HAL
open science

Uniform tightness for time-inhomogeneous particle systems and for conditional distributions of time-inhomogeneous diffusion processes

Denis Villemonais

To cite this version:

Denis Villemonais. Uniform tightness for time-inhomogeneous particle systems and for conditional distributions of time-inhomogeneous diffusion processes. Markov Processes And Related Fields, 2013, 19 (3), pp.543 - 562. hal-00681601

HAL Id: hal-00681601

https://hal.science/hal-00681601

Submitted on 22 Mar 2012

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Uniform tightness for time-inhomogeneous particle systems and for conditional distributions of time-inhomogeneous diffusion processes

Denis Villemonais *

March 22, 2012

Abstract

In this article, we consider time-inhomogeneous diffusive particle systems, whose particles jump from the boundary of a bounded open subset of $\mathbb{R}^d$, $d \geq 1$. We give a sufficient criterion for the family of empirical distributions of such systems to be uniformly tight, independently of the jump location of the particles. As an application, we show that the conditional distribution of a family of time-inhomogeneous and environment-dependent diffusions conditioned not to hit the boundary of a bounded open subset of $\mathbb{R}^d$ is uniformly tight.

Keywords: Particle systems, random measures, conditional distributions, uniform tightness

MSC 2000 subject : Primary 82C22, 65C50, 60K35; Secondary: 60J60

1 Introduction

Particle systems have received a lot of interest in the past decades and have proved to be useful in several domains, as rare events simulation and conditional distribution approximations. In this paper, we consider a large class of particle systems with fixed size $N \geq 2$, whose particles evolve as independent diffusion processes in a given bounded open subset D of $\mathbb{R}^d$ and jump when they hit the boundary ∂D or after some exponential clocks. The distribution of the jump location depends on the whole system of particles, so that the particles are interacting with each others. The particles of such system evolve in the open set D, thus their empirical distribution is a random probability measure on D. In this paper, we are concerned with the uniform tightness of the law of the empirical distributions of particle systems with jump from a boundary.

Such general interacting particle systems with jump from the boundary are directly inspired by the Fleming-Viot type system introduced by Burdzy, Holyst, Ingerman and

*TOSCA project-team, INRIA Nancy – Grand Est, IECN – UMR 7502, Université Henri Poincaré Nancy 1 B.P. 70239, 54506 Vandoeuvre-lès-Nancy Cedex, France

March [3] in 1996. In this paper, the authors describe a particle system whose particles evolve as independent Brownian motions until one of them reaches the boundary of a fixed open subset D of $\mathbb{R}^d$; at this time, the hitting particle instantaneously jumps to the position of an other particle; then the particles evolve as independent Brownian motions, until one particle hits the boundary ∂D and so on. They suggested that the empirical distribution of the process was closely related to the distribution of a Brownian motion conditioned not to hit the boundary of D and that the long-time behavior of its empirical distribution was an approximation of the so-called *quasi-stationary distribution* of the Brownian motion absorbed at ∂D.

Two questions naturally arise in the study of interacting particle systems with jump from the boundary. The first one (and probably the most challenging one) concerns the non-explosion of the number of jumps in finite time. This question has been studied by Burdzy, Hołyst and March [4], by Löbus [11] and by Bieniek, Burdzy and Finch [1]. Different dynamics than the Brownian one have also been studied by Grigorescu and Kang [8], where diffusive particles with C^∞ coefficients are considered, and by Villemonais [15], [14] (where time-inhomogeneous and environment-dependent diffusive particles are considered). Let us emphasize that, in the last three articles, different jump mechanisms are also allowed: each hitting particle can jump from the boundary to a position which follows a distribution that depends on the position of the other particles (and on time and on the environment for the last paper) in a general way, which is also the setting of the present paper. Concerning the non-explosion of the number of jumps, it is remarkable that non-trivial counter-example (cases with explosion of the number of jumps in finite time) remained unkown up to the recent results of Bieniek, Burdzy and Pal [2].

