

Two-sided bounds for degenerate processes with densities supported in subsets of \mathbb{R}^N

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Abstract: We obtain two-sided bounds for the density of stochastic processes satisfying a weak Hörmander condition. In particular we consider the cases when the support of the density is not the whole space and when the density has various asymptotic regimes depending on the starting/final points considered (which are as well related to the number of brackets needed to span the space in Hörmander's theorem). The proofs of our lower bounds are based on Harnack inequalities for positive solutions of PDEs whereas the upper bounds derive from the probabilistic representation of the density given by the Malliavin calculus.

Keywords: Harnack inequality, Malliavin Calculus, Hörmander condition, two-sided bounds.

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1 Introduction

We present a methodology to derive two-sided bounds for the density of some \mathbb{R}^N -valued degenerate processes of the form

$$X_t = x + \sum_{i=1}^n \int_0^t Y_i(X_s) \circ dW_s^i + \int_0^t Y_0(X_s) ds \quad (1.1)$$

where the $(Y_i)_{i \in [0, n]}$ are smooth vector fields defined on \mathbb{R}^N , $((W_t^i)_{t \geq 0})_{i \in [1, n]}$ stand for n -standard monodimensional independent Brownian motions defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions. Also $\circ dW_t$ denotes the Stratonovitch integral. The above stochastic differential equation is associated to the Kolmogorov operator

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^n Y_i^2 + Z, \quad Z = Y_0 - \partial_t. \quad (1.2)$$

We assume that the Hörmander condition holds:

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$$[\text{H}] \quad \text{Rank}(\text{Lie}\{Y_1, \dots, Y_n, Z\}(x)) = N + 1, \quad \forall x \in \mathbb{R}^N.$$

We will particularly focus on processes satisfying a *weak* Hörmander condition, that is $\text{Rank}(\text{Lie}\{Y_1, \dots, Y_n, -\partial_t\}(x)) < N + 1, \forall x \in \mathbb{R}^N$. This means that the first order vector field Y_0 (or equivalently the drift term of the SDE) is needed to span all the directions.

As leading examples we have in mind processes of the form

$$X_t^i = x_i + W_t^i, \quad \forall i \in \llbracket 1, n \rrbracket, \quad X_t^{n+1} = x_{n+1} + \int_0^t |X_s^{1,n}|^k ds, \quad (1.3)$$

where $X_s^{1,n} = (X_s^1, \dots, X_s^n)$ (and correspondingly for every $x \in \mathbb{R}^{n+1}$, $x_{1,n} := (x_1, \dots, x_n)$), k is any *even* positive integer and $|\cdot|$ denotes the Euclidean norm of \mathbb{R}^n . Note that we only consider even exponents in (1.3) in order to keep Y_0 smooth. Our approach also applies to

$$X_t^i = x_i + W_t^i, \quad \forall i \in \llbracket 1, n \rrbracket, \quad X_t^{n+1} = x_{n+1} + \int_0^t \sum_{i=1}^n (X_s^i)^k ds, \quad (1.4)$$

for any given positive integer k .

It is easily seen that the above class of processes satisfies the weak Hörmander condition. Also for equation (1.3), the density $p(t, x, \cdot)$ of X_t is supported on $\mathbb{R}^n \times (x_{n+1}, +\infty)$ for any $t > 0$. Analogously, for equation (1.4), the support of $p(t, x, \cdot)$ is \mathbb{R}^{n+1} when k is odd and $\mathbb{R}^n \times (x_{n+1}, +\infty)$ when k is even.

Let us now briefly recall some known results concerning these two examples. First of all, for $k = 1$, equation (1.4) defines a Gaussian process. The explicit expression of the density goes back to Kolmogorov [25] and writes for all $t > 0$, $x, \xi \in \mathbb{R}^{n+1}$:

$$p_K(t, x, \xi) = \frac{\sqrt{3}}{(2\pi)^{\frac{n+1}{2}} t^{\frac{n+3}{2}}} \exp\left(-\left\{ \frac{1}{4} \frac{|\xi_{1,n} - x_{1,n}|^2}{t} + 3 \frac{|\xi_{n+1} - x_{n+1} - \frac{\sum_{i=1}^n (x_i + \xi_i)}{2} t|^2}{t^3} \right\}\right). \quad (1.5)$$

We already observe the two time scales associated respectively to the Brownian motion (of order $t^{1/2}$) and to its integral (of order $t^{3/2}$) which give the global diagonal decay of order $t^{n/2+3/2}$. The additional term $\frac{x_1 + \xi_1}{2} t$ in the above estimate is due to the transport of the initial condition by the unbounded drift. We also refer to the works of Cinti and Polidoro [17] and Delarue and Menozzi [19] for similar estimates in the more general framework of variable coefficients, including non linear drift terms with linear growth.

For equation (1.3) and $k = 2$, $n = 1$, a representation of the density of X_t has been obtained from the seminal works of Kac on the Laplace transform of the integral of the square of the Brownian motion [23]. We can refer to the monograph of Borodin and Salminen [10] for an explicit expression in terms of special functions. We can also mention the work of Tolmatz [33] concerning the distribution function of the square of the Brownian bridge already characterized in the early work of Smirnov [31]. Anyhow, all these explicit representations are very much linked to Liouville type problems and this approach can hardly be extended to higher dimensions for the underlying Brownian motion. Also, it seems difficult from the expressions of [10] to derive explicit quantitative bounds on the density.

Some related examples have been addressed by Ben Arous and Léandre [4] who obtained asymptotic expansions for the density on the diagonal for the process $X_t^1 = x_1 + W_t^1$, $X_t^2 = x_2 + \int_0^t (X_s^1)^m dW_s^2 + \int_0^t (X_s^1)^k ds$. Various asymptotic regimes are deduced depending on m and k . Anyhow, the strong Hörmander condition is really required in their approach, i.e. the stochastic integral is needed in X^2 .

Our approach to derive two-sided estimates for the above examples is the following. The lower bounds are obtained using local Harnack estimates for positive solutions of $\mathcal{L}u = 0$ with \mathcal{L} defined in (1.2). Once the Harnack inequality is established, the lower bound for $p(t, x, \xi)$ is derived applying it recursively along a suitable path joining x to ξ in time t . The set of points of the path to which the Harnack inequality is applied is commonly called a *Harnack chain*. For $k = 1$ in (1.4) the path can be chosen as the solution to the deterministic controllability problem associated to (1.4), that is taking the points of the Harnack chain along the path γ where

$$\gamma'_i(s) = \omega_i(s), \forall i \in \llbracket 1, n \rrbracket, \gamma'_{n+1}(s) = \sum_{i=1}^n \gamma_i(s), \gamma(0) = x, \gamma(t) = \xi.$$

and $\omega : L^2([0, t]) \rightarrow \mathbb{R}^n$ achieves the minimum of $\int_0^t |\omega(s)|^2 ds$, see e.g. Boscain and Polidoro [11], Carciola *et al.* [13] and Delarue and Menozzi [19].

In the more general case $k > 1$ it is known that uniqueness fails for the associated control problem, i.e. when $\gamma'_{n+1}(s) = \sum_{i=1}^n (\gamma_i(s))^k$ in the above equation (see e.g. Trélat [34]). Therefore, there is not a single natural choice for the path γ . Actually, we will consider suitable paths in order to derive homogeneous two-sided bounds. After the statement of our main results, we will see in Remark 2.2 that the paths we consider allow to obtain a cost similar to the one found in [34] for the abnormal extremals of the value function associated to the control problem.

Anyhow, the crucial point in this approach is to obtain a Harnack inequality *invariant* w.r.t. scale and translation. These properties imply that the dimension of the Lie algebra is the same at every point of \mathbb{R}^{n+1} . Therefore, they cannot hold for $k > 1$. In this work, we are mainly interested in these cases that exhibit different regimes for the density. Hence, we need to consider a *lifting* procedure of \mathcal{L} in (1.2) analogous to the one introduced by Bonfiglioli and Lanconelli [6] (see also Rotschild and Stein [30]). The strategy then consists in obtaining an invariant Harnack inequality for the lifted operator $\widetilde{\mathcal{L}}$. We then conclude applying the previous Harnack inequality to \mathcal{L} -harmonic functions (which are also $\widetilde{\mathcal{L}}$ -harmonic). A first attempt to achieve the whole procedure to derive a lower bound for (1.4) and odd k can be found in Cinti and Polidoro [16].

Concerning the upper bounds, we rely on the representation of the density obtained by the Malliavin calculus, see e.g. Nualart [28]. The main issues then consist in controlling the tails of the random variables at hand and the L^p norm of the Malliavin covariance matrix for $p \geq 1$. The tails can be controlled thanks to some fine properties of the Brownian motion or bridge and its local time. The behavior of the Malliavin covariance matrix has to be carefully analyzed introducing a dichotomy between the case for which the final and starting points of the Brownian motion in (1.3)-(1.4) are close to zero w.r.t. the characteristic time-scale, i.e.

$|x_{1,n}| \vee |\xi_{1,n}| \leq Kt^{1/2}$ for a given $K > 0$, which means that the non-degenerate component is in diagonal regime, and the complementary set. In the first case, we will see that the characteristic time scales of the system (1.3), (1.4) and the probabilistic approach to the proof of Hörmander theorem, see e.g. Norris [26] will lead to the expected bound on the Malliavin covariance matrix whereas in the second case a more subtle analysis is required in order to derive a diagonal behavior of the density similar to the Gaussian case (1.5). Intuitively, when the magnitude of either the starting or the final point of the Brownian motion is above the characteristic time-scale, then only one bracket is needed to span the space and the Gaussian regime prevails in small time.

From the applicative point of view, equations with quadratic growth naturally appear in some turbulence models, see e.g. the chapter concerning the dyadic model in Flandoli [20]. This model is derived from the formulation of the Euler equations on the torus in Fourier series after a simplification consisting in considering a nearest neighbour interaction in the wave space. This operation leads to consider an infinite system of differential equations whose coefficients have quadratic growth. In order to obtain some uniqueness properties, a Brownian noise is usually added on each component. We wanted to investigate from a quantitative viewpoint what could be said for a drastic reduction of this simplified model, that is when considering 2 equations only, when the noise only acted on one component and was transmitted through the system thanks to the Hörmander condition.

The article is organized as follows. We state our main results in Section 2. In Section 3, we recall some aspects of abstract potential theory needed to derive the invariant Harnack inequality. We also give a geometric characterization of the set where the inequality holds. Section 4 is devoted to the proof of our main results. We construct the Harnack chains in a suitable lifted space and derive the lower bounds in Section 4.1. We recall some basic facts of Malliavin calculus and obtain the upper bounds as well as a diagonal lower bound in Gaussian regime in Section 5.

2 Main Results

Let us first recall that $p(t, x, \cdot)$ stands for the density of X in (1.3) or (1.4) at time t starting from x . Our main result is given by the following theorem.

Theorem 2.1 *Let $x = (x_{n+1}, x_{n+1}) \in \mathbb{R}^{n+1}$, and $\xi = (\xi_{1,n}, \xi_{n+1}) \in \mathbb{R}^n \times (x_{n+1}, +\infty)$ for k even, and $\xi \in \mathbb{R}^{n+1}$ for k odd, be given. Define*

$$\Psi(x_{1,n}, \xi_{1,n}) := \begin{cases} \frac{2^{k-1}}{k+1} (|x_{1,n}|^k + |\xi_{1,n}|^k), & \text{for (1.3),} \\ \frac{2^{k-1}}{k+1} \sum_{i=1}^n \{(x_i)^k + (\xi_i)^k\}, & \text{for (1.4).} \end{cases}$$

i) Assume $\frac{|\xi_{n+1} - x_{n+1} - ct(|x_{1,n}|^k + |\xi_{1,n}|^k)|}{t^{3/2}(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1})} \geq \bar{C}$ where $c := c(k) = 2 + \frac{2^{k-1}}{k+1}$ and \bar{C} is fixed.

Then there exists a constant $C_1 := C_1(n, k, \bar{C}) \geq 1$ s.t. for every $t > 0$,

$$\frac{C_1^{-1}}{t^{\frac{n+k}{2}+1}} \exp\left(-C_1 I(t, x, \xi)\right) \leq p(t, x, \xi) \leq \frac{C_1}{t^{\frac{n+k}{2}+1}} \exp\left(-C_1^{-1} I(t, x, \xi)\right), \quad (2.1)$$

$$I(t, x, \xi) := \frac{|\xi_{1,n} - x_{1,n}|^2}{t} + \frac{|\xi_{n+1} - x_{n+1} - \Psi(x_{1,n}, \xi_{1,n})t|^{2/k}}{t^{1+2/k}}.$$

ii) Assume $\frac{|\xi_{n+1} - x_{n+1} - ct(|x_{1,n}|^k + |\xi_{1,n}|^k)|}{t^{3/2}(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1})} \leq \bar{C}$ (with c, \bar{C} as in point i)) and $|x_{1,n}| \vee |\xi_{1,n}|/t^{1/2} \geq K$, with K sufficiently large. Then, there exists $C_2 := C_2(n, k, K, \bar{C}) \geq 1$ s.t. for every $t > 0$:

$$\frac{C_2^{-1} \exp(-C_2 I(t, x, \xi))}{(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1}) t^{\frac{n+3}{2}}} \leq p(t, x, \xi) \leq \frac{C_2 \exp(-C_2^{-1} I(t, x, \xi))}{(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1}) t^{\frac{n+3}{2}}}, \quad (2.2)$$

$$I(t, x, \xi) := \frac{|\xi_{1,n} - x_{1,n}|^2}{t} + \frac{|\xi_{n+1} - x_{n+1} - \Psi(x_{1,n}, \xi_{1,n})t|^2}{(|x_{1,n}|^{(k-1)} + |\xi_{1,n}|^{(k-1)})^2 t^3}.$$

iii) For $t > 0$, assume $|\xi_{n+1} - x_{n+1}| \leq Kt^{1+k/2}$ for sufficiently small K . Then, there exists $C_3 := C_3(n, k, K) \geq 1$ s.t. we have:

$$\frac{C_3^{-1}}{t^{\frac{n+k}{2}+1}} \exp(-C_3 I(t, x, \xi)) \leq p(t, x, \xi) \leq \frac{C_3}{t^{\frac{n+k}{2}+1}} \exp(-C_3^{-1} I(t, x, \xi)),$$

$$I(t, x, \xi) := \frac{|x_{1,n}|^{2+k} + |\xi_{1,n}|^{2+k}}{|\xi_{n+1} - x_{n+1}|} + \frac{t^{1+2/k}}{|\xi_{n+1} - x_{n+1}|^{2/k}}. \quad (2.3)$$

Let us point out that processes of the form (1.3) or (1.4) do not have a single regime anymore for $k > 1$. This aspect can be intuitively justified by the following expansion from the dynamics of (1.4):

$$\int_0^t \sum_{i=1}^n (x_i + W_s^i)^k ds = \sum_{i=1}^n (x_i)^k t + k \sum_{i=1}^n (x_i)^{k-1} \int_0^t W_s^i ds + \dots + \int_0^t \sum_{i=1}^n (W_s^i)^k ds. \quad (2.4)$$

For $x_{1,n} = 0$, only the last integral remains. It has an intrinsic scale of order $t^{k/2+1}$. Together with the characteristic scale of the n -dimensional Brownian motion (of order $t^{n/2}$) this justifies the diagonal exponent in (2.1). The off-diagonal bound can be explained by the fact that every component $\int_0^t (W_s^i)^k ds$, $i \in \llbracket 1, n \rrbracket$, belongs to the Wiener chaos of order k . The tails of the distribution function for such random variables have been characterized in Janson [22] and are homogeneous to the non Gaussian term in (2.1). The same arguments apply for (1.3).

On the other hand, as $x_{1,n} \neq 0$ and $t^{1/2}$ is “small” with respect to $|x_{1,n}|$, then a scaling argument in (2.4) shows that the highest order fluctuation is the Gaussian one. This explains the Gaussian bound of (2.2) which is homogeneous to the Kolmogorov density (1.5). From a PDE viewpoint, this difference of regime can also be explained by the fact that for the origin $x_{1,n} = 0$, exactly k commutators are needed to fulfill the Hörmander condition [H], whereas for $x_{1,n} \neq 0$ only one commutator is required.

Eventually, let us specify that when $C^{-1}\sqrt{t} \leq |x_i| \leq C\sqrt{t}$, $\forall i \in \llbracket 1, n \rrbracket$, $C \geq 1$, then all the terms in (2.4) have the same order and then a global estimate of type (2.1) (resp. of type (2.2)) holds for the upper bound (resp. lower bound) in both cases (1.3) and (1.4). Observe also that in this case (2.1) and (2.2) give the same global diagonal decay of order $t^{(k+n)/2+1}$.

Remark 2.2 As already mentionned in the introduction, for $k = 2, n = 1$, we observe from (2.3) that the off-diagonal bound is homogeneous to the asymptotic expansion of the value

function associated to the control problem at its abnormal extremals, see Example 4.2 in [34]. The optimal cost is asymptotically equivalent to $\frac{1}{4} \frac{\xi_1^4}{\xi_2^4}$ when $x = (0, 0)$ as ξ is close to $(0, 0)$.

Remark 2.3 Fix $|\xi_{n+1} - x_{n+1}|$ small, $t \in [K^{-1}|\xi_{n+1} - x_{n+1}|^{2-\varepsilon}, K|\xi_{n+1} - x_{n+1}|^{2-\varepsilon}]$ for given $K \geq 1, \varepsilon > 0$. We then get from (2.3) that there exist $\tilde{c} := \tilde{c}(n, k), \tilde{C} := \tilde{C}(n, k, T)$ s.t. $p(t, x, \xi) \leq \tilde{C} \exp(-\tilde{c}/|\xi_{n+1} - x_{n+1}|^\varepsilon)$. This estimate can be compared to the exponential decay on the diagonal proved by Ben Arous and Léandre in [4, Theorem 1.1].

3 Potential Theory and PDEs

In this section we are interested in proving Harnack inequalities for non-negative solutions to

$$\mathcal{L}u(z) = 0, \quad z = (x, t) \in \mathbb{R}^{N+1}, \quad (3.1)$$

with \mathcal{L} defined in (1.2). Specifically, we consider any open set $\mathcal{O} \subseteq \mathbb{R}^{N+1}$, and any $z \in \mathcal{O}$, and we aim to show that there exists a compact $K \subset \mathcal{O}$ and a positive constant C_K such that

$$\sup_K u \leq C_K u(z), \quad (3.2)$$

for every positive solution u to $\mathcal{L}u = 0$. We say that a set $\{z_0, z_1, \dots, z_k\} \subset \mathcal{O}$ is a *Harnack chain of length k* if

$$u(z_j) \leq C_j u(z_{j-1}), \quad \text{for } j = 1, \dots, k,$$

for every positive solution u of $\mathcal{L}u = 0$, so that we get

$$u(z_k) \leq C_1 C_2 \dots C_k u(z_0). \quad (3.3)$$

In order to construct Harnack chains, and to have an explicit lower bound for the densities considered in this article, we will prove *invariant* Harnack inequalities w.r.t. a suitable Lie group structure. By exploiting the properties of homogeneity and translation invariance of the Lie group, we will find Harnack chains with the property that every C_j in (3.3) agrees with the constant C_K in (3.2). As a consequence we find $u(z_k) \leq C_K^k u(z_0)$, and the bound will depend only on the length of the Harnack chain connecting z_0 to z_k .

Let us now recall some basic notations concerning homogeneous Lie groups (we refer to the monograph [7] by Bonfiglioli, Lanconelli and Uguzzoni for an exhaustive treatment). Let \circ be a given group law on \mathbb{R}^{N+1} and suppose that the map $(z, \zeta) \mapsto \zeta^{-1} \circ z$ is smooth. Then $\mathbb{G} = (\mathbb{R}^{N+1}, \circ)$ is called a *Lie group*. Moreover, \mathbb{G} is said *homogeneous* if there exists a family of dilations $(\delta_\lambda)_{\lambda>0}$ which defines an automorphism of the group, *i.e.*,

$$\delta_\lambda(z \circ \zeta) = (\delta_\lambda z) \circ (\delta_\lambda \zeta), \quad \text{for all } z, \zeta \in \mathbb{R}^{N+1} \text{ and } \lambda > 0.$$

We also make the following assumption.

[L] \mathcal{L} is *Lie-invariant* with respect to the Lie group $\mathbb{G} = (\mathbb{R}^{N+1}, \circ, (\delta_\lambda)_{\lambda>0})$, *i.e.*

i) Y_1, \dots, Y_n and Z are left-invariant with respect to the composition law of \mathbb{G} , i.e.

$$\begin{aligned} Y_j(u(\zeta \circ \cdot)) &= (Y_j u)(\zeta \circ \cdot), & j = 1, \dots, n, \\ Z(u(\zeta \circ \cdot)) &= (Zu)(\zeta \circ \cdot), \end{aligned}$$

for every function $u \in C^\infty(\mathbb{R}^{N+1})$, and for any $\zeta \in \mathbb{R}^{N+1}$;

ii) Y_1, \dots, Y_n are δ_λ -homogeneous of degree one and Z is δ_λ -homogeneous of degree two:

$$\begin{aligned} Y_j(u(\delta_\lambda z)) &= \lambda(Y_j u)(\delta_\lambda z), & j = 1, \dots, n, \\ Z(u(\delta_\lambda z)) &= \lambda^2(Zu)(\delta_\lambda z), \end{aligned}$$

for every function $u \in C^\infty(\mathbb{R}^{N+1})$, and for any $z \in \mathbb{R}^{N+1}$, $\lambda > 0$.

To illustrate Property [L] we recall the Lie group structure of the Kolmogorov operator corresponding to $k = 1$ in (1.4).

Example 3.1 (KOLMOGOROV OPERATORS) $\mathcal{L} := \frac{1}{2}\Delta_{x_{1,n}} + \sum_{i=1}^n x_i \partial_{x_{2i}} - \partial_t$. The Kolmogorov group is $\mathbb{K} = (\mathbb{R}^{n+2}, \circ, \delta_\lambda)$, where

$$(x, t) \circ (\xi, \tau) = \left(x_{1,n} + \xi_{1,n}, x_{n+1} + \xi_{n+1} - \sum_{i=1}^n x_i \tau, t + \tau\right), \quad \delta_\lambda(x, t) = (\lambda x_{1,n}, \lambda^3 x_{n+1}, \lambda^2 t).$$

Clearly, \mathcal{L} can be written as in (1.2) with $Y_i = \partial_{x_i}$, $i \in \llbracket 1, n \rrbracket$, and $Z = \sum_{i=1}^n x_i \partial_{x_{n+1}} - \partial_t$, and satisfies [L].

It is known that the composition law \circ is always a sum with respect to the t variable (see Proposition 10.2 in [24]). Moreover, the family $(\delta_\lambda)_{\lambda>0}$ acts on \mathbb{R}^{N+1} as follows:

$$\delta_\lambda(x_1, x_2, \dots, x_N, t) = (\lambda^{\sigma_1} x_1, \lambda^{\sigma_2} x_2, \dots, \lambda^{\sigma_N} x_N, \lambda^2 t), \quad \text{for every } (x, t) \in \mathbb{R}^{N+1},$$

where $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_N) \in \mathbb{N}^N$ is a multi-index. The natural number $Q = \sum_{k=1}^N \sigma_k + 2$ is called the *homogeneous dimension* of \mathbb{G} with respect to δ_λ . We shall assume that $Q \geq 3$. Observe that the diagonal decay of the heat kernel on the homogeneous Lie group is given by the characteristic time scale $t^{-(Q-2)/2}$. For the above example we have $Q = n + 3 + 2$, matching the diagonal exponent in (1.5) $(Q - 2)/2 = (n + 3)/2$.

Write the operator \mathcal{L} as follows

$$\mathcal{L} = \sum_{i,j=1}^N a_{i,j}(x) \partial_{x_i x_j} + \sum_{j=1}^N b_j(x) \partial_{x_j} - \partial_t,$$

for suitable smooth coefficients $a_{i,j}$'s and b_j 's only depending on the vector fields Y_0, \dots, Y_n . As $n < N$, \mathcal{L} is strictly degenerate, since the rank $(A(x)) \leq n$ at every x (here $A(x) := (a_{i,j}(x))_{i,j \in \llbracket 1, n \rrbracket}$). In Example 3.1 we see that rank (A) never vanishes. We say that \mathcal{L} is *not totally degenerate* if

[B] for every $x \in \mathbb{R}^N$ there exists $\nu \in \mathbb{R}^N \setminus \{0\}$ such that $\langle A(x)\nu, \nu \rangle > 0$.

This property holds for a more general class of operators. Indeed, if \mathcal{L} satisfies [H] and [L], then there exists a $\nu \in \mathbb{R}^N \setminus \{0\}$ such that

$$\langle A(x)\nu, \nu \rangle > 0, \quad \text{for every } x \in \mathbb{R}^N. \quad (3.4)$$

We refer to Section 1.3 in the monograph [7] for the proof of this statement.

Fix now $T > 0$ and define $I := [0, T]$. We call *diffusion trajectory* any absolutely continuous curve on I such that

$$\gamma'(s) = \sum_{k=1}^n \omega_k(s) Y_k(\gamma(s)), \quad \text{for every } s \in I, \quad (3.5)$$

where $\omega_1, \dots, \omega_n$ are piecewise constant real functions. A *drift trajectory* is any positively oriented integral curve of Z . We say that a curve $\gamma : [0, T] \rightarrow \mathbb{R}^{N+1}$ is \mathcal{L} -admissible if it is absolutely continuous and is a sum of a finite number of diffusion and drift trajectories.

Let \mathcal{O} be any open subset of \mathbb{R}^{N+1} , and let $z_0 \in \mathcal{O}$. We define the *attainable set* $\mathcal{A}_{z_0} := \overline{A_{z_0}}$ as the closure in \mathcal{O} of the following set

$$A_{z_0} = \left\{ z \in \mathcal{O} : \text{there exists an } \mathcal{L}\text{-admissible path} \right. \\ \left. \gamma : [0, T] \rightarrow \mathcal{O} \text{ such that } \gamma(0) = z_0, \gamma(T) = z \right\}. \quad (3.6)$$

The main result of the section is the following

Theorem 3.2 *Let \mathcal{L} be an operator in the form (3.1) satisfying [H] and [L], let $\mathcal{O} \subseteq \mathbb{R}^{N+1}$ be an open set, and let $z_0 \in \mathcal{O}$. Then,*

$$\text{for every compact set } K \subset \text{Int}(\mathcal{A}_{z_0}), \quad \sup_K u \leq C_K u(z_0), \quad (3.7)$$

for any non-negative solutions u to $\mathcal{L}u = 0$ in \mathcal{O} . Here C_K is a positive constant depending on \mathcal{O}, K, z_0 and on \mathcal{L} .

We recall that a Harnack inequality for operators satisfying [H] and [B] is due to Bony (see [9]). Another result analogous to Theorem 3.2 is given in [15, Theorem 1.1] by Cinti, Nystrom and Polidoro, assuming [L] and the following *controllability* condition:

[C] for every $(x, t), (\xi, \tau) \in \mathbb{R}^{N+1}$ with $t > \tau$, there exists an \mathcal{L} -admissible path $\gamma : [0, T] \rightarrow \mathbb{R}^{N+1}$ such that $\gamma(0) = (x, t), \gamma(T) = (\xi, \tau)$.

Our Theorem 3.2 improves Bony's one in that it gives an explicit geometric description of the set K in (3.7). Also, it is more general than the one in [15], since [L] and [C] imply [H] (see Proposition 10.1 in [24]).

The proof of Theorem 3.2 is based on a general result from Potential Theory. In Section 3.1 we recall the basic results of Potential Theory needed in our work, then we apply them

to operators \mathcal{L} satisfying [H] and [L]. We explicitly remark that condition [L] is not satisfied by the Kolmogorov operators

$$\mathcal{L} = \frac{1}{2}\Delta_{x_{1,n}} + |x_{1,n}|^k \partial_{x_{n+1}} - \partial_t \quad (3.8)$$

and

$$\mathcal{L} = \frac{1}{2}\Delta_{x_{1,n}} + \sum_{j=1}^n x_j^k \partial_{x_{n+1}} - \partial_t \quad (3.9)$$

of the stochastic systems (1.3) and (1.4) respectively. Indeed, in both cases k commutators are needed to fulfill Hörmander condition [H] at $x_{1,n} = 0$, while only one commutator is sufficient to span all the directions as $x_{1,n} \neq 0$, and this fact contradicts [L]-i). On the other hand, the operators in (3.8) and (3.9) can be *lifted* to suitable operators $\widetilde{\mathcal{L}} = \widetilde{Y}_1^2 + \widetilde{Z}$, satisfying both [H] and [L] (see (4.3)). We refer to Section 4 for more details, and we note that our Harnack-type inequality for \mathcal{L} , and the asymptotic lower bounds, are obtained in Section 4 by the application of Theorem 3.2 to $\widetilde{\mathcal{L}}$.

3.1 Potential Theory

For the rest of the section, we assume \mathcal{L} to be a general abstract parabolic differential operator satisfying [B] and [L].

Let \mathcal{O} be any open subset of \mathbb{R}^{N+1} . If $u : \mathcal{O} \rightarrow \mathbb{R}$ is a smooth function such that $\mathcal{L}u = 0$ in \mathcal{O} , we say that u is \mathcal{L} -harmonic in \mathcal{O} . We denote by $\mathcal{H}(\mathcal{O})$ the linear space of functions which are \mathcal{L} -harmonic in \mathcal{O} .

Let V be a bounded open subset of \mathbb{R}^{N+1} with Lipschitz-continuous boundary. We say that V is \mathcal{L} -regular if, for every $z_0 \in \partial V$, there exists a neighborhood U of z_0 and a smooth function $w : U \rightarrow \mathbb{R}$ satisfying

$$w(z_0) = 0, \quad \mathcal{L}w(z_0) < 0, \quad w > 0 \text{ in } \overline{V} \cap U \setminus \{z_0\}.$$

Note that the function $\psi(x, t) = \frac{1}{2} + \frac{1}{\pi} \arctan t$ verifies

$$0 \leq \psi \leq 1, \quad \mathcal{L}\psi < 0 \text{ in } \mathbb{R}^{N+1}. \quad (3.10)$$

As a first consequence of (3.10), the classical Picone's maximum principle holds on any bounded open set $\mathcal{O} \subset \mathbb{R}^{N+1}$. Precisely, if $u \in C^2(\mathcal{O})$ satisfies

$$\mathcal{L}u \geq 0 \text{ in } \mathcal{O}, \quad \limsup_{z \rightarrow \zeta} u(z) \leq 0 \text{ for every } \zeta \in \partial\mathcal{O},$$

then $u \leq 0$ in \mathcal{O} (see e.g. Bonfiglioli and Uguzzoni [8]). Then, for every \mathcal{L} -regular open set $V \subset \mathbb{R}^{N+1}$, and for any $\varphi \in C(\partial V)$ there exists a unique function H_φ^V satisfying

$$H_\varphi^V \in \mathcal{H}(V), \quad \lim_{z \rightarrow \zeta} H_\varphi^V(z) = \varphi(\zeta) \text{ for every } \zeta \in \partial V. \quad (3.11)$$

Moreover, $H_\varphi^V \geq 0$ whenever $\varphi \geq 0$ (see Bauer [3] and Constantinescu and Cornea [18]). Hence, if V is \mathcal{L} -regular, for every fixed $z \in V$ the map $\varphi \mapsto H_\varphi^V(z)$ defines a linear positive

functional on $C(\partial V, \mathbb{R})$. Thus, the Riesz representation theorem implies that there exists a Radon measure μ_z^V , supported in ∂V , such that

$$H_\varphi^V(z) = \int_{\partial V} \varphi(\zeta) d\mu_z^V(\zeta), \quad \text{for every } \varphi \in C(\partial V, \mathbb{R}). \quad (3.12)$$

We will refer to μ_z^V as the \mathcal{L} -harmonic measure defined with respect to V and z .

A lower semi-continuous function $u : \mathcal{O} \rightarrow]-\infty, \infty]$ is said to be \mathcal{L} -superharmonic in \mathcal{O} if $u < \infty$ in a dense subset of \mathcal{O} and if

$$u(z) \geq \int_{\partial V} u(\zeta) d\mu_z^V(\zeta),$$

for every open \mathcal{L} -regular set $V \subset \bar{V} \subset \mathcal{O}$ and for every $z \in V$. We denote by $\bar{\mathcal{S}}(\mathcal{O})$ the set of \mathcal{L} -superharmonic functions in \mathcal{O} , and by $\bar{\mathcal{S}}^+(\mathcal{O})$ the set of the functions in $\bar{\mathcal{S}}(\mathcal{O})$ which are non-negative. A function $v : \mathcal{O} \rightarrow]-\infty, \infty[$ is said to be \mathcal{L} -subharmonic in \mathcal{O} if $-v \in \bar{\mathcal{S}}(\mathcal{O})$ and we write $\underline{\mathcal{S}}(\mathcal{O}) := -\bar{\mathcal{S}}(\mathcal{O})$. Since the collection of \mathcal{L} -regular sets is a basis for the Euclidean topology (as we will see in a moment), we have $\bar{\mathcal{S}}(\mathcal{O}) \cap \underline{\mathcal{S}}(\mathcal{O}) = \mathcal{H}(\mathcal{O})$.

This last property and Picone's maximum principle are the main tools in order to show the following criterion of \mathcal{L} -superharmonicity for functions of class C^2 (a proof can be found in the monograph [7, Proposition 7.2.5]).

