equality between ultrafilters, even when the ultrafilters are not convergent. In [13], Labib Haddad introduced the following equivalence relation (\mathcal{L}) on ultrafilters. Given two ultrafilters \mathcal{U}, \mathcal{V} we write $\mathcal{U}(\mathcal{L})\mathcal{V}$ if there exists an ultrafilter \mathcal{W} such that every closed set H lying in \mathcal{W} also lies in \mathcal{U} and similarly, every closed set H lying in \mathcal{W} also lies in \mathcal{V} . So, Relation (\mathcal{L}) is clearly thinner than Relation (\mathcal{S}) . However, it is shown that two ultrafilters \mathcal{U}, \mathcal{V} satisfying $\mathcal{U}(\mathcal{L})\mathcal{V}$ may be distinct without converging.

Basic results:

First, we will show that Relation (\mathcal{S}) may be also defined in terms of clopens.

Theorem 1: *Two ultrafilters \mathcal{U}, \mathcal{V} are stucked if and only if for any two clopens $H \in \mathcal{U}, L \in \mathcal{V}$, we have $H \cap L \neq \emptyset$.*

Theorem 2 may be viewed as a particular version of a theorem due to Urysohn, although it is not a direct consequence of a theorem of Urysohn because Urysohn's theorem only concerns functions with values in $[0, 1]$.

Theorem 2: *Let \mathcal{U}, \mathcal{V} be two ultrafilters on E that are not stucked. There exist $H \in \mathcal{U}, L \in \mathcal{V}$ and $f \in A$ such that $f(x) = 1 \forall x \in H, f(x) = 0 \forall x \in L$.*

Theorem 3: *Let \mathcal{U}, \mathcal{V} be two ultrafilters on E that are not contiguous. There exist $H \in \mathcal{U}, L \in \mathcal{V}$ and $f \in B$ such that $f(x) = 1 \forall x \in H, f(x) = 0 \forall x \in L$.*

Notation: Let T be a K -algebra of bounded functions from E to K .

Given a filter \mathcal{F} on E , we will denote by $\mathcal{I}(\mathcal{F}, T)$ the ideal of the $f \in T$ such that $\lim_{\mathcal{F}} f(x) = 0$. We will denote by $\mathcal{I}^*(\mathcal{F}, T)$ the ideal of the $f \in T$ such that there exists a subset $L \in \mathcal{F}$ such that $f(x) = 0 \forall x \in L$.

Given $a \in E$ we will denote by $\mathcal{I}(a, T)$ the ideal of the $f \in T$ such that $f(a) = 0$.

We will denote by $Max(T)$ the set of maximal ideals and by $Max_E(T)$ the set of maximal ideals of the form $\mathcal{I}(a, T)$, $a \in E$.

Proposition A is easy:

Proposition A: *Let T be a K -algebra of bounded functions from E to K . Given an ultrafilter \mathcal{U} on E , $\mathcal{I}(\mathcal{U}, T)$, $\mathcal{I}^*(\mathcal{U}, T)$ are prime ideals.*

Theorem 4: *Let \mathcal{U} , \mathcal{V} be two ultrafilters on E . Then $\mathcal{I}(\mathcal{U}, A) = \mathcal{I}(\mathcal{V}, A)$ if and only if \mathcal{U} and \mathcal{V} are sticked. Further, $\mathcal{I}(\mathcal{U}, B) = \mathcal{I}(\mathcal{V}, B)$ if and only if \mathcal{U} and \mathcal{V} are contiguous.*

Corollary 4.1: *Both Relations (\mathcal{S}) , (\mathcal{T}) are equivalence relations on $U(E)$.*

Remark 7 : Relations (\mathcal{S}) , (\mathcal{T}) are not transitive when applying to the set of all filters on E . However, given a topological space X satisfying the normality axiom, (i.e. any two closed disjoint subsets H , L admit disjoint open neighborhoods), then (\mathcal{S}) is transitive for ultrafilters and therefore is an equivalence relation on $U(X)$. Similarly, given a metric space X , then (\mathcal{T}) is transitive for ultrafilters and therefore is an equivalence relation on $U(X)$ [13].

Notation: We will denote by $Y_{(\mathcal{S})}(E)$ the set of equivalence classes on $U(E)$ with respect to Relation (\mathcal{S}) and by $Y_{(\mathcal{T})}(E)$ the set of equivalence classes on $U(E)$ with respect to Relation (\mathcal{T}) .

Let $f \in A$ and let ϵ be > 0 . We set $D(f, \epsilon) = \{x \in E \mid |f(x)| \leq \epsilon\}$.

We will need the following Proposition that is immediate:

Proposition B: *Let \mathcal{M} be a maximal ideal of A (resp. of B) of the form $\mathcal{I}(\mathcal{U}, A)$ (resp. $\mathcal{I}(\mathcal{U}, B)$). If \mathcal{U} converges in E then \mathcal{M} is of codimension 1.*

Main Theorems:

Theorem 5 looks like certain Bezout-Corona statements [10], [14]:

Theorem 5: *Let $f_1, \dots, f_q \in A$ (resp. $f_1, \dots, f_q \in B$) satisfy*

$$\inf_{x \in E} (\max_{1 \leq j \leq q} |f_j(x)|) > 0.$$

Then there exists $g_1, \dots, g_q \in A$ (resp. $g_1, \dots, g_q \in B$) such that

$$\sum_{j=1}^q f_j(x)g_j(x) = 1 \quad \forall x \in E.$$

Corollary 5.1: *Let I be an ideal of A (resp. of B) different from A (resp. from B). The family $D(f, \epsilon)$, $f \in I$, $\epsilon > 0$, generates a filter $\mathcal{F}_{I,A}$ (resp. $\mathcal{F}_{I,B}$) on E such that $I \subset \mathcal{I}(\mathcal{F}_{I,A}, A)$ (resp. $I \subset \mathcal{I}(\mathcal{F}_{I,B}, B)$).*

By Proposition B, we now have Corollary 5.2:

Corollary 5.2: *Let \mathcal{M} be a maximal ideal of A . There exists an ultrafilter \mathcal{U} on E such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, A)$. Moreover, if \mathcal{U} converges in E , then \mathcal{M} is of codimension 1.*

Corollary 5.3: *Suppose E is complete. Let \mathcal{M} be a maximal ideal of A and let \mathcal{U} be an ultrafilter on E such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, A)$. If \mathcal{U} is a Cauchy ultrafilter, then \mathcal{M} is of codimension 1.*

Corollary 5.4: *For every maximal ideal \mathcal{M} of A , there exists a unique $\mathcal{H} \in Y_{(\mathcal{S})}(E)$ such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, A) \forall \mathcal{U} \in \mathcal{H}$.*

Moreover, the mapping Φ that associates to each $\mathcal{M} \in \text{Max}(A)$ the unique $\mathcal{H} \in Y_{(\mathcal{S})}(E)$ such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, A) \forall \mathcal{U} \in \mathcal{H}$, is a bijection from $\text{Max}(A)$ onto $Y_{(\mathcal{S})}(E)$.

In the particular case when we consider the discrete topology on E , we have Corollary 5.5:

Corollary 5.5: *For every maximal ideal \mathcal{M} of the Banach K -algebra T of all bounded functions on E , there exists a unique ultrafilter \mathcal{U} on E such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, T)$.*

Moreover, the mapping Φ that associates to each $\mathcal{M} \in \text{Max}(T)$ the unique \mathcal{U} such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, T) \forall \mathcal{U} \in \mathcal{H}$, is a bijection from $\text{Max}(T)$ onto $U(E)$.