The second challenging question which naturally arises in this setting is: when the particle system is well defined (that is when the number of jumps doesn't explode), do the particles degenerate to the boundary in the long term? These question has already been treated in the case of the Fleming-Viot type particle system described above when the particles evolve as Brownian motions [4] or Brownian motions with drift [15]. However, the question remains open for more general jump mechanisms and different dynamics. In the present paper, we answer this question by providing a criterion ensuring that the laws of the empirical distribution of a general particle system are uniformly tight. This criterion is directly derived from the non-explosion criterion provided in [14], which allows various choices of the dynamics and jump mechanism of the particles.

As it has been suggested in [3], the particular case of the Fleming-Viot type particle systems (with jump from the boundary to the position of another particle) provides an approximation method for the conditional distribution of diffusion processes conditioned not to hit a boundary. This property, which has been proved by Grigorescu and Kang in [7] for Brownian particles, by Villemonais [15] for Brownian particles with drift, has been proved in all generality in [14] for general Markov processes. Numerical implementations of this method have been used in [15] to provide numerical approximation of quasi-stationary distributions and in [12] to compute the distance between the quasi-

stationary distribution of a process and its conditional distribution. In the present paper, we use this convergence result to obtain theoretical informations on the distribution of diffusion processes conditioned not to reach the boundary of an open subset D of $\mathbb{R}^d$. More precisely, we derive the uniform tightness of the distribution of such diffusion processes from the uniform tightness of the empirical distributions of a sequence of approximating particle systems.

The paper is divided into three parts. In Section 2, we describe precisely the dynamic of the particle systems we are interested in. In particular, we recall a criterion taken from [14] which ensures that the particle systems are well defined (*i.e.* that the number of jumps of the system remains bounded in finite time almost surely). In Section 3, we state and prove our main result, providing a criterion which ensures that the laws of the empirical distributions of the particle systems are uniformly tight. In Section 4, we apply this result to the Fleming-Viot type process introduced in [3], which allows us to derive the uniform tightness of the distribution of diffusion processes conditioned not to reach the boundary of a bounded open subset of $\mathbb{R}^d$.

2 Definition of the particle systems

For each $N \geq 2$, we define an N-particles system whose particles evolve as independent time-inhomogeneous environment-dependent diffusion processes between their jumps. In a first time we define the diffusion processes which will drive the particles between the jumps; in a second time, we define the jump measures that will give the jump positions of the particles. Finally, we recall a recent result [14] which ensures that the particle system is well define for any time $t \geq 0$.

Let E and D be two bounded open subsets of $\mathbb{R}^d$ and $\mathbb{R}^{d'}$ respectively. For each $N \geq 2$, let $\mathcal{Z}^{1,N}, ..., \mathcal{Z}^{N,N}$ be a family of N strong Markov processes, each of them being equal to a 3-tuple $(t, e_t^{i,N}, Z_t^{i,N})_{t\in[0,\tau_\partial[}$ which evolves in $\mathbb{R}_+ \times E \times D$ as a time-inhomogeneous environment-dependent diffusion process. In the 3-tuple $(t, e_t^{i,N}, Z_t^{i,N})$, the parameter t denotes the time, $e_t^{i,N} \in E$ denotes the state of the environment and $Z_t^{i,N} \in D$ denotes the actual position of the diffusion. Each diffusion process $\mathcal{Z}^{i,N}$ will be used to define the dynamic of the ith particle of the system between its jumps. By a *time-inhomogeneous environment-dependent diffusion process*, we mean that, for any $N \geq 2$ and any $i \in \{1, ..., N\}$, there exist four measurable functions

$$
\begin{aligned}
s_i^N &: [0, +\infty[\times E \times D \mapsto \mathbb{R}^d \times \mathbb{R}^d \\
m_i^N &: [0, +\infty[\times E \times D \mapsto \mathbb{R}^d \\
\sigma_i^N &: [0, +\infty[\times E \times D \mapsto \mathbb{R}^{d'} \times \mathbb{R}^{d'} \\
\eta_i^N &: [0, +\infty[\times E \times D \mapsto \mathbb{R}^{d'},
\end{aligned}
$$

such that $\mathcal{Z}^{i,N} = (., e^{i,N}, Z^{i,N})$ is solution to the stochastic differential system