Remark 3.3 *Let $u \in C^2(\mathcal{O})$. Then u is \mathcal{L} -superharmonic if and only if $\mathcal{L}u \leq 0$ in \mathcal{O} .*

With the terminology of Potential Theory (we refer to the monographs [3, 18]), the map $\mathbb{R}^{N+1} \supseteq \mathcal{O} \mapsto \mathcal{H}(\mathcal{O})$ is said *harmonic sheaf* and $(\mathbb{R}^{N+1}, \mathcal{H})$ is said *harmonic space*. Since the constant functions are \mathcal{L} -harmonic, the last statement is a consequence of the following properties:

- the \mathcal{L} -regular sets form a basis for the Euclidean topology (by (3.4), \mathcal{L} is a not totally degenerate operator, so that this statement is a consequence of [9, Corollaire 5.2]);
- \mathcal{H} satisfies the *Doob convergence property*, i.e., the pointwise limit u of any increasing sequence $\{u_n\}_n$ of \mathcal{L} -harmonic functions, on any open set V , is \mathcal{L} -harmonic whenever u is finite in a dense set $T \subseteq V$ (as in [24, Proposition 7.4], we can rely on the weak Harnack inequality due to Bony stated in [9, Theoreme 7.1]);
- the family $\bar{\mathcal{S}}(\mathbb{R}^{N+1})$ separates the points of \mathbb{R}^{N+1} , i.e., for every $z, \zeta \in \mathbb{R}^{N+1}$, $z \neq \zeta$, there exists $u \in \bar{\mathcal{S}}(\mathbb{R}^{N+1})$ such that $u(z) \neq u(\zeta)$.

This last *separation property* is proved in Lemma 3.5, by adapting the argument in [14, Proposition 7.1]. Furthermore, we will show a stronger result: actually, the family $\bar{\mathcal{S}}^+(\mathbb{R}^{N+1}) \cap C(\mathbb{R}^{N+1})$ separates the points of \mathbb{R}^{N+1} . A harmonic space $(\mathbb{R}^{N+1}, \mathcal{H})$ satisfying this property is said to be a \mathfrak{B} -harmonic space.

In order to prove the separation property we use a fundamental solution Γ of \mathcal{L} . Bonfiglioli and Lanconelli prove in [6, Theorem 1.5] that such Γ exists. They assume [H], [B],

the existence of a function ψ satisfying (3.10) and the existence of an increasing sequence $\{V_n\}_{n \in \mathbb{N}}$ of \mathcal{L} -regular open sets such that $\bigcup_{n \in \mathbb{N}} V_n = \mathbb{R}^{N+1}$.

In order to apply Theorem 1.5 in [6] we only need to fulfill the last requirement. Since [H] and [L] yield [B], there exists a \mathcal{L} -regular open set V_0 containing the origin, a small $r_0 > 0$ and a large $\lambda_0 > 1$ such that

$$U_{r_0} \subseteq V_0 \subseteq \delta_{\lambda_0}(U_{r_0}), \quad U_{r_0} = \{(x_1, \dots, x_N, t) \in \mathbb{R}^{N+1} : |x_i| < r_0, |t| < r_0\}. \quad (3.13)$$

Then, the δ_λ -homogeneity of \mathcal{L} yields that the sequence $\{\delta_{\lambda_0^n}(V_0)\}_{n \in \mathbb{N}}$ has the required property (see Proposition 3.7 in [6] for more details).

Hence, from Theorem 1.5 in [6] it follows that there exists a function Γ with the following properties:

- i) the map $(z, \zeta) \mapsto \Gamma(z, \zeta)$ is defined, non-negative and smooth away from the set $\{(z, \zeta) \in \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} : z \neq \zeta\}$;
- ii) for any $z \in \mathbb{R}^{N+1}$, $\Gamma(\cdot, z)$ and $\Gamma(z, \cdot)$ are locally integrable;
- iii) for every $\phi \in C_0^\infty(\mathbb{R}^{N+1})$ and $z \in \mathbb{R}^{N+1}$ we have

$$\mathcal{L} \int_{\mathbb{R}^{N+1}} \Gamma(z, \zeta) \phi(\zeta) d\zeta = \int_{\mathbb{R}^{N+1}} \Gamma(z, \zeta) \mathcal{L} \phi(\zeta) d\zeta = -\phi(z);$$

- iv) $\mathcal{L} \Gamma(\cdot, \zeta) = -\delta_\zeta$ (Dirac measure supported at ζ);
- v) if we define $\Gamma^*(z, \zeta) := \Gamma(\zeta, z)$, then Γ^* is the fundamental solution for the formal adjoint \mathcal{L}^* of \mathcal{L} , satisfying the dual statements of *iii*), *iv*);
- vi) $\Gamma(x, t, \xi, \tau) = 0$ if $t < \tau$;
- vii) $\Gamma(z, \zeta) = \Gamma(\alpha \circ z, \alpha \circ \zeta)$ for every $\alpha, z, \zeta \in \mathbb{R}^{N+1}$, $z \neq \zeta$.

Definition 3.4 A function Γ satisfying the above properties (i)–(vii) is said a fundamental solution for \mathcal{L} .

Note that property *vi*) follows from Proposition 3.9 in [6], as \mathcal{L} can be written in coordinate form $\mathcal{L} = \sum_{j=1}^N a_{i,j}(x) \partial_{x_i, x_j} + \sum_{j=1}^N b_j(x) \partial_{x_j} - \partial_t$ with a negative coefficient for ∂_t . It is also known that

$$[\text{C}] \text{ and } t > \tau \quad \Rightarrow \quad \Gamma(x, t, \xi, \tau) > 0. \quad (3.14)$$

This property is not true in general, as we will see in Remark 4.7.

The last statement *vii*) is due to the left-translation invariance of \mathcal{L} . We remark that the Lebesgue measure is also left invariant on \mathbb{G} as a consequence of the δ_λ -homogeneity (see, e.g., [7, Proposition 1.3.21]). In particular,

$$\Gamma(z, \zeta) = \Gamma(\zeta^{-1} \circ z, 0) =: \Gamma(\zeta^{-1} \circ z) \quad z, \zeta \in \mathbb{R}^{N+1}, z \neq \zeta. \quad (3.15)$$

Next we show that Γ is invariant as well with respect to the dilations $(\delta_\lambda)_{\lambda>0}$, namely

$$\Gamma(\delta_\lambda(z), \delta_\lambda(\zeta)) = \lambda^{-Q+2}\Gamma(z, \zeta), \quad z, \zeta \in \mathbb{R}^{N+1}, z \neq \zeta, \lambda > 0. \quad (3.16)$$

We prove (3.16) by using the *Green function* G_n related to \mathcal{L} and to the \mathcal{L} -regular open set $V_n = \delta_{\lambda_0^n}(V_0)$ introduced in (3.13). In the proof of Theorem 1.5 in [6], Γ is defined as

$$\Gamma(z, \zeta) := \lim_{n \rightarrow \infty} G_n(z, \zeta), \quad z, \zeta \in \mathbb{R}^{N+1}, z \neq \zeta.$$

We recall that G_n is the (unique) function of class C^∞ in $\{(z, \zeta) \in V_n \times V_n : z \neq \zeta\}$ such that

- i) $G_n \geq 0$ and $G_n(z, \zeta) \rightarrow 0$ as $z \rightarrow z_0$, for every $z_0 \in \partial V_n$ and every $\zeta \in V_n$;
- ii) for any fixed $z \in V_n$, the function $G_n(z, \cdot)$ belongs to $L^1_{\text{loc}}(V_n)$;
- iii) for any $\varphi \in C_0^\infty(V_n)$, the function $u(\cdot) = \int_{V_n} G_n(\cdot, \zeta)\varphi(\zeta) d\zeta$ defined in V_n is smooth and solves the problem

$$\mathcal{L}u = -\varphi \text{ in } V_n, \quad \lim_{z \rightarrow z_0} u(z) = 0 \text{ for every } z_0 \in \partial V_n.$$

The Green function for \mathcal{L} related to the \mathcal{L} -regular open set $V_n = \delta_{\lambda_0^n}(V_0)$ in (3.13) is given by

$$G_n(z, \zeta) = \lambda_0^{n(-Q+2)} G_1(\delta_{\lambda_0^{-n}}(z), \delta_{\lambda_0^{-n}}(\zeta)),$$

where G_1 is the Green function related to V_0 . We obtain (3.16) by letting $n \rightarrow \infty$ in the above identity (we refer to the proof of [24, Proposition 2.8-(i)] for more details).

With (3.16) at hands, it is easy to show that Γ is unbounded. Precisely, it holds

$$\limsup_{z \rightarrow \zeta} \Gamma(z, \zeta) = \infty, \quad \text{for every } \zeta \in \mathbb{R}^{N+1}. \quad (3.17)$$

Indeed, as there exists at least a point $z_0 = (x_0, t_0)$ with $t_0 > 0$ such that $\Gamma(z_0) > 0$, by using (3.15) and (3.16) we get

$$\limsup_{z \rightarrow \zeta} \Gamma(z, \zeta) = \limsup_{w \rightarrow 0} \Gamma(w) \geq \lim_{\lambda \rightarrow 0} \Gamma(\delta_\lambda(z_0)) = \lim_{\lambda \rightarrow 0} \lambda^{-Q+2}\Gamma(z_0) = \infty,$$

recalling that $Q \geq 3$. We are now in position to prove the following

Lemma 3.5 *For every $z_1, z_2 \in \mathbb{R}^{N+1}$, $z_1 \neq z_2$, there exists a function $u \in \overline{\mathcal{S}}^+(\mathbb{R}^{N+1}) \cap C(\mathbb{R}^{N+1})$ such that $u(z_1) \neq u(z_2)$.*

Proof. Let us denote $z_i = (x_i, t_i)$ for $i = 1, 2$. First we suppose that $t_1 < t_2$. The properties of Γ yield that there exists $z_0 = (x_0, t_0)$ with $t_0 > 0$ such that $\Gamma(z_0) > 0$. By the smoothness of Γ , there exists $\varepsilon > 0$ such that $\Gamma > 0$ in the set $z_0 \circ U_\varepsilon$ (see (3.13) for the definition of U_ε). For a fixed $\lambda \in]0, \sqrt{\frac{t_2 - t_1}{2(t_0 + \varepsilon)}}[$ and a non-negative function $\varphi \in C_0^\infty(z_2 \circ (\delta_\lambda(z_0 \circ U_\varepsilon))^{-1} \cap \{t < t_2\})$, we set

$$u_\varphi(z) = \int_{\mathbb{R}^{N+1}} \Gamma(z, \zeta) \varphi(\zeta) d\zeta, \quad z \in \mathbb{R}^{N+1}. \quad (3.18)$$

Hence, we obtain $u_\varphi \in C^\infty(\mathbb{R}^{N+1})$, $u_\varphi \geq 0$ and $\mathcal{L}u_\varphi = -\varphi \leq 0$, so that, by Remark 3.3, $u_\varphi \in \overline{\mathcal{S}}(\mathbb{R}^{N+1})$. Moreover the choice of φ implies that $u_\varphi(z_1) = 0$ and $u_\varphi(z_2) > 0$.

In the case $t_1 = t_2$, $x_1 \neq x_2$, we consider the sequence

$$\mathcal{O}_n(z_2) = \{\zeta \in \mathbb{R}^{N+1} : \Gamma(z_2, \zeta) > n^{Q-2}\}, \quad n \in \mathbb{N}. \quad (3.19)$$

We note that $\mathcal{O}_n(z_2)$ shrinks to $\{z_2\}$ as $n \rightarrow \infty$, by (3.17). For any $\varphi_n \in C_0^\infty(\mathcal{O}_n(z_2))$ such that $\int \varphi_n = 1$ and $\varphi_n \geq 0$, we define u_{φ_n} as in (3.18). Then, u_{φ_n} is a smooth non-negative function in \mathbb{R}^{N+1} satisfying $\mathcal{L}u_{\varphi_n} \leq 0$, and so u_{φ_n} is \mathcal{L} -superharmonic. It holds

$$\begin{aligned} u_{\varphi_n}(z_2) &= \int_{\mathbb{R}^{N+1}} \Gamma(z_2, \zeta) \varphi_n(\zeta) d\zeta \geq n^{Q-2} \quad \text{for every } n \in \mathbb{N}; \\ u_{\varphi_n}(z_1) &\leq \max_{\zeta \in \mathcal{O}_1(z_2)} \Gamma(z_1, \zeta) = C, \end{aligned}$$

where C is a real positive constant independent of n . This ends the proof. \square

We summarize the above facts in the following

Proposition 3.6 *Let \mathcal{L} be an operator in the form (3.1) and assume that [H] and [L] are satisfied. The map \mathcal{H} which associates any open set $\mathcal{O} \subseteq \mathbb{R}^{N+1}$ with the linear space of the \mathcal{L} -harmonic functions in \mathcal{O} is a harmonic sheaf, and $(\mathbb{R}^{N+1}, \mathcal{H})$ is a \mathfrak{B} -harmonic space.*

A remarkable feature of a \mathfrak{B} -harmonic space is that the *Wiener resolutivity theorem* holds (see [3, 18]). In order to state it, we introduce some additional notations. We recall that if $\mathcal{O} \subset \mathbb{R}^{N+1}$ is a bounded open set, then an extended real function $f : \partial\mathcal{O} \rightarrow [-\infty, \infty]$ is called *resolutive* if

$$\inf \overline{\mathcal{U}}_f^\mathcal{O} = \sup \underline{\mathcal{U}}_f^\mathcal{O} =: H_f^\mathcal{O} \in \mathcal{H}(\mathcal{O}),$$

where

$$\begin{aligned} \overline{\mathcal{U}}_f^\mathcal{O} &= \{u \in \overline{\mathcal{S}}(\mathcal{O}) : \inf u > -\infty \text{ and } \liminf_{z \rightarrow \zeta} u(z) \geq f(\zeta), \forall \zeta \in \partial\mathcal{O}\}, \\ \underline{\mathcal{U}}_f^\mathcal{O} &= \{u \in \underline{\mathcal{S}}(\mathcal{O}) : \sup u < \infty \text{ and } \limsup_{z \rightarrow \zeta} u(z) \leq f(\zeta), \forall \zeta \in \partial\mathcal{O}\}. \end{aligned}$$

We say that $H_f^\mathcal{O}$ is the *generalized solution in the sense of Perron-Wiener-Brelot* to the problem

$$u \in \mathcal{H}(\mathcal{O}), \quad u = f \quad \text{on } \partial\mathcal{O}.$$

The Wiener resolutivity theorem yields that any $f \in C(\partial\mathcal{O}, \mathbb{R})$ is resolutive. The map $C(\partial\mathcal{O}, \mathbb{R}) \ni f \mapsto H_f^\mathcal{O}(z)$ defines a linear positive functional for every $z \in \mathcal{O}$. Again, there exists a Radon measure $\mu_z^\mathcal{O}$ on $\partial\mathcal{O}$ such that

$$H_f^\mathcal{O}(z) = \int_{\partial\mathcal{O}} f(\zeta) d\mu_z^\mathcal{O}(\zeta). \quad (3.20)$$

We call $\mu_z^\mathcal{O}$ the *\mathcal{L} -harmonic measure* relative to \mathcal{O} and z , and when \mathcal{O} is \mathcal{L} -regular this definition coincides with the one in (3.12). Finally, a point $\zeta \in \partial\mathcal{O}$ is called *\mathcal{L} -regular* for \mathcal{O} if

$$\lim_{\mathcal{O} \ni z \rightarrow \zeta} H_f^\mathcal{O}(z) = f(\zeta), \quad \text{for every } f \in C(\partial\mathcal{O}, \mathbb{R}). \quad (3.21)$$

Obviously, \mathcal{O} is \mathcal{L} -regular if and only if every $\zeta \in \partial\mathcal{O}$ is \mathcal{L} -regular.

3.2 Harnack inequalities

Let $\mathcal{O} \subset \mathbb{R}^{N+1}$ be an open set. A closed subset F of \mathcal{O} is called an *absorbent set* if, for any $z \in F$ and any \mathcal{L} -regular neighborhood $V \subset \bar{V} \subset \mathcal{O}$ of z , it holds $\mu_z^V(\partial V \setminus F) = 0$. For any given $z_0 \in \mathcal{O}$ we set

$$\mathcal{F}_{z_0} = \{F \subset \mathcal{O} : F \ni z_0, F \text{ is an absorbent set}\}.$$

Then,

$$\mathcal{O}_{z_0} = \bigcap_{F \in \mathcal{F}_{z_0}} F \tag{3.22}$$

is the *smallest* absorbent set containing z_0 . The Potential Theory provides us with the following Harnack inequality. Let $(\mathbb{R}^{N+1}, \mathcal{H})$ be a \mathfrak{B} -harmonic space, let \mathcal{O} be an open subset of \mathbb{R}^{N+1} and let $z_0 \in \mathcal{O}$. Then,

$$\text{for every compact set } K \subset \text{Int}(\mathcal{O}_{z_0}), \quad \sup_K u \leq C_K u(z_0), \tag{3.23}$$

for any non-negative function $u \in \mathcal{H}(\mathcal{O})$. Here C_K is a positive constant depending on \mathcal{O}, K, z_0 . We refer to Theorem 1.4.4 in [3] and Proposition 6.1.5 in [18]. Proposition 3.6 implies that (3.23) applies to our operator \mathcal{L} . We summarize the above argument in the following

Proposition 3.7 *Let \mathcal{L} be an operator in the form (3.1) satisfying [H] and [L], let $\mathcal{O} \subseteq \mathbb{R}^{N+1}$ be an open set, and let $z_0 \in \mathcal{O}$. Then,*

$$\text{for every compact set } K \subset \text{Int}(\mathcal{O}_{z_0}), \quad \sup_K u \leq C_K u(z_0),$$

for any non-negative solutions u to $\mathcal{L}u = 0$ in \mathcal{O} . Here C_K is a positive constant depending on \mathcal{O}, K, z_0 and on \mathcal{L} .

In order to prove Theorem 3.2 we give the following

Lemma 3.8 *Let \mathcal{L} be an operator as in (3.1) satisfying [H] and [L], and let \mathcal{O} be an open subset of \mathbb{R}^{N+1} . For any given $z_0 \in \mathcal{O}$, we have $\mathcal{A}_{z_0} \subseteq \mathcal{O}_{z_0}$ with \mathcal{A}_{z_0} defined in (3.6).*

Proof. Since \mathcal{O}_{z_0} is a closed set, and \mathcal{A}_{z_0} is the closure of the set A_{z_0} defined in (3.6), it is sufficient to show that $A_{z_0} \subseteq \mathcal{O}_{z_0}$. By contradiction, assume that $\bar{z} \in A_{z_0} \setminus \mathcal{O}_{z_0}$. Then, there exists an \mathcal{L} -admissible path $\gamma : [0, T] \rightarrow \mathcal{O}$ such that $\gamma(0) = z_0, \gamma(T) = \bar{z}$.

We set

$$t_1 := \inf\{t > 0 : \gamma([t, T]) \cap \mathcal{O}_{z_0} = \emptyset\}.$$

Note that, since $\mathcal{O} \setminus \mathcal{O}_{z_0}$ is an open set containing \bar{z} and γ is a continuous curve, there exists an open neighborhood $U \subseteq \mathcal{O}$ of \bar{z} such that $U \cap \mathcal{O}_{z_0} = \emptyset$, and a positive σ satisfying $\gamma([T-\sigma, T]) \subseteq U$. Hence, $t_1 \in [0, T[$ is well defined and we have $\gamma(t) \notin \mathcal{O}_{z_0}$ for every $t \in]t_1, T[$. Again, by the continuity of γ , we have

$$z_1 = \gamma(t_1) \in \mathcal{O}_{z_0}.$$

Let $V \subset \bar{V} \subset \mathcal{O}$ be a \mathcal{L} -regular neighborhood of z_1 with $\bar{z} \notin \bar{V}$. Arguing as above, we can find $t_2 \in]t_1, T[$ such that $\gamma([t_1, t_2]) \subset V$ and $z_2 = \gamma(t_2) \in \partial V$. Consider any neighborhood W of z_2 , such that $W \subset \mathcal{O} \setminus \mathcal{O}_{z_0}$. Let $\varphi \in C(\partial V)$ be any non-negative function, supported in $W \cap \partial V$, and such that $\varphi(z_2) > 0$. Recalling that the harmonic function H_φ^V is non-negative, we aim to show that

$$H_\varphi^V(z_1) > 0. \quad (3.24)$$

By contradiction, we suppose that H_φ^V vanishes at z_1 . In other terms, H_φ^V attains its minimum value at z_1 , then Bony's minimum principle implies $H_\varphi^V \equiv 0$ in $\gamma([t_1, t_2])$. As a consequence, since H_φ^V satisfies (3.11),

$$\lim_{t \rightarrow t_2^-} H_\varphi^V(\gamma(t)) = 0. \quad (3.25)$$

On the other hand, by the choice of φ

$$\lim_{V \ni z \rightarrow z_2} H_\varphi^V(z) = \varphi(z_2) > 0.$$

This contradicts (3.25) and proves (3.24). By using representation (3.12) of H_φ^V in terms of the \mathcal{L} -harmonic measure, (3.24) reads as follows

$$H_\varphi^V(z_1) = \int_{\partial V \cap W} \varphi(\zeta) d\mu_{z_1}^V(\zeta) > 0, \quad \text{then} \quad \mu_{z_1}^V(\partial V \cap W) > 0. \quad (3.26)$$

On the other hand, z_1 belongs to the absorbent set \mathcal{O}_{z_0} , so that $\mu_{z_1}^V(\partial V \setminus \mathcal{O}_{z_0}) = 0$. But this clashes with (3.26), being $W \subseteq \mathcal{O} \setminus \mathcal{O}_{z_0}$. This accomplishes the proof. \square

Proof of Theorem 3.2. It is a plain consequence of Proposition 3.7 and Lemma 3.8 \square

As the following proposition shows, we are able to give a complete characterization of the set \mathcal{O}_{z_0} if \mathcal{A}_{z_0} is an absorbent set as well.

Proposition 3.9 *Let \mathcal{L} be an operator as in (3.1) satisfying [H] and [L], let $\mathcal{O} \subseteq \mathbb{R}^{N+1}$ be an open set, and let $z_0 \in \mathcal{O}$. If \mathcal{A}_{z_0} is an absorbent set, then $\mathcal{A}_{z_0} \equiv \mathcal{O}_{z_0}$.*

Proof. The claim directly follows from Lemma 3.8, recalling the definition of \mathcal{O}_{z_0} . \square

The first statement in next proposition is a classical result in abstract potential theory (see e.g. [3, Theorem 1.4.1] and [18, Proposition 6.1.1]). For the convenience of the reader, we explicitly give here its simple proof.

Proposition 3.10 *Let \mathcal{L} be an operator as in (3.1) satisfying [H] and [L], let $\mathcal{O} \subseteq \mathbb{R}^{N+1}$ be an open set, and let $z_0 \in \mathcal{O}$. Assume that there exists a solution $u \geq 0$ to $\mathcal{L}u = 0$ in \mathcal{O} such that $u \equiv 0$ in \mathcal{A}_{z_0} and $u > 0$ in $\mathcal{O} \setminus \mathcal{A}_{z_0}$. Then \mathcal{A}_{z_0} is an absorbent set, and $\mathcal{A}_{z_0} \equiv \mathcal{O}_{z_0}$.*

Proof. Since u is continuous and non-negative,

$$\mathcal{A}_{z_0} = \{z \in \mathcal{O} : u(z) \leq 0\}$$

is a closed subset of \mathcal{O} . Let $z \in \mathcal{A}_{z_0}$, and let $V \subset \bar{V} \subset \mathcal{O}$ be a \mathcal{L} -regular neighborhood of z . As $u \in \mathcal{H}(\mathcal{O})$, we have

$$0 \geq u(z) = \int_{\partial V} u(\zeta) d\mu_z^V(\zeta) \geq 0, \quad \text{so that} \quad \mu_z^V(\partial V \setminus \mathcal{A}_{z_0}) = 0.$$

Hence \mathcal{A}_{z_0} is an absorbent set. The last statement plainly follows from Proposition 3.9. \square

4 Proof of the main results I: Harnack chains and lower bounds

4.1 Harnack chains and lower bounds for $k = 2$

We first consider the stochastic system (1.3) for $k = 2$. Note that, in this case, it is equivalent to (1.4). The relevant Kolmogorov operator is

$$\mathcal{L} = \frac{1}{2}\Delta_{x_{1,n}} + |x_{1,n}|^2\partial_{x_{n+1}} - \partial_t, \quad (4.1)$$

and is homogeneous with respect to the following dilation

$$\delta_\lambda(x, t) = (\lambda x_{1,n}, \lambda^4 x_{n+1}, \lambda^2 t). \quad (4.2)$$

Even if \mathcal{L} does not satisfy [L]-i), it has a fundamental solution Γ which shares several properties of the usual heat kernels. We remark that, since \mathcal{L} does not satisfy the controllability condition [C], the support of Γ is strictly contained in the half space $\{t < \tau\}$. We refer to Remark 4.7.

We next show that \mathcal{L} can be *lifted* to a suitable operator $\tilde{\mathcal{L}}$ in the form (3.1) satisfying both [H] and [L]. By adding a new variable $y = y_{1,n} \in \mathbb{R}^n$, we define the following vector fields on \mathbb{R}^{2n+2}

$$\tilde{Y}_i = Y_i = \partial_{x_i}, i \in \llbracket 1, n \rrbracket, \quad \tilde{Z} = |x_{1,n}|^2\partial_{x_{n+1}} + \sum_{i=1}^n x_i\partial_{y_i} - \partial_t. \quad (4.3)$$

Clearly, if we denote $v(x, y, t) = u(x, t)$ for any $u \in C^\infty(\mathbb{R}^{n+2})$, we have

$$\tilde{Y}_i v(x, y, t) = Y_i u(x, t), \quad \forall i \in \llbracket 1, n \rrbracket, \quad \tilde{Z} v(x, y, t) = Z u(x, t),$$

then, if we consider the *lifted* operator $\tilde{\mathcal{L}} = \frac{1}{2}\sum_{i=1}^n \tilde{Y}_i^2 + \tilde{Z}$, we find $\tilde{\mathcal{L}}v(x, y, t) = \mathcal{L}u(x, t)$.

Note that the Lie algebra of Y_1, \dots, Y_n, Z has dimension $2n + 2$ at any point of \mathbb{R}^{2n+2} . A result by Bonfiglioli and Lanconelli (Theorem 1.1, in [6]) thus yields the existence of a homogeneous Lie group $\mathbb{G} = (\mathbb{R}^{2n+2}, \circ, (\tilde{\delta}_\lambda)_{\lambda>0})$ such that $\tilde{\mathcal{L}}$ is \mathbb{G} -Lie-invariant. By a standard procedure (see e.g., [7, Chapter 1]), in our case we can explicitly write the group law \circ :

$$(x, y, t) \circ (\xi, \eta, \tau) = (x_{1,n} + \xi_{1,n}, x_{n+1} + \xi_{n+1} + 2\langle x_{1,n}, \eta_{1,n} \rangle - \tau|x_{1,n}|^2, y_{1,n} + \eta_{1,n} - \tau x_{1,n}, t + \tau), \quad (4.4)$$

and the dilation $\tilde{\delta}_\lambda$:

$$\tilde{\delta}_\lambda(x, y, t) = (\lambda x_{1,n}, \lambda^4 x_{n+1}, \lambda^3 y_{1,n}, \lambda^2 t). \quad (4.5)$$

Therefore, the lifted operator $\tilde{\mathcal{L}}$ satisfies [H] and [L].

In the sequel we will consider admissible paths in the following form

$$\tilde{\gamma}'(s) = \sum_{j=1}^n \omega_j \tilde{Y}_j(\tilde{\gamma}(s)) + \tilde{Z}(\tilde{\gamma}(s)), \quad s \in [0, \tilde{\tau}],$$

for some constant vector $\omega = (\omega_1, \dots, \omega_n)$, $\tilde{\gamma}(0) = (x, y, t)$. Its explicit expression is

$$\tilde{\gamma}(s) = \left(x_{1,n} + s\omega, x_{n+1} + s|x_{1,n}|^2 + s^2\langle x_{1,n}, \omega \rangle + \frac{s^3}{3}|\omega|^2, y + sx_{1,n} + \frac{s^2}{2}\omega, t - s \right). \quad (4.6)$$

In order to prove an invariant Harnack inequality for the non-negative solutions to $\tilde{\mathcal{L}}v = 0$, we describe the sets \mathcal{O}_{z_0} and \mathcal{A}_{z_0} in the case when z_0 is the origin and

$$\mathcal{O} = \left\{ (x, y, t) \in \mathbb{R}^{2n+2} \mid |x_{1,n}| < 1, -1 < x_{n+1} < 1, |y| < 1, -1 < t < 1 \right\}. \quad (4.7)$$

Lemma 4.1 *Let \mathcal{O} be the open set defined in (4.7), and let $z_0 = (0, 0, 0)$. Then*

$$\mathcal{A}_{z_0} = \left\{ (x, y, t) \in \mathcal{O} \mid 0 \leq x_{n+1} \leq -t, |y|^2 \leq -tx_{n+1} \right\}, \quad (4.8)$$

and $\mathcal{O}_{z_0} = \mathcal{A}_{z_0}$.

Proof. In order to prove (4.8), we consider any $\tilde{\mathcal{L}}$ -admissible curve γ in \mathcal{O} . In our setting, the components $x_{n+1}, y_{1,n}$ and t of every diffusion trajectory are constant functions. Moreover, any drift trajectory $\gamma : [0, T] \rightarrow \mathcal{O}$ starting from $(\bar{x}, \bar{y}, \bar{t})$ is given by

$$\gamma(s) = (\bar{x}_{1,n}, \bar{x}_{n+1} + s|\bar{x}_{1,n}|^2, \bar{y} + s\bar{x}_{1,n}, \bar{t} - s). \quad (4.9)$$

Hence, any $\tilde{\mathcal{L}}$ -admissible curve $\gamma : [0, T] \rightarrow \mathcal{O}$ with $\gamma(0) = (0, 0, 0)$ is given by

$$\gamma(s) = \left(x_{1,n}(s), \int_0^s \sum_{k=1}^m |c_k|^2 \mathbb{I}_{I_k}(r) dr, \int_0^s \sum_{k=1}^m c_k \mathbb{I}_{I_k}(r) dr, -\int_0^s \sum_{k=1}^m \mathbb{I}_{I_k}(r) dr \right), \quad s \in [0, T].$$

Here I_1, \dots, I_m are disjoint intervals contained in $[0, T]$ and \mathbb{I}_{I_k} denotes the characteristic function of I_k . The function $x_{1,n}$ is constant on every I_k , and any c_k is a constant vector such that $|c_k| \leq 1$ for $k = 1, \dots, m$. As a consequence of the Hölder inequality we find

$$\mathcal{A}_{z_0} \subseteq \left\{ (x, y, t) \in \mathcal{O} \mid 0 \leq x_{n+1} \leq -t, |y|^2 \leq -tx_{n+1} \right\}.$$

In order to prove the opposite inclusion, we consider any point

$$(\bar{x}, \bar{y}, \bar{t}) \in \left\{ (x, y, t) \in \mathcal{O} \mid 0 < x_{n+1} < -t, |y|^2 < -tx_{n+1} \right\}, \quad \bar{y} \neq 0,$$

and we show that there exists a $\tilde{\mathcal{L}}$ -admissible curve $\gamma = \gamma_1 + \gamma_2 + \dots + \gamma_5$ contained in \mathcal{O} , which steers $(0, 0, 0)$ to $(\bar{x}, \bar{y}, \bar{t})$. To this aim, we fix a small positive ε , that will be specified in the sequel, and we set

$$s_\varepsilon = \frac{-\bar{t}\bar{x}_{n+1} - |\bar{y}|^2}{\bar{x}_{n+1} - 2|\bar{y}|(1 - \varepsilon) - \bar{t}(1 - \varepsilon)^2}.$$

Note that $-\bar{t}\bar{x}_{n+1} + 2|\bar{y}|\bar{t}(1 - \varepsilon) + \bar{t}^2(1 - \varepsilon)^2 \geq (|\bar{y}| + \bar{t}(1 - \varepsilon))^2$, so that $0 < s_\varepsilon < -\bar{t}$. We set $\tilde{x}_{1,n} = \frac{1-\varepsilon}{|\bar{y}|}\bar{y}$ and we choose γ_1 as a diffusion trajectory connecting $(0, 0, 0)$ to $(\tilde{x}_{1,n}, 0, 0, 0)$,

and $\gamma_2 : [0, s_\varepsilon] \rightarrow \mathbb{R}^{2n+2}$ as a drift trajectory starting from $(\tilde{x}_{1,n}, 0, 0, 0)$. Hence, according to (4.9), we find $\gamma_2(s_\varepsilon) = (\tilde{x}_{1,n}, s_\varepsilon(1-\varepsilon)^2, s_\varepsilon \frac{1-\varepsilon}{|\bar{y}|} \bar{y}, -s_\varepsilon)$. Then, by a diffusion trajectory γ_3 , we connect $\gamma_2(s_\varepsilon)$ to the point $\left(\frac{|\bar{y}| - s_\varepsilon(1-\varepsilon)}{(-t-s_\varepsilon)|\bar{y}|} \bar{y}, s_\varepsilon(1-\varepsilon)^2, s_\varepsilon \frac{1-\varepsilon}{|\bar{y}|} \bar{y}, -s_\varepsilon \right)$. We next consider a drift path $\gamma_4 : [0, -\bar{t} - s_\varepsilon] \rightarrow \mathbb{R}^{2n+2}$ which, by (4.9), and by our choice of s_ε , steers the end point of γ_3 to $\left(\frac{|\bar{y}| - s_\varepsilon(1-\varepsilon)}{(-t-s_\varepsilon)|\bar{y}|} \bar{y}, \bar{x}_{n+1}, \bar{y}, \bar{t} \right)$. Finally, we can find a diffusion path γ_5 connecting $\gamma_4(-\bar{t} - s_\varepsilon)$ to $(\bar{x}, \bar{y}, \bar{t})$.