Theorem 6: *Let \mathcal{M} be a maximal ideal of B . There exists an ultrafilter \mathcal{U} on E such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, B)$. Moreover, if \mathcal{U} is a cauchy ultrafilter, then \mathcal{M} is of codimension 1.*

Corollary 6.1: *For every maximal ideal \mathcal{M} of B there exists a unique $\mathcal{H} \in Y_{(\mathcal{T})}(E)$ such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, B) \forall \mathcal{U} \in \mathcal{H}$.*

Moreover, the mapping Ψ that associates to each $\mathcal{M} \in \text{Max}(B)$ the unique $\mathcal{H} \in Y_{(\mathcal{T})}(E)$ such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, B) \forall \mathcal{U} \in \mathcal{H}$, is a bijection from $\text{Max}(B)$ onto $Y_{(\mathcal{T})}(E)$.

Theorem 7: *Let K be a locally compact field. Then every maximal ideal of A (resp. B) is of codimension 1.*

The Banaschewski compactification of E is directly linked to $\text{Max}(A)$.

Definition and notation: Let $\mathcal{B}(E)$ be the boolean ring of clopens provided with the classical laws Δ (the symmetrical difference taking place of the addition) and \cap (taking place of the multiplication).

We will denote by $\Sigma(E)$ the set of homomorphisms from $\mathcal{B}(E)$ to \mathbb{F}_2 : $\Sigma(E)$ is also called *the Stone space of $\mathcal{B}(E)$* and is provided with the topology of simple convergence, while \mathbb{F}_2 is provided with the discrete topology, so $\Sigma(E)$ is compact in the compact space $\mathbb{F}_2^{\mathcal{B}(E)}$.

Given $a \in E$, we denote by ζ_a the ring homomorphism from $\mathcal{B}(E)$ to \mathbb{F}_2 defined as $\zeta_a(O) = 1 \forall O \in \mathcal{B}(E)$ such that $a \in O$ and $\zeta_a(O) = 0 \forall O \in \mathcal{B}(E)$ such that $a \notin O$.

We will denote by $\Sigma_E(E)$ the set of the ζ_a , $a \in E$.

We know the following proposition [15]:

Proposition C : *There is a natural bijection between $\Sigma(E)$ and $Max(A)$. Moreover, $\Sigma(E)$ is compact and $\Sigma_E(E)$ is dense in $\Sigma(E)$.*

Here we can describe more precisely this bijection thanks to the following theorem:

Theorem 8 : *Let \mathcal{U}, \mathcal{V} be sticked ultrafilters on E . Then a clopen belongs to \mathcal{U} if and only if it belongs to \mathcal{V} .*

Corollary 8.1 *For every maximal ideal \mathcal{M} of A and $\mathcal{U}, \mathcal{V} \in \Phi(\mathcal{M})$, then a clopen belongs to \mathcal{U} if and only if it belongs to \mathcal{V} .*

Notation: Given $\mathcal{M} \in Max(A)$ we will denote by $\Xi(\mathcal{M})$ the mapping from $\mathcal{B}(E)$ to \mathbb{F}_2 defined as $\Xi(\mathcal{M})(O) = 1$ whenever any $\mathcal{U} \in \Phi(\mathcal{M})$ is secant with O and $\Xi(\mathcal{M})(O) = 0$ whenever any $\mathcal{U} \in \Phi(\mathcal{M})$ is not secant with O .

Theorem 9: *Given $\mathcal{M} \in Max(A)$, $\Xi(\mathcal{M})$ is a ring homomorphism from $\mathcal{B}(E)$ onto \mathbb{F}_2 .*

Theorem 10: *Ξ is a bijection from $Max(A)$ onto $\Sigma(E)$. Given $a \in E$ then $\Xi(\mathcal{I}(a, A))$ is ζ_a defined above. The restriction of Ξ to $Max_E(A)$ is a bijection from $Max_E(A)$ onto $\Sigma_E(E)$ and $\Sigma_E(E)$ is dense in $\Sigma(E)$.*

Definition: $\Sigma(E)$ is called the *Banaschewski compactification* of E .

We will now examine prime closed ideals of A and B .

Theorem 11: *Let \mathcal{U} be an ultrafilter on E and let \mathcal{P} be a prime ideal included in $\mathcal{I}(\mathcal{U}, A)$ (resp. $\mathcal{I}(\mathcal{U}, B)$). Let $L \in \mathcal{U}$ be a clopen (resp. let $L \in \mathcal{U}$ be uniformly open) and let $H = E \setminus L$. Let u be the function defined on E by $u(x) = 1 \forall x \in H$, $u(x) = 0 \forall x \in L$. Then u belongs to \mathcal{P} .*

Corollary 11.1: *Let \mathcal{U} be an ultrafilter on E and let $\mathcal{I}^{**}(\mathcal{U}, A)$ the ideal of the $f \in A$ such that there exists a clopen $H \in \mathcal{U}$ such that $f(x) = 0 \forall x \in H$. Then $\mathcal{I}^{**}(\mathcal{U}, A)$ is included in every prime ideal $\mathcal{P} \subset \mathcal{I}(\mathcal{U}, A)$.*

Corollary 11.2: *Let \mathcal{U} be an ultrafilter on E and let T be the Banach K -algebra of all bounded functions on E . Then $\mathcal{I}^*(\mathcal{U}, T)$ is the smallest prime ideal among all prime ideals $\mathcal{P} \subset \mathcal{I}(\mathcal{U}, T)$.*

Corollary 11.3: *Let \mathcal{U} be an ultrafilter on E and let $\mathcal{I}^{***}(\mathcal{U}, B)$ the ideal of the $f \in B$ such that there exists a uniformly open subset $H \in \mathcal{U}$ such that $f(x) = 0 \forall x \in H$. Then $\mathcal{I}^{***}(\mathcal{U}, B)$ is included in every prime ideal $\mathcal{P} \subset \mathcal{I}(\mathcal{U}, B)$.*

Theorem 12: *The closure of a prime ideal of A (resp. of B) is a maximal ideal.*

Corollary 12.1: *Let \mathcal{P} be a prime ideal of A (resp. of B). There exists a unique maximal ideal \mathcal{M} of A (resp. of B) containing \mathcal{P} .*

Corollary 12.2: *Every prime closed ideal of A (resp. of B) is a maximal ideal.*

Now, since the kernel of a continuous multiplicative semi-norm is a closed prime ideal, we will show Corollary 12.3:

Notation and definition: Let T be a normed commutative K -algebra with unity. We denote by $Mult(T, \| \cdot \|)$ the set of multiplicative semi-norms of T provided with the topology of simple convergence. Given $\phi \in Mult(T, \| \cdot \|)$, the set of the $x \in T$ such that $\phi(x) = 0$ is a closed prime ideal and is called *the kernel of ϕ* . It is denoted by $Ker(\phi)$.

We denote by $Mult_m(T, \| \cdot \|)$ the set of multiplicative semi-norms of T whose kernel is a maximal ideal and by $Mult_1(T, \| \cdot \|)$ the set of multiplicative semi-norms of T whose kernel is a maximal ideal of codimension 1.

Suppose now T is a K -algebra of bounded functions from E to K normed by the norm of uniform convergence on E . Let $a \in E$. The mapping φ_a from T to \mathbb{R} defined by $\varphi_a(f) = |f(a)|$ belongs to $Mult(T, \| \cdot \|)$.

Let \mathcal{U} be an ultrafilter on E . By Urysohn's Theorem, given $f \in T$, the mapping from E to \mathbb{R} that sends x to $|f(x)|$ admits a limit along \mathcal{U} . We set $\varphi_{\mathcal{U}}(f) = \lim_{\mathcal{U}} |f(x)|$.

Propositions D, E below are immediate and well known:

Proposition D: *Let $T = A$ or B and let $a \in E$. Then $\mathcal{I}(a, T)$ is a maximal ideal of T of codimension 1 and φ_a belongs to $Mult_1(T, \| \cdot \|)$.*

Notation: Let $T = A$ or B . We denote by $Mult_E(T, \| \cdot \|)$ the set of multiplicative semi-norms of T of the form φ_a , $a \in E$. Consequently, by definition, $Mult_E(T, \| \cdot \|)$ is a subset of $Mult_1(T, \| \cdot \|)$.