$$\begin{aligned} de_t^{i,N} &= s_i^N(t, e_t^{i,N}, Z_t^{i,N})d\beta_t^{i,N} + m_i(t, e_t^{i,N}, Z_t^{i,N})dt, \qquad e_0^{i,N} \in E, \\ dZ_t^{i,N} &= \sigma_i^N(t, e_t^{i,N}, Z_t^i)dB_t^{i,N} + \eta^{i,N}(t, e_t^{i,N}, Z_t^{i,N})dt, \qquad Z_0^{i,N} \in D, \end{aligned}$$

where the $(\beta^{i,N}, B^{i,N})$ are independent standard $d+d'$ Brownian motions. Each process $\mathcal{Z}^{i,N}$ is killed when $Z_t^{i,N}$ hits ∂D (*hard killing*) and with a rate of killing $\kappa_i^N(t, e_t^{i,N}, Z_t^{i,N}) \geq 0$ (*soft killing*), where

$$\kappa_i^N : [0, +\infty[\times E \times D \mapsto \mathbb{R}_+$$

is a uniformly bounded measurable function. We emphasize that each process $\mathcal{Z}^{i,N}$ evolves in the same state space $\mathbb{R}_+ \times E \times D$, for any i, N.

Let us now define the jump measures, given by two measurable functions

$$\mathcal{S}^N : [0, +\infty[\times E^N \times D^N \to \mathcal{M}_1(E^N \times D^N)$$

and

$$\mathcal{H}^N : [0, +\infty[\times E^N \times \partial(D^N) \to \mathcal{M}_1(E^N \times D^N),$$

where $\mathcal{M}_1(F)$ denotes the space of probability measures on F, for any set F.

The position of the particle system at time t will be denoted by $(t, \mathbb{O}_t^N, \mathbb{X}_t^N)$, where t is the time, $\mathbb{O}_t^N = (o_t^{1,N}, \cdots, o_t^{N,N}) \in E^N$ is the vector of environments and $\mathbb{X}_t^N = (X_t^{1,N}, \cdots, X_t^{N,N}) \in D^N$ is the vector of positions of the particles. In particular, the ith particle at time $t \geq 0$ is given by $(t, o_t^{i,N}, X_t^{i,N})$. We are now able to describe precisely the dynamic of our particle system.

Dynamic of the particle system The particles of the system evolve as independent copies of $\mathcal{Z}^{i,N}$, $i = 1, ..., N$, until one of them is killed. If the killed particle is softly killed, then the whole system jumps instantaneously with respect to the jump measure $\mathcal{S}^N(t, \mathbb{O}_t^N, \mathbb{X}_t^N)$; if the killed particle is hardly killed, it jumps with respect to $\mathcal{H}^N(t, \mathbb{O}_t^N, \mathbb{X}_t^N)$. Then the particles evolve as independent copies of $\mathcal{Z}^{i,N}$, $i = 1, ..., N$, until one of them is killed and so on.

We denote the successive jump times of the particle system by

$$\tau_1^N < \tau_2^N < ... < \tau_n^N < ...,$$

and we set $\tau_\infty^N = \lim_{n\to\infty} \tau_n^N$. It is clear that the N-particles system is well defined at any time t such that $t < \tau_\infty^N$, but there is no natural way to extend the definition of the particle system beyond the explosion time τ_∞^N. As a consequence, it is of first important to be able to decide whether the number of jumps of the process explodes in finite time or not, that is to decide wether $\mathbb{P}(\tau_\infty^N = +\infty) < 1$ or $= 1$. The two following hypotheses ensure that $\mathbb{P}(\tau_\infty^N = +\infty) = 1$ holds for any $N \geq 2$. As mentioned in the introduction, different criterions ensuring this non-explosion property exist, but

they do not allow time-inhomogeneous (and *a fortiori* environment-dependent) diffusive particles; moreover, they are far more restrictive in the choice of the jump measures.