Clearly, $\gamma = \gamma_1 + \gamma_2 + \dots + \gamma_5$ is a $\widetilde{\mathcal{L}}$ -admissible curve of \mathbb{R}^{2n+2} connecting $(0, 0, 0)$ to $(\bar{x}, \bar{y}, \bar{t})$. Next we prove that, for sufficiently small ε , the trajectory γ is contained in \mathcal{O} . To this aim, as the set \mathcal{O} is convex and the paths $\gamma_1, \gamma_2, \dots, \gamma_5$ are segments, we only need to show that the end-points of $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ belong to \mathcal{O} . The inequalities $-1 < \frac{|\bar{y}| - s_\varepsilon(1-\varepsilon)}{-t-s_\varepsilon} < 1$ directly follow from the definition of s_ε , for sufficiently small positive ε . The other inequalities are a plain consequence of the fact that $0 < s_\varepsilon < -\bar{t} < 1$, as previously noticed. Since \mathcal{A}_{z_0} is the closure of the set of the points that can be reached by a $\widetilde{\mathcal{L}}$ -admissible path, we get

$$\{(x, y, t) \in \mathcal{O} \mid 0 \leq x_{n+1} \leq -t, |y|^2 \leq -t x_{n+1}\} \subseteq \mathcal{A}_{z_0}.$$

This concludes the proof of (4.8).

To complete the proof, by Proposition 3.10 it is sufficient to find a non-negative solution v of $\widetilde{\mathcal{L}}v = 0$, such that $v \equiv 0$ in \mathcal{A}_{z_0} , and $v > 0$ in $\mathcal{O} \setminus \mathcal{A}_{z_0}$. Let φ be any function in $C(\partial\mathcal{O})$, such that $\varphi \equiv 0$ in $\partial\mathcal{O} \cap \mathcal{A}_{z_0}$ and $\varphi > 0$ in $\partial\mathcal{O} \setminus \mathcal{A}_{z_0}$. Then the Perron-Wiener-Brelot solution $v := H_\varphi^\mathcal{O}$ of the following Cauchy-Dirichlet problem

$$\begin{cases} \widetilde{\mathcal{L}}v = 0 & \text{in } \mathcal{O} \\ v = \varphi & \text{in } \partial\mathcal{O} \end{cases}$$

is non-negative. Next we prove that $v > 0$ in $\mathcal{O} \setminus \mathcal{A}_{z_0}$. By contradiction, let $(x, y, t) \in \mathcal{O} \setminus \mathcal{A}_{z_0}$ be such that $v(x, y, t) = 0$. Then (x, y, t) is a minimum for v , so that from Bony's minimum principle [9, Théorème 3.2] it follows that $v(\tilde{x}_{1,n}, x_{n+1}, y, t) = \varphi(\tilde{x}_{1,n}, x_{n+1}, y, t) = 0$, for every $\tilde{x}_{1,n} \in \partial([-1, 1]^n)$. Since every point $(\tilde{x}_{1,n}, x_{n+1}, y, t)$ is regular for the Dirichlet problem, and belongs to $\partial\mathcal{O} \setminus \mathcal{A}_{z_0}$, we find a contradiction with our assumption on φ . Suppose now that there exists $(x, y, t) \in \mathcal{A}_{z_0}$ such that $v(x, y, t) > 0$. Since every point of the set $\partial\mathcal{O} \cap \mathcal{A}_{z_0}$ is $\widetilde{\mathcal{L}}$ -regular, v is continuous in \mathcal{A}_{z_0} . Hence there exists a $(\bar{x}, \bar{y}, \bar{t}) \in \mathcal{A}_{z_0}$ such that $v(\bar{x}, \bar{y}, \bar{t}) = \max_{\mathcal{A}_{z_0}} v > 0$. By Bony's minimum principle we have $v(\tilde{x}_{1,n}, \bar{x}_{n+1}, \bar{y}, \bar{t}) = \varphi(\tilde{x}_{1,n}, \bar{x}_{n+1}, \bar{y}, \bar{t}) > 0$, for any $\tilde{x}_{1,n} \in \partial([-1, 1]^n)$, and this fact contradicts our assumption on φ . \square

Next we introduce some notations to state a Harnack inequality which is invariant with respect to the group law \circ defined in (4.4) and the dilation $\widetilde{\delta}_r$ introduced in (4.5). Consider the box $Q_r =]-r, r[\times]-r^4, r^4[\times]-r^3, r^3[\times]-r^2, 0[$, and note that $Q_r = \widetilde{\delta}_r Q_1$. For every compact set $K \subseteq Q_1$, for any positive r and for any $z_0 \in \mathbb{R}^{2n+2}$ we denote by

$$Q_r(z_0) = z_0 \circ \widetilde{\delta}_r Q_1 = \{z_0 \circ \widetilde{\delta}_r \zeta \mid \zeta \in Q_1\}, \quad K_r(z_0) = z_0 \circ \widetilde{\delta}_r K. \quad (4.10)$$

Corollary 4.2 For every compact set $K \subseteq \{(x, y, t) \in Q_1 \mid 0 < x_{n+1} < -t, |y|^2 < -tx_{n+1}\}$, $r > 0$ and $z_0 \in \mathbb{R}^{2n+2}$ there exists a positive constant C_K , depending only on \mathcal{L} and K , such that

$$\sup_{K_r(z_0)} v \leq C_K v(z_0),$$

for every non-negative solution v of $\mathcal{L}v = 0$ on any open set containing $\overline{Q}_r(z_0)$.

Proof. Consider the function $w(z) = v(z_0 \circ \delta_r z)$. By the invariance with respect to δ_r and \circ , we have $\mathcal{L}w = 0$ in Q_1 . Aiming to apply Theorem 3.2, we consider the open set \mathcal{O} defined in (4.7), and we note that $\mathcal{O} \cap \{t < 0\} \subset Q_1$. Then w is defined as a continuous function on $\partial\mathcal{O} \cap \{t < 0\}$. We extend w to a continuous function on $\partial\mathcal{O}$, and we solve the boundary value problem $\mathcal{L}\tilde{w} = 0$ in \tilde{Q}_1 , with $\tilde{w} = w$ in $\partial\mathcal{O}$. Then we apply Theorem 3.2 and Lemma 4.1, and we get $\sup_K \tilde{w} \leq C_K \tilde{w}(0, 0, 0)$. By the comparison principle we have $\tilde{w} = w$ in $\mathcal{O} \cap \{t \leq 0\}$, then the claim plainly follows from the inclusion $K \subset \mathcal{O} \cap \{t < 0\}$. \square

In the sequel, we will apply the above result to the set

$$K = \left\{ (x, y, t) \in \mathbb{R}^{2n+2} \mid |x_{1,n}| \leq \frac{1}{2}, \frac{1}{32} \leq x_{n+1} \leq \frac{1}{4}, |y| \leq \frac{1}{8}, t = -\frac{1}{2} \right\} \quad (4.11)$$

which is a compact subset of $\{(x, y, t) \in Q_1 \mid 0 < x_{n+1} < -t, |y|^2 < -tx_{n+1}\}$ introduced in (4.8).

Lemma 4.3 Let $\mathcal{O} \subseteq \mathbb{R}^{2n+2}$ be any open set, let $z = (x, y, t) \in \mathcal{O}$, and let $\tilde{\tau} > 0$. Consider the path $\tilde{\gamma} : [0, \tilde{\tau}] \rightarrow \mathbb{R}^{2n+2}$ satisfying $\tilde{\gamma}(0) = z$, and

$$\tilde{\gamma}'(s) = \sum_{j=1}^n \omega_j \tilde{Y}_j(\tilde{\gamma}(s)) + \tilde{Z}(\tilde{\gamma}(s)), \quad s \in [0, \tilde{\tau}],$$

for some constant vector $\omega = (\omega_1, \dots, \omega_n)$ such that $\tilde{\tau}|\omega|^2 \geq 2$. Set $r = \frac{1}{|\omega|}$ and suppose that

$$\overline{Q}_r(\tilde{\gamma}(s)) \subseteq \mathcal{O} \quad \text{for every } s \in [0, \tilde{\tau}].$$

Then, there exists a constant $C > 0$, only depending on \mathcal{L} , such that

$$v(\tilde{\gamma}(\tilde{\tau})) \leq \exp(C(\tilde{\tau}|\omega|^2 + 1)) v(x, y, t),$$

for every non-negative solution v to $\mathcal{L}v = 0$ in \mathcal{O} .

Proof. To prove our claim we apply the Harnack inequality stated in Corollary 4.2 to a suitable set of points z_1, \dots, z_k lying on $\tilde{\gamma}([0, \tilde{\tau}])$. Specifically, we let k be the positive integer such that $k - 1 < 2\tilde{\tau}|\omega|^2 \leq k$, we set $\tilde{s} = \frac{\tilde{\tau}}{k}$, $\tilde{r} = \sqrt{2\tilde{s}}$ and we define $z_j = \tilde{\gamma}(j\tilde{s})$ for $j \in \llbracket 1, k \rrbracket$. According with (4.4), (4.5) and (4.6), we find

$$z_j = z_{j-1} \circ \left(\tilde{s}\omega, \frac{|\omega|^2 \tilde{s}^3}{3}, \frac{\tilde{s}^2}{2}\omega, -\tilde{s} \right) = z_{j-1} \circ \delta_{\tilde{r}} \left(\sqrt{\frac{\tilde{s}}{2}}\omega, \frac{|\omega|^2 \tilde{s}}{12}, \frac{\sqrt{\tilde{s}}}{4\sqrt{2}}\omega, -\frac{1}{2} \right), \quad j \in \llbracket 1, k \rrbracket.$$

Note that, being $\tilde{\tau}|\omega|^2 \geq 2$, we have $k \geq 4$, so that $\frac{3}{8} \leq \tilde{s}|\omega|^2 \leq \frac{1}{2}$, then

$$z_j \in K_{\tilde{r}}(z_{j-1}), \quad j \in \llbracket 1, k \rrbracket,$$

where $K_{\tilde{r}}(z_{j-1})$ is defined in (4.10), and K in (4.11). Moreover $0 < \tilde{r} \leq r$, then $Q_{\tilde{r}}(z_j) \subset \mathcal{O}$ for $j \in \llbracket 0, k \rrbracket$. Then, by Corollary 4.2, there exists a constant $C_K > 1$ such that $v(z_j) \leq C_K v(z_{j-1})$ for every $j \in \llbracket 1, k \rrbracket$. In particular, being $k < 2\tilde{\tau}|\omega|^2 + 1$, $z_0 = z$, $z_k = \tilde{\gamma}(\tilde{\tau})$, we find

$$v(\tilde{\gamma}(\tilde{\tau})) \leq C_K^{2\tilde{\tau}|\omega|^2+1} v(x, y, t).$$

The conclusion then follows by choosing $C := \log(C_K)$. \square

Note that, whenever Lemma 4.3 applies, we have $x_{1,n}(\tilde{\tau}) = x_{1,n} + \tilde{\tau}\omega \neq x_{1,n}$. The next result gives a bound along a trajectory $\tilde{\gamma}$ such that $x_{1,n}(s) = 0$ for any $s \in [0, \tilde{\tau}]$.

Lemma 4.4 *Let $\mathcal{O} \subseteq \mathbb{R}^{2n+2}$ be any open set, and let $z = (x, y, t) \in \mathcal{O}$, with $x_{1,n} = 0$. Set $\tilde{\tau} > 0, \tilde{\xi}_{n+1} \in]0, \frac{13}{15}\tilde{\tau}^2]$, $r = 2\sqrt{\frac{\tilde{\xi}_{n+1}}{\tilde{\tau}}}$, define*

$$\tilde{\gamma}(s) = \left(0, \dots, 0, x_{n+1} + \frac{s\tilde{\xi}_{n+1}}{\tilde{\tau}}, y, t - s \right), \quad 0 \leq s \leq \tilde{\tau},$$

and suppose that $\overline{Q_r(\tilde{\gamma}(s))} \subset \mathcal{O}$ for any $s \in [0, \tilde{\tau}]$. Then, there exists a constant $C > 0$, only depending on \mathcal{L} , such that

$$v(\tilde{\gamma}(\tilde{\tau})) \leq \exp\left(C\left(\frac{\tilde{\tau}^2}{\tilde{\xi}_{n+1}} + 1\right)\right) v(x, y, t),$$

for every non-negative solution v to $\tilde{\mathcal{L}}v = 0$ in \mathcal{O} .

Proof. Let k be the unique positive integer such that $\frac{\tilde{\tau}^2}{\tilde{\xi}_{n+1}} - 1 < k \leq \frac{\tilde{\tau}^2}{\tilde{\xi}_{n+1}}$, set $\tilde{s} = \frac{\tilde{\tau}}{k}$, $\tilde{r} = \sqrt{2\tilde{s}}$ and define $z_j = \tilde{\gamma}(j\tilde{s})$ for $j \in \llbracket 1, k \rrbracket$. Note that our assumption $\tilde{\xi}_{n+1} \leq \frac{13}{15}\tilde{\tau}^2$ yields $\frac{\tilde{\tau}^2}{\tilde{\xi}_{n+1}} - 1 > 0$. According with (4.4) and (4.5) we find

$$z_j = z_{j-1} \circ \left(0, \frac{\tilde{s}\tilde{\xi}_{n+1}}{\tilde{\tau}}, 0, -\tilde{s} \right) = z_{j-1} \circ \delta_{\tilde{r}} \left(0, \frac{k\tilde{\xi}_{n+1}}{4\tilde{\tau}^2}, 0, -\frac{1}{2} \right), \quad j \in \llbracket 1, k \rrbracket.$$

From $\tilde{\xi}_{n+1} \leq \frac{13}{15}\tilde{\tau}^2$ it follows that $\frac{1}{30} \leq \frac{k\tilde{\xi}_{n+1}}{4\tilde{\tau}^2} \leq \frac{1}{4}$ and that $\tilde{r} \leq r$. Then $z_j \in K_{\tilde{r}}(z_{j-1})$, and $\overline{Q_{\tilde{r}}(z_j)} \subset \mathcal{O}$, for any $j \in \llbracket 0, k \rrbracket$. Hence the set $\{z_0, \dots, z_k\}$ is a Harnack chain and, being $z_0 = z, z_k = \tilde{\gamma}(\tilde{\tau})$, we find $v(\tilde{\gamma}(\tilde{\tau})) \leq C_K^k v(z)$. The conclusion follows immediately from the definition of k , and (4.4). \square

We next obtain a bound for the positive solutions to $\mathcal{L}u = 0$ as a corollary of Lemmas 4.3 and 4.4.

Proposition 4.5 *Let T_1, τ, t, T_2 be such that $T_1 < \tau < t < T_2$, and assume that there exists a path $\gamma : [0, t - \tau] \rightarrow \mathbb{R}^{n+1} \times]T_1, T_2[$ satisfying*

$$\gamma'(s) = \sum_{j=1}^n \omega_j Y_j(\gamma(s)) + Z(\gamma(s)), \quad \gamma(0) = (x, t), \quad \gamma(t - \tau) = (\xi, \tau), \quad (4.12)$$

for some constant vector $\omega = (\omega_1, \dots, \omega_n)$ such that $|\omega|^2 \geq \max\{\frac{2}{t-\tau}, \frac{1}{\tau-T_1}\}$. Then

$$u(\xi, \tau) \leq \exp(C((t - \tau)|\omega|^2 + 1)) u(x, t),$$

for every non-negative solution u to $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1} \times]T_1, T_2[$. Moreover

$$u(0, \dots, 0, x_{n+1} + \tilde{\xi}_{n+1}, \tau) \leq \exp\left(C\left(\frac{(t - \tau)^2}{\tilde{\xi}_{n+1}} + 1\right)\right) u(0, \dots, 0, x_{n+1}, t).$$

for every $(x, t) \in \mathbb{R}^{n+1} \times]T_1, T_2[, \tau \in]T_1, t[$ and $\tilde{\xi}_{n+1} \in]0, \min\{\frac{13}{15}(t - \tau)^2, \frac{(t - \tau)(\tau - T_1)}{4}\}]$, for every non-negative solution u to $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1} \times]T_1, T_2[$. In the above inequalities, C is a positive constant only depending on \mathcal{L} .

Proof. Let $u : \mathbb{R}^{n+1} \times]T_1, T_2[\rightarrow \mathbb{R}$ be a non-negative solution to $\mathcal{L}u = 0$, and let $\gamma : [0, t - \tau] \rightarrow \mathbb{R}^{n+1} \times]T_1, T_2[$ be as in the statement. Consider the lifted operator $\tilde{\mathcal{L}}$ and define the function $v : \mathbb{R}^{2n+1} \times]T_1, T_2[\rightarrow \mathbb{R}$ by setting $v(x, y, t) = u(x, t)$ for every $(x, y, t) \in \mathbb{R}^{2n+1} \times]T_1, T_2[$. Then v is a non-negative solution to $\tilde{\mathcal{L}}v = 0$. Next, we denote by $\tilde{\gamma} : [0, t - \tau] \rightarrow \mathbb{R}^{2n+1} \times]T_1, T_2[$ the solution of the Cauchy problem

$$\begin{cases} \tilde{\gamma}'(s) = \sum_{j=1}^n \omega_j \tilde{Y}_j(\tilde{\gamma}(s)) + \tilde{Z}(\tilde{\gamma}(s)), & s \in [0, t - \tau], \\ \tilde{\gamma}(0) = (x, 0, t), \end{cases}$$

where ω is the constant vector appearing in (4.12). Note that $Q_r(\tilde{\gamma}(s)) \subseteq \mathbb{R}^{2n+1} \times]T_1, T_2[$ for every $s \in [0, t - \tau]$, with $r = \frac{1}{|\omega|}$. Then, by applying Lemma 4.3 with $\tilde{\tau} = t - \tau$, we find

$$\begin{aligned} u(\xi, \tau) &= v(\xi, \eta, \tau) \leq \exp(C((t - \tau)|\omega|^2 + 1)) v(x, 0, t) \\ &= \exp(C((t - \tau)|\omega|^2 + 1)) u(x, t). \end{aligned}$$

where $\eta \in \mathbb{R}^n$ is such that $\tilde{\gamma}(t - \tau) = (\xi, \eta, \tau)$. This accomplishes the proof of the first statement. The same token and Lemma 4.4 give the proof of the second one. \square

Corollary 4.6 *Let $u : \mathbb{R}^{n+1} \times]T_1, T_2[\rightarrow \mathbb{R}$ be a non-negative solution to $\mathcal{L}u = 0$, and let $t, \tau \in \mathbb{R}$ be such that $T_1 < \tau < t < T_2$, and $t - \tau \leq 2(\tau - T_1)$. Then there exists a positive constant C_1 , only depending on \mathcal{L} , such that*

(i) for any $x, \xi \in \mathbb{R}^{n+1}$ such that $\xi_{n+1} - x_{n+1} \geq (t - \tau)(|x_{1,n}|^2 + |\xi_{1,n}|^2) + (t - \tau)^2$ we have

$$u(\xi, \tau) \leq \exp\left(C_1\left(\frac{|x_{1,n} - \xi_{1,n}|^2}{t - \tau} + \frac{\xi_{n+1} - x_{n+1} - \frac{2}{3}(|x_{1,n}|^2 + |\xi_{1,n}|^2)}{(t - \tau)^2} + 1\right)\right) u(x, t);$$

(ii) for any $x, \xi \in \mathbb{R}^{n+1}$ such that $0 < \xi_{n+1} - x_{n+1} \leq \frac{13(t-\tau)^2}{15}$ we have

$$u(\xi, \tau) \leq \exp \left(C_1 \left(\frac{|x_{1,n}|^4 + |\xi_{1,n}|^4 + (t-\tau)^2}{\xi_{n+1} - x_{n+1}} + 1 \right) \right) u(x, t).$$

Proof. In order to simplify the proof we assume, as it is not restrictive, that $n \geq 2$.

We first prove (i). We will find a path $\gamma : [0, t-\tau] \rightarrow \mathbb{R}^{n+1} \times]T_1, T_2[$ satisfying (4.12) for some piecewise constant function ω , then we will apply the first statement of Proposition 4.5. In our construction, the length of every interval where ω is constant will be greater or equal to $\frac{t-\tau}{4}$, and $|\omega(s)|^2 \geq \frac{8}{t-\tau}$. Our assumption $t-\tau \leq 2(\tau-T_2)$ implies that $\max \left\{ \frac{2}{t-\tau}, \frac{1}{\tau-T_1} \right\} = \frac{2}{t-\tau}$, and the claims follow by applying Proposition 4.5. We divide the proof into two steps. In the first one we choose $\omega(s)$, $s \in [0, \frac{t-\tau}{2}]$ such that $x_{1,n}(\frac{t-\tau}{2}) = \xi_{1,n}$, in the second step we complete the proof.

Consider first $x, \xi \in \mathbb{R}^n$ such that

$$|x_{1,n} - \xi_{1,n}|^2 \geq t - \tau. \quad (4.13)$$

As a first step, we set $\omega(s) = \frac{2}{t-\tau}(\xi_{1,n} - x_{1,n})$, for $s \in [0, \frac{t-\tau}{2}]$. According with (4.6), we have

$$\gamma(s) = \left(x_{1,n} + s\omega, x_{n+1} + s|x_{1,n}|^2 + s^2 \langle x_{1,n}, \omega \rangle + \frac{s^3}{3} |\omega|^2, t-s \right).$$

Proposition 4.5 then gives

$$\begin{aligned} u(\gamma((t-\tau)/2)) &\leq \exp \left(C \left(2 \frac{|x_{1,n} - \xi_{1,n}|^2}{t-\tau} + 1 \right) \right) u(x, t), \\ \gamma((t-\tau)/2) &= \left(\xi_{1,n}, x_{n+1} + \frac{t-\tau}{6} (|x_{1,n}|^2 + \langle x_{1,n}, \xi_{1,n} \rangle + |\xi_{1,n}|^2), \frac{t+\tau}{2} \right). \end{aligned} \quad (4.14)$$

In the second step, we define ω by choosing any vector $\bar{\omega} \in \mathbb{R}^n$ with $|\bar{\omega}| = 1$ and $\bar{\omega} \perp \xi_{1,n}$. This is always possible, since $n \geq 2$. We set

$$\begin{aligned} m &= 4 \sqrt{6 \frac{\xi_{n+1} - x_{n+1}}{(t-\tau)^3} - \frac{|x_{1,n}|^2 + \langle x_{1,n}, \xi_{1,n} \rangle + 4|\xi_{1,n}|^2}{(t-\tau)^2}} \\ \omega(s) &= m\bar{\omega}, \text{ for } s \in]\frac{t-\tau}{2}, \frac{3}{4}(t-\tau)], \quad \omega(s) = -m\bar{\omega}, \text{ for } s \in]\frac{3}{4}(t-\tau), t-\tau]. \end{aligned} \quad (4.15)$$

We find $\gamma(t-\tau) = (\xi, \tau)$. Moreover, from $\xi_{n+1} \geq x_{n+1} + (t-\tau)(|x_{1,n}|^2 + |\xi_{1,n}|^2) + (t-\tau)^2$ it follows that $|x_{1,n}|^2 + \langle x_{1,n}, \xi_{1,n} \rangle + 4|\xi_{1,n}|^2 \leq \frac{9}{2}(|x_{1,n}|^2 + |\xi_{1,n}|^2) \leq \frac{9}{2} \frac{\xi_{n+1} - x_{n+1}}{(t-\tau)}$, then $|\omega(s)|^2 = m^2 \geq \frac{24}{t-\tau}$, for $s \in]\frac{t-\tau}{2}, t-\tau]$. Hence, by Proposition 4.5, we get

$$u(\gamma(t-\tau)) \leq \exp \left(2C \left(\frac{(t-\tau)m^2}{4} + 1 \right) \right) u(\gamma((t-\tau)/2)).$$

This inequality and (4.14) prove our claim if (4.13) holds, since $\frac{(t-\tau)m^2}{4} \leq 24 \frac{\xi_{n+1} - x_{n+1}}{(t-\tau)^2}$.

If $\xi_{1,n} = x_{1,n}$ it is sufficient to skip the first step and to consider the controls of (4.15) on $[0, t - \tau]$ instead of $[\frac{t-\tau}{2}, t - \tau]$, with $m = \sqrt{12 \left(\frac{\xi_{n+1} - x_{n+1}}{(t-\tau)^3} - \frac{|\xi_{1,n}|^2}{(t-\tau)^2} \right)}$.

We are left with the case

$$0 < |x_{1,n} - \xi_{1,n}|^2 < t - \tau. \quad (4.16)$$

In the first step we find a point $\tilde{x}_{1,n}$ such that $|x_{1,n} - \tilde{x}_{1,n}|^2 \geq \frac{t-\tau}{2}$ and $|\tilde{x}_{1,n} - \xi_{1,n}|^2 \geq \frac{t-\tau}{2}$, and we argue as in the case (4.13). To this aim, we choose any vector $\tilde{\omega} \in \mathbb{R}^n$ with $|\tilde{\omega}| = 1$ and $\tilde{\omega} \perp (\xi_{1,n} - x_{1,n})$. Then we define

$$\tilde{x}_{1,n} = \frac{1}{2}(\xi_{1,n} + x_{1,n}) + \sqrt{\frac{t-\tau}{2}}\tilde{\omega}.$$

Note that

$$\frac{t-\tau}{2} \leq |\tilde{x}_{1,n} - x_{1,n}|^2 < t - \tau, \quad \text{and} \quad \frac{t-\tau}{2} \leq |\tilde{x}_{1,n} - \xi_{1,n}|^2 < t - \tau, \quad (4.17)$$

then, using the same argument as in the proof of (4.14), choosing $\omega(s) = \frac{4}{t-\tau}(\tilde{x}_{1,n} - x_{1,n})$ for $0 \leq s \leq \frac{t-\tau}{4}$, and $\omega(s) = \frac{4}{t-\tau}(\xi_{1,n} - \tilde{x}_{1,n})$ for $\frac{t-\tau}{4} < s \leq \frac{t-\tau}{2}$, we find a path γ such that

$$\begin{aligned} \gamma((t-\tau)/4) &= \left(\tilde{x}_{1,n}, x_{n+1} + \frac{t-\tau}{12} (|x_{1,n}|^2 + \langle x_{1,n}, \tilde{x}_{1,n} \rangle + |\tilde{x}_{1,n}|^2), \frac{3t+\tau}{4} \right), \\ \gamma((t-\tau)/2) &= \left(\xi_{1,n}, x_{n+1} + \frac{t-\tau}{12} (|x_{1,n}|^2 + \langle x_{1,n} + \xi_{1,n}, \tilde{x}_{1,n} \rangle + |\xi_{1,n}|^2 + 2|\tilde{x}_{1,n}|^2), \frac{t+\tau}{2} \right), \\ u(\gamma((t-\tau)/2)) &\leq \exp(10C) u(x, t). \end{aligned}$$

Last inequality follows from Proposition 4.5, since (4.17) yields $\frac{t-\tau}{4}|\omega(s)|^2 = \frac{4}{t-\tau}|\tilde{x}_{1,n} - x_{1,n}|^2 \leq 4$ for any $s \in [0, \frac{t-\tau}{4}]$, and $\frac{t-\tau}{4}|\omega(s)|^2 = \frac{4}{t-\tau}|\tilde{x}_{1,n} - \xi_{1,n}|^2 \leq 4$ for any $s \in]\frac{t-\tau}{4}, \frac{t-\tau}{2}]$.

In the second step we argue as in the case (4.13), by setting

$$\begin{aligned} m &= 4\sqrt{6 \frac{\xi_{n+1} - x_{n+1}}{(t-\tau)^3} - \frac{|x_{1,n}|^2 + \langle x_{1,n} + \xi_{1,n}, \tilde{x}_{1,n} \rangle + 7|\xi_{1,n}|^2 + 2|\tilde{x}_{1,n}|^2}{2(t-\tau)^2}} \\ \omega(s) &= m\bar{\omega}, \text{ for } s \in]\frac{t-\tau}{2}, \frac{3}{4}(t-\tau)], \quad \omega(s) = -m\bar{\omega}, \text{ for } s \in]\frac{3}{4}(t-\tau), t-\tau], \end{aligned}$$

for some vector $\bar{\omega} \in \mathbb{R}^n$ with $|\bar{\omega}| = 1$ and $\bar{\omega} \perp \xi_{1,n}$. We find $\gamma(t-\tau) = (\xi, \tau)$, and

$$u(\xi, \tau) \leq \exp \left(C \left(\frac{\xi_{n+1} - x_{n+1}}{(t-\tau)^2} - \frac{|x_{1,n}|^2 + \langle x_{1,n} + \xi_{1,n}, \tilde{x}_{1,n} \rangle + 7|\xi_{1,n}|^2 + 2|\tilde{x}_{1,n}|^2}{12(t-\tau)} + 2 \right) \right) u(\gamma((t-\tau)/2)).$$

Note that, by (4.17), we have $|\tilde{x}_{1,n}| < |x_{1,n}| + \sqrt{t-\tau}$ and $|\tilde{x}_{1,n}| < |\xi_{1,n}| + \sqrt{t-\tau}$. Hence

$$\begin{aligned} &- (|x_{1,n}|^2 + \langle x_{1,n} + \xi_{1,n}, \tilde{x}_{1,n} \rangle + 7|\xi_{1,n}|^2 + 2|\tilde{x}_{1,n}|^2) \leq \\ &- (|x_{1,n}|^2 + 7|\xi_{1,n}|^2 + 2|\tilde{x}_{1,n}|^2) + \frac{3}{2} (|x_{1,n}|^2 + |\xi_{1,n}|^2) + (t-\tau) \leq \frac{1}{2}|x_{1,n}|^2 - \frac{11}{2}|\xi_{1,n}|^2 + (t-\tau). \end{aligned}$$

From $\xi_{n+1} - x_{n+1} \geq (t - \tau) (|x_{1,n}|^2 + |\xi_{1,n}|^2) + (t - \tau)^2$, we find

$$u(\xi, \tau) \leq \exp \left(C \left(9 \frac{\xi_{n+1} - x_{n+1} - \frac{2}{3} (|x_{1,n}|^2 + |\xi_{1,n}|^2)}{2(t - \tau)^2} + \frac{25}{12} \right) \right) u(x, t).$$

This inequality is equivalent to our claim because of (4.16). This concludes the proof of (i).

We next prove (ii) by using again Proposition 4.5. However, if we only consider a path γ satisfying (4.12) for a piecewise constant function ω , we find $\gamma_{n+1}(t - \tau) \geq x_{n+1} + \frac{t - \tau}{6} (|x_{1,n}|^2 + |\xi_{1,n}|^2)$, and we cannot consider any point (ξ, τ) with ξ_{n+1} close to x_{n+1} . To avoid this obstruction we will apply the second statement of Proposition 4.5 in a suitable interval $[\tau + t_2, t - t_1] \subsetneq [\tau, t]$. In the remaining intervals $[t - t_1, t]$ and $[\tau, \tau + t_2]$ we will rely on the first statement of Proposition 4.5.

We first suppose that $x_{1,n} \neq 0$, $\xi_{1,n} \neq 0$, we set

$$\begin{aligned} t_1 &= \min \left\{ \frac{|x_{1,n}|^2}{2}, \frac{\xi_{n+1} - x_{n+1}}{|x_{1,n}|^2}, \frac{t - \tau}{3} \right\}, & \omega_1 &= -\frac{1}{t_1} x_{1,n}, \\ t_2 &= \min \left\{ \frac{|\xi_{1,n}|^2}{2}, \frac{\xi_{n+1} - x_{n+1}}{|\xi_{1,n}|^2}, \frac{t - \tau}{3} \right\}, & \omega_2 &= \frac{1}{t_2} \xi_{1,n}, \end{aligned}$$

and we consider the paths

$$\begin{aligned} \gamma_1(s) &= \left(\left(1 - \frac{s}{t_1} \right) x_{1,n}, x_{n+1} + \frac{(s - t_1)^3 + t_1^3}{3t_1^2} |x_{1,n}|^2, t - s \right), & s &\in [0, t_1], \\ \gamma_2(s) &= \left(\frac{s}{t_2} \xi_{1,n}, \xi_{n+1} + \frac{s^3 - t_2^3}{3t_2^2} |\xi_{1,n}|^2, \tau + t_2 - s \right), & s &\in [0, t_2]. \end{aligned}$$

Note that $|\omega_1|^2 \geq \max \left\{ \frac{2}{t_1}, \frac{1}{t - t_1 - T_1} \right\}$, and $|\omega_2|^2 \geq \max \left\{ \frac{2}{t_2}, \frac{1}{\tau - T_1} \right\}$, then Proposition 4.5 yields

$$\begin{aligned} u(\gamma_1(t_1)) &\leq \exp \left(C \left(\max \left\{ \frac{|x_{1,n}|^4}{\xi_{n+1} - x_{n+1}}, 3 \frac{|x_{1,n}|^2}{t - \tau}, 2 \right\} + 1 \right) \right) u(x, t), \\ u(\xi, \tau) &\leq \exp \left(C \left(\max \left\{ \frac{|\xi_{1,n}|^4}{\xi_{n+1} - x_{n+1}}, 3 \frac{|\xi_{1,n}|^2}{t - \tau}, 2 \right\} + 1 \right) \right) u(\gamma_2(0)), \end{aligned} \tag{4.18}$$

with

$$\gamma_1(t_1) = (0, x_{n+1} + \frac{t_1}{3} |x_{1,n}|^2, t - t_1), \quad \gamma_2(0) = (0, \xi_{n+1} - \frac{t_2}{3} |\xi_{1,n}|^2, \tau + t_2).$$

We next compare $u(\gamma_1(t_1))$ with $u(\gamma_2(0))$ by using the second statement of Proposition 4.5. We set $\tilde{\xi}_{n+1} = \xi_{n+1} - x_{n+1} - \frac{t_1}{3} |x_{1,n}|^2 - \frac{t_2}{3} |\xi_{1,n}|^2$, and we recall that $\xi_{n+1} - x_{n+1} \leq \frac{13(t - \tau)^2}{15}$. Hence

$$\frac{t - \tau}{3} < (t - t_1) - (\tau + t_2) < t - \tau, \quad \frac{\xi_{n+1} - x_{n+1}}{3} \leq \tilde{\xi}_{n+1} \leq \frac{13(t - \tau)^2}{15} \tag{4.19}$$

As a consequence $0 < \tilde{\xi}_{n+1} \leq \min \left\{ \frac{13}{15} (t - \tau - t_2 - t_1)^2, \frac{(t - \tau - t_2 - t_1)(\tau + t_2 - T_1)}{4} \right\}$, then, by applying Proposition 4.5, we get

$$u(\gamma_2(0)) \leq \exp \left(C \left(\frac{(t - t_1 - t_2 - \tau)^2}{\xi_{n+1} - x_{n+1} - \frac{t_1}{3} |x_{1,n}|^2 - \frac{t_2}{3} |\xi_{1,n}|^2} + 1 \right) \right) u(\gamma(t_1)),$$

and using again (4.19) we find

$$u(\gamma_2(0)) \leq \exp\left(3C\left(\frac{(t-\tau)^2}{\xi_{n+1}-x_{n+1}}+1\right)\right)u(\gamma(t_1)).$$

The above inequality, with (4.18), gives

$$u(\xi, \tau) \leq \exp\left(C'\left(\frac{|x_{1,n}|^4+|\xi_{1,n}|^4+(t-\tau)^2}{\xi_{n+1}-x_{n+1}}+\frac{|x_{1,n}|^2+|\xi_{1,n}|^2}{t-\tau}+1\right)\right)u(x, t),$$

for some positive constant C' only depending on C . The claim then follows from the elementary inequality

$$2\frac{|x_{1,n}|^2+|\xi_{1,n}|^2}{t-\tau} \leq 1 + \frac{|x_{1,n}|^4+|\xi_{1,n}|^4}{(t-\tau)^2} \leq 1 + \frac{13}{5}\frac{|x_{1,n}|^4+|\xi_{1,n}|^4}{\xi_{n+1}-x_{n+1}},$$

using once again (4.19) for the last inequality. We finally note that, if $x_{1,n} = 0$, then we skip the construction of γ_1 , and we rely on γ_2 and on the application of the second statement of Proposition 4.5 in the interval $[\tau + t_2, t]$. Analogously, if $\xi_{1,n} = 0$, we skip the construction of γ_2 . This concludes the proof. \square

We end this section with a remark about the Fundamental Solution Γ of \mathcal{L} , with $k = 2$, as characterized in Definition 3.4.