Proposition E is immediate:

Proposition E : *Let $T = A$ or B and let $a \in E$ and let \mathcal{U} be an ultrafilter on E . Then $\varphi_{\mathcal{U}}$ belongs to the closure of $Mult_E(T, \| \cdot \|)$.*

Now, Corollaries 12.3 is an immediate consequence of Theorem 12 and Propositions D, E:

Corollary 12.3 : *$Mult(A, \| \cdot \|) = Mult_m(A, \| \cdot \|)$, $Mult(B, \| \cdot \|) = Mult_m(B, \| \cdot \|)$. Further, if K is locally compact then $Mult(A, \| \cdot \|) = Mult_1(A, \| \cdot \|)$, $Mult(B, \| \cdot \|) = Mult_1(B, \| \cdot \|)$.*

Remark 8: Suppose K is locally compact and E is a disk in an algebraically closed complete ultrametric field. There do exist ultrafilters that do not converge. Let \mathcal{U} be such an ultrafilter. Then $\varphi_{\mathcal{U}}$ belongs to $Mult_1(B, \| \cdot \|)$ but does not belong to $Mult_E(B, \| \cdot \|)$.

Remark 9: In $\mathcal{H} \in Y_{(\mathcal{S})}(E)$, the various ultrafilters $\mathcal{U} \in \mathcal{H} \in Y_{(\mathcal{S})}(E)$ define various prime ideals of A . It is not clear whether these ideals are minimal among the set of prime ideals of A . Similarly, in $\mathcal{H} \in Y_{(\mathcal{T})}(E)$ the various ultrafilters $\mathcal{U} \in \mathcal{H} \in Y_{(\mathcal{T})}(E)$ define various prime ideals of B and it is not clear whether these ideals are minimal among the set of prime ideals of B .

Remark 10: The ideal $\mathcal{I}^{**}(\mathcal{U}, A)$ is not a prime ideal of A , as the following example shows.

Suppose E is the disk $d(0, 1)$ in the field K and let (a_n) be a sequence of limit 0 such that $|a_n| < |a_{n+1}|$. Let \mathcal{U} be an ultrafilter of limit 0. Let $r_n = |a_n|$, $n \in \mathbb{N}$, let $H = \bigcup_{n=0}^{\infty} d(a_n, r_n^-)$ and let $H' = E \setminus H$. Let $f(x) = x \forall x \in H$, $f(x) = 0 \forall x \in H'$ and let $g(x) = x - f(x)$. Then $f(x)g(x) = 0 \forall x \in E$. However, neither f nor g is identically zero on any clopen belonging to \mathcal{U} because such a clopen must contain the origin that is on the boundary of both H and H' .

Similarly, $\mathcal{I}^{***}(\mathcal{U}, B)$ is not a prime ideal of B .

Theorem 13: $\Phi \circ \Xi^{-1}$ is a homeomorphism from $\Sigma(E)$ onto $Mult(A, \| \cdot \|)$.

Corollary 13.1: The topology of $Mult(A, \| \cdot \|)$ and this of the Banaschewski compactification induce the same topology on $Max(A)$.

Corollary 13.2: The topology of $Mult(A, \| \cdot \|)$ does not depend on the field K .

Remark 11: Let F be a set compact for two topologies, admitting a subset E dense for both topologies. In general, we may not conclude that the two topologies are identical, as shows the following example.

Let $F = [0, 1]$ be provided with the topology \mathcal{N} induced by this of \mathbb{R} and let $E =]0, 1[$. Now, let \mathcal{Q} be the topology on F defined as follows:

For $a \in E$, a neighborhood of a is a subset of F containing an open interval included in E .

A neighborhood of 0 is a subset of F containing a subset of the form $\{0\} \cup]1 - \epsilon, 1[$.

A neighborhood of 1 is a subset of F containing a subset of the form $\{1\} \cup]0, \epsilon[$.

So we have defined \mathcal{Q} a topology on F . Of course, \mathcal{Q} is different from \mathcal{N} . Then E is obviously dense in F for \mathcal{Q} . Next, we can check that F is compact for \mathcal{Q} .

Theorem 14: Let $T = A$ or B . The topology induced on E by this of $Mult_E(T, \| \cdot \|)$ is equivalent to the metric topology defined by δ .

Theorem 15 was proved in [15] for the algebra B . We can find it again for A and B in a different way.

Theorem 15: Let $T = A$ or B and let \mathcal{M} be a maximal ideal of T . Let T' be the field $\frac{T}{\mathcal{M}}$. Let θ be the canonical surjection from T onto T' . Given any ultrafilter \mathcal{U} such that $\mathcal{I}(\mathcal{U}, T) = \mathcal{M}$, the quotient norm $\| \cdot \|'$ on T' is defined by $\|\theta(f)\|' = \lim_{\mathcal{U}} |f(s)|$ and hence is multiplicative.

Definition: Recall that given a commutative Banach K -algebra with unity T , every maximal ideal of T is the kernel of at least one continuous multiplicative semi-norm [4]. T is said to be *multibjective* if every maximal ideal is the kernel of only one continuous multiplicative semi-norm.

Remark 12: There exist ultrametric Banach K -algebras that are not multibjective [2], [4], [5].

Theorem 16: A, B are multibjective.

Corollary 16.1: The K -algebra of all bounded functions from a set X to K is multibjective.

By Corollaries 5.2, 5.5 and Theorem 16, we have Corollary 16.2:

Corollary 16.2: For every $\phi \in \text{Mult}(A, \| \cdot \|)$, there exists an ultrafilter \mathcal{U} on E such that $\phi(f) = \lim_{\mathcal{U}} |f(x)| \forall f \in A$.

Moreover, the mapping $\tilde{\Phi}$ that associates to each $\phi \in \text{Mult}(A, \| \cdot \|)$ the unique $\mathcal{H} \in Y_{(\mathcal{S})}(E)$ such that $\phi(f) = \lim_{\mathcal{U}} |f(x)| \forall f \in A, \forall \mathcal{U} \in \mathcal{H}$, is a bijection from $\text{Mult}(A, \| \cdot \|)$ onto $Y_{(\mathcal{S})}(E)$.

Corollary 16.3: Let T be the Banach algebra of all bounded functions from E to K . For every $\phi \in \text{Mult}(T, \| \cdot \|)$ there exists a unique ultrafilter \mathcal{U} on F such that $\phi(f) = \lim_{\mathcal{U}} |f(x)| \forall f \in T$. The mapping $\tilde{\Phi}$ that associates to each $\phi \in \text{Mult}(T, \| \cdot \|)$ the unique ultrafilter \mathcal{U} such that $\phi(f) = \lim_{\mathcal{U}} |f(x)| \forall f \in T$, is a bijection from $\text{Mult}(T, \| \cdot \|)$ onto the set of ultrafilters on F .

Corollary 16.4: For every $\phi \in \text{Mult}(B, \| \cdot \|)$ there exists a unique $\mathcal{H} \in Y_{(\mathcal{T})}(E)$ such that $\phi(f) = \lim_{\mathcal{U}} |f(x)| \forall f \in B, \forall \mathcal{U} \in \mathcal{H}$.

Moreover, the mapping $\tilde{\Psi}$ that associates to each $\phi \in \text{Mult}(B, \| \cdot \|)$ the unique $\mathcal{H} \in Y_{(\mathcal{T})}(E)$ such that $\phi(f) = \lim_{\mathcal{U}} |f(x)| \forall f \in B, \forall \mathcal{U} \in \mathcal{H}$, is a bijection from $\text{Mult}(B, \| \cdot \|)$ onto $Y_{(\mathcal{T})}(E)$.