The first assumption concerns the processes $\mathcal{Z}^{i,N}$, $i = 1, \ldots, N$, which give the dynamics of the particles between the jumps. We denote by ϕ_D the Euclidean distance to the boundary ∂D, defined for all $x \in \mathbb{R}^{d'}$ by

$$\phi_D(x) = \inf_{y \in \partial D} \|x - y\|_2,$$

where $\|.\|_2$ denotes the Euclidean norm of $\mathbb{R}^{d'}$. For any $a > 0$, the boundary's neighbourhood D^a of ∂D is defined by

$$D^a = \{x \in D,\ \phi_D(x) < a\}.$$

Hypothesis 1. *We assume that there exist five positive constants a_0, A, k_g, c_0 and C_0 such that*

1. *ϕ_D is of class C^2 on D^{a_0}, with uniformly bounded derivatives,*
2. *for any $N \geq 2$, κ_i^N is uniformly bounded by A on $[0, +\infty[\times E \times D$ and s_i^N, σ_i^N, m_i^N and η_i^N are uniformly bounded by A on $[0, +\infty[\times E \times D^{a_0}$,*
3. *for any $N \geq 2$ and any $i \in \{1, ..., N\}$, there exist two measurable functions $f_i^N : [0, +\infty[\times E \times D^{a_0} \to \mathbb{R}_+$ and $g_i^N : [0, +\infty[\times E \times D^{a_0} \to \mathbb{R}$ such that $\forall (t, e, z) \in [0, +\infty[\times E \times D^{a_0}$,*

$$\sum_{k,l} \frac{\partial \phi_D}{\partial x_k}(z) \frac{\partial \phi_D}{\partial x_l}(z) [\sigma_i^N \sigma_i^{N*}]_{kl}(t, e, z) = f_i^N(t, e, z) + g_i^N(t, e, z), \tag{1}$$

   and such that

   (a) *f_i^N is of class C^1 in time and of class C^2 in environment/space, and the derivatives of f_i^N are uniformly bounded by A in $[0, +\infty[\times E \times D^{a_0}$.*

   (b) *for all $(t, e, z) \in [0, +\infty[\times E \times D^{a_0}$,*

$$|g_i^N(t, e, z)| \leq k_g \phi_D(z),$$

   (c) *for all $(t, e, z) \in [0, +\infty[\times E \times D^{a_0}$,*

$$c_0 < f_i^N(t, e, z) < C_0 \text{ and } c_0 < f_i^N(t, e, z) + g_i^N(t, e, z) < C_0.$$

The second assumption below concerns the jump measures $\mathcal{H}^N$ and $\mathcal{S}^N$. Let us first remark that, when the system hits the boundary $\partial(D^N)$, at most one particle hits ∂D. This implies that the whole set of particles hits one and only one of the sets $\mathcal{D}_i^N$, $i = 1, \cdots, N$, defined by

$$\mathcal{D}_i = \left\{(x_1, \cdots, x_N) \in \partial(D^N),\ x_i \in \partial D \text{ and } x_j \in D, \forall j \neq i\right\}.$$

With this definition, it is clear that the ith particle is hardly killed if and only if $(\cdot, \mathbb{O}^N, \mathbb{X}^N)$ hits $[0, +\infty[\times E^N \times \mathcal{D}_i^N$. In particular, the behavior of $\mathcal{H}^N$ outside these sets does not present any interest.

Hypothesis 2. *We assume that, for any $N \geq 2$,*

1. *There exists a non-decreasing continuous function $h^N : \mathbb{R}_+ \to \mathbb{R}_+$ vanishing only at 0 such that, $\forall i \in \{1, ..., N\}$,*

$$\inf_{(t,e,(x_1,...,x_N))\in[0,+\infty[\times E^N\times\mathcal{D}_i^N} \mathcal{H}^N(t, e, x_1, ..., x_N)(E^N \times A_i^N) \geq p_0^N,$$

where $p_0^N > 0$ is a positive constant and $A_i^N \subset D^N$ is the set defined by

$$A_i^N = \left\{(y_1, ..., y_N) \in D^N \,|\, \exists j \neq i \text{ such that } \phi_D(y_i) \geq h^N(\phi_D(y_j))\right\}.$$

2. *We have*

$$\inf_{(t,e,(x_1,...,x_N))\in[0,+\infty[\times E\times\mathcal{D}_i} \mathcal{H}(t, e, x_1, ..., x_N)(E \times B_{x_1,...,x_n}) = 1,$$

where

$$B_{x_1,...,x_n} = \left\{(y_1, ..., y_N) \in D^N \,|\, \forall i,\ \phi_i(y_i) \geq \phi_i(x_i)\right\}$$

Let us explain the meaning of each point of the last hypothesis.