Remark 4.7 Consider the operator \mathcal{L} with $k = 2$. Its fundamental solution Γ is homogeneous with respect to δ_λ defined in (4.2):

$$\Gamma(\lambda x_{1,n}, \lambda^4 x_{n+1}, \lambda^2 t, \lambda \xi_{1,n}, \lambda^4 \xi_{n+1}, \lambda^2 \tau) = \lambda^{-5} \Gamma(x_{1,n}, x_{n+1}, t, \xi_1, \xi_{n+1}, \tau), \quad (4.20)$$

for any $(x_{1,n}, x_{n+1}, t), (\xi_{1,n}, \xi_{n+1}, \tau) \in \mathbb{R}^{n+2}$, such that $(x_{1,n}, x_{n+1}, t) \neq (\xi_{1,n}, \xi_{n+1}, \tau)$, and $\lambda > 0$. Furthermore,

$$\Gamma(x_{1,n}, x_{n+1}, t, \xi_{1,n}, \xi_{n+1}, \tau) = 0 \quad \text{if } x_{n+1} > \xi_{n+1} \text{ or } t < \tau. \quad (4.21)$$

There exist two positive constants c and C only depending on \mathcal{L} , such that

i) for any $x \in \mathbb{R}^{n+1}$ such that $x_{n+1} \leq -t|x_{1,n}|^2 - t^2$ we have

$$\Gamma(x, t) \geq \frac{c}{t^{\frac{n+4}{2}}} \exp\left(-C\left(\frac{|x_{1,n}|^2}{t} + \frac{-x_{n+1} - \frac{2}{3}|x_{1,n}|^2 t}{t^2} + 1\right)\right);$$

ii) for any $x \in \mathbb{R}^{n+1}$ such that $-\frac{13t^2}{15} \leq x_{n+1} < 0$ we have

$$\Gamma(x, t) \geq \frac{c}{t^{\frac{n+4}{2}}} \exp\left(-C\left(\frac{|x_{1,n}|^4 + t^2}{-x_{n+1}} + 1\right)\right).$$

Proof. The existence of a fundamental solution has been proved just before in Definition 3.4. Hence, there exists an increasing sequence $\{V_k\}_{k \in \mathbb{N}}$ of \mathcal{L} -regular open sets such that $\bigcup_{k \in \mathbb{N}} V_k = \mathbb{R}^{n+2}$ (see Section 2).

In order to prove (4.21), we recall that there exists an increasing sequence $\{V_k\}_{k \in \mathbb{N}}$ of \mathcal{L} -regular open sets such that $\bigcup_{k \in \mathbb{N}} V_k = \mathbb{R}^{n+2}$ (see Section 2). Fix $\zeta = (\xi_{1,n}, \xi_{n+1}, \tau) \in \mathbb{R}^{n+2}$. By property *vi*) of Γ , it is enough to show that $\Gamma(\cdot, \zeta)$ vanishes at any point $z = (x_{1,n}, x_{n+1}, t)$ with $x_{n+1} > \xi_{n+1}$. We argue as in the proof of [6, Proposition 3.9]. Let k_0 be such that $\zeta \in V_k$ for every $k \geq k_0$. For any $h \in \mathbb{R}$ we consider the set $V_{k,h} = V_k \cap \{(x, t) \in \mathbb{R}^{n+2} : x_{n+1} > h\}$. If G_k denotes the Green function for \mathcal{L} related to V_k , our claim is proved by showing that

$$G_k(z, \zeta) = 0 \quad \text{for any } z \in V_{k, \xi_{n+1}}. \quad (4.22)$$

Indeed, we have $G_k \rightarrow \Gamma$ as k goes to infinity, and $\bigcup_{k \in \mathbb{N}} V_{k, \xi_{n+1}} = \mathbb{R}^n \times]\xi_{n+1}, +\infty[\times \mathbb{R}$. For some fixed positive ε and σ , we define the function

$$u_\varepsilon(z) = G_k(z, \zeta) - \varepsilon(x_{n+1} - \xi_{n+1} - \sigma)^{-1}, \quad z \in V_{k, \xi_{n+1} + \sigma}.$$

Recall that $\mathcal{L}G_k(\cdot, \zeta) = -\delta_\zeta$, for any positive integer k . Here δ_ζ stands for the Dirac measure supported at ζ . Then,

$$\mathcal{L}u_\varepsilon(z) = \varepsilon|x_{1,n}|^2(x_{n+1} - \xi_{n+1} - \sigma)^{-2} \geq 0 \quad \text{in } V_{k, \xi_{n+1} + \sigma}.$$

Moreover,

$$\limsup_{z \rightarrow z_0} u_\varepsilon(z) \leq 0 \quad \text{for every } z_0 \in \partial V_{k, \xi_{n+1} + \sigma}.$$

The maximum principle then gives $u_\varepsilon \leq 0$ in $V_{k, \xi_{n+1} + \sigma}$. Letting ε and σ go to zero, and recalling that $G_k \geq 0$, we obtain (4.22).

We finally prove *(i)* and *(ii)*. By (4.20) we have

$$\Gamma(x, t) = \left(\frac{2}{t}\right)^{\frac{n+4}{2}} \Gamma\left(\sqrt{\frac{2}{t}}x_{1,n}, \frac{4x_{n+1}}{t^2}, 2\right), \quad (4.23)$$

then it is not restrictive to assume $t = 2$. The function $u(x, t) = \Gamma(x, t + 1)$ is a solution to $\mathcal{L}u = 0$ in $\mathbb{R}^n \times]-1, +\infty[\times \mathbb{R}$. The claim then directly follows from Corollary 4.6. \square

4.2 Lifting and Harnack inequalities for $k > 2$

We next consider the stochastic system (1.3) and (1.4) for $k > 2$. The Kolmogorov operators of (1.3) is

$$\mathcal{L} = \frac{1}{2} \Delta_{x_{1,n}} + |x_{1,n}|^k \partial_{x_{n+1}} - \partial_t, \quad \text{for any even positive integer } k,$$

while the Kolmogorov operators of (1.4) is,

$$\mathcal{L} = \frac{1}{2} \Delta_{x_{1,n}} + \sum_{j=1}^n x_j^k \partial_{x_{n+1}} - \partial_t, \quad \text{for any } k \in \mathbb{N}^*.$$

We next show that, in both cases, \mathcal{L} can be lifted to a suitable operator $\widetilde{\mathcal{L}}$ in the form (3.1) satisfying [H] and [L]. We introduce a new variable $y \in \mathbb{R}^{(k-1)n}$, that will be denoted as follows $y = (y_1, y_2, \dots, y_{(k-1)})$, with $y_j = (y_{j1}, \dots, y_{jn}) \in \mathbb{R}^n$ for $j \in \llbracket 1, k-1 \rrbracket$. We then define the lifted vector fields on \mathbb{R}^{kn+2} :

$$\widetilde{Y}_i = Y_i = \partial_{x_i}, \quad i \in \llbracket 1, n \rrbracket, \quad \widetilde{Z} = Z + \sum_{i=1}^{k-1} \sum_{j=1}^n x_j^i \partial_{y_{ij}}; \quad (4.24)$$

where $Z = |x_{1,n}|^k \partial_{x_{n+1}} - \partial_t$ for the system (1.3), and $Z = \sum_{j=1}^n x_j^k \partial_{x_{n+1}} - \partial_t$ for (1.4). If we denote $v(x, y, t) = u(x, t)$ for any $u \in C^\infty(\mathbb{R}^{n+2})$, we have

$$\widetilde{Y}_i v(x, y, t) = Y_i u(x, t), \quad \forall i \in \llbracket 1, n \rrbracket, \quad \widetilde{Z} v(x, y, t) = Z u(x, t).$$

Then, setting $\widetilde{\mathcal{L}} = \frac{1}{2} \sum_{i=1}^n \widetilde{Y}_i^2 + \widetilde{Z}$, we plainly find $\widetilde{\mathcal{L}} v(x, y, t) = \mathcal{L} u(x, t)$.

Since $\dim(\text{Lie}\{\widetilde{Y}_1, \dots, \widetilde{Y}_n, \widetilde{Z}\}) = kn + 2$ and $\text{rank}(\text{Lie}\{\widetilde{Y}_1, \dots, \widetilde{Y}_n, \widetilde{Z}\}(x, y, t)) = kn + 2$ at every point $(x, y, t) \in \mathbb{R}^{kn+2}$, a result by Bonfiglioli and Lanconelli in [5] yields the existence of a homogeneous Lie group $\mathbb{G} = (\mathbb{R}^{kn+2}, \circ, (\widetilde{\delta}_\lambda)_{\lambda>0})$ such that $\widetilde{\mathcal{L}}$ is Lie-invariant on \mathbb{G} . Therefore, the lifted operators $\widetilde{\mathcal{L}}$ satisfy [H] and [L]. The dilation $\widetilde{\delta}_\lambda$ acts as follows:

$$\widetilde{\delta}_\lambda(x, y, t) = \left(\lambda x_{1,n}, \lambda^{k+2} x_{n+1}, \lambda^3 y_1, \dots, \lambda^{k+1} y_{k-1}, \lambda^2 t \right), \quad (4.25)$$

for every $(x, y, t) \in \mathbb{R}^{kn+2}$, and $\lambda > 0$. We next aim to apply Theorem 3.2 in order to prove a Harnack inequality on the lifted space \mathbb{R}^{kn+2} . As in the previous section, for any $\omega \in L^2([0, T], \mathbb{R}^n)$ for every $(x, y, t) \in \mathbb{R}^{kn+2}$ and $T > 0$, we denote by $\widetilde{\gamma} : [0, T] \rightarrow \mathbb{R}^{kn+2}$ the solution of the Cauchy problem

$$\begin{cases} \widetilde{\gamma}'(s) = \sum_{j=1}^n \omega_j(s) \widetilde{Y}_j(\widetilde{\gamma}(s)) + \widetilde{Z}(\widetilde{\gamma}(s)), & s \in [0, T], \\ \widetilde{\gamma}(0) = (x, y, t). \end{cases} \quad (4.26)$$

In order to simplify the notation, in the sequel we will denote the solution of (4.26) as

$$\gamma(s) = (x_{1,n}(s), x_{n+1}(s), y(s), t(s)), \quad s \in [0, T]. \quad (4.27)$$

Note that $t(s) = t - s$ for every $s \in [0, T]$.

The composition law “ \circ ” of G is related to (4.26) as follows: if $(\bar{x}, \bar{y}, \bar{t}) = \widetilde{\gamma}(T)$ is the end point of the path $\widetilde{\gamma}$ defined by (4.26) with $\widetilde{\gamma}(0) = (0, 0, 0)$ and $(\tilde{x}, \tilde{y}, \tilde{t}) = \widetilde{\gamma}(T)$ is the end point of the path $\widetilde{\gamma}$ defined by (4.26) with $\widetilde{\gamma}(0) = (\xi, \eta, \tau)$, then

$$(\tilde{x}, \tilde{y}, \tilde{t}) = (\xi, \eta, \tau) \circ (\bar{x}, \bar{y}, \bar{t}), \quad (4.28)$$

(see for instance Corollary 1.2.24 in [7]).

We next consider the attainable set \mathcal{A}_{z_0} of the unit cylinder

$$\mathcal{O} = \left\{ (x, y, t) \in \mathbb{R}^{kn+2} \mid |x_{1,n}| < 1, -1 < x_{n+1} < 1, |y| < 1, -1 < t < 1 \right\}, \quad (4.29)$$

with respect to the point $z_0 = (0, 0, 0)$. Here $|x_{1,n}|$ and $|y|$ denote, respectively, the Euclidean norm of the vectors $x_{1,n} \in \mathbb{R}^n$ and $y \in \mathbb{R}^{(k-1)n}$. Unlike in the case $k = 2$, as $k > 2$ we are not able to give a complete characterization of the sets \mathcal{A}_{z_0} and \mathcal{O}_{z_0} as we did in Lemma 4.1. We will consider instead the differential of the *end point map* related to (4.26) to find some interior points of \mathcal{A}_{z_0} . With obvious meaning of the notations, we set $(x(T), y(T), t(T)) = \tilde{\gamma}(T)$, we note that $t(T) = t - T$, and we define

$$E : L^2([0, T]) \rightarrow \mathbb{R}^{kn+1}, \quad E(\omega) = (x(T), y(T)). \quad (4.30)$$

We refer to the classical literature (see *e.g.* [12, Theorem 3.2.6]) for the differentiability properties of E . We next show that the differential $DE(\omega)$ of E , computed at some given $\omega \in L^2([0, T])$ is surjective. Hence $E(\omega)$ is an interior point of \mathcal{A}_{z_0} , so that we can apply Theorem 3.2.

Lemma 4.8 *Let \bar{w} be any given vector of \mathbb{R}^n such that $\bar{w}_j \neq 0$ for every $j \in \llbracket 1, n \rrbracket$. Consider the solution $\tilde{\gamma}$ to the problem (4.26), with $\omega \equiv \bar{w}$. Then $DE(\omega)$ is surjective.*

Proof. By the invariance of the vector fields $\tilde{Y}_i, i \in \llbracket 1, n \rrbracket$, and \tilde{Z} with respect to the homogeneous Lie group \mathbb{G} , is not restrictive to assume $(x, y, t) = (0, 0, 0)$ and $T = 1$. To prove our claim, we compute

$$DE(\omega)\tilde{\omega} = \lim_{h \rightarrow 0} \frac{1}{h} (E(\omega + h\tilde{\omega}) - E(\omega)),$$

where

$$\begin{aligned} \tilde{\omega}(s) &= \frac{1}{b-a}v \quad \text{for } s \in [a, b], \quad a, b \in [0, 1], \quad a < b, \quad v \text{ is any vector of } \mathbb{R}^n, \\ \tilde{\omega}(s) &= 0 \quad \text{for } s \notin [a, b]. \end{aligned} \quad (4.31)$$

In the sequel, we denote by $\tilde{\gamma}^h(s) = (x^h(s), y^h(s), t^h(s))$ the solution of (4.26) relevant to $\omega + h\tilde{\omega}$. Clearly, $t^h(s) = -s$, and $x^h(1) = \bar{w} + hv$, so that

$$\lim_{h \rightarrow 0} \frac{1}{h} (x_{1,n}^h(1) - x_{1,n}(1)) = v. \quad (4.32)$$

We next show that, for every $j \in \llbracket 1, n \rrbracket$ and $i \in \llbracket 1, k-1 \rrbracket$, we have

$$\lim_{h \rightarrow 0} \frac{y_{ij}^h(1) - y_{ij}(1)}{h} = \left(\frac{i}{i+1} \frac{b^{i+1} - a^{i+1}}{b-a} - a \frac{b^i - a^i}{b-a} + 1 - b^i \right) \bar{w}_j^{i-1} v_j. \quad (4.33)$$

Indeed, we have

$$\begin{aligned} y_{ij}^h(1) &= \int_0^a (t\bar{w}_j)^i dt + \int_a^b \left(t\bar{w}_j + h \frac{t-a}{b-a} v_j \right)^i dt + \int_b^1 (t\bar{w}_j + hv_j)^i dt \\ &= \int_0^a (t\bar{w}_j)^i dt + \int_a^b (t\bar{w}_j)^i dt + \int_b^1 (t\bar{w}_j)^i dt \\ &\quad + ih\bar{w}_j^{i-1} v_j \left(\int_a^b t^{i-1} \frac{t-a}{b-a} dt + \int_b^1 t^{i-1} dt \right) + o(h), \quad \text{as } h \rightarrow 0, \\ &= y_{ij}(1) + \left(\frac{i}{i+1} \frac{b^{i+1} - a^{i+1}}{b-a} - a \frac{b^i - a^i}{b-a} + 1 - b^i \right) \bar{w}_j^{i-1} v_j h + o(h), \quad \text{as } h \rightarrow 0, \end{aligned}$$

where $o(h)$ vanishes as h goes to zero. This proves (4.33). Analogously,

$$\lim_{h \rightarrow 0} \frac{x_{n+1}^h(1) - x_{n+1}(1)}{h} = \left(\frac{k}{k+1} \frac{b^{k+1} - a^{k+1}}{b-a} - a \frac{b^k - a^k}{b-a} + 1 - b^k \right) |\bar{w}|^{k-2} \langle \bar{w}, v \rangle, \quad (4.34)$$

when considering system (1.3), and

$$\lim_{h \rightarrow 0} \frac{x_{n+1}^h(1) - x_{n+1}(1)}{h} = \left(\frac{k}{k+1} \frac{b^{k+1} - a^{k+1}}{b-a} - a \frac{b^k - a^k}{b-a} + 1 - b^k \right) \sum_{j=1}^n \bar{w}_j^{k-1} v_j, \quad (4.35)$$

in the case of (1.4). Note that for all $i \in \llbracket 1, k \rrbracket$ one has

$$\frac{i}{i+1} \frac{b^{i+1} - a^{i+1}}{b-a} - a \frac{b^i - a^i}{b-a} = O(b-a), \quad \text{as } b-a \rightarrow 0, \quad (4.36)$$

for any $i \in \llbracket 1, k \rrbracket$. Then, from (4.32), (4.33), (4.34), in the case (1.3), it follows that

$$\begin{aligned} DE(\omega)\tilde{\omega} = & \left(v, (1-b^k)|\bar{w}|^{k-2} \langle \bar{w}, v \rangle, (1-b)v, (1-b^2)\bar{w}_1 v_1, \dots, (1-b^2)\bar{w}_n v_n, \right. \\ & \left. \dots, (1-b^{k-1})\bar{w}_1^{k-2} v_1, \dots, (1-b^{k-1})\bar{w}_n^{k-2} v_n \right) + O(b-a), \end{aligned} \quad (4.37)$$

as $b-a \rightarrow 0$. We next choose $b_0, \dots, b_k \in]0, 1]$ such that $b_i \neq b_m$ if $i \neq m$ and we let v be any unit vector e_j of the canonical basis of \mathbb{R}^n . Then the j -th, the $n+j+1-th \dots$, the $(k-1)n+j+1-th$ components of $DE(\omega)\tilde{\omega}$ are

$$\left(1, 1-b_i, (1-b_i^2)w_j, \dots, (1-b_i^{k-1})\bar{w}_j^{k-2} \right),$$

while the $n+1-th$ component is $(1-b_i^k)|\bar{w}|^{k-2}\bar{w}_j$. By our assumption, $\bar{w}_j \neq 0$, and the following $(k+1) \times (k+1)$ matrix

$$M(b_0, b_1, \dots, b_k) = \begin{pmatrix} 1 & 1-b_0 & 1-b_0^2 & \dots & 1-b_0^k \\ 1 & 1-b_1 & 1-b_1^2 & \dots & 1-b_1^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1-b_k & 1-b_k^2 & \dots & 1-b_k^k \end{pmatrix}.$$

is non singular, since

$$\det M(b_0, b_1, \dots, b_k) = (-1)^k \prod_{i \neq m} (b_i - b_m) \neq 0,$$

because of our choice of the b_i 's. Thus, if we choose $v = e_j$ and each a_i sufficiently close to b_i , then (4.37) restores $k+1$ linearly independent vectors. In conclusion, it is possible to find $v_1, \dots, v_n, b_0, \dots, b_k, a_0, \dots, a_k$, such that the vectors $DE(\omega)\tilde{\omega}$ defined by using v_j, a_i, b_i in (4.31), span \mathbb{R}^{kn+1} . This proves our claim for system (1.3). The proof in the case (1.4) is analogous, we only need to replace (4.34) by (4.35). We omit the details. \square

We next obtain, as a corollary, a Harnack inequality which is invariant with respect to the Lie group $\mathbb{G} = (\mathbb{R}^{kn+2}, \circ, (\tilde{\delta}_\lambda)_{\lambda>0})$. For every compact subset K of the unit cylinder \mathcal{O} defined in (4.29), any positive r and any $z_0 = (x_0, y_0, t_0) \in \mathbb{R}^{kn+2}$ we set

$$\mathcal{O}_r(z_0) = z_0 \circ \tilde{\delta}_r \mathcal{O} = \{z_0 \circ \tilde{\delta}_r \zeta \mid \zeta \in \mathcal{O}\}, \quad K_r(z_0) = z_0 \circ \tilde{\delta}_r K. \quad (4.38)$$

We also introduce the point $\bar{z} = (\bar{x}, \bar{y}, \bar{t})$ such that:

$$\begin{aligned} \bar{t} &= -\frac{1}{2}, \quad \bar{x}_{1,n} = 0, \\ \bar{x}_{n+1} &= \frac{4n^{k/2}}{k+1} 8^{-(k+1)} \text{ for (1.3),} \\ \bar{x}_{n+1} &= \frac{4n}{k+1} 8^{-(k+1)} \text{ for (1.4) and } k \text{ even,} \\ \bar{x}_{n+1} &= 0 \text{ for (1.4) and } k \text{ odd,} \\ \bar{y}_{2i-1,j} &= 0, \quad \bar{y}_{2i,j} = \frac{4}{2i+1} 8^{-(2i+1)} \quad 2i \in \llbracket 2, k-1 \rrbracket, j \in \llbracket 1, n \rrbracket. \end{aligned} \quad (4.39)$$

Proposition 4.9 *Let $\tilde{\mathcal{L}}$ be the lifted operator of the Kolmogorov operator \mathcal{L} of the system (1.3) or (1.4). Then there exists a compact neighborhood K of the point $\bar{z} = (\bar{x}, \bar{y}, \bar{t})$ in (4.39), and a positive constant C_K , only depending on \mathcal{O}, K and on $\tilde{\mathcal{L}}$, such that*

$$\sup_{K_r(z_0)} u \leq C_K u(z_0),$$

for every positive solution \tilde{u} of $\tilde{L}\tilde{u} = 0$ in $\mathcal{O}_r(z_0)$.

Proof. By the invariance of $\tilde{\mathcal{L}}$ with respect to the homogeneous Lie group \mathbb{G} , it is not restrictive to assume $(x_0, y_0, t_0) = (0, 0, 0)$ and $r = 1$. For $j \in \llbracket 1, n \rrbracket$, set $\omega_j(s) = 1$ for $s \in [0, \frac{1}{8}] \cup [\frac{3}{8}, \frac{1}{2}]$, $\omega_j(s) = -1$ for $s \in]\frac{1}{8}, \frac{3}{8}[$, and consider the path $\tilde{\gamma} : [0, \frac{1}{2}] \rightarrow \mathbb{R}^{kn+2}$ defined by (4.26), starting from $(0, 0, 0)$. A direct computation shows that $\tilde{\gamma}(s) \in \mathcal{O}$ for every $s \in [0, \frac{1}{2}]$, and $\tilde{\gamma}(\frac{1}{2})$ equals $(\bar{x}, \bar{y}, \bar{t})$ in (4.39), then $(\bar{x}, \bar{y}, \bar{t}) \in \mathcal{A}_{(0,0,0)}$. We next show that $(\bar{x}, \bar{y}, \bar{t})$ is an interior point of $\mathcal{A}_{(0,0,0)}$. Indeed, by Lemma 4.8 there exists a neighborhood $V \subset \mathbb{R}^{kn+1}$ of (\bar{x}, \bar{y}) such that $(x, y, \bar{t}) \in \mathcal{A}_{(0,0,0)}$ for any $(x, y) \in V$. Using again the invariance of the vector fields $\tilde{Y}_i, i \in \llbracket 1, n \rrbracket$, and \tilde{Z} with respect to the dilations of the Lie group \mathbb{G} , and the continuity of $\tilde{\delta}_\lambda$, we also have that $\tilde{\delta}_\lambda(x, y, \bar{t}) \in \mathcal{A}_{(0,0,0)}$ for any $(x, y) \in V$ and $\lambda \in]0, \lambda_0[$, for some $\lambda_0 \in]1, 2[$. This proves that there exists a compact neighborhood K of $(\bar{x}, \bar{y}, \bar{t})$ contained in the interior of $\mathcal{A}_{(0,0,0)}$. The conclusion then follows from Theorem 3.2. \square

4.3 Lower bounds for System (1.3)

We first consider system (1.3) with $k > 2$. We recall the relevant Kolmogorov equation (3.8)

$$\mathcal{L} = \frac{1}{2} \Delta_{x_{1,n}} + |x_{1,n}|^k \partial_{x_{n+1}} - \partial_t.$$

Our first result extends Proposition 4.5.

Proposition 4.10 *Let \mathcal{L} be the operator defined in (3.8) and let k be a positive even integer. Let T_1, τ, t, T_2 be such that $T_1 < \tau < t < T_2$ and $t - \tau \leq \tau - T_1$, and assume that there exists a path $\gamma : [0, t - \tau] \rightarrow \mathbb{R}^{n+1} \times]T_1, T_2[$ satisfying*

$$\gamma'(s) = \sum_{j=1}^n \omega_j Y_j(\gamma(s)) + Z(\gamma(s)), \quad \gamma(0) = (x, t), \quad \gamma(t - \tau) = (\xi, \tau), \quad (4.40)$$

for some constant vector $\omega = (\omega_1, \dots, \omega_n)$ such that $|\omega|^2 \geq \frac{2}{t - \tau}$. Then

$$u(\xi, \tau) \leq \exp(C((t - \tau)|\omega|^2 + 1)) u(x, t),$$

for every non-negative solution u to $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1} \times]T_1, T_2[$. Moreover

$$u\left(0, \dots, 0, x_{n+1} + \tilde{\xi}_{n+1}, \tau\right) \leq \exp\left(C\left(\frac{(t - \tau)^{1+2/k}}{\tilde{\xi}_{n+1}^{2/k}} + 1\right)\right) u(0, \dots, 0, x_{n+1}, t).$$

for every $(x, t) \in \mathbb{R}^{n+1} \times]T_1, T_2[, \tau \in]T_1, t[$ with $\tilde{\xi}_{n+1} \in \left]0, \frac{(t - \tau)^{1+k/2}}{(k+1)4^{k+1}}\right]$, for every non-negative solution u to $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1} \times]T_1, T_2[$. In the above inequalities, C is a positive constant only depending on \mathcal{L} .

Proof. We first prove a local Harnack inequality in \mathbb{R}^{kn+2} , then we construct a Harnack chain in \mathbb{R}^{kn+2} and we conclude the proof by going back to \mathbb{R}^{n+2} . We preliminarily note that Theorem 3.2 requires $\omega_j \neq 0$ for every $j \in \llbracket 1, n \rrbracket$. On the other hand, the Kolmogorov equation (3.8) is invariant with respect to the rotation of the variable $x_{1,n}$, then it is not restrictive to assume $\omega = \frac{|\omega|}{\sqrt{n}}(1, \dots, 1)$. After this change of variable, we add the variable $y \in \mathbb{R}^{(k-1)n}$ and we lift the vector fields Y_1, \dots, Y_n and Z according with equation (4.24).

For any positive c we denote $w_c = \tilde{\gamma}\left(\frac{1}{2}\right)$, where $\tilde{\gamma} : \left[0, \frac{1}{2}\right] \rightarrow \mathbb{R}^{kn+2}$ is the path starting at $(0, 0, 0)$, and defined by (4.26) with $v = \frac{c}{\sqrt{n}}(1, \dots, 1)$, then we set

$$K = \left\{w_c \mid \frac{1}{4} \leq c \leq \frac{1}{2}\right\}. \quad (4.41)$$

With the same notation used in Proposition 4.9, we claim that there exists a positive constant C_K such that

$$\sup_{K_r(z_0)} \tilde{u} \leq C_K \tilde{u}(z_0), \quad (4.42)$$

for every positive solution \tilde{u} to $\tilde{\mathcal{L}}\tilde{u} = 0$ in $\mathcal{O}_r(z_0)$. By the invariance with respect to the homogeneous Lie group on \mathbb{R}^{kn+2} , it is sufficient to prove (4.42) for $z_0 = (0, 0, 0)$ and $r = 1$. A direct computation shows that $\tilde{\gamma}(s) \in \mathcal{O}$ for every $s \in \left[0, \frac{1}{2}\right]$, and $\tilde{\gamma}\left(\frac{1}{2}\right)$ is an interior point of $\mathcal{A}_{(0,0,0)}$ for every $c \in \left[\frac{1}{2}, 1\right]$, because of Lemma 4.8. Moreover, K is a compact subset of \mathcal{O} , by the continuity of the end point map $v \mapsto \tilde{\gamma}(s)\left(\frac{1}{2}\right)$. The bound (4.42) then follows from Theorem 3.2, by a plain compactness argument.

The conclusion of the proof of the first statement follows from (4.42) and by arguments similar to those used in the proof of Lemma 4.3 and Proposition 4.5 in the case $k = 2$.

Let $u : \mathbb{R}^{n+1} \times]T_1, T_2[\rightarrow \mathbb{R}$ be a non-negative solution to $\mathcal{L}u = 0$, and $\gamma : [0, t - \tau] \rightarrow \mathbb{R}^{n+1} \times]T_1, T_2[$ be as prescribed in (4.40), with $\omega = \frac{|\omega|}{\sqrt{n}}(1, \dots, 1)$. Consider the lifted operator $\widetilde{\mathcal{L}}$ and define the function $\tilde{u} : \mathbb{R}^{kn+1} \times]T_1, T_2[\rightarrow \mathbb{R}$ by setting $\tilde{u}(x, y, t) = u(x, t)$ for every $(x, y, t) \in \mathbb{R}^{kn+2} \times]T_1, T_2[$. Then \tilde{u} is a non-negative solution to $\widetilde{\mathcal{L}}\tilde{u} = 0$. Next, we denote by $\tilde{\gamma} : [0, t - \tau] \rightarrow \mathbb{R}^{kn+1} \times]T_1, T_2[$ the solution of the Cauchy problem

$$\begin{cases} \tilde{\gamma}'(s) = \sum_{j=1}^n \omega_j \tilde{Y}_j(\tilde{\gamma}(s)) + \tilde{Z}(\tilde{\gamma}(s)), & s \in [0, t - \tau], \\ \tilde{\gamma}(0) = (x, 0, t), \end{cases}$$

where ω is the constant vector in (4.40). We next apply the Harnack inequality in (4.42) to a suitable set of points z_1, \dots, z_m lying on $\tilde{\gamma}([0, t - \tau])$. Let m be the positive integer such that $m - 1 < 2(t - \tau)|\omega|^2 \leq m$, set $\tilde{s} = \frac{t - \tau}{m}$, $\tilde{r} = \sqrt{2\tilde{s}}$ and define $z_j = \tilde{\gamma}(j\tilde{s})$ for $j \in \llbracket 1, m \rrbracket$. A direct computation shows

$$z_j = z_{j-1} \circ \delta_{\tilde{r}} w_c, \quad c = \sqrt{\frac{\tilde{s}}{2}} |\omega|, \quad j \in \llbracket 1, m \rrbracket.$$

Note that from our assumption $(t - \tau)|\omega|^2 \geq 2$ it follows that $m \geq 4$, then $\frac{3}{16} \leq \tilde{s}|\omega|^2 \leq \frac{1}{4}$. As a consequence w_c belongs to the set K defined in (4.42), thus

$$z_j \in K_{\tilde{r}}(z_{j-1}), \quad j \in \llbracket 1, m \rrbracket.$$

Moreover we have $0 < \tilde{r} < \frac{1}{|\omega|}$, then from our assumption $|\omega|^2 \geq \frac{1}{\tau - T_1}$ it follows that $K_{\tilde{r}}(z_j) \subset \mathcal{O}$ for $j \in \llbracket 0, m \rrbracket$. Then, we can apply (4.42), which yields $\tilde{u}(z_j) \leq C_K \tilde{u}(z_{j-1})$ for every $j \in \llbracket 1, m \rrbracket$. In particular, being $m < 2(t - \tau)|\omega|^2 + 1$, $z_0 = z$, $z_m = \tilde{\gamma}(t - \tau)$, we find

$$\tilde{u}(\tilde{\gamma}(t - \tau)) \leq C_K^{2(t - \tau)|\omega|^2 + 1} \tilde{u}(x, y, t).$$

By choosing $C := \frac{1}{2} \log(C_K)$, we finally find

$$\begin{aligned} u(\xi, \tau) &= \tilde{u}(\xi, \eta, \tau) \leq \exp\left(C\left((t - \tau)|\omega|^2 + \frac{1}{2}\right)\right) \tilde{u}(x, 0, t) \\ &\leq \exp\left(C\left((t - \tau)|\omega|^2 + 1\right)\right) u(x, t), \end{aligned}$$

where $\eta \in \mathbb{R}^{(k-1)n}$ is such that $\tilde{\gamma}(t - \tau) = (\xi, \eta, \tau)$. This accomplishes the proof of the first statement.