Remark 13: Consider two ultrafilters \mathcal{U}, \mathcal{V} which are contiguous but not sticked. They define the same maximal ideal and the same multiplicative semi-norm on B , but not on A . This means that for every bounded uniformly continuous fonction f from E to K , we have

$\lim_{\mathcal{U}} |f(x)| = \lim_{\mathcal{V}} |f(x)|$. But there exist bounded continuous functions g from E to K such that $\lim_{\mathcal{U}} |g(x)| \neq \lim_{\mathcal{V}} |g(x)|$. Actually, by Theorem 1, we can find a bounded continuous function u such that $\lim_{\mathcal{U}} |u(x)| = 1$, $\lim_{\mathcal{V}} |u(x)| = 0$.

Now, by Propositions D and E, we have Corollary 16.5:

Corollary 16.5: *$Mult_E(A, \|\cdot\|)$ is dense in $Mult(A, \|\cdot\|)$, $Mult_E(B, \|\cdot\|)$ is dense in $Mult(B, \|\cdot\|)$.*

Remark 14: In [8], it is showed that in the algebra of bounded analytic functions in the open unit disk of a complete ultrametric algebraically closed field, any maximal ideal which is not defined by a point of the open unit disk is of infinite codimension. Here, we may ask whether the same holds. In the general case no answer is obvious. We can only answer a particular case:

Theorem 17: *Suppose K is algebraically closed. Let \mathcal{U} be an ultrafilter on K and let $P \in K[x]$, $P \neq 0$ satisfy $\lim_{\mathcal{U}} P(x) = 0$. Then \mathcal{U} is a principal ultrafilter.*

As a consequence, we have Theorem 18:

Theorem 18: *Suppose K is algebraically closed. Let F be a closed bounded subset of K with infinitely many points and let \mathcal{M} be a maximal ideal of A (resp. B) which is not principal. Then \mathcal{M} is of infinite codimension.*

Remark 15: Suppose K is algebraically closed and let $E = K$. Then the algebras A , B contain no polynomial. In such a case, it is not clear whether maximal ideals not defined by points of K are of infinite codimension.

Remark 15: Concerning uniformly continuous functions, it has been shown that two ultrafilters that are not contiguous define two distinct continuous multiplicative semi-norms.

Now, concerning bounded analytic functions inside the disk $F = \{x \in K \mid |x| < 1\}$, in [9], it was shown that the same property holds for a large set of ultrafilters on F . However, the question remains whether it holds for all ultrafilters on F .

Let us recall some results on the Shilov boundary of an ultrametric normed algebra:

Proposition F [5], [6]: *Let T be a normed K -algebra whose norm is $\|\cdot\|$. For each $x \in T$, let $\|x\|_{si} = \lim_{n \rightarrow \infty} \|x^n\|^{\frac{1}{n}}$. Then $\|\cdot\|_{si}$ is power multiplicative semi-norm on T .*

Definitions: Let T be a normed K -algebra whose norm is $\|\cdot\|$. We call *spectral semi-norm of T* the semi-norm defined by Proposition F.

We call *Shilov boundary of T* a closed subset S of $\text{Mult}(T, \|\cdot\|)$ that is minimum with respect to inclusion, such that, for every $x \in T$, there exists $\phi \in S$ such that $\phi(x) = \|x\|_{si}$.

Proposition G [5], [7] : *Every normed K -algebra admits a Shilov boundary.*

Theorem 19: *The Shilov boundary S of A (resp. B) is equal to $\text{Mult}(A, \|\cdot\|)$ (resp. $\text{Mult}(B, \|\cdot\|)$).*

The Proofs:

Lemma 1 is classical due to the ultrametric distance of E :

Lemma 1: *For every $r > 0$, E admits a partition of the form $(d(a_i, r^-))_{i \in I}$.*

Definition and notation: A function f from E to K will be said to be *uniformly locally constant* if there exists $r > 0$ such that for every $a \in E$, $f(x)$ is constant in $d(a, r^-)$.

Lemma 2: *The set of bounded uniformly locally constant functions from E to K is a K -subalgebra of B and is dense in B*

Proof: It is obvious that S is a K -algebra and is included in B . We will check that S is dense in B . Let $f \in B$ and let ϵ be > 0 . There exists $r > 0$ such that $|f(x) - f(y)| \leq \epsilon$ for all $x, y \in E$ such that $\delta(x, y) \leq \epsilon$. Now, by Lemma 1, E admits a partition of the form $(d(a_i, r^-))_{i \in I}$. Let h be the function defined by $h(x) = f(a_i) \forall x \in d(a_i, r^-)$. Clearly, $\|f - h\| \leq \epsilon$.

Remark 16: Lemma 2 suggests that in our general study, we can't find an interesting complete subalgebra of B .

Notation: We will denote by $|\cdot|_\infty$ the Archimedean absolute value of \mathbb{R} .

Lemma 3: *Let $m, M \in \mathbb{R}_+^*$ and let $f \in A$. Then the sets $H = \{x \in E \mid |f(x) - m|_\infty \geq M\}$, $L = \{x \in E \mid |f(x) - m|_\infty \leq M\}$, are clopen. Moreover, if $f \in B$, then H, L are uniformly open.*

Lemma 4: *Let H be a clopen. Then the characteristic function u of H belongs to A . Moreover, if H is uniformly open, then u belongs to B .*

Given a bounded function in the set E , $|f(x)|$ obviously takes values in a compact of \mathbb{R} , therefore the following Lemma 5 comes from Urysohn's Theorem [1].

Lemma 5: *Let \mathcal{U} be an ultrafilter on E . Let f be a bounded function from E to K . The function $|f|$ from E to \mathbb{R}_+ defined as $|f|(x) = |f(x)|$ admits a limit along \mathcal{U} . Moreover, if K is locally compact, then $f(x)$ admits a limit along \mathcal{U} .*

Lemma 6: *Let \mathcal{U}, \mathcal{V} be sticked (resp. contiguous) ultrafilters on E and let $f \in A$ (resp. let $f \in B$). Then $\lim_{\mathcal{U}} |f(x)| = \lim_{\mathcal{V}} |f(x)|$.*

Lemma 7 is immediate:

Lemma 7: *Let $f \in B$ and let \tilde{E} be the completion of E . Then f has continuation to a function \tilde{f} uniformly continuous on \tilde{E} .*

Proof of Theorem 1: If \mathcal{U}, \mathcal{V} are sticked, then by definition, given a clopen $H \in \mathcal{U}$ and a clopen $L \in \mathcal{V}$ we have $H \cap L \neq \emptyset$.

Now, suppose that two ultrafilters \mathcal{U}, \mathcal{V} are not sticked. We can find closed subsets $F \in \mathcal{U}, G \in \mathcal{V}$ of E such that $F \cap G = \emptyset$. For each $x \in F$, let $r(x)$ be the distance from x to G and let $H = \bigcup_{x \in F} d(x, (\frac{1}{2}r(x))^-)$. So, H is open. Suppose H is not closed and let $(c_n)_{n \in \mathbb{N}}$ be a sequence of H converging to a point $c \in E \setminus H$. Since the distance is ultrametric, each point c_n belongs to a ball $d(a_n, r(a_n))^-$ with $a_n \in F$. Suppose the sequence $(r(a_n))_{n \in \mathbb{N}}$ does not tend to 0. There exists a subsequence $(r(a_{q(m)}))_{m \in \mathbb{N}}$ and $s > 0$ such that $r(a_{q(m)}) \geq s \forall m \in \mathbb{N}$ and consequently, c belongs to one of the balls $d(a_{q(m)}, (\frac{1}{2}r(a_{q(m)}))^-)$, a contradiction. Thus, the sequence $(r(a_n))_{n \in \mathbb{N}}$ must tend to 0. But since a_n belongs to F and since F is closed, clearly c lies in F , a contradiction. Thus, H is a clopen. By Construction, H belongs to \mathcal{U} and satisfies $H \cap G = \emptyset$, hence H does not belong to \mathcal{V} . Now, let $L = E \setminus H$. Then L also is a clopen that does not belong to \mathcal{U} . But since \mathcal{V} is an ultrafilter that is not secant with H , it is secant with L and hence L belongs to \mathcal{V} , which ends the proof.