- The set A_i^N is a subset of D^N such that if the ith component of an element $(y_1, \cdots, y_n) \in A_i^N$ is near the boundary ∂D (that is if $\phi_D(y_i) \ll 1$), then at least one another component y_j fulfils $h^N(\phi_D(y_j)) \ll 1$ and thus $\phi_D(y_j) \ll 1$. This implies that if the ith particle jumps after a hard killing to a position located near the boundary ∂D, then, with a probability lowered by p_0^N, at least one of the other particles is located near the boundary.

- The set $B_{x_1,...,x_n}$ is a subset of D^N such that for any $(y_1, \cdots, y_n) \in B_{x_1,...,x_n}$, the components y_i are respectively farther from the boundary than x_i. This means that after a hard killing, each particle jumps to a position farther from the boundary after the jump than before the jump.

Let us now recall the result which will ensure that the interacting particle systems are well defined, for any $N \geq 2$.

Theorem 1 (V. 2011 in [14])**.** *Assume that Hypotheses 1 and 2 are fulfilled, then, for any $N \geq 2$, the number of jumps of the N-particles system doesn't explode in finite time almost surely. Equivalently, we have $\tau_\infty^N = +\infty$ almost surely, for all $N \geq 2$.*

Before turning to the next step of our study, let us emphasize that one could consider a more complicated situation where the diffusion process $\mathcal{Z}^{i,N}$ is reflected on ∂D and killed when its local time on the boundary reaches the value of an independent exponential random variable. Such processes have been studied in deep details in the one-dimensional situation $d_0' = 1$ (see for instance [10] and references therein) and in the multi-dimensional situation (see for instance [16]). The technics and calculous used in [14] would still be valid, as would be the rest of the present paper. However, the case under study is complicated enough, so that we only consider killing boundaries without reflection.

3 Uniform tightness for time inhomogeneous particle systems

In the present section, we consider a family of interacting particle systems and prove a sufficient criterion for the uniform tightness of the laws of the empirical distributions of the particle system, at any time time t.

For any $N \geq 2$, let $(\cdot, \mathbb{O}^{(N)}, \mathbb{X}^{(N)})$ be a N-particles system defined as in the previous section, driven by the time-inhomogeneous environment-dependent diffusion processes $\mathcal{Z}^{i,N}$, $i = 1, ..., N$, with environment state space E and diffusion state space D. We also denote by $\mathcal{S}^N$ and $\mathcal{H}^N$ the jump measures of the interacting particle system.

For any time $t < \tau_\infty^N$, we define the empirical distribution of the particle system with N particles $(\cdot, \mathbb{O}^{(N)}, \mathbb{X}^{(N)})$ at time t by

$$\mu_t^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^{i,N}},$$

which is a probability measure on D (we emphasize that the empirical distribution doesn't take into account the value of the environment). We're now able to state our tightness result (which will be used in the next section to prove that the family of conditional distributions of time-homogeneous environment-dependent diffusion processes is uniformly tight).

Theorem 2. *Assume that Hypotheses 1 and 2 are fulfilled and fix $t_0 > 0$. Then, for any $\epsilon > 0$, there exists $a_\epsilon > 0$ and $N_\epsilon \geq 2$ such that, independently of the initial distribution of the particle systems,*

$$\mathbb{E}\left(\mu_T^N(D^{a_\epsilon})\right) \leq \epsilon, \ \forall N \geq N_\epsilon, \ \forall T \geq t_0.$$

In particular, for any sequence of deterministic times $(t_N)_{N\geq 2}$ such that $\inf_{N\geq 2} t_N > 0$, the family of laws of the random measures $\mu_{t_N}^N(dx)$, $N \geq 2$, is uniformly tight.