To prove the second assertion, we argue as in Lemma 4.4. We first prove a Harnack inequality analogous to (4.42). For any positive c we denote $w_c = \tilde{\gamma}(\frac{1}{2})$, where $\tilde{\gamma} : [0, \frac{1}{2}] \rightarrow \mathbb{R}^{kn+2}$ is the path starting at $(0, 0, 0)$, and defined by (4.26) with $\omega(s) = \frac{c}{\sqrt{n}}(1, \dots, 1)$ for $s \in [0, \frac{1}{4}]$, $\omega(s) = -\frac{c}{\sqrt{n}}(1, \dots, 1)$ for $s \in]\frac{1}{4}, \frac{1}{2}[$. It is easy to check that the x and t components of w_c are

$$\left(x_{1,n}\left(\frac{1}{2}\right), x_{n+1}\left(\frac{1}{2}\right), t\left(\frac{1}{2}\right)\right) = \left(0, c^k a, -\frac{1}{2}\right), \quad \text{where } a = \frac{2}{k+1} 4^{-(k+1)}.$$

Then we set

$$K = \left\{ w_c \mid \frac{1}{\sqrt{2}} \leq c \leq 1 \right\}. \quad (4.43)$$

and we note that the $n + 1$ -th component of $w_c \in K$ belongs to $\left[\frac{a}{2^{k/2}}, a\right]$. By the same argument used in the proof of (4.42) it follows that there exists a positive constant C_K such that

$$\sup_{K_r(z_0)} \tilde{u} \leq C_K \tilde{u}(z_0), \quad (4.44)$$

for every positive solution \tilde{u} to $\mathcal{L}\tilde{u} = 0$ in $\mathcal{O}_r(z_0)$.

Let m be the unique positive integer such that

$$2(2a)^{2/k} \frac{(t - \tau)^{1+2/k}}{\tilde{\xi}_{n+1}^{2/k}} - 1 < m \leq 2(2a)^{2/k} \frac{(t - \tau)^{1+2/k}}{\tilde{\xi}_{n+1}^{2/k}}.$$

Next, we set

$$c = \sqrt{\frac{m \tilde{\xi}_{n+1}^{2/k}}{2(2a)^{2/k} (t - \tau)^{1+2/k}}}, \quad \tilde{s} = \frac{t - \tau}{2m}, \quad \text{and} \quad r = 2\sqrt{\tilde{s}}.$$

Note that our assumption $\tilde{\xi}_{n+1}^2 \leq 4a^2(t - \tau)^{k+2}$ implies $m \geq 1$ and $\frac{1}{\sqrt{2}} \leq c \leq 1$. We finally consider the path $\tilde{\gamma} : [0, t - \tau] \rightarrow \mathbb{R}^{kn+2}$ defined by (4.26) with $\tilde{\gamma}(0) = (x, y, t)$ and $\omega(s) = \frac{c}{r\sqrt{n}}(1, \dots, 1)$ for $0 \leq s < \tilde{s}$, $\omega(s) = -\frac{c}{r\sqrt{n}}(1, \dots, 1)$ for $\tilde{s} \leq s < 2\tilde{s}$, and recursively $\omega(s) = \omega(s + 2\tilde{s})$ as $2\tilde{s} \leq s < t - \tau$. We finally set $z_0 = (x, y, t)$, $z_j = \tilde{\gamma}(j\tilde{s})$ for $j \in \llbracket 1, m \rrbracket$. We have $\mathcal{O}_r(z_j) \subset \mathbb{R}^{kn+1} \times]T_1, T_2[$ for every $j \in \llbracket 1, m - 1 \rrbracket$, and the x and t components of z_m are $(0, \dots, 0, x_{n+1} + \tilde{\xi}_{n+1})$ and τ , respectively.

According with (4.28) and (4.25), we see that $z_{j+1} \in K_{\tilde{r}}(z_j)$, and $\overline{Q_{\tilde{r}}(z_j)} \subset \mathbb{R}^{kn+1} \times]T_1, T_2[$, for any $j \in \llbracket 0, m - 1 \rrbracket$, with K defined in (4.43). Then, by (4.44), we get $v(\tilde{\gamma}(\tilde{\tau})) \leq C_K^m v(z)$ for every non-negative solution \tilde{u} to $\mathcal{L}\tilde{u} = 0$ in $\mathbb{R}^{kn+1} \times]T_1, T_2[$. Thus, from the definition of $\tilde{\gamma}$ and m it then follows that

$$\tilde{u}(\tilde{\gamma}(t - \tau)) \leq \exp\left(C\left(\frac{(t - \tau)^{1+2/k}}{\tilde{\xi}_{n+1}^{2/k}} + 1\right)\right) \tilde{u}(x, y, t),$$

where $C = 2(2a)^{2/k} \log(C_K)$.

Finally, if u is a non-negative solution of $Lu = 0$ in $\mathbb{R}^{n+1} \times]T_1, T_2[$, we apply the above inequality to the function $\tilde{u}(x, y, t) = u(x, t)$ and we get our second claim. \square

Corollary 4.11 *Let \mathcal{L} be the operator defined in (3.8) and let k be a positive even integer. Let $u : \mathbb{R}^{n+1} \times]T_1, T_2[\rightarrow \mathbb{R}$ be a non-negative solution to $\mathcal{L}u = 0$, and let $t, \tau \in \mathbb{R}$ be such that $T_1 < \tau < t < T_2$, and $t - \tau \leq 2(\tau - T_1)$. Then there exists a positive constant C_1 , only depending on \mathcal{L} , such that*

i) *for any $x, \xi \in \mathbb{R}^{n+1}$ such that $\xi_{n+1} - x_{n+1} \geq 2^k(t - \tau)(|x_{1,n}|^k + |\xi_{1,n}|^k) + 2^k(t - \tau)^{1+k/2}$ we have*

$$u(\xi, \tau) \leq \exp\left(C_1\left(\frac{|x_{1,n} - \xi_{1,n}|^2}{t - \tau} + \frac{(\xi_{n+1} - x_{n+1} - \frac{2^{k-1}}{k+1}(|x_{1,n}|^k + |\xi_{1,n}|^k)t)^{2/k}}{(t - \tau)^{1+2/k}} + 1\right)\right) u(x, t);$$

ii) for any $x, \xi \in \mathbb{R}^{n+1}$ such that $0 < \xi_{n+1} - x_{n+1} \leq \frac{(t-\tau)^{1+k/2}}{4^{k+1}(k+1)}$ we have

$$u(\xi, \tau) \leq \exp \left(C_1 \left(\frac{|x_{1,n}|^{k+2} + |\xi_{1,n}|^{k+2}}{\xi_{n+1} - x_{n+1}} + \frac{(t-\tau)^{1+2/k}}{(\xi_{n+1} - x_{n+1})^{2/k}} + 1 \right) \right) u(x, t).$$

Proof. We follow the same argument used in the proof of Corollary 4.6. We first prove (i), assuming that $x, \xi \in \mathbb{R}^n$ are such that

$$|x_{1,n} - \xi_{1,n}|^2 \geq t - \tau. \quad (4.45)$$

We set $\omega = \frac{2}{t-\tau}(\xi_{1,n} - x_{1,n})$, and we apply Proposition 4.10. According with notation (4.27), we find

$$\begin{aligned} u(\gamma(\frac{t-\tau}{2})) &\leq \exp \left(C \left(2 \frac{|x_{1,n} - \xi_{1,n}|^2}{t-\tau} + 1 \right) \right) u(x, t), \\ \gamma(\frac{t-\tau}{2}) &= \left(\xi_{1,n}, x_{n+1} + \int_0^{\frac{t-\tau}{2}} \left| x_{1,n} + \frac{2s}{t-\tau}(\xi_{1,n} - x_{1,n}) \right|^k ds, \frac{t+\tau}{2} \right). \end{aligned} \quad (4.46)$$

Note that, by the convexity of the norm, we have

$$x_{n+1}(\frac{t-\tau}{2}) - x_{n+1} = \int_0^{\frac{t-\tau}{2}} \left| x_{1,n} + \frac{2s}{t-\tau}(\xi_{1,n} - x_{1,n}) \right|^k ds \leq \frac{t-\tau}{2} (|x_{1,n}|^k + |\xi_{1,n}|^k),$$

then, by our assumption $\xi_{n+1} - x_{n+1} \geq 2^k(t-\tau)(|x_{1,n}|^k + |\xi_{1,n}|^k) + 2^k(t-\tau)^{1+k/2}$, we find

$$\xi_{n+1} - x_{n+1}(\frac{t-\tau}{2}) \geq 2^k \frac{t-\tau}{2} (|x_{1,n}|^k + |\xi_{1,n}|^k) + 2^k(t-\tau)^{1+k/2}. \quad (4.47)$$

We next choose any vector $\bar{\omega} \in \mathbb{R}^n$ such that $|\bar{\omega}| = 1$ and $\bar{\omega} \perp \xi_{1,n}$, and a real parameter $m \geq \sqrt{\frac{8}{t-\tau}}$, that will be specified later. We consider the path $\gamma_1 : [\frac{t-\tau}{2}, \frac{3}{4}(t-\tau)] \rightarrow \mathbb{R}^{n+1} \times]T_1, T_2[$, starting from $\gamma(\frac{t-\tau}{2})$ and defined as in (4.40) with $\omega = m\bar{\omega}$, and the path $\gamma_2 : [\frac{3}{4}(t-\tau), t-\tau] \rightarrow \mathbb{R}^{n+1} \times]T_1, T_2[$, starting from $\gamma_1(\frac{3}{4}(t-\tau))$ and defined as in (4.40) with $\omega = -m\bar{\omega}$. From Proposition 4.10 it then follows

$$u(\xi_{1,n}, \varphi(m), \tau) \leq \exp \left(2C \left(\frac{(t-\tau)m^2}{4} + 1 \right) \right) u(\gamma(\frac{t-\tau}{2})), \quad (4.48)$$

where

$$\varphi(m) = x_{n+1}(\frac{t-\tau}{2}) + 2 \int_0^{\frac{t-\tau}{4}} |\xi_{1,n} + sm\bar{\omega}|^k ds$$

is an increasing continuous function of $m \in [\sqrt{\frac{8}{t-\tau}}, +\infty[$. An elementary computation shows that

$$2 \int_0^{\frac{t-\tau}{4}} |\xi_{1,n} + sm\bar{\omega}|^k ds \leq 2^k \frac{t-\tau}{2} |\xi_{1,n}|^k + \frac{m^k}{k+1} \frac{(t-\tau)^{k+1}}{2^{k+1}},$$

then

$$\varphi \left(\sqrt{\frac{8}{t-\tau}} \right) \leq x_{n+1} \left(\frac{t-\tau}{2} \right) + 2^k \frac{t-\tau}{2} |\xi_{1,n}|^k + \frac{2^{k/2}}{k+1} \frac{(t-\tau)^{1+k/2}}{2} < \xi_{n+1},$$

by (4.47). Moreover, since $\bar{\omega} \perp \xi_{1,n}$, by the same argument we also find

$$\varphi(m) \geq x_{n+1} \left(\frac{t-\tau}{2} \right) + \frac{m^k}{k+1} \frac{(t-\tau)^{k+1}}{2^{k+1}},$$

so that there exists $c := c(k)$ s.t.

$$\varphi \left(2^{1+1/k} (k+1)^{1/k} \frac{(\xi_{n+1} - x_{n+1} - c(|x_{1,n}|^k + |\xi_{1,n}|^k)(t-\tau))^{1/k}}{(t-\tau)^{1+1/k}} \right) \geq \xi_{n+1}.$$

Hence, there exists a unique $m \in \left[\sqrt{\frac{8}{t-\tau}}, 2^{1+1/k} (k+1)^{1/k} \frac{(\xi_{n+1} - x_{n+1} - c(|x_{1,n}|^k + |\xi_{1,n}|^k)(t-\tau))^{1/k}}{(t-\tau)^{1+1/k}} \right]$ such that $\varphi(m) = \xi_{n+1}$. For such a m equation (4.48) yields

$$\begin{aligned} u(\xi, \tau) &\leq \exp \left(4^{1+1/k} (k+1)^{2/k} C \left(\frac{(\xi_{n+1} - x_{n+1} - c(|x_{1,n}|^k + |\xi_{1,n}|^k)(t-\tau))^{2/k}}{(t-\tau)^{1+2/k}} + 1 \right) \right) \\ &\quad \times u \left(\gamma \left(\frac{t-\tau}{2} \right) \right). \end{aligned}$$

The above inequality, combined with (4.46), proves the claim (i) if $|x_{1,n} - \xi_{1,n}|^2 \geq t - \tau$.

As in the proof of Corollary 4.6, to prove (i) when $\xi_{1,n} = x_{1,n}$ it is sufficient to skip the first step of the previous argument.

Finally, if $0 < |x_{1,n} - \xi_{1,n}|^2 < t - \tau$, we still proceed as in the proof of Corollary 4.6. We choose a point $\tilde{x}_{1,n}$ such that $|x_{1,n} - \tilde{x}_{1,n}|^2 \geq \frac{t-\tau}{2}$ and $|\tilde{x}_{1,n} - \xi_{1,n}|^2 \geq \frac{t-\tau}{2}$, and we apply twice Proposition 4.10, then we follow the second step of the above argument. We omit the other details.

As in the proof of Corollary 4.6, we prove (ii) by applying the second inequality of Proposition 4.10 in a suitable interval $[\tau + t_2, t - t_1] \subsetneq [t, \tau]$, and the first inequality of Proposition 4.10 in the remaining intervals $[t - t_1, t]$ and $[\tau, \tau + t_2]$. We first suppose that $x_{1,n} \neq 0$, $\xi_{1,n} \neq 0$, we set

$$\begin{aligned} t_1 &= \min \left\{ \frac{|x_{1,n}|^2}{2}, \frac{\xi_{n+1} - x_{n+1}}{|x_{1,n}|^k}, \frac{t-\tau}{3} \right\}, & \omega_1 &= -\frac{1}{t_1} x_{1,n}, \\ t_2 &= \min \left\{ \frac{|\xi_{1,n}|^2}{2}, \frac{\xi_{n+1} - x_{n+1}}{|\xi_{1,n}|^k}, \frac{t-\tau}{3} \right\}, & \omega_2 &= \frac{1}{t_2} \xi_{1,n}, \end{aligned}$$

and we consider the paths

$$\begin{aligned} \gamma_1(s) &= \left(\left(1 - \frac{s}{t_1} \right) x_{1,n}, x_{n+1} + \frac{(s-t_1)^{k+1} + t_1^{k+1}}{(k+1)t_1^k} |x_{1,n}|^k, t-s \right), & s &\in [0, t_1] \\ \gamma_2(s) &= \left(\frac{s}{t_2} \xi_{1,n}, \xi_{n+1} + \frac{s^{k+1} - t_2^{k+1}}{(k+1)t_2^k} |\xi_{1,n}|^k, \tau + t_2 - s \right), & s &\in [0, t_2]. \end{aligned}$$

Note that $|\omega_1|^2 \geq \frac{2}{t_1}$, and $|\omega_2|^2 \geq \frac{2}{t_2}$, then Proposition 4.10 yields

$$\begin{aligned} u(\gamma_1(t_1)) &\leq \exp\left(C\left(\max\left\{\frac{|x_{1,n}|^{k+2}}{\xi_{n+1}-x_{n+1}}, 3\frac{|x_{1,n}|^2}{t-\tau}, 2\right\}+1\right)\right)u(x,t), \\ u(\xi,\tau) &\leq \exp\left(C\left(\max\left\{\frac{|\xi_{1,n}|^{k+2}}{\xi_{n+1}-x_{n+1}}, 3\frac{|\xi_{1,n}|^2}{t-\tau}, 2\right\}+1\right)\right)u(\gamma_2(0)), \end{aligned} \quad (4.49)$$

with

$$\gamma_1(t_1) = \left(0, x_{n+1} + \frac{t_1}{k+1}|x_{1,n}|^k, t-t_1\right), \quad \gamma_2(0) = \left(0, \xi_{n+1} - \frac{t_2}{k+1}|\xi_{1,n}|^k, \tau+t_2\right).$$

We set $\tilde{\xi}_{n+1} = \xi_{n+1} - x_{n+1} - \frac{t_1}{k+1}|x_{1,n}|^k - \frac{t_2}{k+1}|\xi_{1,n}|^k$, and we recall that $\xi_{n+1} - x_{n+1} \leq \frac{t-\tau}{2(k+1)}(|x_{1,n}|^k + |\xi_{1,n}|^k) + \frac{(t-\tau)^{1+k/2}}{4^{k+1}(k+1)}$ and $|x_{1,n}| \vee |\xi_{1,n}| \leq (t-\tau)^{1/2}$. Thus

$$\frac{t-\tau}{3} < (t-t_1) - (\tau+t_2) < t-\tau, \quad \frac{k-1}{k+1}(\xi_{n+1}-x_{n+1}) \leq \tilde{\xi}_{n+1} \leq \frac{(t-\tau)^{1+k/2}}{k+1} \frac{1}{4^{k+1}}. \quad (4.50)$$

Then, by the second statement of Proposition 4.5, we find

$$\begin{aligned} u(\gamma_2(0)) &\leq \exp\left(C\left(\frac{(t-t_1-t_2-\tau)^{1+2/k}}{(\tilde{\xi}_{n+1})^{2/k}}+1\right)\right)u(\gamma(t_1)) \\ &\leq \exp\left(C\left(\left(\frac{k+1}{k-1}\right)^{2/k} \frac{(t-\tau)^{1+2/k}}{(\xi_{n+1}-x_{n+1})^{2/k}}+1\right)\right)u(\gamma(t_1)). \end{aligned} \quad (4.51)$$

From inequalities (4.49) and (4.51) it follows that

$$u(\xi,\tau) \leq \exp\left(C'\left(\frac{|x_{1,n}|^{k+2} + |\xi_{1,n}|^{k+2}}{\xi_{n+1}-x_{n+1}} + \frac{(t-\tau)^{1+2/k}}{(\xi_{n+1}-x_{n+1})^{2/k}} + \frac{|x_{1,n}|^2 + |\xi_{1,n}|^2}{t-\tau} + 1\right)\right)u(x,t),$$

for some positive constant C' only depending on C and on k . Note that the last term in the above expression is bounded by the first one. Indeed, the inequality

$$\frac{|x_{1,n}|^2 + |\xi_{1,n}|^2}{t-\tau} \leq \frac{2}{k+2} \frac{|x_{1,n}|^{k+2} + |\xi_{1,n}|^{k+2}}{(t-\tau)^{1+k/2}} + \frac{k}{k+2}$$

combined with (4.50), gives

$$\frac{|x_{1,n}|^2 + |\xi_{1,n}|^2}{t-\tau} \leq \frac{2}{4^{k+1}(k+2)(k-1)} \frac{|x_{1,n}|^{k+2} + |\xi_{1,n}|^{k+2}}{\xi_{n+1}-x_{n+1}} + \frac{k}{k+2}.$$

This concludes the proof of (ii) when $x_{1,n} \neq 0$, and $\xi_{1,n} \neq 0$.

If $x_{1,n} = 0$, we simply omit the construction of γ_1 , and we rely on γ_2 and on the application of the second statement of Proposition 4.10 in the interval $[\tau+t_2, t]$. Analogously, if $\xi_{1,n} = 0$, we avoid the construction of γ_2 . This concludes the proof. \square

We next consider system (1.4) with $k > 2$. We recall here the relevant Kolmogorov equation (3.9)

$$\mathcal{L} = \frac{1}{2}\Delta_{x_{1,n}} + \sum_{j=1}^n x_j^k \partial_{x_{n+1}} - \partial_t.$$

Proposition 4.12 *Let \mathcal{L} be the operator defined in (3.9) and let k be a positive integer. Let T_1, τ, t, T_2 be such that $T_1 < \tau < t < T_2$ and $t - \tau \leq \tau - T_1$, and assume that there exists a path $\gamma : [0, t - \tau] \rightarrow \mathbb{R}^{n+1} \times]T_1, T_2[$ satisfying*

$$\gamma'(s) = \sum_{j=1}^n \omega_j Y_j(\gamma(s)) + Z(\gamma(s)), \quad \gamma(0) = (x, t), \quad \gamma(t - \tau) = (\xi, \tau), \quad (4.52)$$

for some constant vector $\omega = (\omega_1, \dots, \omega_n)$ such that $|\omega|^2 \geq \frac{2}{t - \tau}$. Then

$$u(\xi, \tau) \leq \exp(C((t - \tau)|\omega|^2 + 1)) u(x, t),$$

for every non-negative solution u to $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1} \times]T_1, T_2[$.

Moreover, if k is even, we have

$$u\left(0, \dots, 0, x_{n+1} + \tilde{\xi}_{n+1}, \tau\right) \leq \exp\left(C\left(\frac{(t - \tau)^{1+2/k}}{\tilde{\xi}_{n+1}^{2/k}} + 1\right)\right) u(0, \dots, 0, x_{n+1}, t).$$

for every $(x, t) \in \mathbb{R}^{n+1} \times]T_1, T_2[, \tau \in]T_1, t[$ with $\tilde{\xi}_{n+1} \in \left]0, \frac{(t - \tau)^{1+k/2}}{(k+1)4^{k+1}}\right]$, for every non-negative solution u to $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1} \times]T_1, T_2[$. In the above inequalities, C is a positive constant only depending on \mathcal{L} .

Proof. We follow the argument used in the proof of Proposition 4.10. We consider the lifted operator $\tilde{\mathcal{L}}$ and the function defined by $\tilde{u}(x, y, t) = u(x, t)$ for every $(x, y, t) \in \mathbb{R}^{kn+2} \times]T_1, T_2[$, which is a solution of $\tilde{\mathcal{L}}\tilde{u} = 0$. We first prove a local Harnack inequality in \mathbb{R}^{kn+2} , then we construct a Harnack chain in \mathbb{R}^{kn+2} and we conclude the proof by going back to \mathbb{R}^{n+2} . However, equation (3.9) is not invariant with respect to the rotation in the $x_{1,n}$ variable, then we cannot assume that $\omega = \frac{|\omega|}{\sqrt{n}}(1, \dots, 1)$, as we did at the beginning of the proof of Proposition 4.10. In order to apply Theorem 3.2, which requires $\omega_j \neq 0$ for every $j \in \llbracket 1, n \rrbracket$, we proceed as follows. For any $\eta \in \mathbb{R}^n$ we consider the path $\tilde{\gamma}$ defined by (4.52), starting at $(0, 0, 0)$, with $\omega = \eta$, then we denote $w_\eta = \tilde{\gamma}\left(\frac{1}{2}\right)$ and we set

$$K = \left\{w_\eta \mid \frac{1}{8\sqrt{n}} \leq |\eta_j| \leq 1, j = 1, \dots, n\right\}. \quad (4.53)$$

By the same argument used in the proof of (4.42), there exist a positive constant C_K such that

$$\sup_{K_r(z_0)} \tilde{u} \leq C_K \tilde{u}(z_0), \quad (4.54)$$

for every positive solution \tilde{u} to $\tilde{\mathcal{L}}\tilde{u} = 0$ in $\mathcal{O}_r(z_0)$.

We next consider the vector ω in the statement of Proposition 4.12. We assume, as it is not restrictive, that $\omega_j \geq 0$ for every $j \in \llbracket 1, n \rrbracket$ and $\omega_1 = \max\{\omega_1, \dots, \omega_n\}$. Note that, as a consequence, $\frac{2}{n(t - \tau)} \leq \frac{|\omega|^2}{n} \leq \omega_1^2 \leq |\omega|^2$. For every $j \in \llbracket 1, n \rrbracket$ we set $\tilde{\omega}_j = \frac{\omega_j}{2}$ if $\omega_j \geq \frac{\omega_1}{2}$, $\tilde{\omega}_j = \omega_1$ if $0 \leq \omega_j < \frac{\omega_1}{2}$. In both cases we have

$$\frac{|\omega|}{4\sqrt{n}} \leq \tilde{\omega}_j \leq |\omega|, \quad \text{and} \quad \frac{|\omega|}{4\sqrt{n}} \leq |\omega_j - \tilde{\omega}_j| \leq |\omega|, \quad \text{for every } j \in \llbracket 1, n \rrbracket. \quad (4.55)$$

We denote by $\tilde{\gamma}_1 : [0, \frac{t-\tau}{2}] \rightarrow \mathbb{R}^{kn+1} \times]T_1, T_2[$ the solution of the Cauchy problem

$$\begin{cases} \tilde{\gamma}'(s) = \sum_{j=1}^n \tilde{\omega}_j \tilde{Y}_j(\tilde{\gamma}(s)) + \tilde{Z}(\tilde{\gamma}(s)), & s \in [0, \frac{t-\tau}{2}], \\ \tilde{\gamma}(0) = (x, 0, t), \end{cases}$$

and by $\gamma_2 : [\frac{t-\tau}{2}, t-\tau] \rightarrow \mathbb{R}^{kn+1} \times]T_1, T_2[$ the solution of the Cauchy problem

$$\begin{cases} \tilde{\gamma}'(s) = \sum_{j=1}^n (\omega_j - \tilde{\omega}_j) \tilde{Y}_j(\tilde{\gamma}(s)) + \tilde{Z}(\tilde{\gamma}(s)), & s \in [\frac{t-\tau}{2}, t-\tau], \\ \tilde{\gamma}(0) = \gamma_1(\frac{t-\tau}{2}). \end{cases}$$

Following the same argument used in Proposition 4.10, we let m be the positive integer such that $m-1 < \frac{t-\tau}{2} |\tilde{\omega}|^2 \leq m$, we set $\tilde{s} = \frac{t-\tau}{2m}$, $\tilde{r} = \sqrt{2\tilde{s}}$ and we define $z_j = \tilde{\gamma}(j\tilde{s})$ for $j \in \llbracket 1, m \rrbracket$. A direct computation shows

$$z_j = z_{j-1} \circ \delta_{\tilde{r}} w_\eta, \quad \eta = \sqrt{\frac{\tilde{s}}{2}} \tilde{\omega}, \quad j \in \llbracket 1, m \rrbracket.$$

Note that $\frac{1}{8\sqrt{n}} \leq |\eta_j| \leq 1$, then from (4.55) it follows that z_j belongs to the compact set $K_{\tilde{r}}(z_{j-1})$ defined in (4.53). Then, by the Harnack inequality (4.54), we find

$$u(\gamma_1(\frac{t-\tau}{2})) \leq \exp(C((t-\tau)|\omega|^2 + 1)) u(x, t).$$

By the same token, we also find

$$u(\gamma_2(t-\tau)) \leq \exp(C((t-\tau)|\omega|^2 + 1)) u(\gamma_1(\frac{t-\tau}{2})),$$

with, together with the previous inequality, concludes the proof of the first statement of Proposition 4.12.

The argument used in the proof of the second statement of Proposition 4.10 applies, without changes, to the proof the second statement of Proposition 4.12. In both cases we can consider the path (4.40) defined by the vector $\bar{\omega} \in \mathbb{R}^n$ such that $\bar{\omega}_j = \frac{1}{\sqrt{n}}$, for any $j \in \llbracket 1, n \rrbracket$. The equivalence of the estimates depends on the fact that, for even k , we have

$$\int_0^t |s\bar{\omega}|^k ds = \frac{1}{\sqrt{n}} \int_0^t \sum_{j=1}^n (s\bar{\omega}_j)^k ds.$$

We omit the other details of the proof. □

Corollary 4.13 *Let \mathcal{L} be the operator defined in (3.9) and let k be a positive integer. Let $u : \mathbb{R}^{n+1} \times]T_1, T_2[\rightarrow \mathbb{R}$ be a non-negative solution to $\mathcal{L}u = 0$, and let $t, \tau \in \mathbb{R}$ be such that $T_1 < \tau < t < T_2$, and $t - \tau \leq 2(\tau - T_1)$. Then there exists a positive constant C_1 , only depending on \mathcal{L} , such that*

i) *if k is even, then for any $x, \xi \in \mathbb{R}^{n+1}$ such that $\xi_{n+1} - x_{n+1} \geq 2^k(t-\tau) \sum_{j=1}^n (x_j^k + \xi_j^k) + 2^k(t-\tau)^{1+k/2}$ we have*

$$u(\xi, \tau) \leq \exp\left(C_1 \left(\frac{|x_{1,n} - \xi_{1,n}|^2}{t-\tau} + \frac{(\xi_{n+1} - x_{n+1} - \frac{2^{k-1}}{k+1} \sum_{j=1}^n (x_j^k + \xi_j^k)(t-\tau))^{2/k}}{(t-\tau)^{1+2/k}} + 1 \right)\right) \times u(x, t);$$

ii) if k is even, then for any $x, \xi \in \mathbb{R}^{n+1}$ such that $0 < \xi_{n+1} - x_{n+1} \leq \frac{(t-\tau)^{1+k/2}}{4^{k+1}(k+1)}$ we have

$$u(\xi, \tau) \leq \exp \left(C_1 \left(\frac{|x_{1,n}|^{k+2} + |\xi_{1,n}|^{k+2}}{\xi_{n+1} - x_{n+1}} + \frac{(t-\tau)^{1+2/k}}{(\xi_{n+1} - x_{n+1})^{2/k}} + 1 \right) \right) u(x, t);$$

iii) if k is odd, we have

$$u(\xi, \tau) \leq \exp \left(C_1 \left(\frac{|x_{1,n} - \xi_{1,n}|^2}{t - \tau} + \frac{|\xi_{n+1} - x_{n+1} - \frac{2^{k-1}}{k+1} \sum_{j=1}^n (x_j^k + \xi_j^k)(t - \tau)|^{2/k}}{(t - \tau)^{1+2/k}} + 1 \right) \right) \times u(x, t).$$

Proof. As in the proof of Corollary 4.6, it is sufficient to prove our claims when $x, \xi \in \mathbb{R}^n$ are such that $|x_{1,n} - \xi_{1,n}|^2 \geq t - \tau$. We set $\omega = \frac{2}{t-\tau}(\xi_{1,n} - x_{1,n})$, and we apply Proposition 4.10. According with notation (4.27), we find

$$\begin{aligned} u(\gamma(\frac{t-\tau}{2})) &\leq \exp \left(C \left(2 \frac{|x_{1,n} - \xi_{1,n}|^2}{t - \tau} + 1 \right) \right) u(x, t), \\ \gamma(\frac{t-\tau}{2}) &= \left(\xi_{1,n}, x_{n+1} + \int_0^{\frac{t-\tau}{2}} \sum_{j=1}^n \left(x_j + \frac{2s}{t-\tau} (\xi_j - x_j) \right)^k ds, \frac{t+\tau}{2} \right), \end{aligned} \quad (4.56)$$

thus

$$|x_{n+1}(\frac{t-\tau}{2}) - x_{n+1}| \leq \frac{t-\tau}{2} \sum_{j=1}^n \left(|x_j|^k + |\xi_j|^k \right). \quad (4.57)$$

We next proceed with the proof of (i). From our assumption $\xi_{n+1} - x_{n+1} \geq 2^k(t - \tau) \sum_{j=1}^n (|x_j|^k + |\xi_j|^k) + 2^k(t - \tau)^{1+k/2}$ and (4.57) it follows that

$$\xi_{n+1} - x_{n+1}(\frac{t-\tau}{2}) \geq 2^k \frac{t-\tau}{2} \sum_{j=1}^n \left(|x_j|^k + |\xi_j|^k \right) + 2^k(t - \tau)^{1+k/2}. \quad (4.58)$$

We next choose a real parameter m , that will be specified later, and the vector $\bar{\omega} \in \mathbb{R}^n$ such that

$$\bar{\omega}_j = \frac{1}{\sqrt{n}} \text{ if } \xi_j \geq 0, \quad \bar{\omega}_j = -\frac{1}{\sqrt{n}} \text{ otherwise } \quad j \in \llbracket 1, n \rrbracket. \quad (4.59)$$

We consider the path $\gamma_1 : [\frac{t-\tau}{2}, \frac{3}{4}(t - \tau)] \rightarrow \mathbb{R}^{n+1} \times]T_1, T_2[$, starting from $\gamma(\frac{t-\tau}{2})$ and defined as in (4.52) with $\omega = m\bar{\omega}$, and the path $\gamma_2 : [\frac{3}{4}(t - \tau), t - \tau] \rightarrow \mathbb{R}^{n+1} \times]T_1, T_2[$, starting from $\gamma_1(\frac{3}{4}(t - \tau))$ and with $\omega = -m\bar{\omega}$. If $|m| \geq \sqrt{\frac{8}{t-\tau}}$, then Proposition 4.12 yields

$$u(\xi_{1,n}, \varphi(m), \tau) \leq \exp \left(2C \left(\frac{(t-\tau)m^2}{4} + 1 \right) \right) u(\gamma(\frac{t-\tau}{2})), \quad (4.60)$$

where

$$\varphi(m) = x_{n+1}(\frac{t-\tau}{2}) + 2 \sum_{j=1}^n \int_0^{\frac{t-\tau}{4}} \left(\xi_j + s \frac{m}{\sqrt{n}} \right)^k ds \quad (4.61)$$

is an increasing continuous function of $m \in \left[\sqrt{\frac{8}{t-\tau}}, +\infty \right[$ such that $\varphi(m) = \xi_{n+1}$ for a unique $m \in \left[\sqrt{\frac{8}{t-\tau}}, 2^{1+1/k}(k+1)^{1/k} \frac{(\xi_{n+1}-x_{n+1}-c\sum_{j=1}^n(x_j^k+\xi_j^k)(t-\tau))^{1/k}}{(t-\tau)^{1+1/k}} \right]$. For such a choice of m , equation (4.60) gives

$$u(\xi, \tau) \leq \exp \left(4^{1+1/k}(k+1)^{2/k} C \left(\frac{(\xi_{n+1} - x_{n+1} - c \sum_{j=1}^n (x_j^k + \xi_j^k)(t-\tau))^{2/k}}{(t-\tau)^{1+2/k}} + 1 \right) \right) u \left(\gamma \left(\frac{t-\tau}{2} \right) \right).$$

The above inequality, combined with (4.56), proves (i).

The proof of (ii) of Corollary 4.11, without changes, gives assertion (ii) of Corollary 4.13. We omit the details.