Proof of Theorem 2: Since \mathcal{U}, \mathcal{V} are not sticked, by Theorem 1 we can find clopens $H \in \mathcal{U}, L \in \mathcal{V}$ of E such that $H \cap L = \emptyset$. Then the set $H' = E \setminus H$ also is a clopen. Let u be the characteristic function of H . Since H and H' are open, u is continuous, which ends the proof.

Proof of Theorem 3: Since \mathcal{U} and \mathcal{V} are not contiguous, there exist $H \in \mathcal{U}, L \in \mathcal{V}$ such that $\delta(H, L) = \mu > 0$. Let $H' = \{x \in E \mid \delta(x, H) \leq \frac{\mu}{2}\}$. Then H' is a clopen containing H and by ultrametricity, we can check that $\delta(H', L) \geq \mu$. Let u be the function defined in E by $u(x) = 1 \forall x \in H'$ and $f(x) = 0 \forall x \in E \setminus H'$. Since u is constant in any ball of diameter $\frac{\mu}{2}$, u belongs to B .

Proof of Proposition A: It is obvious and well known that $\mathcal{I}(\mathcal{U}, T)$ is prime. Let us check that so is $\mathcal{I}^*(\mathcal{U}, T)$. Suppose $\mathcal{I}^*(\mathcal{U}, T)$ is not prime. There exists $f, g \notin \mathcal{I}^*(\mathcal{U}, T)$ such that $fg \in \mathcal{I}^*(\mathcal{U}, T)$. Thus, there exists $L \in \mathcal{U}$ such that $f(x)g(x) = 0 \forall x \in L$, but neither f nor g are identically zero on L . Let F be the set of the $x \in L$ such that $f(x) = 0$ and let G be the set of the $x \in L$ such that $g(x) = 0$. Then $F \cup G = L$, hence \mathcal{U} is secant at least with one of the two sets F and G . Suppose it is secant with F . The intersection of \mathcal{U} with F is a filter thinner than \mathcal{U} , hence it is \mathcal{U} . Thus, f is identically zero on a set $F \in \mathcal{U}$, a contradiction. And similarly if it is secant with G .

Proof of Theorem 4: First, if \mathcal{U} and \mathcal{V} are not sticked, by Theorem 2 we have $\mathcal{I}(\mathcal{F}, A) \neq \mathcal{I}(\mathcal{G}, A)$.

Now, suppose that \mathcal{U}, \mathcal{V} are sticked and let $f \in \mathcal{I}(\mathcal{U}, A)$. Let ϵ be > 0 and let $H \in \mathcal{U}$ be such that $|f(x)| \leq \epsilon \forall x \in H$. Since $|f|$ has a limit l along \mathcal{V} , we can find $L \in \mathcal{V}$ such that $|f(x) - l| \leq \epsilon \forall x \in L$. Since f is continuous, it satisfies $|f(x)| \leq \epsilon \forall x \in \overline{H}$ and $|f(x) - l| \leq \epsilon \forall x \in \overline{L}$. But since \mathcal{U}, \mathcal{V} are sticked, there exists $a \in \overline{H} \cap \overline{L}$, hence $l \leq 2\epsilon$. And since ϵ is arbitrary, then $l = 0$ and hence $f \in \mathcal{I}(\mathcal{V}, A)$. Thus, $\mathcal{I}(\mathcal{U}, A) \subset \mathcal{I}(\mathcal{V}, A)$. And symmetrically, we have $\mathcal{I}(\mathcal{V}, A) \subset \mathcal{I}(\mathcal{U}, A)$, hence the two ideals are equal.

Suppose now that \mathcal{U}, \mathcal{V} are not contiguous. By Theorem 3, there exist $H \in \mathcal{U}, L \in \mathcal{V}$ and $f \in B$ such that $f(x) = 1 \forall x \in H, f(x) = 0 \forall x \in L$. Consequently, f belongs to $\mathcal{I}(\mathcal{U}, B)$ but does not belong to $\mathcal{I}(\mathcal{V}, B)$. Thus, $\mathcal{I}(\mathcal{F}, B) \neq \mathcal{I}(\mathcal{G}, B)$.

Finally, suppose that \mathcal{U}, \mathcal{V} are contiguous. Let $f \in \mathcal{I}(\mathcal{U}, B)$. Let $l = \lim_{\mathcal{V}} |f(x)|$, suppose $l > 0$ and let $L \in \mathcal{V}$ be such that $||f(x)| - l|_{\infty} \leq \frac{l}{3} \forall x \in L$, hence $|f(x)| \geq \frac{2l}{3} \forall x \in L$. Let $H \in \mathcal{U}$ be such that $|f(x)| \leq \frac{l}{3} \forall x \in H$. Since $f \in B$, there exists $\rho > 0$ such that $\delta(x, y) \leq \rho$ implies $|f(x) - f(y)| \leq \frac{l}{4}$. And since f is uniformly continuous, there exist $a \in H, b \in L$ such that $\delta(a, b) \leq \rho$, hence $|f(a) - f(b)| \leq \frac{l}{4}$, a contradiction because $|f(a)| \leq \frac{l}{3}$ and $|f(b)| \geq \frac{2l}{3}$.

Proof of Proposition B: Suppose that \mathcal{U} converges to a point a . Given $f \in A$ (resp. $f \in B$), we have $\lim_{\mathcal{U}} f(x) = f(a)$. So, the mapping θ from A (resp. B) to K defined as $\theta(f) = f(a)$ admits \mathcal{M} for kernel and hence \mathcal{M} is of codimension 1.

Proof of Theorem 5: Let $M = \inf_{x \in E} (\max_{1 \leq j \leq q} |f_j(x)|)$. Let $E_j = \{x \in E \mid |f_j(x)| \geq M\}$, $j = 1, \dots, q$ and let $F_j = \bigcup_{m=1}^j E_m$, $j = 1, \dots, q$. Let $g_1(x) = \frac{1}{f_1(x)} \forall x \in E_1$ and $g_1(x) = 0 \forall x \in E \setminus E_1$. Since $|f_1(x)| \geq M \forall x \in E_1$, $|g_1(x)|$ is clearly bounded.

Suppose first $f_1, \dots, f_q \in A$. Since E is ultrametric, each E_j is obviously a clopen and so is each F_j . And since f_1 is continuous g_1 is continuous, hence belongs to A .

Now, suppose that $f_1, \dots, f_q \in B$. By Lemma 3, E_1 has a strictly positive codiameter ρ and so does $E \setminus E_1$. Then g_1 is obviously uniformly continuous in $E \setminus E_1$. And, since $|f_1(x)| \geq M \forall x \in E_1$, g_1 is uniformly continuous in E_1 . Hence it is uniformly continuous in E . Thus g_1 belongs to B .

Suppose now we have constructed $g_1, \dots, g_k \in A$ (resp. $g_1, \dots, g_k \in B$) satisfying $\sum_{j=1}^k f_j g_j(x) = 1 \forall x \in F_k$ and $\sum_{j=1}^k f_j g_j(x) = 0 \forall x \in E \setminus F_k$. Let g_{k+1} be defined on E by $g_{k+1}(x) = \frac{1}{f_{k+1}(x)} \forall x \in F_{k+1} \setminus F_k$ and $g_{k+1}(x) = 0 \forall x \in E \setminus (F_{k+1} \setminus F_k)$. Then g_{k+1} is bounded.