We emphasize that it is not required for the laws of the initial empirical distributions $(\mu_0^N)_{N\geq 2}$ to be uniformly tight.

We also recall that the uniform tightness of the laws of the empirical distributions is of particular interest since it implies that the given sequence is weakly sequentially compact, as a sequence of random probability measures.

Proof of Theorem 2. By Theorem 1, the particle system $(\cdot, \mathbb{O}_\cdot^{(N)}, \mathbb{X}_\cdot^{(N)})$ is well defined at any time for any $N \geq 2$.

The proof of the uniform tightness is based on the following criterion, due to Jakubowski in [9]: if, for any $\epsilon > 0$, there exists a compact subset K_ϵ of D such that $\mathbb{E}(\mu_{t_N}^N(K_\epsilon)) \geq 1-\epsilon$

for all $N \geq 2$, then the family of laws of the probability measures $\mu^N_{t_N}$ is uniformly tight. Taking $t_0 = \inf_{N\geq 2} t_N > 0$, the first part of Theorem 2 and Jakubowski's criterion clearly implies the second part of Theorem 2. Thus it remains us to prove the first part of the result.

Fix $\epsilon > 0$ and $t_0 > 0$. In a first time, we assume that the killing rate κ^N_i is equal to 0, for all $N \geq 2$ and $i \in \{1, ..., N\}$. For any $N \geq 2$ and any $a > 0$, we have

$$\begin{aligned}
E\left(\mu^N_{t_0}(D^a)\right)^2 &\leq E\left(\mu^N_{t_0}(D^a)^2\right), \\
&\leq E\left(\frac{1}{N^2}\sum_{i=1}^N \left(\delta_{X^{i,N}_{t_0}}(D^a)\right)^2 + \frac{1}{N^2}\sum_{1\leq i\neq j\leq N} \delta_{X^{i,N}_{t_0}}(D^a)\delta_{X^{j,N}_{t_0}}(D^a)\right) \\
&\leq \frac{1}{N}E\left(\mu^N_{t_0}(D^a)\right) + \frac{1}{N^2}\sum_{1\leq i\neq j\leq N} E\left(\delta_{X^{i,N}_{t_0}}(D^a)\delta_{X^{j,N}_{t_0}}(D^a)\right) \\
&\leq \frac{1}{N}E\left(\mu^N_{t_0}(D^a)\right) + \max_{1\leq i\neq j\leq N} P\left(\phi_D(X^{i,N}_{t_0}) \leq a \text{ and } \phi_D(X^{j,N}_{t_0}) \leq a\right).
\end{aligned}$$

Fix $N \geq 2$ and $i \neq j \in \{1, ..., N\}$. For all $\gamma \in [0, \frac{a_0}{2}]$ (where a_0 is taken from Hypothesis 1), we define the stopping time

$$S^{i,N}_\gamma = \inf\{t \geq 0,\ \phi_D(X^{i,N}_t) \geq \gamma\},$$

which is the first time at which the distance between the ith particle and the boundary ∂D is greater than γ. Let us now state the following useful Lemma, whose proof is postponed to the end of this section.

Lemma 3. *There exists a positive constant $C > 0$, independent of i, j, N and γ, such that for all $a \in [0, \gamma\sqrt{\frac{c_0}{C_0}}[$ (where c_0 and C_0 are taken from Hypothesis 1),*

$$P\left(\exists t \in [S^{i,N}_\gamma, t_0],\ \sqrt{\phi_D(X^{i,N}_t)^2 + \phi_D(X^{j,N}_t)^2} \leq a\right) \leq \frac{Ct_0}{\log\left(\frac{\gamma}{a}\sqrt{\frac{c_0}{C_0}}\right)}.$$

We immediately deduce from Lemma 3 that, for any $\gamma < a_0/2$ and any $a \in [0, \gamma\sqrt{\frac{c_0}{C_0}}[$,

$$P\left(\phi_D(X^{i,N}_{t_0}) \leq \frac{a}{\sqrt{2}} \text{ and } \phi_D(X^{j,N}_{t_0}) \leq \frac{a}{\sqrt{2}}\right) \leq \frac{Ct_0}{\log\left(\frac{\gamma}{a}\sqrt{\frac{c_0}{C_0}}\right)} + P(S^{i,N}_\gamma > t_0).$$