The proof of (iii) follows from the same argument used in the proof of (i). We rely on inequality (4.56), then we consider the function φ in (4.61). Note that, in the case of k odd, the sign of the expression $\sum_{j=1}^n \int_0^{\frac{t-\tau}{4}} \left(\xi_j + s \frac{m}{\sqrt{n}} \right)^k ds$ depends on $m \in \mathbb{R}$, then the function φ is surjective on \mathbb{R} , and it is not necessary to make use of the last inequality in Proposition 4.12. However, if the solution m of $\varphi(m) = \xi_{n+1}$ belongs to $\left] -\sqrt{\frac{8}{t-\tau}}, \sqrt{\frac{8}{t-\tau}} \right[$, we cannot apply Proposition 4.12 to conclude that (4.56) holds. In this case we argue as follows.

Suppose that $\xi_{n+1} > x_{n+1} \left(\frac{t-\tau}{2} \right)$. Consider the vector $\bar{\omega}$ defined in (4.59), the path $\gamma_1 : \left[\frac{t-\tau}{2}, \frac{5}{8}(t-\tau) \right] \rightarrow \mathbb{R}^{n+1} \times]T_1, T_2[$, starting from $\gamma \left(\frac{t-\tau}{2} \right)$ and defined as in (4.52) with $\omega = \frac{4}{\sqrt{t-\tau}} \bar{\omega}$, and the path $\gamma_2 : \left[\frac{3}{8}(t-\tau), \frac{3}{4}(t-\tau) \right] \rightarrow \mathbb{R}^{n+1} \times]T_1, T_2[$, starting from $\gamma_1 \left(\frac{3}{4}(t-\tau) \right)$ and with $\omega = -\frac{4}{\sqrt{t-\tau}} \bar{\omega}$. Then we consider the path $\gamma_3 : \left[\frac{3}{4}(t-\tau), \frac{7}{8}(t-\tau) \right] \rightarrow \mathbb{R}^{n+1} \times]T_1, T_2[$, starting from $\gamma_2 \left(\frac{3}{4}(t-\tau) \right)$ and defined with $\omega = -m \bar{\omega}$, and the path $\gamma_4 : \left[\frac{7}{8}(t-\tau), t-\tau \right] \rightarrow \mathbb{R}^{n+1} \times]T_1, T_2[$, starting from $\gamma_3 \left(\frac{7}{8}(t-\tau) \right)$ and with $\omega = m \bar{\omega}$. A simple computation shows that there exist $m \in \left] \frac{4}{\sqrt{t-\tau}}, \frac{8}{\sqrt{t-\tau}} \right[$ such that $\gamma_4(t-\tau) = \xi_{n+1}$. From Proposition 4.12 we finally find

$$u(\xi_{1,n}, \xi_{n+1}, \tau) \leq \exp \left(4C \left(\frac{(t-\tau)m^2}{8} + 1 \right) \right) u \left(\gamma \left(\frac{t-\tau}{2} \right) \right).$$

This inequality and inequality (4.56) conclude the proof in the case $\xi_{n+1} \geq x_{n+1} \left(\frac{t-\tau}{2} \right)$. If $\xi_{n+1} > x_{n+1} \left(\frac{t-\tau}{2} \right)$ it is sufficient to repeat last argument with $\bar{\omega}$ replaced by $-\bar{\omega}$ in the definition of the paths $\gamma_1, \gamma_2, \gamma_3$ and γ_4 . \square

We conclude this section observing that Corollaries 4.11 and 4.13 give the lower bounds in cases (i) and (ii) for Theorem 2.1.

5 Proof of the main results II: Malliavin calculus and upper bounds

5.1 Representation of densities through Malliavin calculus

In this section, we state some basic facts and notations concerning the Malliavin calculus. We refer to the monograph of Nualart [27], from which we borrow the notations, or Chapter 5 in Ikeda and Watanabe [21], for further details.

Let us consider an n -dimensional Brownian motion W on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and a given $T > 0$. Define for $h \in L^2(\mathbb{R}^+, \mathbb{R}^n)$, $W(h) = \int_0^T \langle h(s), dW_s \rangle$. We denote by \mathcal{S} the space of simple functionals of the Brownian motion W , that is the subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ consisting of real valued random variables F having the form

$$F = f(W(h_1), \dots, W(h_m)),$$

for some $m \in \mathbb{N}$, $h_i \in L^2(\mathbb{R}^+, \mathbb{R}^n)$, and where $f : \mathbb{R}^m \rightarrow \mathbb{R}$ stands for a smooth function with polynomial growth.

For $F \in \mathcal{S}$, we define the Malliavin derivative $(D_t F)_{t \in [0, T]}$ as the \mathbb{R}^n -dimensional (non adapted) process

$$D_t F = \sum_{i=1}^m \partial_{x_i} f(W(h_1), \dots, W(h_m)) h_i(t).$$

For any $q \geq 1$, the operator $D : \mathcal{S} \rightarrow L^q(\Omega, L^2(0, T))$ is closable. We denote its domain by $\mathbb{D}^{1, q}$ which is actually the completion of \mathcal{S} w.r.t. the norm

$$\|F\|_{1, q} := \left\{ \mathbb{E}[|F|^q] + \mathbb{E}[|DF|_{L^2(0, T)}^q] \right\}^{1/q}.$$

Writing $D_t^j F$ for the j^{th} component of $D_t F$, we define the k^{th} order derivative as the random vector on $[0, T]^k \times \Omega$ with coordinates:

$$D_{t_1, \dots, t_k}^{j_1, \dots, j_k} F := D_{t_k}^{j_k} \dots D_{t_1}^{j_1} F.$$

We then denote by $\mathbb{D}^{N, q}$ the completion of \mathcal{S} w.r.t. the norm

$$\|F\|_{N, q} := \left\{ \mathbb{E}[|F|^q] + \sum_{k=1}^N \mathbb{E}[|D^k F|_{L^2((0, T)^k)}^q] \right\}^{1/q}.$$

Also, $\mathbb{D}^\infty := \bigcap_{q \geq 1} \bigcap_{j \geq 1} \mathbb{D}^{j, q}$.

To state the main tool used in our proofs, i.e. the integration by parts formula, we need to introduce a last operator. Namely, the Ornstein-Uhlenbeck operator L which for $F \in \mathcal{S}$ writes:

$$LF = \langle \nabla f(W(h)), W(h) \rangle - \text{Tr}(D^2 f(W(h)) \langle h, h^* \rangle_{L^2(0, T)}), \quad W(h) = (W(h_1), \dots, W(h_m)).$$

This operator is also closable and \mathbb{D}^∞ is included in its domain.

For $F = (F_1, \dots, F_p) \in (\mathbb{D}^\infty)^p$, we define the Malliavin covariance matrix γ_F by

$$\gamma_F^{i, j} := \langle DF^i, DF^j \rangle_{L^2(0, T)}, \quad \forall (i, j) \in \llbracket 1, p \rrbracket^2.$$

Let us now introduce the non-degeneracy condition

(ND) We say that the random vector $F = (F_1, \dots, F_p)$ satisfies the non degeneracy condition if γ_F is a.s. invertible and $\det(\gamma_F)^{-1} \in \bigcap_{q \geq 1} L^q(\Omega)$.

This non degeneracy condition guarantees the existence of a smooth density, i.e. C^∞ , for the random variable F , see e.g. Corollary 2.1.2 in [27] or Theorem 9.3 in [21].

The following Proposition will be crucial in the derivation of an explicit representation of the density.

Proposition 5.1 (Integration by parts) *Let $F = (F_1, \dots, F_p) \in (\mathbb{D}^\infty)^p$ satisfy the non-degeneracy condition **(ND)**. Then, for all smooth function φ with polynomial growth, $G \in \mathbb{D}^\infty$ and all multi-index α ,*

$$\begin{aligned}\mathbb{E}[\partial_\alpha \varphi(F)G] &= \mathbb{E}[\varphi(F)H_\alpha(F, G)], \\ H_i(F, G) &= -\sum_{j=1}^p \{G \langle D\Gamma_F^{ij}, DF^j \rangle_{L^2(0,T)} + \Gamma_F^{ij} \langle DG, DF^j \rangle_{L^2(0,T)} - \Gamma_F^{ij} GLF^j\}, \quad \forall i \in \llbracket 1, p \rrbracket, \\ H_\alpha(F, G) &= H_{(\alpha_1, \dots, \alpha_k)}(F, G) = H_{\alpha_k}(F, H_{(\alpha_1, \dots, \alpha_{k-1})}(F, G)).\end{aligned}$$

Also, for all $q > 1$, and all multi-index α , there exists $(C, k, N, r, N', r') := (C, k, N, r, N', r')(q, \alpha)$ s.t.

$$\mathbb{E}[|H_\alpha(F, G)|^q]^{1/q} \leq C \|\Gamma_F\|_k \|G\|_{N, r} \|F\|_{N', r'}, \quad \Gamma_F = \gamma_F^{-1}, \quad \|\Gamma_F\|_k := \mathbb{E}[|\Gamma_F|^k]^{1/k}. \quad (5.1)$$

For the first part of the proposition we refer to Section V-9 of [21]. Concerning equation (5.1), it can be directly derived from the Meyer inequalities on $\|LF\|_q$ and the explicit definition of H , see also Proposition 2.4 in Bally and Talay [2].

Corollary 5.1 (Representation of the density and associated upper bound) *Let $F = (F_1, \dots, F_p) \in (\mathbb{D}^\infty)^p$ satisfy the nondegeneracy condition **(ND)**. The random vector F admits a density on \mathbb{R}^p . Fix $y \in \mathbb{R}^p$. Introduce $\forall (u, v) \in \mathbb{R}^2$, $\varphi_0^u(v) = \mathbb{I}_{v > u}$, $\varphi_1^u(v) = \mathbb{I}_{v \leq u}$. For all multi-index $\beta = (\beta_1, \dots, \beta_p) \in \{0, 1\}^p$ the density writes:*

$$p_F(y) = \mathbb{E}\left[\prod_{i=1}^p \varphi_{\beta_i}^{y_i}(F_i) H_\alpha(F, 1)\right] (-1)^{|\beta|}, \quad \alpha = (1, \dots, p), \quad |\beta| := \sum_{i=1}^p \beta_i. \quad (5.2)$$

As a consequence of (5.2) and (5.1) we get for all multi-index $\beta \in \{0, 1\}^p$:

$$\exists C > 0, \quad p_F(y) \leq C \prod_{i=1}^p \mathbb{E}[\varphi_{\beta_i}^{y_i}(F_i)]^{\gamma(i)} \|H_\alpha(F, 1)\|_2, \quad \gamma(i) = 2^{-(i+1)}. \quad (5.3)$$

Proof. Let $B := \prod_{i=1}^p [a_i, b_i], \forall i \in \llbracket 1, p \rrbracket$, $a_i < b_i$. Denote for all $u \in \mathbb{R}$, $I_0(u) := (-\infty, u)$, $I_1(u) := [u, \infty)$. Set finally, for all multi-index $\beta \in \{0, 1\}^p$, $\forall y \in \mathbb{R}^p$, $\Psi_B^\beta(y) = \int_{\prod_{i=1}^p I_{\beta_i}(y_i)} \mathbb{I}_B(x) dx$.

Proposition 5.1 applied with $\alpha = (1, \dots, p)$ and Ψ_B^β yields

$$\mathbb{E}[\partial_\alpha \Psi_B^\beta(F)] = \mathbb{E}[\Psi_B^\beta(F) H_\alpha(F, 1)]. \quad (5.4)$$

Now, the r.h.s. of equation (5.4) writes

$$\begin{aligned}\mathbb{E}[\Psi_B^\beta(F) H_\alpha(F, 1)] &= \mathbb{E}\left[\int_{\prod_{i=1}^p I_{\beta_i}(F_i)} \mathbb{I}_B(y) dy H_\alpha(F, 1)\right] = \int_B \mathbb{E}\left[\prod_{i=1}^p \mathbb{I}_{y_i \in I_{\beta_i}(F_i)} H_\alpha(F, 1)\right] dy \\ &= \int_B \mathbb{E}\left[\prod_{i=1}^p \varphi_{\beta_i}^{y_i}(F_i) H_\alpha(F, 1)\right] dy.\end{aligned} \quad (5.5)$$

The application of Fubini's theorem for the last but one equality is justified thanks to the integrability condition (5.1) of Proposition 5.1. On the other hand, the l.h.s. in (5.4) writes

$$\mathbb{E}[\partial_\alpha \Psi_B^\beta(F)] = \mathbb{E}\left[\prod_{i=1}^p \mathbb{I}_{F_i \in [a_i, b_i]} (-1)^{\beta_i}\right] = (-1)^{|\beta|} \int_B p_F(y) dy. \quad (5.6)$$

Equation (5.2) is now a direct consequence of (5.4), (5.5), (5.6). Equation (5.3) is then simply derived applying iteratively the Cauchy-Schwarz inequality. \square

5.2 Estimates on the Malliavin derivatives

We here concentrate on the particular case of the process (1.3) (indeed the estimates concerning (1.4) can be derived in a similar way). From Theorem 2.3.2 in [27], since condition [H] is satisfied, assumption **(ND)** is fulfilled and therefore the process $(X_s)_{s \geq 0}$ admits a smooth density $p(t, x, \cdot)$ at time $t > 0$. Our goal is to derive quantitative estimates on this density, emphasizing as well that we have different regimes in function of the starting point.

The first step is to rewrite the density conditioning w.r.t. to the Brownian component. Namely, for all $(t, x, \xi) \in \mathbb{R}^{+*} \times (\mathbb{R}^{n+1})^2$:

$$\begin{aligned} p(t, x, \xi) &= p_{X^{1,n}}(t, x_{1,n}, \xi_{1,n}) p_{X^{n+1}}(t, x_{n+1}, \xi_{n+1} | X_0^{1,n} = x_{1,n}, X_t^{1,n} = \xi_{1,n}) \\ &= \frac{\exp\left(-\frac{|\xi_{1,n} - x_{1,n}|^2}{2t}\right)}{(2\pi t)^{n/2}} p_{Y_t}(\xi_{n+1} - x_{n+1}), Y_t := \int_0^t \left| x_{1,n} \frac{t-u}{t} + \xi_{1,n} \frac{u}{t} + W_u^{0,t} \right|^k du, \end{aligned} \quad (5.7)$$

where $(W_u^{0,t})_{u \in [0,t]}$ is the standard n -dimensional Brownian bridge on the interval $[0, t]$. The idea is then to take advantage of the Malliavin representation of the density of Y_t to derive some estimates on $p(t, x, \xi)$. The most convenient way to proceed, in order to deal with functionals of the Brownian increments as in the previous paragraph, is to exploit that $(W_u^{0,t})_{u \in [0,t]} \stackrel{(\text{law})}{=} \left((t-u) \int_0^u \frac{dW_v}{t-v} \right)_{u \in [0,t]}$. Set now $m(u, t, x_{1,n}, \xi_{1,n}) := x_{1,n} \frac{t-u}{t} + \xi_{1,n} \frac{u}{t}$. To compute the Malliavin derivative of Y_t we first rewrite:

$$\begin{aligned} Y_t &= \int_0^t du \{ |m(u, t, x_{1,n}, \xi_{1,n}) + W_u^{0,t}|^2 \}^{k/2} \\ &= \int_0^t du \{ |m(u, t, x_{1,n}, \xi_{1,n})|^2 + |W_u^{0,t}|^2 + 2\langle m(u, t, x_{1,n}, \xi_{1,n}), W_u^{0,t} \rangle \}^{k/2} \\ &= \sum_{i=0}^{k/2} C_{k/2}^i \int_0^t du |m(u, t, x_{1,n}, \xi_{1,n})|^{k-2i} \{ |W_u^{0,t}|^2 + 2\langle m(u, t, x_{1,n}, \xi_{1,n}), W_u^{0,t} \rangle \}^i. \end{aligned} \quad (5.8)$$

Recalling that for $(u, s) \in [0, t]^2$, $D_s W_u^{0,t} = \mathbb{I}_{s \leq u} \frac{t-u}{t-s}$, we derive that the Malliavin derivative of Y_t (seen as a column vector) and the ‘‘covariance’’ matrix (that is in our case a scalar)

write:

$$\begin{aligned}
D_s Y_t &= \sum_{i=1}^{k/2} C_{k/2}^i \int_s^t du |m(u, t, x_{1,n}, \xi_{1,n})|^{k-2i} \{ |W_u^{0,t}|^2 + 2\langle m(u, t, x_{1,n}, \xi_{1,n}), W_u^{0,t} \rangle \}^{i-1} \\
&\quad \times 2 \frac{t-u}{t-s} (W_u^{0,t} + m(u, t, x_{1,n}, \xi_{1,n})) := \sum_{i=1}^{k/2} M_i(s, t, x_{1,n}, \xi_{1,n}), \\
\gamma_{Y_t} &= \int_0^t ds |D_s Y_t|^2. \tag{5.9}
\end{aligned}$$

5.2.1 “Gaussian” regime

In this section we assume that $|x_{1,n}| \vee |\xi_{1,n}| \geq Kt^{1/2}$, for $K := K(n, d)$ sufficiently large. That is we suppose that the starting or the final point of the non-degenerate component has greater norm than the characteristic time-scale $t^{1/2}$. In this case, we show below that the dominating term in the Malliavin derivative is the one associated to the non-random part of the term M_1 in (5.9). This term corresponds to the Malliavin derivative of a Gaussian process. This justifies the terminology “Gaussian” regime. Let us now split M_1 into two terms:

$$\begin{aligned}
M_1(s, t, x_{1,n}, \xi_{1,n}) &:= k \int_s^t du |m(u, t, x_{1,n}, \xi_{1,n})|^{k-2} \frac{t-u}{t-s} m(u, t, x_{1,n}, \xi_{1,n}) \\
&\quad + M_1^R(s, t, x_{1,n}, \xi_{1,n}) := (M_1^D + M_1^R)(s, t, x_{1,n}, \xi_{1,n}). \tag{5.10}
\end{aligned}$$

With these notations we rewrite from (5.9),

$$\begin{aligned}
D_s Y_t &= (M_1^D + M_1^R)(s, t, x_{1,n}, \xi_{1,n}) + \sum_{i=2}^{k/2} M_i(s, t, x_{1,n}, \xi_{1,n}) := (M_1^D + R)(s, t, x_{1,n}, \xi_{1,n}), \\
\gamma_{Y_t} &= \int_0^t ds |(M_1^D + R)(s, t, x_{1,n}, \xi_{1,n})|^2. \tag{5.11}
\end{aligned}$$

In order to give precise asymptotics on the density of Y_t , the crucial step consists in controlling the norm of $\Gamma_{Y_t} := \gamma_{Y_t}^{-1}$ in $L^p(\Omega)$, $p \in [1, +\infty)$ spaces.

Lemma 5.1 (Estimates on the Malliavin covariance) *Assume that $|x_{1,n}| \vee |\xi_{1,n}| \geq Kt^{1/2}$. Then, for all $p \in [1, +\infty)$ there exists $C_p := C_p(n, k) \geq 1$ s.t.*

$$\frac{C_p^{-1}}{(|x_{1,n}|^{2(k-1)} + |\xi_{1,n}|^{2(k-1)})t^3} \leq \|\Gamma_{Y_t}\|_p \leq \frac{C_p}{(|x_{1,n}|^{2(k-1)} + |\xi_{1,n}|^{2(k-1)})t^3}.$$

Proof. Set $M_t := \int_0^t ds |M_1^D(s, t, x_{1,n}, \xi_{1,n})|^2$. W.l.o.g. we can assume that $|\xi_{1,n}| \geq |x_{1,n}|/2$. Indeed, our proof in some sense strongly exploits that the norm of the final point is “far” from 0 for the characteristic time scale $t^{1/2}$. Because of the symmetry of the Brownian Bridge and its reversibility in time, if $|\xi_{1,n}| < |x_{1,n}|/2$ we can perform the computations w.r.t. to

the Brownian bridge $(\bar{W}_u^{0,t})_{u \in [0,t]} := (W_{t-u}^{0,t})_{u \in [0,t]}$ using the sensitivity w.r.t. to the Brownian motion $(\bar{W}_u)_{u \in [0,t]} := (W_{t-u} - W_t)_{u \in [0,t]}$.

Step 1: Decomposition of the expectation. To give the L^p estimates on the Malliavin derivative we use the following partition:

$$\mathbb{E}[|\Gamma_{Y_t}|^p] = \sum_{m \in \mathbb{N}} \mathbb{E}[|\Gamma_{Y_t}|^p \mathbb{I}_{\Gamma_{Y_t} \in [\frac{4m}{M_t}, \frac{4(m+1)}{M_t}]}] \leq \left(\frac{4}{M_t}\right)^p + \sum_{m \geq 2} \left(\frac{4(m+1)}{M_t}\right)^p \mathbb{P}[\gamma_{Y_t} \leq \frac{M_t}{4m}]. \quad (5.12)$$

Now we have to give estimates on M_t and to control the probabilities $\mathbb{P}[\gamma_{Y_t} \leq \frac{M_t}{4m}]$.

Step 2: Controls on the ‘‘Gaussian contribution’’ M_t . On the one hand, usual computations involving convexity inequalities yield that there exists $C := C(k, n) \geq 1$ s.t.

$$M_t \leq Ct^3(|x_{1,n}|^{2(k-1)} + |\xi_{1,n}|^{2(k-1)}). \quad (5.13)$$

On the other hand to prove that a lower bound at the same order also holds for M_t one has to be a little more careful. Since we have assumed $|\xi_{1,n}| \geq \frac{1}{2}|x_{1,n}|$ we also have $|\xi_{1,n}|_\infty \geq \frac{1}{2n^{1/2}}|x_{1,n}|_\infty$. Let $i_0 \in \llbracket 1, n \rrbracket$ be the index s.t. $|\xi_{1,n}|_\infty := |\xi_{i_0}|$, then $|\xi_{i_0}| \geq \frac{1}{2n^{1/2}}|x_{i_0}|$. Let us now write

$$M_t \geq k^2 \int_0^t ds \left(\int_s^t du |m(u, t, x_{1,n}, \xi_{1,n})|^{k-2} \left(\frac{t-u}{t} x_{i_0} + \frac{u}{t} \xi_{i_0} \right) \frac{t-u}{t-s} \right)^2.$$

Observe now that for $s \geq \frac{2n^{1/2}}{2n^{1/2}+1}t$ we have that $\forall u \in [s, t]$, $\frac{t-u}{t}x_{i_0} + \frac{u}{t}\xi_{i_0}$ has the sign of ξ_{i_0} . Hence,

$$M_t \geq k^2 \int_{\frac{2n^{1/2}}{2n^{1/2}+1}t}^t ds \left(\int_s^t du \left| \frac{t-u}{t}x_{i_0} + \frac{u}{t}\xi_{i_0} \right|^{k-1} \frac{t-u}{t-s} \right)^2. \quad (5.14)$$

Now, for $s \geq t \left(\frac{2n^{1/2}}{2n^{1/2}+2-1} \right)$, we have for all $u \in [s, t]$:

$$\begin{aligned} \left| \frac{t-u}{t}x_{i_0} + \frac{u}{t}\xi_{i_0} \right|^{k-1} &\geq \left(\frac{u}{t}\right)^{k-1} \frac{|\xi_{i_0}|^{k-1}}{2^{k-2}} - \left(\frac{t-u}{t}\right)^{k-1} |x_{i_0}|^{k-1} \\ &\geq |\xi_{i_0}|^{k-1} \left(\frac{n^{1/2}}{2n^{1/2}+2-1} \right)^{k-1}. \end{aligned} \quad (5.15)$$

We therefore derive from (5.14), (5.15) and (5.13) that there exists $C := C(k, n) \geq 1$ s.t.

$$C^{-1}t^3(|x_{1,n}|^{2(k-1)} + |\xi_{1,n}|^{2(k-1)}) \leq M_t \leq Ct^3(|x_{1,n}|^{2(k-1)} + |\xi_{1,n}|^{2(k-1)}). \quad (5.16)$$

Step 3. Flatness of the process. We now give estimates on $\mathbb{P}[\gamma_{Y_t} \leq \frac{M_t}{4m}]$, $m \geq 2$ in the spirit of Bally [1].

Introduce $t_m := \inf\{v \in [0, t] : \int_v^t ds |M_1^D(s, t, x_{1,n}, \xi_{1,n})|^2 \leq M_t/m\}$. We first show that there exists $m_0 \in \mathbb{N}^* \setminus \{1\}$ and $\bar{C} := \bar{C}(n, k)$ s.t. for all $m \geq m_0$, $t_m \geq t(1 - \bar{C}m^{-1/3})$. From

(5.16) we obtain

$$\begin{aligned} t_m &\geq \inf\{v \in [0, t] : \int_v^t ds |M_1^D(s, t, x_{1,n}, \xi_{1,n})|^2 \leq Ct^3(|x_{1,n}|^{2(k-1)} + |\xi_{1,n}|^{2(k-1)})/m\} \\ &\geq \inf\{v \in [0, t] : \int_v^t ds |M_1^D(s, t, x_{1,n}, \xi_{1,n})|^2 \leq C_1 t^3 |\xi_{1,n}|^{2(k-1)}/m\} := \bar{t}_m, \end{aligned}$$

with $C_1 := C(1 + 2^{2(k-1)}) := C_1(n, k)$, recalling we have assumed $|\xi_{1,n}| \geq \frac{|x_{1,n}|}{2}$ for the last inequality. Equations (5.14) and (5.15) also yield that there exists $C_2 := C_2(n, k)$ s.t. for all $v \geq \frac{2n^{1/2}}{2n^{1/2}+2^{-1}}t$,

$$\int_v^t ds |M_1^D(s, t, x_{1,n}, \xi_{1,n})|^2 \geq C_2(t-v)^3 |\xi_{1,n}|^{2(k-1)}.$$

Hence setting $\bar{C} := (C_1/C_2)^{1/3}$, for $m \geq \lfloor \bar{C}^3 \rfloor \vee 2 := m_0$, we derive that $\bar{t}_m \geq t(1 - \bar{C}m^{-1/3}) \geq 0$. Now for $m \geq m_0$, we get:

$$\begin{aligned} \mathbb{P}\left[\gamma_{Y_t} \leq \frac{M_t}{4m}\right] &\leq \mathbb{P}\left[\int_{t_m}^t ds |(M_1^D + R)(s, t, x_{1,n}, \xi_{1,n})|^2 \leq \frac{M_t}{4m}\right] \\ &\leq \mathbb{P}\left[\frac{1}{2} \int_{t_m}^t ds |M_1^D(s, t, x_{1,n}, \xi_{1,n})|^2 - \int_{t_m}^t ds |R(s, t, x_{1,n}, \xi_{1,n})|^2 \leq \frac{M_t}{4m}\right] \\ &\leq \mathbb{P}\left[\frac{M_t}{4m} \leq \int_{t_m}^t ds |R(s, t, x_{1,n}, \xi_{1,n})|^2\right] \\ &\leq \mathbb{P}\left[\frac{M_t}{4m} \leq \int_{t(1-\bar{C}m^{-1/3})}^t ds |R(s, t, x_{1,n}, \xi_{1,n})|^2\right]. \end{aligned} \quad (5.17)$$

Let us recall from equations (5.11) and (5.10) that the remainder term writes: $R(s, t, x_{1,n}, \xi_{1,n}) := (M_1^R + \sum_{i=2}^{k/2} M_i)(s, t, x_{1,n}, \xi_{1,n})$. From (5.17) and the convexity inequality $|R(s, t, x_{1,n}, \xi_{1,n})|^2 \leq \frac{k}{2}(|M_1^R(s, t, x_{1,n}, \xi_{1,n})|^2 + \sum_{i=2}^{k/2} |M_i(s, t, x_{1,n}, \xi_{1,n})|^2)$ we thus derive:

$$\begin{aligned} \mathbb{P}\left[\gamma_{Y_t} \leq \frac{M_t}{4m}\right] &\leq \mathbb{P}\left[\frac{M_t}{2mk} \leq \int_{t(1-\bar{C}m^{-1/3})}^t ds |M_1^R(s, t, x_{1,n}, \xi_{1,n})|^2 \right. \\ &\quad \left. + \sum_{i=2}^{k/2} \int_{t(1-\bar{C}m^{-1/3})}^t ds |M_i(s, t, x_{1,n}, \xi_{1,n})|^2\right] \\ &\leq \mathbb{P}\left[\frac{M_t}{mk^2} \leq \int_{t(1-\bar{C}m^{-1/3})}^t ds |M_1^R(s, t, x_{1,n}, \xi_{1,n})|^2\right] \\ &\quad + \sum_{i=2}^{k/2} \mathbb{P}\left[\frac{M_t}{mk^2} \leq \int_{t(1-\bar{C}m^{-1/3})}^t ds |M_i(s, t, x_{1,n}, \xi_{1,n})|^2\right] := \sum_{i=1}^{k/2} P_i. \end{aligned} \quad (5.18)$$

Now, one gets that there exists $C_3 := C_3(n, k), C_4 := C_4(n, k)$ s.t.:

$$\begin{aligned}
|M_1^R(s, t, x_{1,n}, \xi_{1,n})|^2 &\leq C_3(t-s)^2 \sup_{u \in [s, t]} |W_u^{0,t}|^2 |\xi_{1,n}|^{2(k-2)}, \\
\forall i \in \llbracket 2, k/2 \rrbracket, |M_i(s, t, x_{1,n}, \xi_{1,n})|^2 &\leq C_4(t-s)^2 \{ |\xi_{1,n}|^{2(k-2i)} \sup_{u \in [s, t]} |W_u^{0,t}|^{2(2i-1)} \\
+ |\xi_{1,n}|^{2(k-2i+1)} \sup_{u \in [s, t]} |W_u^{0,t}|^{4(i-1)} + |\xi_{1,n}|^{2(k-i)} \sup_{u \in [s, t]} |W_u^{0,t}|^{2(i-1)} + |\xi_{1,n}|^{2(k-i-1)} \sup_{u \in [s, t]} |W_u^{0,t}|^{2i} \}.
\end{aligned} \tag{5.19}$$

From equations (5.18), (5.19) and (5.16) we get:

$$\begin{aligned}
P_1 &\leq \mathbb{P} \left[\frac{M_t}{mk^2} \leq C_3(t\bar{C})^3 m^{-1} \sup_{u \in [t(1-\bar{C}m^{-1/3}), t]} |W_u^{0,t}|^2 |\xi_{1,n}|^{2(k-2)} \right] \\
&\leq \mathbb{P} \left[C_5 |\xi_{1,n}|^2 \leq \sup_{u \in [0, \bar{C}tm^{-1/3}]} \left| \bar{W}_u - \frac{u}{t} \bar{W}_t \right|^2 \right], \quad C_5 := C_5(n, k),
\end{aligned}$$

where we recall that $\bar{W}_u := W_{t-u} - W_t, u \in [0, t]$, is also a Brownian motion, so that $\bar{W}_u - \frac{u}{t} \bar{W}_t$ is a Brownian Bridge on $[0, t]$ as well. Let us now recall Lévy's identity in law (see e.g. Chapter 6 in [29]). Let $(B_t)_{t \geq 0}$ be a standard scalar Brownian motion. Then:

$$\sup_{u \in [0, s]} B_u \stackrel{(\text{law})}{=} |B_s|, \forall s > 0. \tag{5.20}$$

Since $\sup_{u \in [0, \bar{C}tm^{-1/3}]} |\bar{W}_u - \frac{u}{t} \bar{W}_t| \leq \sup_{u \in [0, \bar{C}tm^{-1/3}]} |\bar{W}_u| + \frac{\bar{C}}{m^{1/3}} |\bar{W}_t|$, we derive that there exists $C_6 := C_6(n, k) \geq 1$ s.t.:

$$\begin{aligned}
P_1 &\leq \mathbb{P} \left[\frac{C_5}{4} |\xi_{1,n}|^2 \leq \bar{C}tm^{-1/3} |\mathcal{N}(0, 1)|^2 \right] + \mathbb{P} \left[\frac{C_5}{4} |\xi_{1,n}|^2 \leq \bar{C}^2 tm^{-2/3} |\mathcal{N}(0, 1)|^2 \right] \\
&\leq C_6 \exp \left(-C_6^{-1} \frac{|\xi_{1,n}|^2 m^{1/3}}{t} \right).
\end{aligned} \tag{5.21}$$

For all $i \in \llbracket 2, \frac{k}{2} \rrbracket$, the previous arguments yield

$$P_i \leq C_7 \exp \left(-C_7^{-1} \frac{|\xi_{1,n}|^2 m^{1/3}}{t} \right), \tag{5.22}$$

with $C_7 := C_7(n, k) \geq 1$. Plugging (5.22) and (5.21) into (5.18), one gets that there exists $C_8 := C_8(n, k) \geq 1$ s.t. for all $m \geq m_0$

$$\mathbb{P}[\gamma_{Y_t} \leq \frac{M_t}{m}] \leq C_8 \exp \left(-C_8^{-1} \frac{|\xi_{1,n}|^2 m^{1/3}}{t} \right). \tag{5.23}$$

On the other hand for $m \in \llbracket 2, m_0 \rrbracket$, we write

$$\begin{aligned}
\mathbb{P} \left[\gamma_{Y_t} \leq \frac{M_t}{4m} \right] &\leq \mathbb{P} \left[\int_0^t ds |(M_1^D + R)(s, t, x_{1,n}, \xi_{1,n})|^2 \leq \frac{M_t}{4m} \right] \\
&\leq \mathbb{P} \left[M_t \left(\frac{1}{2} - \frac{1}{4m} \right) \leq \int_0^t ds |R(s, t, x_{1,n}, \xi_{1,n})|^2 \right] \\
&\leq \mathbb{P} \left[\frac{3}{8} C^{-1} t^3 (|x_{1,n}|^{2(k-1)} + |\xi_{1,n}|^{2(k-1)}) \leq \int_0^t ds |R(s, t, x_{1,n}, \xi_{1,n})|^2 \right] \\
&\leq \mathbb{P} \left[\frac{3}{8} C^{-1} t^3 |\xi_{1,n}|^{2(k-1)} \leq \int_0^t ds |R(s, t, x_{1,n}, \xi_{1,n})|^2 \right]
\end{aligned}$$

using (5.16) for the last but one inequality.