Now we can check that $\sum_{j=1}^{k+1} f_j g_j(x) = 1 \forall x \in F_{k+1}$ and $\sum_{j=1}^k f_j g_j(x) = 0 \forall x \in E \setminus F_{k+1}$. So, by an immediate recurrence, we can get bounded functions g_1, \dots, g_q such that

$$\sum_{j=1}^q f_j g_j(x) = 1 \quad \forall x \in E.$$

Now suppose that $f_1, \dots, f_q \in A$. Since F_k and F_{k+1} are clopens, so is $E \setminus (F_{k+1} \setminus F_k)$ and consequently, g_{k+1} is continuous. Similarly as for g_1 , since $|f_{k+1}(x)| \geq M \quad \forall x \in E_{k+1}$, $|g_{k+1}(x)|$ is clearly bounded, hence belongs to A . And similarly, if $g_1, \dots, g_k \in B$, g_{k+1} belongs to B for the same reason as g_1 above. So, by induction, we can get $g_1, \dots, g_q \in B$ such that $\sum_{j=1}^q f_j g_j(x) = 1 \quad \forall x \in E$.

Proof of Theorem 6: By Theorem 5, there exists an ultrafilter \mathcal{U} on E such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, B)$. Now, suppose that \mathcal{M} is a Cauchy ultrafilter. Since the functions of B are uniformly continuous, by Lemma 7 they have continuation to the completion \tilde{E} of E and \mathcal{U} defines an ultrafilter that converges in \tilde{E} to a point a . Given $f \in B$, let \tilde{f} be the continuation of f in \tilde{E} : we have $\lim_{\mathcal{U}} f(x) = \tilde{f}(a)$. So by Proposition B \mathcal{M} is of codimension 1.

Proof of Theorem 7: Let \mathcal{M} be a maximal ideal of A (resp. B). By Corollary 4.2 there exists an ultrafilter \mathcal{U} such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, A)$ (resp. $\mathcal{M} = \mathcal{I}(\mathcal{U}, B)$). Let $f \in A$ (resp. $f \in B$). By Lemma 2, the function f has a limit $\chi(f)$ along \mathcal{U} . Thus, the mapping χ from A (resp. B) to K is a K -algebra homomorphism and therefore \mathcal{M} is of codimension 1.

Proof of Theorem 8: Let \mathcal{U}, \mathcal{V} be stucked ultrafilters and let O be a clopen that belongs to \mathcal{U} . Suppose it does not belong to \mathcal{V} . Then \mathcal{V} is secant with $E \setminus O$. But since \mathcal{V} is an ultrafilter, $E \setminus O$ belongs to \mathcal{V} . But $E \setminus O$ is a clopen, hence it has a non-empty intersection with O (because \mathcal{U} and \mathcal{V} are stucked), a contradiction. Thus, O belongs to \mathcal{V} , which proves Theorem 8.

Lemma 8 : *Let O be a clopen and let \mathcal{U} be an ultrafilter that is not secant with O . There exists a clopen L that belongs to \mathcal{U} and satisfies $L \cap O = \emptyset$.*

Proof : Let H be a clopen that belongs to \mathcal{U} and let $L = H \setminus O$. Since \mathcal{U} is not secant with O , it is secant with L . But since both H, O are clopen, so is L . And by definition, $L \cap O = \emptyset$.

Proof of Theorem 9: Let $\mathcal{U} \in \Phi(\mathcal{M})$. Let O_1, O_2 be clopen and set $\theta = \Xi(\mathcal{M})$. We first have to check that $\theta(O_1 \Delta O_2) = \theta(O_1) + \theta(O_2)$ in \mathbb{F}_2 . Let $\mathcal{U} \in \Phi(\mathcal{M})$.

If O_1 belongs to \mathcal{U} and if $O_2 \notin \mathcal{U}$, the conclusion is immediate. Similarly, so is it whenever $O_1 \notin \mathcal{U}$ and $O_2 \in \mathcal{U}$. Now, consider the case when $O_1 \in \mathcal{U}$ and $O_2 \in \mathcal{U}$.

Then $O_1 \cap O_2$ belongs to \mathcal{U} . But since \mathcal{U} is an ultrafilter, it cannot be secant with $(O_1 \cup O_2) \setminus (O_1 \cap O_2)$. Consequently, $\theta(O_1 \Delta O_2) = 0$. So we have checked that $\theta(O_1 \Delta O_2) = \theta(O_1) + \theta(O_2)$.

Concerning $\theta(O_1 \cap O_2)$, clearly, $\theta(O_1 \cap O_2) = 1$ if and only if both O_1, O_2 belong to \mathcal{U} i.e. $\theta(O_1) = \theta(O_2) = 1$, hence $\theta(O_1 \cap O_2) = \theta(O_1)\theta(O_2)$. This finishes proving that θ is a ring homomorphism.

Proof of Theorem 10: Let us check that Ξ is injective. Suppose $\mathcal{M}_1, \mathcal{M}_2$ are two distinct maximal ideals such that $\Xi(\mathcal{M}_1) = \Xi(\mathcal{M}_2)$. Let $\mathcal{U}_1 \in \Phi(\mathcal{M}_1), \mathcal{U}_2 \in \Phi(\mathcal{M}_2)$. Since $\mathcal{U}_1, \mathcal{U}_2$ are not sticked, by Theorem 1 there exists a clopen $O \in \mathcal{U}_1$ that does not belong to \mathcal{U}_2 . Consequently, $\Xi(\mathcal{M}_1)(O) = 1, \Xi(\mathcal{M}_2)(O) = 0$, which proves that Ξ is injective.

Now, let us check that Ξ is surjective. Let $\theta \in \Sigma(E)$. The family of clopens O satisfying $\theta(O) = 1$ clearly generates a filter \mathcal{F} . Let \mathcal{U} be an ultrafilter thinner than \mathcal{F} and let $\mathcal{M} = \mathcal{I}(\mathcal{U}, A)$. We will check that $\theta = \Xi(\mathcal{M})$. Let O be clopen that belongs to \mathcal{U} . Then \mathcal{F} is secant with O , hence $\theta(O) = 1$. Now, let \mathcal{V} be an ultrafilter sticked to \mathcal{U} . By Theorem 8, the clopens that belong to \mathcal{U} are the same as those which belong to \mathcal{V} . Consequently, $\theta(O) = \Xi(\mathcal{M})(O)$ for every clopen which belongs to any $\mathcal{V} \in \Phi(\mathcal{M})$.

And now, let O be a clopen that does not belong to any $\mathcal{V} \in \Phi(\mathcal{M})$. Let us take again $\mathcal{U} \in \Phi(\mathcal{M})$. By Lemma 8, there exists a clopen $L \in \mathcal{V}$ such that $O \cap L = \emptyset$. Then $O \Delta L = O \cup L$ belongs to \mathcal{U} . Hence $\theta(L \cup O) = 1 = \theta(L)$ and consequently, $\theta(O) = 0$, which finishes proving that $\theta = \Xi(\mathcal{M})$. So, Ξ is surjective.

$\Sigma(E)$ is compact because it is closed in $\mathbb{F}_2^{\mathcal{B}(E)}$. By definition, we have $\Xi(\mathcal{I}(a, A)) = \zeta_a \forall a \in E$. Let us check that $\Sigma_E(E)$ is dense in $\Sigma(E)$. Let $\theta = \Xi(\mathcal{M})$ ($\mathcal{M} \in \text{Max}(A)$) and let O_1, \dots, O_q be clopens. We may assume that $\theta(O_j) = 1 \forall j = 1, \dots, k$ and $\theta(O_j) = 0 \forall j = k + 1, \dots, q$. Let $\mathcal{U} \in \Phi(\mathcal{M})$. Then \mathcal{U} is secant with O_1, \dots, O_k and is not with O_{k+1}, \dots, O_q . Let $a \in \bigcap_{j=1}^k O_j$, hence clearly $\theta(O_j) = \zeta_a(O_j) \forall j = 1, \dots, q$. This finishes the proof of Theorem 10.