In particular, replacing a by $a\sqrt{2}$, we deduce that, for any $\gamma < a_0/2$ and any $a \in [0, \gamma\sqrt{\frac{c_0}{2C_0}}[$,

$$E\left(\mu^N_{t_0}(D^a)\right)^2 \leq \frac{1}{N}E\left(\mu^N_{t_0}(D^a)\right) + \frac{Ct_0}{\log\left(\frac{\gamma}{a}\sqrt{\frac{c_0}{2C_0}}\right)} + \max_{1\leq i\leq N} P\left(S^{i,N}_\gamma > t_0\right). \tag{2}$$

The proof of the following lemma is also postponed to the end of this section.

Lemma 4. *For any $\epsilon > 0$, there exists a constant $\gamma_\epsilon > 0$ such that, for all $N \geq 2$ and all $i \in \{1, ..., N\}$, we have*

$$P\left(S^{i,N}_{\gamma_\epsilon} > t_0\right) \leq \epsilon/3, \tag{3}$$

independently of the sequence of initial distributions.

By (2) and Lemma 4, we deduce that, for all $a \in [0, \gamma_\epsilon \sqrt{\frac{c_0}{2C_0}}[$,

$$E\left(\mu^N_{t_0}(D^a)\right)^2 \leq \frac{1}{N} E\left(\mu^N_{t_0}(D^a)\right) + \frac{Ct_0}{\log\left(\frac{\gamma_\epsilon}{a}\sqrt{\frac{c_0}{2C_0}}\right)} + \frac{\epsilon}{3}.$$

Fixing an integer $N_\epsilon \geq 2 \vee \frac{3}{\epsilon}$, we thus have, for all $N \geq N_\epsilon$ and all $a > 0$,

$$\frac{1}{N} E\left(\mu^N_{t_0}(D^a)\right) \leq \frac{\epsilon}{3}.$$

Let $a_\epsilon > 0$ be a positive constant such that $\log\left(\frac{\gamma_\epsilon}{a_\epsilon}\sqrt{\frac{c_0}{2C_0}}\right) \leq 3(Ct_0\epsilon)^{-1}$. We then have

$$E\left(\mu^N_{t_0}(D^{a_\epsilon})\right)^2 \leq \epsilon, \ \forall N \geq N_\epsilon,$$

independently of the sequence of initial distributions. We deduce that the first part of Theorem 2 is fulfilled for $T = t_0$.

Fix $T > t_0$. Since the previous inequality doesn't depend on the distribution of the initial position $(X^{1,N}_0, ..., X^{N,N}_0)$, it can be applied to the process initially distributed with the same distribution as $(X^{1,N}_{T-t_0}, ..., X^{2,N}_{T-t_0})$. By the Markov property of the particle system, we thus obtain

$$E\left(\mu^N_T(D^{a_\epsilon})\right)^2 \leq \epsilon, \ \forall N \geq N_\epsilon.$$

This allows us to conclude the proof of the first part of Theorem 2 when $\kappa^N_i = 0$ for all $N \geq N_\epsilon$ and $i \in \{1, ..., N\}$.

Fix $N \geq 2$ and assume now that $(\kappa^N_i)_{i\in\{1,...,N\}}$ isn't equal to 0. Fix $i, j \in \{1, ..., N\}$. For any $\gamma > 0$, we define the stopping time $S^{i,N}_\gamma$ as above. We also denote by τ^{soft}_γ the first soft killing time of $X^{i,N}$ or $X^{j,N}$ after $S^{i,N}_\gamma$, that is

$$\tau^{soft}_\gamma = \inf\{t \geq S^{i,N}_\gamma, \ X^{i,N} \text{ or } X^{j,N} \text{ is softly killed at time } t\}.$$