Hence, similarly to (5.17)-(5.21), we derive that up to a modification of C_8 , (5.23) is valid for all $m \geq 2$. Plugging this control into (5.12), using once again (5.16) recalling $|\xi_{1,n}| \geq \frac{|x_{1,n}|}{2}$, we derive that there exists $(n, k), C_9 := C_9(n, k, p)$ s.t.:

$$\begin{aligned}
\mathbb{E}[|\Gamma_{Y_t}|^p] &\leq \left(\frac{C}{t^3 |\xi_{1,n}|^{2(k-1)}} \right)^p + C_8 \sum_{m \geq 2} \left(\frac{4C(m+1)}{t^3 |\xi_{1,n}|^{2(k-1)}} \right)^p \exp \left(-C_8^{-1} \frac{|\xi_{1,n}|^2 m^{1/3}}{t} \right) \\
&\leq \left(\frac{C}{t^3 |\xi_{1,n}|^{2(k-1)}} \right)^p + \left(\frac{C_8 (4C \times \frac{3}{2})^p}{|\xi_{1,n}|^{(2(k-1)+6)p}} \right) \sum_{m \geq 2} \left(\frac{m^{1/3} |\xi_{1,n}|^2}{t} \right)^{3p} \exp \left(-C_8^{-1} \frac{|\xi_{1,n}|^2 m^{1/3}}{t} \right) \\
&\leq \left(\frac{C}{t^3 |\xi_{1,n}|^{2(k-1)}} \right)^p + \frac{C_9}{|\xi_{1,n}|^{(2k+4)p}} \sum_{m \geq 2} \exp \left(-C_9^{-1} \frac{|\xi_{1,n}|^2 m^{1/3}}{t} \right) \\
&\leq \left(\frac{C}{t^3 |\xi_{1,n}|^{2(k-1)}} \right)^p + \frac{C_9}{|\xi_{1,n}|^{(2k+4)p}} \frac{t^3}{|\xi_{1,n}|^6},
\end{aligned}$$

which for $|\xi_{1,n}| \geq Kt^{1/2}$ gives the upper bound of the lemma.

Let us now turn to the lower bound for $\|\Gamma_{Y_t}\|_{L^p(\mathbb{P})}$. Put $\mathcal{R}_t := \int_0^t ds |R(s, t, x_{1,n}, \xi_{1,n})|^2$. Write:

$$\mathbb{E}[\Gamma_{Y_t}^p] \geq \mathbb{E}[\Gamma_{Y_t}^p \mathbb{1}_{\gamma_{Y_t} \leq 3M_t}] \geq \frac{1}{(3M_t)^p} \mathbb{P}[\gamma_{Y_t} \leq 3M_t] \geq \frac{1}{(3M_t)^p} (1 - \mathbb{P}[\gamma_{Y_t} > 3M_t]).$$

From equations (5.9)-(5.11) one has $\mathbb{P}[\gamma_{Y_t} > 3M_t] \leq \mathbb{P}[2M_t + 2\mathcal{R}_t > 3M_t] = \mathbb{P}[\mathcal{R}_t > \frac{1}{2}M_t]$. Now, similarly to (5.21) and (5.22), one gets $\mathbb{P}[\gamma_{Y_t} > 3M_t] \leq C_8 \exp \left(-C_8^{-1} \frac{|\xi_{1,n}|^2}{t} \right)$. Therefore, for $|\xi_{1,n}| \geq Kt^{1/2}$ and K large enough, we get $\mathbb{E}[\Gamma_{Y_t}^p] \geq \frac{1}{2(3M_t)^p}$, which thanks to (5.16) completes the proof.

Controls of the weight for the integration by parts.

From Proposition 5.1 and Corollary 5.1, we derive

$$\begin{aligned}
p_{Y_t}(\xi_{n+1} - x_{n+1}) &= \mathbb{E}[H_t \mathbb{1}_{Y_t > \xi_{n+1} - x_{n+1}}], \\
H_t &= -\langle D\Gamma_{Y_t}, DY_t \rangle_{L^2(0,t)} + \Gamma_{Y_t} LY_t = \gamma_{Y_t}^{-2} \langle D\gamma_{Y_t}, DY_t \rangle_{L^2(0,t)} + \Gamma_{Y_t} LY_t \\
&:= H_t^1 + H_t^2,
\end{aligned} \tag{5.24}$$

using the chain rule (see e.g. Proposition 1.2.3 in [27]) for the last but one identity.

We have the following

Proposition 5.2 (Estimates for the Malliavin weight) *Assume that $|x_{1,n}| \vee |\xi_{1,n}| \geq Kt^{1/2}$ for K large enough. Then, for all $p \in [1, +\infty)$ there exists $C_p := C(p, n, k, c, K) \geq 1$ s.t.*

$$\|H_t\|_p \leq \frac{C_p}{(|x_{1,n}|^{(k-1)} + |\xi_{1,n}|^{(k-1)})t^{3/2}}.$$

Proof.

Control of H_t^1 . From (5.24) we get for all given $p \geq 1$,

$$\begin{aligned}
\|H_t^1\|_p &:= \mathbb{E}[\gamma_{Y_t}^{-2p} |\langle D\gamma_{Y_t}, DY_t \rangle_{L^2(0,t)}|^p]^{1/p} \leq \mathbb{E}[\gamma_{Y_t}^{-4p}]^{1/2p} \mathbb{E}[|\langle D\gamma_{Y_t}, DY_t \rangle_{L^2(0,t)}|^{2p}]^{1/2p} \\
&\leq \frac{C_p}{t^6 (|\xi_{1,n}|^{2(k-1)} + |x_{1,n}|^{2(k-1)})^2} \times \mathbb{E}[|\langle D\gamma_{Y_t}, DY_t \rangle_{L^2(0,t)}|^{2p}]^{1/2p} \\
&\leq \frac{C_p}{t^6 (|\xi_{1,n}|^{2(k-1)} + |x_{1,n}|^{2(k-1)})^2} \mathbb{E}[|D\gamma_{Y_t}|_{L^2(0,t)}^{4p}]^{1/4p} \mathbb{E}[|DY_t|_{L^2(0,t)}^{4p}]^{1/4p}, \tag{5.25}
\end{aligned}$$

using Lemma 5.1 for the last but one inequality. Now, from equations (5.9), (5.11), using the notations of Lemma 5.1,

$$\begin{aligned}
\mathbb{E}[|DY_t|_{L^2(0,t)}^{4p}]^{1/4p} &= \mathbb{E}\left[\left(\int_0^t ds |D_s Y_t|^2\right)^{2p}\right]^{1/4p} := \mathbb{E}[\gamma_{Y_t}^{2p}]^{1/4p} \\
&\leq \left(4^{2p-1} \left\{M_t^{2p} + \mathbb{E}[\mathcal{R}_t^{2p}]\right\}\right)^{1/4p} \leq C_p \left\{M_t^{1/2} + \mathbb{E}[\mathcal{R}_t^{2p}]^{1/4p}\right\}.
\end{aligned}$$

On the one hand equation (5.16) in Lemma 5.1 readily gives $M_t^{1/2} \leq Ct^{3/2}(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1})$. On the other hand, we derive from (5.11)

$$\begin{aligned}
\mathbb{E}[\mathcal{R}_t^{2p}]^{1/4p} &\leq C(k, p) \left(\mathbb{E}\left[\left(\int_0^t ds |M_1^R(s, t, x_{1,n}, \xi_{1,n})|^2\right)^{2p}\right] \right. \\
&\quad \left. + \sum_{i=2}^{k/2} \mathbb{E}\left[\left(\int_0^t ds |M_i(s, t, x_{1,n}, \xi_{1,n})|^2\right)^{2p}\right] \right)^{1/4p}.
\end{aligned}$$

Assuming w.l.o.g. that $|\xi_{1,n}| \geq \frac{|x_{1,n}|}{2}$, (5.19) yields

$$\begin{aligned} \mathbb{E}[|\mathcal{R}_t|^{2p}]^{1/4p} &\leq C(k, p) \left(t^{3/2} |\xi_{1,n}|^{k-1} \left\{ \frac{\mathbb{E}[\sup_{u \in [0,t]} |W_u^{0,t}|^{4p}]^{1/4p}}{|\xi_{1,n}|} + \right. \right. \\ &\quad + \sum_{i=2}^{k/2} \left[\frac{\mathbb{E}[\sup_{u \in [0,t]} |W_u^{0,t}|^{(2i-1) \times 4p}]^{1/4p}}{|\xi_{1,n}|^{2i-1}} + \frac{\mathbb{E}[\sup_{u \in [0,t]} |W_u^{0,t}|^{2(i-1) \times 4p}]^{1/4p}}{|\xi_{1,n}|^{2(i-1)}} \right. \\ &\quad \left. \left. + \frac{\mathbb{E}[\sup_{u \in [0,t]} |W_u^{0,t}|^{(i-1) \times 4p}]^{1/4p}}{|\xi_{1,n}|^{i-1}} + \frac{\mathbb{E}[\sup_{u \in [0,t]} |W_u^{0,t}|^{i \times 4p}]^{1/4p}}{|\xi_{1,n}|^i} \right] \right\} \right). \end{aligned}$$

From Lévy's identity (5.20), one derives that there exists $\bar{C} := \bar{C}(n, k, p)$ s.t. for all $i \in \llbracket 0, k-1 \rrbracket$, $\mathbb{E}[\sup_{u \in [0,t]} |W_u^{0,t}|^{4p}] \leq \bar{C} t^{i2p}$. Recalling that $|\xi_{1,n}| \vee |x_{1,n}| \geq K t^{1/2}$, $K \geq 1$ we obtain:

$$\mathbb{E}[|\mathcal{R}_t|^{2p}]^{1/4p} \leq C(k, p) \left(t^{3/2} |\xi_{1,n}|^{k-1} K^{-1} k/2 \right).$$

Hence, there exists $C_{10} := C_{10}(n, k, p, K)$ s.t.

$$\mathbb{E}[|DY_t|_{L^2(0,t)}^{4p}]^{1/4p} = \mathbb{E}[\gamma_{Y_t}^{2p}]^{1/4p} \leq C_{10} t^{3/2} (|\xi_{1,n}|^{k-1} + |x_{1,n}|^{k-1}). \quad (5.26)$$

Thus, from (5.25), in order to get a bound for $\|H_t^1\|_p$, it remains to control $\mathbb{E}[|D\gamma_{Y_t}|_{L^2(0,t)}^{4p}]^{1/4p}$. Equation (5.9) and the chain rule yield that for all $u_2 \in [0, t]$, $D_{u_2} \gamma_{Y_t} = 2 \int_0^t du_1 D_{u_2} D_{u_1} Y_t \times D_{u_1} Y_t$. We get

$$\begin{aligned} \mathbb{E}[|D\gamma_{Y_t}|_{L^2(0,t)}^{4p}]^{1/4p} &\leq \mathbb{E}[\gamma_{Y_t}^{4p}]^{1/8p} \mathbb{E}\left[\left(\int_0^t du_1 \int_0^t du_2 |D_{u_2, u_1} Y_t|^2\right)^{4p}\right]^{1/8p} \\ &\leq C_{10} t^{3/2} (|\xi_{1,n}|^{k-1} + |x_{1,n}|^{k-1}) \mathbb{E}[|D^2 Y_t|_{L^2(0,t)}^{8p}]^{1/8p}, \quad (5.27) \end{aligned}$$

using (5.26) for the last inequality.

We recall from equations (5.10), (5.11) that for all $u_1 \in [0, t]$,

$$\begin{aligned} D_{u_1} Y_t &:= \sum_{i=1}^{k/2} C_{k/2}^i \int_{u_1}^t dv |m(v, t, x_{1,n}, \xi_{1,n})|^{k-2i} 2i \{ |W_v^{0,t}|^2 + 2\langle m(v, t, x_{1,n}, \xi_{1,n}), W_v^{0,t} \rangle \}^{i-1} \\ &\quad \times \frac{t-v}{t-u_1} (W_v^{0,t} + m(v, t, x_{1,n}, \xi_{1,n})) := \sum_{i=1}^{k/2} M_i(u_1, t, x_{1,n}, \xi_{1,n}) := \sum_{i=1}^{k/2} \bar{M}_i(u_1, t), \end{aligned}$$

for notational convenience.

Observe now that for all $i \in \llbracket 2, k/2 \rrbracket$, $u_2 \in [0, t]$,

$$\begin{aligned} D_{u_2} \bar{M}_i(u_1, t) &= C_{k/2}^i \int_{u_1 \vee u_2}^t dv |m(v, t, x_{1,n}, \xi_{1,n})|^{k-2i} \{ |W_v^{0,t}|^2 + 2\langle m(v, t, x_{1,n}, \xi_{1,n}), W_v^{0,t} \rangle \}^{i-2} \\ &\quad \times 2i \frac{(t-v)^2}{(t-u_1)(t-u_2)} \{ 2(i-1)(W_v^{0,t} + m(v, t, x_{1,n}, \xi_{1,n})) \otimes (W_v^{0,t} + m(v, t, x_{1,n}, \xi_{1,n})) \\ &\quad \quad \quad + \{ |W_v^{0,t}|^2 + 2\langle m(v, t, x_{1,n}, \xi_{1,n}), W_v^{0,t} \rangle \} I_n \}, \\ D_{u_2} \bar{M}_1(u_1, t) &= k \int_{u_1 \vee u_2}^t dv |m(v, t, x_{1,n}, \xi_{1,n})|^{k-2} \frac{(t-v)^2}{(t-u_1)(t-u_2)} I_n. \end{aligned}$$

From the above equations, assuming once again w.l.o.g. $|\xi_{1,n}| \geq \frac{|x_{1,n}|}{2}$, we get

$$\begin{aligned} \mathbb{E}[|D^2 Y_t|_{L^2((0,t)^2)}^{8p}]^{1/8p} &\leq C \mathbb{E} \left[\int_{[0,t]^2} du_1 du_2 (t - u_1 \vee u_2)^2 |\xi_{1,n}|^{2(k-2)} \right. \\ &\quad \times \left(1 + \sum_{i=2}^{k/2} |\xi_{1,n}|^{4(1-i)} \sup_{u \in [0,t]} |W_u^{0,t}|^{4(i-1)} \right)^{4p}]^{1/8p} \\ &\leq C t^2 |\xi_{1,n}|^{k-2} \left(1 + \sum_{i=2}^{k/2} |\xi_{1,n}|^{2(1-i)} \mathbb{E} \left[\sup_{u \in [0,t]} |W_u^{0,t}|^{16p(i-1)} \right]^{1/8p} \right) \\ &\leq C t^2 |\xi_{1,n}|^{k-2} \left(1 + \sum_{i=2}^{k/2} \left(\frac{t^{1/2}}{|\xi_{1,n}|} \right)^{2(i-1)} \right), \end{aligned}$$

where $C := C(n, k, p)$ may change from line to line. Recalling that $|\xi_{1,n}| \vee |x_{1,n}| \geq K t^{1/2}$ we obtain

$$\mathbb{E}[|D^2 Y_t|_{L^2((0,t)^2)}^{8p}]^{1/8p} \leq C t^2 |\xi_{1,n}|^{k-2}, \quad C := C(n, k, p, K).$$

Plugging the above equation into (5.27) we derive that

$$\mathbb{E}[|D\gamma_{Y_t}^{4p}|_{L^2(0,t)}]^{1/4p} \leq C t^{7/2} |\xi_{1,n}|^{2k-3},$$

which together with (5.26) and (5.25), eventually yields

$$\|H_t^1\|_p \leq \frac{\bar{C}_1}{t |\xi_{1,n}|^k} \leq \frac{CK^{-1}}{t^{3/2} |\xi_{1,n}|^{k-1}}, \quad \bar{C}_1 := \bar{C}_1(n, k, p, K). \quad (5.28)$$

On the other hand, from (5.24) and Lemma 5.1, for all $p \geq 1$,

$$\|H_t^2\|_p \leq \mathbb{E}[|\Gamma_{Y_t}|^{2p}]^{1/2p} \mathbb{E}[|LY_t|^{2p}]^{1/2p} \leq \frac{C_p}{t^3 (|x_{1,n}|^{2(k-1)} + |\xi_{1,n}|^{2(k-1)})} \mathbb{E}[|LY_t|^{2p}]^{1/2p}. \quad (5.29)$$

Now, since $LY_t = \delta(DY_t)$, the idea is to provide a chaotic representation of DY_t from which the Skorohod integral is obtained (up to a symmetrization) adding an integration w.r.t. the Brownian motion. To do so we use Stroock's formula see [32]. For a given $u_1 \in [0, t]$,

recalling $D_{u_1} Y_t := \sum_{i=1}^{k/2} \bar{M}_i(u_1, t)$ where $\bar{M}_i(u_1, t)$ is a random contribution belonging to the Wiener chaos of order $2i - 1$, one has:

$$\begin{aligned} \bar{M}_i(u_1, t) &= \mathbb{E}[\bar{M}_i(u_1, t)] + \sum_{l=1}^{2i-1} I_l(g_l^i(\cdot, u_1, t)), \\ I_l(g_l^i(\cdot, u_1, t)) &:= \int_0^t dW_{v_1} \int_0^{v_1} dW_{v_2} \cdots \int_0^{v_{l-1}} dW_{v_l} g_l^i(v_1, \dots, v_l, u_1, t), \\ g_l^i(v_1, \dots, v_l, u_1, t) &:= \mathbb{E}[D_{v_1, \dots, v_l} \bar{M}_i(u_1, t)]. \end{aligned}$$

Hence, $D_{u_1} Y_t := g_0(u_1, t) + \sum_{l=1}^{k-1} I_l(g_l(\cdot, u_1, t))$, where $g_0(u_1, t) := \sum_{i=1}^{k/2} \mathbb{E}[\bar{M}_i(u_1, t)]$ and for all

$l \in \llbracket 1, k-1 \rrbracket$, $g_l(v_1, \dots, v_l, u_1, t) := \sum_{i=\lfloor l/2 \rfloor + 1}^{k/2} g_l^i(v_1, \dots, v_l, u_1, t)$, so that

$$LY_t = \int_0^t dW_{u_1} g_0(u_1, t) + \sum_{l=2}^k I_l(g_{l-1}(\cdot, t)) := \sum_{l=1}^k I_l(g_{l-1}(\cdot, t)).$$

From the computations performed to control $\mathbb{E}[|\mathcal{R}_t|^{2p}]^{1/4p}$ that gave the bound (5.26) we obtain that there exists $C := C(n, k)$ s.t. for all $l \in \llbracket 0, k-1 \rrbracket$ and for all $(v_1, \dots, v_l, u_1) \in [0, t]^{l+1}$:

$$|g_l(v_1, \dots, v_l, u_1, t)| \leq Ct(|\xi_{1,n}|^{k-(l+1)} + |x_{1,n}|^{k-(l+1)}).$$

Therefore,

$$\begin{aligned} \mathbb{E}[|LY_t|^{2p}]^{1/2p} &\leq C \sum_{i=l}^k t^{1+l/2} (|\xi_{1,n}|^{k-l} + |x_{1,n}|^{k-l}) \\ &\leq Ct^{3/2} (|\xi_{1,n}|^{k-1} + |x_{1,n}|^{k-1}) \left\{ \sum_{l=1}^k t^{(l-1)/2} (|\xi_{1,n}|^{1-l} + |x_{1,n}|^{1-l}) \right\}, \end{aligned}$$

where $C := C(n, k, p)$ may change from line to line. Recalling that $|\xi_{1,n}| \vee |x_{1,n}| \geq Kt^{1/2}$, we derive from (5.29) that there exists $\bar{C}_2 := \bar{C}_2(n, k, p, K)$ s.t.

$$\|H_t^2\|_p \leq \frac{\bar{C}_2}{t^{3/2} (|\xi_{1,n}|^{k-1} + |x_{1,n}|^{k-1})},$$

which together with (5.28) and (5.24) completes the proof.

5.2.2 Non Gaussian regime

We briefly state that when $|x_{1,n}| \vee |\xi_{1,n}| \leq Kt^{1/2}$, that is when the non-degenerate component is in "diagonal" regime w.r.t. to its characteristic time scale, then the characteristic time-scale of the density $p_{Y_i}(\xi_{n+1} - x_{n+1})$ is actually $t^{1+k/2}$. Namely we have the following result

Proposition 5.3 (Estimates for the Malliavin weight in Non Gaussian regime) *Let $K > 0$ be given and assume that $|x_{1,n}| \vee |\xi_{1,n}| \leq Kt^{1/2}$. For every $p \geq 1$ there exists $C_p := C(n, k, p, K)$ s.t.*

$$\|H_t\|_p \leq \frac{C_p}{t^{1+k/2}}.$$

Proof. For $t > 0$ write:

$$\begin{aligned} Y_t = \int_0^t \left| x_{1,n} \frac{t-u}{t} + \xi_{1,n} \frac{u}{t} + W_u^{0,t} \right|^k du &= t^{1+k/2} \int_0^1 \left| \frac{x_{1,n}}{t^{1/2}} (1-u) + \frac{\xi_{1,n}}{t^{1/2}} u + \frac{W_{ut}^{0,t}}{t^{1/2}} \right|^k du \\ &:= t^{1+k/2} \bar{Y}_1^t. \end{aligned}$$

Thus:

$$\begin{aligned} p_{Y_t}(\xi_{n+1} - x_{n+1}) &:= -\partial_{\xi_{n+1}} \mathbb{P}[Y_t > \xi_{n+1} - x_{n+1}] = -\partial_{\xi_{n+1}} \mathbb{P}[\bar{Y}_1^t > \frac{\xi_{n+1} - x_{n+1}}{t^{1+k/2}}] \\ &= \frac{1}{t^{1+k/2}} p_{\bar{Y}_1^t}(\frac{\xi_{n+1} - x_{n+1}}{t^{1+k/2}}). \end{aligned}$$

From Corollary 5.1 (Malliavin representation of the densities), we obtain:

$$\begin{aligned} p_{Y_t}(\xi_{n+1} - x_{n+1}) &= \mathbb{E}[H(Y_t, 1) \mathbb{I}_{Y_t > \xi_{n+1} - x_{n+1}}] = \frac{1}{t^{1+k/2}} \mathbb{E}[H(\bar{Y}_1^t, 1) \mathbb{I}_{\bar{Y}_1^t > \frac{\xi_{n+1} - x_{n+1}}{t^{1+k/2}}}] \\ &= \frac{1}{t^{1+k/2}} \mathbb{E}[H(\bar{Y}_1^t, 1) \mathbb{I}_{Y_t > \xi_{n+1} - x_{n+1}}], \end{aligned}$$

so that $H_t := H(Y_t, 1) = t^{-(1+k/2)} H(\bar{Y}_1^t, 1) := t^{-(1+k/2)} H_1^{\bar{Y}_1^t}$. Hence, for all $p \geq 1$,

$$\|H_t\|_p \leq \frac{1}{t^{1+k/2}} \|H_1^{\bar{Y}_1^t}\|. \quad (5.30)$$

Now, as a consequence of the Brownian scaling we get $(\frac{W_u^{0,t}}{t^{1/2}})_{u \in [0,1]} \stackrel{(\text{law})}{=} (W_u^{0,1})_{u \in [0,1]}$ so that $\bar{Y}_1^t \stackrel{(\text{law})}{=} t^{1+k/2} \int_0^1 \left| \frac{x_{1,n}}{t^{1/2}}(1-u) + \frac{\xi_{1,n}}{t^{1/2}}u + W_u^{0,1} \right|^k du$. Recalling that $|\frac{x_{1,n}}{t^{1/2}}| \vee |\frac{\xi_{1,n}}{t^{1/2}}| \leq K$ we derive that the usual techniques used to prove the non degeneracy of the Malliavin covariance matrix under Hörmander's condition (see e.g. Norris [26] or Nualart [27]) yield that there exists $C := C(n, p, k, K) \in \mathbb{R}^{+*}$ s.t. $\|H_1^{\bar{Y}_1^t}\| \leq C_p$ which from (5.30) concludes the proof. The crucial tool here is the global scaling.

5.3 Deviation estimates

5.3.1 Off-diagonal bounds

From the Malliavin representation of the density given by (5.24), to derive off-diagonal bounds on the density, it remains to give estimates on $\mathbb{P}[Y_t > \xi_{n+1} - x_{n+1}]$.

Lemma 5.2 (Off diagonal-bounds) *Assume that $U_t^k(x, \xi) := \xi_{n+1} - x_{n+1} - \frac{2^{k-1}}{k+1} (|x_{1,n}|^k + |\xi_{1,n}|^k)t > 0$. Then, there exists $C_{5.2} := C_{5.2}(n, k)$ s.t.*

(i) *If $|x_{1,n}| \vee |\xi_{1,n}| \geq Kt^{1/2}$ for a given $K > 0$,*

$$\begin{aligned} \mathbb{P}[Y_t > \xi_{n+1} - x_{n+1}] &\leq C_{5.2} \left\{ \exp \left(-C_{5.2}^{-1} \frac{U_t^k(x, \xi)^2}{(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1})^2 t^3} \right) \right. \\ &\left. + \exp \left(-C_{5.2}^{-1} \frac{|x_{1,n}|^2 + |\xi_{1,n}|^2}{t} \right) \sum_{i=1}^{k/2} \exp \left(-C_{5.2}^{-1} \frac{U_t^k(x, \xi)^{1/i}}{\{|x_{1,n}|^{k-2i} + |\xi_{1,n}|^{k-2i}\}^{1/i} t^{1+1/i}} \right) \right\}. \end{aligned}$$

(ii) *If $|x_{1,n}| \vee |\xi_{1,n}| \leq Kt^{1/2}$ for the same previous K ,*

$$\mathbb{P}[Y_t > \xi_{n+1} - x_{n+1}] \leq C_{5.2} \left\{ \exp \left(-C_{5.2}^{-1} \frac{U_t^k(x, \xi)^2}{t^{k+2}} \right) + \sum_{i=1}^{k/2} \exp \left(-C_{5.2}^{-1} \frac{U_t^k(x, \xi)^{1/i}}{t^{k/(2i)+1/i}} \right) \right\}.$$

Proof. We only prove point (i), the second point can be derived in a similar way. From equation (5.8), we have

$$\begin{aligned}
\mathbb{P}[Y_t > \xi_{n+1} - x_{n+1}] &= \mathbb{P}\left[\int_0^t |m(u, t, x_{1,n}, \xi_{1,n}) + W_u^{0,t}|^k du > \xi_{n+1} - x_{n+1}\right] \\
&= \mathbb{P}\left[k \int_0^t |m(u, t, x_{1,n}, \xi_{1,n})|^{k-2} \langle m(u, t, x_{1,n}, \xi_{1,n}), W_u^{0,t} \rangle du \right. \\
&\quad \left. + R_t^k(x_{1,n}, \xi_{1,n}) > \xi_{n+1} - x_{n+1} - \int_0^t |m(u, t, x_{1,n}, \xi_{1,n})|^k du\right], \\
R_t^k(x_{1,n}, \xi_{1,n}) &:= \frac{k}{2} \int_0^t |m(u, t, x_{1,n}, \xi_{1,n})|^{k-2} |W_u^{0,t}|^2 du \\
&+ \sum_{i=2}^{k/2} C_{k/2}^i \int_0^t |m(u, t, x_{1,n}, \xi_{1,n})|^{k-2i} (2 \langle m(u, t, x_{1,n}, \xi_{1,n}), W_u^{0,t} \rangle + |W_u^{0,t}|^2)^i du.
\end{aligned}$$

Observe now that all the terms in $R_t^k(x_{1,n}, \xi_{1,n})$ have characteristic time scales that are in small time neglectable w.r.t. the one of the Gaussian contribution:

$$\begin{aligned}
&k \int_0^t |m(u, t, x_{1,n}, \xi_{1,n})|^{k-2} \langle m(u, t, x_{1,n}, \xi_{1,n}), W_u^{0,t} \rangle du \\
&\leq 2^{k-2} (|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1}) t \sup_{u \in [0,t]} |W_u^{0,t}| := M_t^k(x_{1,n}, \xi_{1,n}).
\end{aligned}$$

Since by definition $U_t^k(x, \xi) < \xi_{n+1} - x_{n+1}$ and $Y_t \leq (M_t^k + R_t^k)(x_{1,n}, \xi_{1,n})$, one gets:

$$\begin{aligned}
\mathbb{P}[Y_t > \xi_{n+1} - x_{n+1}] &\leq \mathbb{P}[(M_t^k + R_t^k)(x_{1,n}, \xi_{1,n}) > U_t^k(x, \xi)] \leq \mathbb{P}[2M_t^k(x_{1,n}, \xi_{1,n}) > U_t^k(x, \xi)] \\
&+ \mathbb{P}[(M_t^k + R_t^k)(x_{1,n}, \xi_{1,n}) > U_t^k(x, \xi)]^{1/2} \\
&\quad \times \mathbb{P}[R_t^k(x_{1,n}, \xi_{1,n}) \geq M_t^k(x_{1,n}, \xi_{1,n})]^{1/2}. \tag{5.31}
\end{aligned}$$

Standard computations now give that there exist $C_1 := C_1(k)$, $C_2 := C_2(n, k) \geq 1$ s.t.

$$\begin{aligned}
\mathbb{P}[R_t^k(x_{1,n}, \xi_{1,n}) \geq M_t^k(x_{1,n}, \xi_{1,n})] &\leq (k-1) \mathbb{P}\left[\sup_{u \in [0,t]} |W_u^{0,t}| \geq C_1 \{|x_{1,n}| + |\xi_{1,n}|\}\right] \\
&\leq C_2 \exp\left(-C_2^{-1} \frac{|x_{1,n}|^2 + |\xi_{1,n}|^2}{t}\right), \\
\mathbb{P}[M_t^k(x_{1,n}, \xi_{1,n}) > U_t^k(x, \xi)/2] &\leq C_2 \exp\left(-C_2^{-1} \frac{U_t^k(x, \xi)^2}{(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1})t^3}\right), \tag{5.32}
\end{aligned}$$

using once again Lévy's identity (5.20) for the last two inequalities. On the other hand, we have:

$$\begin{aligned}
&\mathbb{P}[(M_t^k + R_t^k)(x_{1,n}, \xi_{1,n}) \geq U_t^k(x, \xi)] \\
&\leq \mathbb{P}[M_t^k(x_{1,n}, \xi_{1,n}) \geq \frac{1}{2} U_t^k(x, \xi)] + \mathbb{P}[R_t^k(x_{1,n}, \xi_{1,n}) \geq \frac{1}{2} U_t^k(x, \xi)].
\end{aligned}$$

Now,

$$\begin{aligned}
& \mathbb{P}[R_t^k(x_{1,n}, \xi_{1,n}) \geq \frac{1}{2}U_t^k(x, \xi)] \\
& \leq \mathbb{P}\left[\frac{2^{(k-3)\vee 0}k}{2(k-1)}\{|x_{1,n}|^{k-2} + |\xi_{1,n}|^{k-2}\}t \sup_{u \in [0,t]} |W_u^{0,t}|^2 \geq \frac{1}{2(k-1)}U_t^k(x, \xi)\right] \\
& + \sum_{i=2}^{k/2} \left\{ \mathbb{P}\left[C_{k/2}^i \frac{2^{k-i-2}}{k-2i+1} \{|x_{1,n}|^{k-2i} + |\xi_{1,n}|^{k-2i}\}t \sup_{u \in [0,t]} |W_u^{0,t}|^{2i} \geq \frac{1}{2(k-1)}U_t^k(x, \xi)\right] \right. \\
& \quad \left. + \mathbb{P}\left[C_{k/2}^i \frac{2^{k+i-2}}{k-i+1} \{|x_{1,n}|^{k-i} + |\xi_{1,n}|^{k-i}\}t \sup_{u \in [0,t]} |W_u^{0,t}|^i \geq \frac{1}{2(k-1)}U_t^k(x, \xi)\right] \right\} \\
& \qquad \qquad \qquad := P_1 + \sum_{i=2}^{k/2} (P_2^i + P_3^i).
\end{aligned}$$

One gets that there exists $C_3 := C_3(k, n) \geq 1$ s.t.

$$\begin{aligned}
P_1 & \leq C_3 \exp\left(-C_3^{-1} \frac{U_t^k(x, \xi)}{\{|x_{1,n}|^{k-2} + |\xi_{1,n}|^{k-2}\}t^2}\right), \quad \forall i \in \llbracket 2, k/2 \rrbracket, \\
P_2^i & \leq C_3 \exp\left(-C_3^{-1} \frac{U_t^k(x, \xi)^{1/i}}{\{|x_{1,n}|^{k-2i} + |\xi_{1,n}|^{k-2i}\}^{1/i}t^{1+1/i}}\right), \\
P_3^i & \leq C_3 \exp\left(-C_3^{-1} \frac{U_t^k(x, \xi)^{2/i}}{\{|x_{1,n}|^{k-i} + |\xi_{1,n}|^{k-i}\}^{2/i}t^{1+2/i}}\right).
\end{aligned}$$

Hence, from (5.32), there exist $C_4 := C_4(n, k)$, $C_5 := C_5(n, k)$ s.t.

$$\begin{aligned}
\mathcal{P} & := \mathbb{P}[(M_t^k + R_t^k)(x_{1,n}, \xi_{1,n}) > U_t^k(x, \xi)]^{1/2} \mathbb{P}[R_t^k(x_{1,n}, \xi_{1,n}) \geq M_t^k(x_{1,n}, \xi_{1,n})]^{1/2} \\
& \leq C_4 \left\{ \exp\left(-C_4^{-1} \frac{|x_{1,n}|^2 + |\xi_{1,n}|^2}{t}\right) \times \left[\exp\left(-C_4^{-1} \frac{U_t^k(x, \xi)^2}{(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1})^2 t^3}\right) \right. \right. \\
& + \exp\left(-C_4^{-1} \frac{U_t^k(x, \xi)}{\{|x_{1,n}|^{k-2} + |\xi_{1,n}|^{k-2}\}t^2}\right) + \sum_{i=2}^{k/2} \exp\left(-C_4^{-1} \frac{U_t^k(x, \xi)^{1/i}}{\{|x_{1,n}|^{k-2i} + |\xi_{1,n}|^{k-2i}\}^{1/i}t^{1+1/i}}\right) \\
& \quad \left. \left. + \exp\left(-C_4^{-1} \frac{U_t^k(x, \xi)^{2/i}}{\{|x_{1,n}|^{k-i} + |\xi_{1,n}|^{k-i}\}^{2/i}t^{1+2/i}}\right) \right] \right\} \\
& \leq C_5 \left\{ \exp\left(-C_4^{-1} \frac{|x_{1,n}|^2 + |\xi_{1,n}|^2}{t}\right) \times \left[\exp\left(-C_4^{-1} \frac{U_t^k(x, \xi)^2}{(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1})^2 t^3}\right) \right. \right. \\
& \quad \left. \left. + \sum_{i=1}^{k/2} \exp\left(-C_5^{-1} \frac{U_t^k(x, \xi)^{1/i}}{\{|x_{1,n}|^{k-2i} + |\xi_{1,n}|^{k-2i}\}^{1/i}t^{1+1/i}}\right) \right] \right\},
\end{aligned}$$

which together with (5.31), (5.32) gives the claim.