Proof of Theorem 11: We know that $u \in A$ (resp. $u \in B$). By construction, $1 - u$ does not belong to $\mathcal{I}(\mathcal{U}, A)$ (resp. $\mathcal{I}(\mathcal{U}, B)$) because $\lim_{\mathcal{U}} u(x) = 1$. But $u(1 - u) = 0$, hence u belongs to \mathcal{P} because \mathcal{P} is prime.

Proof of Theorem 12: Let \mathcal{P} be a prime ideal of A included in a maximal ideal $\mathcal{M} = \mathcal{I}(\mathcal{U}, A)$, (resp. $\mathcal{M} = \mathcal{I}(\mathcal{U}, B)$). Let f belong to $\mathcal{I}(\mathcal{U}, A)$, (resp. $\mathcal{I}(\mathcal{U}, B)$). Let us take $\epsilon > 0$ and find $h \in \mathcal{P}$ such that $\|f - h\| \leq \epsilon$. By Lemma 3 we can find a clopen $L \in \mathcal{U}$, (resp. a uniformly open subset $L \in \mathcal{U}$) such that $|f(x)| \leq \epsilon \forall x \in L$. Let u be the characteristic function of $E \setminus L$. By Theorem 11 u belongs to \mathcal{P} and hence so does uf . We then check that $\|f - uf\| \leq \epsilon$. Thus, \mathcal{P} is dense in \mathcal{M} .

Notation: Let $T = A$ or B . Given $f_1, \dots, f_q \in T$, $\epsilon > 0$, on $\text{Mult}_E(T, \|\cdot\|)$ we will denote by $W(\varphi_a, f_1, \dots, f_q, \epsilon)$ the set of the φ_x such that $\| |f_j(x)| - |f_j(a)| \|_{\infty} < \epsilon \forall j = 1, \dots, q$.

Definitions and notation: Let $T = A$ or B . Given $\phi \in \text{Mult}(T, \|\cdot\|)$ and $f_1, \dots, f_q \in T$, $\epsilon > 0$, we will denote by $W(\phi, f_1, \dots, f_q, \epsilon)$ the set of the ψ such that $\| |\phi(f_j)| - |\psi(f_j)| \|_{\infty} < \epsilon \forall j = 1, \dots, q$. Such neighborhoods of ϕ will be called *basic neighborhoods of ϕ* .

By definition of the topology of simple convergence on $\text{Mult}(T, \|\cdot\|)$ we know that the set of basic neighborhoods of ϕ makes a fundamental system of neighborhoods of ϕ .

Similarly, given $\zeta \in \Sigma(E)$ and clopens O_1, \dots, O_q we will denote by $Z(\zeta, O_1, \dots, O_q)$ the set of the ξ such that $\zeta(O_j) = \xi(O_j) \forall j = 1, \dots, q$. Such neighborhoods of ζ will be called *basic neighborhoods of ζ* .

Then by definition of the topology of simple convergence on $\Sigma(E)$, we know that the set of basic neighborhoods of ζ makes a fundamental system of neighborhoods of ζ .

Given $\psi \in Mult(A, \|\cdot\|)$, we set $\bar{\psi} = \Xi \circ \Phi^{-1}(\psi)$.

Lemma 9: quad Let $\phi \in Mult(A, \|\cdot\|)$, let O_1, \dots, O_q be clopens and let $\epsilon \in]0, 1[$. Let $D = \bigcap_{j=1}^q O_j$ and let u be the characteristic function of D . Then, given $\psi \in Mult(A, \|\cdot\|)$, ψ belongs to $W(\phi, u, \epsilon)$ if and only if $\bar{\psi}$ belongs to $Z(\bar{\phi}, O_1, \dots, O_q)$.

Proof: Let \mathcal{U} be an ultrafilter such that $Ker(\phi) = \mathcal{I}(\mathcal{U}, A)$. Of course $\phi(u)$ is equal to 0 or 1. Since $\epsilon < 1$, we can see that $\psi(u) = 1$ if and only if \mathcal{U} is secant with D and $\psi(u) = 0$ if and only if \mathcal{U} is not secant with D . But this holds if and only if $\bar{\psi}(O_j) = 1 \forall j = 1, \dots, q$, hence $\bar{\psi}$ belongs to $Z(\bar{\phi}, O_1, \dots, O_q)$.

Proof of Theorem 13: Let $\phi \in Mult(A, \|\cdot\|)$ and consider first a basic neighborhood of $\bar{\phi}$: $Z(\bar{\phi}, O_1, \dots, O_q)$. Since $\bigcap_{j=1}^q O_j$ is a clopen, its characteristic function u belongs to A . Then by Lemma 9, given $\epsilon \in]0, 1[$, a $\psi \in Mult(A, \|\cdot\|)$ belongs to $W(\phi, u, \epsilon)$ if and only if $\bar{\psi}$ belongs to $Z(\bar{\phi}, O_1, \dots, O_q)$. Consequently, any basic neighborhood of $\bar{\phi}$ is the image of a basic neighborhood of ϕ by the bijection $\Xi \circ \Phi^{-1}$. Therefore, the topology of $Mult(A, \|\cdot\|)$ is at least as thin as this of $\Sigma(E)$.

Now, conversely, consider a basic neighborhood of ϕ : $W(\phi, f_1, \dots, f_q, \epsilon)$ with $\epsilon \in]0, 1[$. The set of the $x \in E$ such that $||f_j(x)| - \phi(f_j)|_\infty \leq \epsilon \forall j = 1, \dots, q$ is a clopen D hence its characteristic function u belongs to A . Now, let $\zeta \in \Sigma(E)$. It is of the form $\Xi(\mathcal{M})$ where \mathcal{M} is a maximal ideal $\mathcal{I}(\mathcal{U}, A)$ with \mathcal{U} an ultrafilter on E . And then, given any clopen O , we have $\Xi(\mathcal{M})(O) = 1$ if and only if \mathcal{U} is secant with O . Thus, ζ belongs to $Z(\bar{\phi}, D)$ if and only if ultrafilters \mathcal{U} such that $\Xi^{-1}(\zeta) = \mathcal{I}(\mathcal{U}, A)$ are secant with D . Now, suppose that ζ belongs to $Z(\bar{\phi}, D)$. Of course, $Ker(\Phi \circ \Xi^{-1}(\zeta))$ is equal to $\mathcal{I}(\mathcal{U}, A)$. Set $\hat{\zeta} = \Phi \circ \Xi^{-1}(\zeta)$. Since \mathcal{U} is secant with D , the inequality $|\lim_{\mathcal{U}} |f_j(x)| - \phi(f_j)|_\infty \leq \epsilon$ holds for every $j = 1, \dots, q$. Therefore, $\hat{\zeta}$ belongs to $W(\phi, f_1, \dots, f_q, \epsilon)$. This proves that $\Xi \circ \Phi^{-1}(W(\phi, f_1, \dots, f_q, \epsilon))$ contains a neighborhood of ζ in $\Sigma(E)$ and finishes proving that the two topologies are equivalent.

Proof of Theorem 14: Let $a \in E$. The filter of neighborhoods of a admits for basis the family of balls $d(a, r^-) = \{x \in E \mid \delta(x, a) \leq r\}$, $r > 0$. But we can check that such a ball is induced by a neighborhood of φ_a with respect to both topologies of $Mult_E(A, \|\cdot\|)$ and $Mult_E(B, \|\cdot\|)$. Given $\varphi_a \in Mult_E(T, \|\cdot\|)$, we set $W'(\varphi_a, f_1, \dots, f_q, \epsilon) = W(\varphi_a, f_1, \dots, f_q, \epsilon) \cap Mult_E(T, \|\cdot\|)$. Let $r \in]0, 1[$. By Lemma 4 there exists $u \in B$ such that $u(x) = 0 \forall x \in \mathcal{B}(a, r)$ and $u(x) = 1 \forall x \in E \setminus d(a, r^-)$.

Consequently, $W'(\varphi_a, u, r)$ is the set of the φ_b such that $|b - a| \leq r$. This holds when we consider A as when we consider B and hence the topology of $Mult_E(B, \|\cdot\|)$ as this of $Mult_E(A, \|\cdot\|)$ is thinner or equal to the metric topology of E . Now, since each f_j is continuous, the set of the $x \in E$ such that $|f_j(x) - f_j(a)| \leq \epsilon \forall j = 1, \dots, q$ is open and hence contains a ball $d(a, r^-)$ of E . Consequently, the topology of E is thinner or equal to this of $Mult_E(T, \|\cdot\|)$, which finishes proving that the topology induced on E by $Mult(T, \|\cdot\|)$ coincides with the metric topology of E .

Proof of Theorem 15: Let $t \in T'$ and let $f \in T$ be such that $\theta(f) = t$. Let \mathcal{U} be an ultrafilter such that $\mathcal{I}(\mathcal{U}, T) = \mathcal{M}$. So, $\|t\|' \geq \lim_{\mathcal{U}} |f(s)|$. Conversely, let $W \in \mathcal{U}$ be such that $|f(x)| \leq \lim_{\mathcal{U}} |f(s)| + \epsilon \forall x \in W$. There exists $f_1, \dots, f_q \in \mathcal{M}$ and $\epsilon > 0$ such that

$$\bigcap_{j=1}^q D(f_j, \epsilon) \subset W. \text{ Let } X = \bigcap_{j=1}^q D(f_j, \epsilon).$$

Suppose \mathcal{M} is a maximal ideal of T . There exists $u \in T$ such that $u(x) = 0 \forall x \in X$, $u(x) = 1 \forall x \in F \setminus X$. Then $u(1 - u) = 0$. But $1 - u \notin \mathcal{M}$. Hence, u belongs to \mathcal{M} . Then $\theta(f - uf) = \theta(f) = t$. But by construction, $(f - uf)(x) = 0 \forall x \in F \setminus X$ and $(f - uf)(x) = f(x) \forall x \in X$. Consequently, $\|f - uf\| \leq \lim_{\mathcal{U}} |f(s)| + \epsilon$ and therefore $\|t\|' = \|\theta(f - uf)\|' \leq \lim_{\mathcal{U}} |f(s)| + \epsilon$. This finishes proving the equality $\|\theta(f)\|' = \lim_{\mathcal{U}} |f(s)|$. Now, such a norm defined as $\|\theta(f)\|' = \lim_{\mathcal{U}} |f(s)|$ is obviously multiplicative. The proof concerning A is exactly similar, the set X being then closed and open by Lemma 3 and the proof concerning B is also similar the set X being uniformly open.

Proof of Theorem 16: Let \mathcal{M} be a maximal ideal of A (resp. of B) and let A' be the field $\frac{A}{\mathcal{M}}$ (resp. let B' be the field $\frac{B}{\mathcal{M}}$). By Theorem 15, the quotient norm of A' (resp. of B') is multiplicative. But then, A' (resp. B') admits only one continuous multiplicative semi-norm: its quotient K -algebra norm. Consequently, A (resp. B) admits only one continuous multiplicative semi-norm whose kernel is \mathcal{M} , which proves that A (resp. B) is multibjective.

Proof of Theorem 17: Let $P(x) = \prod_{j=1}^q (x - a_j)$. Let \mathcal{F} be the filter admitting for

basis the sets $\Lambda(r) = \bigcup_{j=1}^q d(a_j, r)$. Suppose first that \mathcal{U} is not secant with \mathcal{F} . There exists $\rho > 0$ and $H \in \mathcal{U}$ such that $\Lambda(\rho) \cap H = \emptyset$. Then $|P(x)| \geq \rho^q \forall x \in H$, a contradiction to the hypothesis $\lim_{\mathcal{U}} P(x) = 0$. Consequently, \mathcal{U} is secant with \mathcal{F} . Hence it is obviously secant with the filter of neighborhoods of one of the points a_j and therefore, it converges to this point.

Proof of Theorem 18: Indeed, by Theorem 17 the ideal \mathcal{M} contains no polynomials, hence $\frac{A}{\mathcal{M}}$, (resp. $\frac{B}{\mathcal{M}}$) contains a subfield isomorphic to $K(x)$.

Proof of Theorem 19: Given $\psi \in Mult(A, \|\cdot\|)$, $f_1, \dots, f_q \in A$, $\epsilon > 0$, we set

$$V(\psi, f_1, \dots, f_q, \epsilon) = \{\phi \in Mult(A, \|\cdot\|) \mid |\phi(f_j) - \psi(f_j)|_\infty \leq \epsilon, j = 1, \dots, q\}.$$

By definition of the topology of simple convergence, the filter of neighborhoods of ψ admits for basis the family of sets

$$V(\psi, f_1, \dots, f_q, \epsilon), q \in \mathbb{N}^*, f_1, \dots, f_q \in A, \epsilon > 0.$$

Henceforth we take $\epsilon \in]0, \frac{1}{2}[$.

Suppose that the Shilov boundary of A is not equal to $Mult(A, \|\cdot\|)$ and let $\psi \in Mult(A, \|\cdot\|) \setminus S$. Since S is a closed subset of $Mult(A, \|\cdot\|)$, the set $Mult(A, \|\cdot\|) \setminus S$ is an open subset of $Mult(A, \|\cdot\|)$ and hence, there exist f_1, \dots, f_q such that $V(\psi, f_1, \dots, f_q, \epsilon) \subset (Mult(A, \|\cdot\|) \setminus S)$. Let $L = \{x \in E \mid |\psi(f_j) - |f_j(x)||_\infty \leq \frac{\epsilon}{2}\}$. By Lemma 3 L is a clopen. Consequently, by Lemma 4 the characteristic function u of L belongs to A and obviously satisfies $\psi(u) = 1$. On the other hand, we have $u(x) = 0 \forall x \notin L$.

Now, there exists $\theta \in S$ such that $\theta(u) = \|u\| = 1$. Consider the neighborhood $V(\theta, f_1, \dots, f_q, u, \frac{\epsilon}{2})$. Since $Mult_1(A, \|\cdot\|)$ is dense in $Mult(A, \|\cdot\|)$, particularly, for every $\varphi_a \in V(\theta, f_1, \dots, f_q, u, \frac{\epsilon}{2})$ we have $||u(a)| - \theta(u)|_\infty \leq \frac{\epsilon}{2}$, hence $|u(a)| \geq 1 - \frac{\epsilon}{2} > 0$. But since $\theta \in S$, we have $V(\theta, f_1, \dots, f_q, \frac{\epsilon}{2}) \cap V(\psi, f_1, \dots, f_q, \frac{\epsilon}{2}) = \emptyset$ and so much the more: $V(\theta, f_1, \dots, f_q, u, \frac{\epsilon}{2}) \cap V(\psi, f_1, \dots, f_q, u, \frac{\epsilon}{2}) = \emptyset$. Let $H = \{a \in E \mid \varphi_a \in V(\theta, f_1, \dots, f_q, u, \frac{\epsilon}{2})\}$. Then $H \cap L = \emptyset$. But by definition of u , we have $u(a) = 0 \forall a \in H$, a contradiction.

The proof of the statement concerning B is similar. Following the same notation with B in place of A , we only have to remark that by lemma 3 here, L is a uniformly open subset of E . Consequently, by Lemma 4 the characteristic function u of L belongs to B . The end of the proof follows the same way as this for A .

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