5.3.2 Auxiliary deviation estimates

Still from the Malliavin representation of the density given by (5.24), when $\xi_{n+1} - x_{n+1}$ is small, that is when for the degenerate component the starting and final points are close, we have to give estimates on $\mathbb{P}[Y_t \leq \xi_{n+1} - x_{n+1}]$ (small and moderate deviations).

Proposition 5.4 (Estimates for the cost) *There exist constants $(c_1, c_2) := (c_1, c_2)(n, k)$ s.t. for all $(x_{1,n}, \xi_{1,n}) \in (\mathbb{R}^n \setminus \{0\})^2$, $\xi_{n+1} > x_{n+1}$ and $t \geq 12 \times 2^k \frac{\xi_{n+1} - x_{n+1}}{|x_{1,n}|^k + |\xi_{1,n}|^k}$:*

$$\mathbb{P}[Y_t \leq \xi_{n+1} - x_{n+1}] \leq c_1 \exp\left(-c_2 \frac{|x_{1,n}|^{2+k} + |\xi_{1,n}|^{2+k}}{\xi_{n+1} - x_{n+1}}\right). \quad (5.33)$$

On the other hand, for a given $K \geq 0$ and $|x_{1,n}| \vee |\xi_{1,n}| \leq Kt^{1/2}$, $t \geq [(\xi_{n+1} - x_{n+1}) \frac{3}{4} (64K)^k]^{2/(k+2)}$, there exist $(\bar{c}_1, \bar{c}_2) := (\bar{c}_1, \bar{c}_2)(n, k)$:

$$\mathbb{P}[Y_t \leq \xi_{n+1} - x_{n+1}] \leq \bar{c}_1 \exp\left(-\bar{c}_2 \frac{t^{1+2/k}}{(\xi_{n+1} - x_{n+1})^{2/k}}\right). \quad (5.34)$$

Proof. We first begin with the proof of (5.33). As in the previous sections, we can assume w.l.o.g. that $|x_{1,n}| \geq \frac{|\xi_{1,n}|}{2}$. For $s \in [0, t]$, we define $\tilde{X}_s := x_{1,n} \frac{t-s}{t} + \xi_{1,n} \frac{s}{t} + W_s^{0,t}$ (where $(W_s^{0,t})_{s \in [0, t]}$ is a standard n -dimensional Brownian Bridge on $[0, t]$), so that $Y_t = \int_0^t |\tilde{X}_s|^k ds$. Let us also set $\tau_{|x_{1,n}|/2} := \inf\{s \geq 0 : |\tilde{X}_s| \leq |x_{1,n}|/2\}$. Consider now the event $A := \{\tau_{|x_{1,n}|/2} \leq 2^k \frac{\xi_{n+1} - x_{n+1}}{|x_{1,n}|^k}\}$ and denote by A^C its complementary. Observe that $\mathbb{P}[\int_0^t |\tilde{X}_s|^k ds \leq \xi_{n+1} - x_{n+1}, A^C] = \mathbb{P}[\int_0^{2^k \frac{\xi_{n+1} - x_{n+1}}{|x_{1,n}|^k}} \left(\frac{|x_{1,n}|}{2}\right)^k ds < \int_0^t |\tilde{X}_s|^k ds \leq \xi_{n+1} - x_{n+1}, A^C] = 0$. Thus, $\mathbb{P}[Y_t \leq \xi_{n+1} - x_{n+1}] = \mathbb{P}[Y_t \leq \xi_{n+1} - x_{n+1}, A] \leq \mathbb{P}[A]$. Now

$$\begin{aligned} \mathbb{P}[A] &\leq \mathbb{P}\left[\inf_{s \in [0, 2^k \frac{\xi_{n+1} - x_{n+1}}{|x_{1,n}|^k}]} |\tilde{X}_s| \leq |x_{1,n}|/2\right] \\ &\leq \mathbb{P}\left[\inf_{s \in [0, 2^k \frac{\xi_{n+1} - x_{n+1}}{|x_{1,n}|^k}]} \left|x_{1,n} \frac{t-s}{t} + \xi_{1,n} \frac{s}{t}\right| + \inf_{s \in [0, 2^k \frac{\xi_{n+1} - x_{n+1}}{|x_{1,n}|^k}]} (-|W_s^{0,t}|) \leq |x_{1,n}|/2\right] \\ &\leq \mathbb{P}\left[|x_{1,n}|/2 + \inf_{s \in [0, 2^k \frac{\xi_{n+1} - x_{n+1}}{|x_{1,n}|^k}]} \left(-\frac{s}{t}\right)\{|x_{1,n}| + |\xi_{1,n}|\} - \sup_{s \in [0, 2^k \frac{\xi_{n+1} - x_{n+1}}{|x_{1,n}|^k}]} |W_s^{0,t}| \leq 0\right] \\ &\leq \mathbb{P}\left[|x_{1,n}|(1/2 - 3 \frac{2^k (\xi_{n+1} - x_{n+1})}{|x_{1,n}|^k t}) \leq \sup_{s \in [0, 2^k \frac{\xi_{n+1} - x_{n+1}}{|x_{1,n}|^k}]} |W_s^{0,t}|\right] \\ &\leq \mathbb{P}\left[|x_{1,n}|/4 \leq \sup_{s \in [0, 2^k \frac{\xi_{n+1} - x_{n+1}}{|x_{1,n}|^k}]} |W_s^{0,t}|\right], \end{aligned}$$

recalling $|x_{1,n}| \geq |\xi_{1,n}|/2$ and $t \geq 12 \times 2^k \frac{\xi_{n+1} - x_{n+1}}{|x_{1,n}|^k}$ for the last two inequalities. From Lévy's identity (5.20) and usual controls on the Gaussian distribution function, we obtain:

$$\mathbb{P}[Y_t \leq \xi_{n+1} - x_{n+1}] \leq \mathbb{P}[A] \leq c_1 \exp\left(-c_2 \frac{|x_{1,n}|^{2+k}}{\xi_{n+1} - x_{n+1}}\right),$$

which from the assumption $|x_{1,n}| \geq |\xi_{1,n}|/2$ gives (5.33) up to a modification of c_2 .

Let us now turn to (5.34). Introduce $I_\beta(t) := \int_0^t \mathbb{I}_{|\tilde{X}_s|^k \leq \beta(\xi_{n+1} - x_{n+1})} ds$ for a parameter $\beta > 0$ to be fixed later on. Define the set $A_\beta := \{I_\beta(t) \geq t/4\}$. Observe that

$$\begin{aligned} & \mathbb{P}\left[\int_0^t |\tilde{X}_s|^k ds \leq \xi_{n+1} - x_{n+1}, A_\beta^C\right] = \\ & \mathbb{P}\left[\int_0^t \mathbb{I}_{|\tilde{X}_s|^k > \beta(\xi_{n+1} - x_{n+1})} |\tilde{X}_s|^k ds \leq \int_0^t |\tilde{X}_s|^k ds \leq \xi_{n+1} - x_{n+1}, A_\beta^C\right] \\ & \leq \mathbb{P}[\beta(\xi_{n+1} - x_{n+1})3t/4 < \int_0^t |\tilde{X}_s|^k ds \leq \xi_{n+1} - x_{n+1}, A_\beta^C]. \end{aligned}$$

Choosing $\beta = \frac{4}{3t}$ we get from the above inequality $\mathbb{P}[\int_0^t |\tilde{X}_s|^k ds \leq \xi_{n+1} - x_{n+1}, A_\beta^C] = 0$. Hence,

$$\begin{aligned} & \mathbb{P}\left[\int_0^t |\tilde{X}_s|^k ds \leq \xi_{n+1} - x_{n+1}\right] = \mathbb{P}\left[\int_0^t |\tilde{X}_s|^k ds \leq \xi_{n+1} - x_{n+1}, A_{\frac{4}{3t}}\right] \leq \mathbb{P}\left[A_{\frac{4}{3t}}\right] \\ & \leq \mathbb{P}\left[\int_0^t \mathbb{I}_{|\tilde{X}_s|^k \leq \frac{3(\xi_{n+1} - x_{n+1})}{4t}} ds > t/4\right] \leq \mathbb{P}\left[\int_0^t \mathbb{I}_{|\tilde{X}_s|^k \leq \frac{3(\xi_{n+1} - x_{n+1})}{4t}} ds > t/4\right] \\ & \leq \mathbb{P}\left[\int_0^t \mathbb{I}_{|x_1 \frac{t-s}{t} + \xi_1 \frac{s}{t} + B_s^{0,t}| \leq c(x,\xi,t,k)} ds > t/4\right], \quad c(x,\xi,t,k) := \left(\frac{3(\xi_{n+1} - x_{n+1})}{4t}\right)^{1/k}, \\ & \leq \mathbb{P}\left[\int_0^{t/2} \mathbb{I}_{|x_1 \frac{t-s}{t} + \xi_1 \frac{s}{t} + B_s^{0,t}| \leq c(x,\xi,t,k)} ds > t/8\right] + \mathbb{P}\left[\int_{t/2}^t \mathbb{I}_{|x_1 \frac{t-s}{t} + \xi_1 \frac{s}{t} + B_s^{0,t}| \leq c(x,\xi,t,k)} ds > t/8\right] \\ & := P_1 + P_2, \end{aligned} \tag{5.35}$$

where $(B_s^{0,t})_{s \in [0,t]}$ stands for a one-dimensional Brownian bridge on $[0,t]$. Observing that $(\bar{B}_s^{0,t}) := (B_{t-s}^{0,t})_{s \in [0,t]}$ is also a Brownian bridge, we get that

$$\begin{aligned} P_2 & := \mathbb{P}\left[\int_0^{t/2} ds \mathbb{I}_{|x_1 \frac{s}{t} + \xi_1 \frac{t-s}{t} + \bar{B}_s^{0,t}| \leq c(x,\xi,t,k)} ds > t/8\right] \\ & = \mathbb{P}\left[\int_0^{t/2} ds \mathbb{I}_{|x_1 \frac{s}{t} + \xi_1 \frac{t-s}{t} + B_s^{0,t}| \leq c(x,\xi,t,k)} ds > t/8\right]. \end{aligned}$$

Since we assumed $|x_1| \vee |\xi_1| \leq Kt^{1/2}$, $|x_1|$ and $|\xi_1|$ have at most the same magnitude so that P_1 and P_2 can be handled exactly in the same way. Let us deal with P_1 . The occupation time formula for semimartingales (see Chapter 6 in [29]) yields

$$\int_0^{t/2} \mathbb{I}_{|x_1 \frac{t-s}{t} + \xi_1 \frac{s}{t} + B_s^{0,t}| \leq c(x,\xi,t,k)} ds = \int_{-c(x,\xi,t,k)}^{c(x,\xi,t,k)} dz L_{t/2}^z,$$

where $L_{t/2}^z$ stands for the local time at level z and time $t/2$ of the process $(x_1 \frac{t-s}{t} + \xi_1 \frac{s}{t} + B_s^{0,t})_{s \in [0,t]}$. From the definition of P_1 in (5.35):

$$\begin{aligned} P_1 &\leq \mathbb{P}\left[\sup_{z \in [-c(x,\xi,t,k), c(x,\xi,t,k)]} L_{t/2}^z \times 2c(x,\xi,t,k) > \frac{t}{8}\right] \\ &= \mathbb{P}\left[\sup_{z \in [-\frac{c(x,\xi,t,k)}{t^{1/2}}, \frac{c(x,\xi,t,k)}{t^{1/2}}]} \bar{L}_{1/2}^z > \frac{t^{1/2}}{16c(x,\xi,t,k)}\right], \end{aligned} \quad (5.36)$$

where $\bar{L}_{1/2}^z$ stands for the local time at level z and time $1/2$ for the scalar process

$$(\bar{X}_u)_{u \in [0,1]} := \left(\frac{x_1}{t^{1/2}}(1-u) + \frac{\xi_1}{t^{1/2}}u + \frac{B_{ut}^{0,t}}{t^{1/2}} \right)_{u \in [0,1]} \stackrel{(\text{law})}{=} \left(\frac{x_1}{t^{1/2}}(1-u) + \frac{\xi_1}{t^{1/2}}u + B_u^{0,1} \right)_{u \in [0,1]}.$$

The last equality in (5.36) is a consequence of the scaling properties of the local time. From Tanaka's formula for semimartingales $\bar{L}_{1/2}^z = |\bar{X}_{1/2} - z| - |\bar{X}_0 - z| - \int_0^{1/2} \text{sgn}(\bar{X}_s - z) d\bar{X}_s$. Denoting with a slight abuse of notation $(\frac{B_{ut}^{0,t}}{t^{1/2}})_{u \in [0,1]} = (B_u^{0,1})_{u \in [0,1]}$, we have the following differential dynamics for \bar{X}_u :

$$d\bar{X}_u = -\frac{x_1 - \xi_1}{t^{1/2}} du + dB_u^{0,1} = -\frac{\bar{X}_u - \xi_1}{1-u} du + dB_u,$$

where $(B_u)_{u \in [0,1]}$ is a standard scalar Brownian motion.

Therefore, from equation (5.36) and the usual differential dynamics for the Brownian bridge:

$$\begin{aligned} P_1 &\leq \mathbb{P}\left[\frac{|\xi_1 - x_1|}{2t^{1/2}} + |B_{1/2}^{0,1}| \right. \\ &\quad \left. + \sup_{z \in [-\frac{c(x,\xi,t,k)}{t^{1/2}}, \frac{c(x,\xi,t,k)}{t^{1/2}}]} \left| \int_0^{1/2} \text{sgn}(\bar{X}_s - z) d\left(-\frac{x_1 - \xi_1}{t^{1/2}} ds + dB_s^{0,1}\right) \right| \geq \frac{t^{1/2}}{8c(x,\xi,t,k)}\right] \\ &\leq \mathbb{P}\left[\frac{|\xi_1 - x_1|}{t^{1/2}} + |B_{1/2}^{0,1}| + \right. \\ &\quad \left. \int_0^{1/2} ds \frac{|B_s^{0,1}|}{1-s} + \sup_{z \in [-\frac{c(x,\xi,t,k)}{t^{1/2}}, \frac{c(x,\xi,t,k)}{t^{1/2}}]} \left| \int_0^{1/2} \text{sgn}(\bar{X}_s - z) dB_s \right| \geq \frac{t^{1/2}}{16c(x,\xi,t,k)}\right] \\ &\leq \mathbb{P}\left[2K + 3 \sup_{s \in [0,1/2]} |B_s^{0,1}| + \sup_{z \in [-\frac{c(x,\xi,t,k)}{t^{1/2}}, \frac{c(x,\xi,t,k)}{t^{1/2}}]} \left| \int_0^{1/2} \text{sgn}(\bar{X}_s - z) dB_s \right| \geq \frac{t^{1/2}}{16c(x,\xi,t,k)}\right]. \end{aligned}$$

Now from the definition of $c(x, \xi, t, k)$ in (5.35), for $t \geq [(\xi_{n+1} - x_{n+1})\frac{3}{4}(64K)^k]^{2/(k+2)}$ one has $\frac{t^{1/2}}{16c(x, \xi, t, k)} - 2K \geq \frac{t^{1/2}}{32c(x, \xi, t, k)}$. Thus

$$P_1 \leq \mathbb{P}\left[3 \sup_{s \in [0, 1/2]} |B_s^{0,1}| \geq \frac{t^{1/2}}{64c(x, \xi, t, k)}\right] \\ + \mathbb{P}\left[\sup_{z \in [-\frac{c(x, \xi, t, k)}{t^{1/2}}, \frac{c(x, \xi, t, k)}{t^{1/2}}]} \left| \int_0^{1/2} \operatorname{sgn}(\bar{X}_s - z) dB_s \right| \geq \frac{t^{1/2}}{64c(x, \xi, t, k)}\right].$$

Setting for all $t \in [0, 1/2]$, $M_t := \int_0^t \operatorname{sgn}(\bar{X}_s - z) dB_s$, $\tilde{M}_t := \tilde{B}_{\langle M \rangle_t} = \tilde{B}_t$ (i.e. \tilde{B} is the Dambis-Dubbins-Schwarz Brownian motion associated to M). Hence, from Lévy's identity (5.20) and usual computations we derive the announced bound for P_1 . Since P_2 can be handled in a similar way, the claim then follows from equation (5.35).

5.4 Final derivation of the upper-bounds in the various regimes

In this section we put together our previous estimates in order to derive the upper bounds of Theorem 2.1 in the various regimes.

5.4.1 Derivation of the Gaussian upper bounds

In this paragraph we assume $|x_{1,n}| \vee |\xi_{1,n}| \geq Kt^{1/2}$ for K large enough. We also suppose $\frac{|\xi_{n+1} - x_{n+1} - ct(|x_{1,n}|^k + |\xi_{1,n}|^k)|}{t^{3/2}(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1})} \leq \bar{C}$ where $c := c(k) = 2 + \frac{2^{k-1}}{k+1}$ and \bar{C} is fixed. From Corollary 5.1 (representation of the density), Proposition 5.2 (controls of the weight in the integration by part) and Lemma 5.2 (deviation bounds), we have that there exists $C := C(n, k, K, \bar{C}) \geq 1$, s.t. setting $U_t^k(x, \xi) := \xi_{n+1} - x_{n+1} - \frac{2^{k-1}}{k+1}(|x_{1,n}|^k + |\xi_{1,n}|^k)t$ as in Lemma 5.2 one has:

$$p(t, x, \xi) \leq \frac{C \exp\left(-\frac{|\xi_{1,n} - x_{1,n}|^2}{2t} - C^{-1} \frac{U_t^k(x, \xi)^2}{(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1})^2 t^3}\right)}{t^{n/2+3/2}(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1})}. \quad (5.37)$$

Remark 5.1 *The above result means that the Gaussian regime holds if the final point $\xi_{1,n}$ of the degenerate component has the same order as the “mean” transport term $m_t(x, \xi) := x_{n+1} + \frac{2^{k-1}}{k+1}(|x_{1,n}|^k + |\xi_{1,n}|^k)t$ (moderate deviations). A similar lower bound holds true, see Lemma 5.3.*

5.4.2 Derivation of the heavy-tailed upper bounds

We here assume $\frac{|\xi_{n+1} - x_{n+1} - ct(|x_{1,n}|^k + |\xi_{1,n}|^k)|}{t^{3/2}(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1})} \geq \bar{C}$ where $c := c(k) = 2 + \frac{2^{k-1}}{k+1}$ and \bar{C} is as in the previous paragraph.

If $|x_{1,n}| \vee |\xi_{1,n}| \leq Kt^{1/2}$ (K being as in the previous paragraph), then Corollary 5.1, Proposition 5.3 and Lemma 5.2 yield that there exists $C := C(n, k) \geq 1$ s.t.

$$p(t, x, \xi) \leq \frac{C}{t^{(n+k)/2+1}} \exp\left(-\frac{|\xi_{1,n} - x_{1,n}|^2}{2t} - C^{-1} \frac{(U_t^k(x, \xi))^{2/k}}{t^{1+2/k}}\right). \quad (5.38)$$

On the other hand if $|x_{1,n}| \vee |\xi_{1,n}| \geq Kt^{1/2}$, then Corollary 5.1, Proposition 5.2 and Lemma 5.2 yield that there exists $\tilde{C} := \tilde{C}(n, k) \geq 1$ s.t.

$$\begin{aligned} p(t, x, \xi) &\leq \frac{\tilde{C}}{t^{n/2+3/2}(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1})} \exp\left(-\frac{|\xi_{1,n} - x_{1,n}|^2}{2t} - \tilde{C}^{-1} \frac{U_t^k(x, \xi)}{t^{1+2/k}}\right) \\ &\leq \frac{\tilde{C}}{K^{k-1}t^{(n+k)/2+1}} \exp\left(-\frac{|\xi_{1,n} - x_{1,n}|^2}{2t} - \tilde{C}^{-1} \frac{U_t^k(x, \xi)}{t^{1+2/k}}\right). \end{aligned}$$

Hence, up to a modification of C , the control given by (5.38) holds for all off-diagonal cases.

5.4.3 Moderate deviations of the degenerate component

In this paragraph we suppose $0 < \xi_{n+1} - x_{n+1} \leq Kt^{1+k/2}$, for K sufficiently small. This means that the deviation of the degenerate component is small w.r.t. its characteristic time scale. From Corollary 5.1, Propositions 5.2 and 5.3 and Proposition 5.4 we derive similarly to the previous paragraph that there exists $C := C(n, k, K)$ s.t.

$$p(t, x, \xi) \leq \frac{C \exp\left(-\frac{|\xi_{1,n} - x_{1,n}|^2}{2t} - C^{-1} \left\{ \frac{|x_{1,n}|^{2+k} + |\xi_{1,n}|^{2+k}}{\xi_{n+1} - x_{n+1}} + \frac{t^{1+2/k}}{(\xi_{n+1} - x_{n+1})^{2/k}} \right\}\right)}{t^{(n+k)/2+1}}.$$

5.5 Gaussian lower bound on the compact sets of the metric

We conclude this section with a proof of a lower bound for the density on the compact sets of the metric associated to the Gaussian regime in Theorem 2.1. A similar feature already appears in the appendix of [19].

Lemma 5.3 *Assume that $|x_{1,n}| \vee |\xi_{1,n}| \geq Kt^{1/2}$, $K \geq K_0 := K_0(n, k)$ and that for a given $\bar{C} \geq 0$ we have $\frac{|\xi_{n+1} - x_{n+1} - ct(|x_{1,n}|^k + |\xi_{1,n}|^k)|}{t^{3/2}(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1})} \leq \bar{C}$ where $c := c(k)$ is fixed. Then, there exists $C_{5.3} := C_{5.3}(n, k, \bar{C})$ s.t.*

$$\frac{C_{5.3}}{(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1})t^{3/2}} \leq p_{Y_t}(\xi_{n+1} - x_{n+1}).$$

Proof. Let us assume w.l.o.g. that $\xi_{n+1} - x_{n+1} - ct(|x_{1,n}|^k + |\xi_{1,n}|^k) \geq 0$. From (5.24) we recall:

$$p_{Y_t}(\xi_{n+1} - x_{n+1}) = \mathbb{E}[H_t \mathbb{1}_{Y_t \geq \xi_{n+1} - x_{n+1}}].$$

As in the proof of Proposition 5.2, from which we use the notations, we assume w.l.o.g. that $|\xi_{1,n}| \geq \frac{|x_{1,n}|}{2}$. One gets:

$$\begin{aligned} p_{Y_t}(\xi_{n+1} - x_{n+1}) &\geq \mathbb{E}[H_t^2 \mathbb{1}_{Y_t \geq \xi_{n+1} - x_{n+1}}] - \mathbb{E}[|H_t^1|] \geq \mathbb{E}\left[\frac{I_1(g_0(\cdot, t))}{\gamma_{Y_t}} \mathbb{1}_{Y_t \geq \xi_{n+1} - x_{n+1}}\right] \\ &\quad - \left\{ \mathbb{E}\left[\frac{|\sum_{l=2}^k I_l(g_{l-1}(\cdot, t))|}{\gamma_{Y_t}}\right] + \frac{\bar{C}_1}{t|\xi_{1,n}|^k} \right\}, \end{aligned}$$

using the bound for $\mathbb{E}[|H_t^1|]$ given by equation (5.28), with $\bar{C}_1 := \bar{C}_1(n, k, 1, K)$, in the last inequality. From the definitions in the proof of Proposition 5.2, we have that there exists $\bar{C}_2 := \bar{C}_2(n, k)$, $\mathbb{E}[|\sum_{l=2}^k I_l(g_{l-1}(\cdot, t))|^2]^{1/2} \leq \bar{C}_2 t^{3/2} |\xi_{1,n}|^{k-1} \sum_{l=2}^k \left(\frac{t^{1/2}}{|\xi_{1,n}|}\right)^{l-1} \leq \frac{(k-1)\bar{C}_2}{K} t^{3/2} |\xi_{1,n}|^{k-1}$, recalling $|\xi_{1,n}| \leq K t^{1/2}$ for the last inequality. Also $I_1(g_0(\cdot, t)) = \int_0^t \mathbb{E}[\bar{M}_1(u, t)] dW_u + R_0^t$ where $\mathbb{E}[|R_0^t|^2]^{1/2} \leq \frac{\bar{C}_2}{K} t^{3/2} |\xi_{1,n}|^{k-1}$. Hence writing from (5.8)

$$\begin{aligned} Y_t &= \int_0^t du |m(u, t, x_{1,n}, \xi_{1,n})|^k + k \int_0^t du |m(u, t, x_{1,n}, \xi_{1,n})|^{k-2} \langle m(u, t, x_{1,n}, \xi_{1,n}), W_u^{0,t} \rangle + R_t^k \\ &:= m_t^k(x_{1,n}, \xi_{1,n}) + G_t^k + R_t^k, \end{aligned}$$

we obtain using Proposition 5.2:

$$\begin{aligned} p_{Y_t}(\xi_{n+1} - x_{n+1}) &\geq \mathbb{E} \left[\frac{\int_0^t \mathbb{E}[\bar{M}_1(u, t)] dW_u}{\gamma_{Y_t}} \mathbb{1}_{m_t^k(x_{1,n}, \xi_{1,n}) + G_t^k + R_t^k \geq \xi_{n+1} - x_{n+1}} \right] \\ &\quad - \left\{ \frac{C_2 \bar{C}_2 k t^{3/2} |\xi_{1,n}|^{k-1}}{K |\xi_{1,n}|^{2(k-1)} t^3} + \frac{\bar{C}_1}{K t^{3/2} |\xi_{1,n}|^{k-1}} \right\} \\ &:= p_{Y_t}^1(\xi_{n+1} - x_{n+1}) - r^1(t, x, \xi). \end{aligned}$$

From the martingale representation theorem and the above computations we identify $G_t^k = \int_0^t \mathbb{E}[\bar{M}_1(u, t)] dW_u$. Still from Proposition 5.2 we get:

$$\begin{aligned} p_{Y_t}(\xi_{n+1} - x_{n+1}) &\geq \mathbb{E} \left[\frac{G_t^k}{\gamma_{Y_t}} \mathbb{1}_{G_t^k + R_t^k \geq \xi_{n+1} - x_{n+1} - m_t^k(x_{1,n}, \xi_{1,n})} \mathbb{1}_{|R_t^k| \leq |G_t^k|/2} \right] \\ &\quad - \frac{\bar{C}_3 \mathbb{P}[|R_t^k| > |G_t^k|/2]^{1/2}}{(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1}) t^{3/2}} - r^1(t, x, \xi) \\ &= p_{Y_t}^2(\xi_{n+1} - x_{n+1}) - r^2(t, x, \xi), \end{aligned}$$

where $\bar{C}_3 := \bar{C}_3(n, k)$. One easily gets that there exists $c := c(k) > 0$, $m_t^k(x_{1,n}, \xi_{1,n}) \geq ct(|x_{1,n}|^k + |\xi_{1,n}|^k)$. Thus, setting $U_t^k(x, \xi) := \xi_{n+1} - x_{n+1} - ct(|x_{1,n}|^k + |\xi_{1,n}|^k)$ and recalling as well that $U_t^k(x, \xi) \geq 0$, one obtains that on the event $\{G_t^k + R_t^k \geq U_t^k(x, \xi), |R_t^k| \leq |G_t^k|/2\}$,

$G_t^k \geq 0$. Hence:

$$\begin{aligned}
p_{Y_t}(\xi_{n+1} - x_{n+1}) &\geq \mathbb{E}\left[\frac{G_t^k}{\gamma_{Y_t}} \mathbb{I}_{G_t^k - |R_t^k| \geq U_t^k(x, \xi) \geq 0} \mathbb{I}_{|R_t^k| \leq |G_t^k|/2}\right] - r^2(t, x, \xi) \\
&\geq \mathbb{E}\left[\frac{G_t^k}{\gamma_{Y_t}} \mathbb{I}_{G_t^k \geq 2U_t^k(x, \xi) \geq 0} \mathbb{I}_{|R_t^k| \leq G_t^k/2}\right] - r^2(t, x, \xi) \\
&\geq \mathbb{E}\left[\frac{G_t^k}{3M_t} \mathbb{I}_{G_t^k \geq 2U_t^k(x, \xi) \geq 0} \mathbb{I}_{|R_t^k| \leq G_t^k/2} \mathbb{I}_{\gamma_{Y_t} \leq 3M_t}\right] - \frac{\bar{C}_3 \mathbb{P}[\gamma_{Y_t} > 3M_t]^{1/2}}{|\xi_{1,n}|^{k-1} t^{3/2}} \\
&\quad - r^2(t, x, \xi) \\
&\geq \mathbb{E}\left[\frac{G_t^k}{3M_t} \mathbb{I}_{G_t^k \geq 2Ct^{3/2}(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1})} \mathbb{I}_{|R_t^k| \leq G_t^k/2} \mathbb{I}_{\gamma_{Y_t} \leq 3M_t}\right] - r^3(t, x, \xi) \\
&\geq \frac{2C\bar{C}\mathbb{P}[G_t^k \geq 2Ct^{3/2}(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1}), |R_t^k| \leq G_t^k/2]}{3t^{3/2}(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1})} \\
&\quad - \frac{\bar{C}_3 \mathbb{P}[\gamma_{Y_t} > 3M_t]^{1/2}}{t^{3/2}|\xi_{1,n}|^{k-1}} - r^3(t, x, \xi), \tag{5.39}
\end{aligned}$$

where we used that $U_t^k(x, \xi) \leq \bar{C}t^{3/2}(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1})$ for the last but one inequality (compact sets of the metric). The constant C is the one appearing in (5.16). To conclude it suffices to prove that

$$P := \mathbb{P}[G_t^k \geq 2Ct^{3/2}(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1}), |R_t^k| \leq |G_t^k|/2] \geq \tilde{C}, \tag{5.40}$$

$$|r^4(t, x, \xi)| := \frac{\bar{C}_3 \mathbb{P}[\gamma_{Y_t} > 3M_t]^{1/2}}{t^{3/2}|\xi_{1,n}|^{k-1}} + r^3(t, x, \xi) \leq \frac{C\bar{C}\tilde{C}}{3t^{3/2}(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1})}. \tag{5.41}$$

Indeed, plugging (5.40) and (5.41) into (5.39) gives the statement. Let us first prove (5.40). Write:

$$\begin{aligned}
P &\geq \mathbb{P}[G_t^k \geq 2Ct^{3/2}(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1})] \\
&\quad - \mathbb{P}[G_t^k \geq 2Ct^{3/2}(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1}), |R_t^k| > |G_t^k|/2] \\
&\geq \mathbb{P}[\mathcal{N}(0, 1) \geq 2\check{C}] - \mathbb{P}[|R_t^k| \geq Ct^{3/2}(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1})], \check{C} := \check{C}(n, k).
\end{aligned}$$

Thus, similarly to the proof of (5.21), (5.22) we can show that there exists $\bar{C}_4 := \bar{C}_4(n, k) \geq 1$ s.t. $\mathbb{P}[|R_t^k| \geq Ct^{3/2}(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1})] \leq \bar{C}_4 \exp\left(-\bar{C}_4^{-1} \frac{|x_{1,n}|^2 + |\xi_{1,n}|^2}{t}\right)$. Under the current assumptions, using standard controls on the Gaussian distribution function, this gives (5.40) for $\tilde{C} := \tilde{C}(n, k)$ for K large enough.

Recall now that $|r^4(t, x, \xi)| \leq \frac{1}{t^{3/2}|\xi_{1,n}|^{k-1}} \left(\frac{\bar{C}_1 + C_2 \bar{C}_2 k}{K} + 2\bar{C}_3 \mathbb{P}[\gamma_{Y_t} > 3M_t]^{1/2} + \bar{C}_3 \mathbb{P}[|R_t^k| > |G_t^k|/2]^{1/2}\right) := \sum_{i=1}^3 r^{4i}(t, x, \xi)$. Under the current assumptions, we derive that for K large enough, $r^{41}(t, x, \xi) \leq \frac{C\bar{C}\tilde{C}}{9t^{3/2}(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1})}$. On the other hand, writing $\mathbb{P}[|R_t^k| > |G_t^k|/2]^{1/2} \leq (\mathbb{P}[|R_t^k| > \frac{\hat{C}}{2}(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1})t^{3/2}] + \mathbb{P}[G_t^k \leq \hat{C}(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1})t^{3/2}])^{1/2}$ we derive similarly to (5.21), (5.22) that $r^{43}(t, x, \xi) \leq \frac{C\bar{C}\tilde{C}}{9t^{3/2}(|x_{1,n}|^{k-1} + |\xi_{1,n}|^{k-1})}$ taking \hat{C} small enough. Eventually, the same control holds true for $r^{42}(t, x, \xi)$, still from arguments similar to those used to derive (5.21), (5.22). This concludes the proof.

Remark 5.2 *Observe that the derivation of the lower bound on the compact sets of the metric is the first step to obtain a global Gaussian lower bound using a chaining argument similar to the one in [19]. In our case this can be done provided all the norms of the points on the curve used to perform the chaining are greater than $K_0 t^{1/2}$.*

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