Similarly to Lemma 3, we have, for any $\gamma < a_0/2$ and any $a \in [0, \gamma\sqrt{\frac{c_0}{C_0}}]$,

$$P\left(\exists t \in [S_\gamma, \tau^{soft}_\gamma \wedge t_0[, \ \sqrt{\phi_D(X^{i,N}_t)^2 + \phi_D(X^{j,N}_t)^2} \leq a\right) \leq \frac{Ct_0}{\log\left(\frac{\gamma}{a}\sqrt{\frac{c_0}{C_0}}\right)}.$$

In particular, we deduce that

$$P\left(\phi_D(X_{t_0}^{i,N}) \leq a \text{ and } \phi_D(X_{t_0}^{j,N}) \leq a\right)$$
$$\leq \frac{C}{\log\left(\frac{\gamma}{a}\sqrt{\frac{c_0}{2C_0}}\right)} + P(S_\gamma > t_0 \text{ and } \tau_\gamma^{soft} \geq t_0) + P(\tau_\gamma^{soft} < t_0).$$

By Hypothesis 1, the killing rates κ_i^N and κ_j^N are uniformly bounded by a constant $A > 0$. As a consequence, there exists $T_0 > 0$ such that, for all $t \leq T_0$ and all $\gamma > 0$,

$$P(\tau_\gamma^{soft} < t) \leq \frac{\epsilon}{4}.$$

We emphasize that T_0 is chosen so that it only depends on the uniform bound A and that we can assume, without loss of generality, that t_0 is smaller than T_0. By the same arguments as in the proof (postponed below) of Lemma 4, we can find $\gamma_\epsilon > 0$ such that

$$P(S_{\gamma_\epsilon}^{i,N} > t_0 \text{ and } \tau_{\gamma_\epsilon}^{soft} \geq t_0) \leq \frac{\epsilon}{4}.$$

Finally, choosing a_ϵ small enough, we deduce that

$$P\left(\phi_D(X_{t_0}^{i,N}) \leq a_\epsilon \text{ and } \phi_D(X_{t_0}^{j,N} \leq a_\epsilon\right) \leq \frac{3\epsilon}{4}.$$

Finally, choosing $N_\epsilon \geq 2 \vee \frac{4}{\epsilon}$ and proceeding as in the first part of the proof, we deduce that Theorem 2 holds for $T = t_0$ and can be extended by the Markov property to any $T > t_0$. □

Proof of Lemma 3. Fix $\gamma \in [0, a_0/2]$ and let us prove that, for all $a \in [0, \gamma[$,

$$P\left(\exists t \in [S_\gamma^{i,N}, t_0],\ \phi_D(X_t^{i,N}) + \phi_D(X_t^{j,N}) \leq a\right) \leq \frac{Ct_0}{\log\left(\frac{\gamma}{a}\right)}.$$

Let $(s_n)_{n\geq 0}$ be the sequence of stopping times defined by

$$s_0 = \inf\{s \in [S_\gamma^{i,N}, t_0],\ \sqrt{\phi_D(X_s^{i,N})^2 + \phi_D(X_s^{j,N})^2} \leq a_0/2\} \wedge t_0$$

and, for all $n \geq 0$,

$$s_{2n+1} = \inf\{s \in [s_{2n}, t_0],\ \sqrt{\phi_D(X_s^{i,N})^2 + \phi_D(X_s^{j,N})^2} \geq a_0\} \wedge t_0$$
$$s_{2n+2} = \inf\{s \in [s_{2n+1}, t_0],\ \sqrt{\phi_D(X_s^{i,N})^2 + \phi_D(X_s^{j,N})^2} \leq a_0/2\} \wedge t_0.$$

It is immediate that s_n converges almost surely to t_0. By construction, we have for all $n \geq 0$,

$$\begin{cases} \phi_D(X_t^{i,N}) < a_0 \text{ and } \phi_D(X_t^{j,N}) < a_0,\ \forall t \in [s_{2n}, s_{2n+1}[, \\ \sqrt{\phi_D(X_t^{i,N})^2 + \phi_D(X_t^{j,N})^2} \geq a_0/2 \text{ otherwise.} \end{cases}$$

In particular, , for all $t \in [s_{2n}, s_{2n+1}[$, ϕ_D is of class C^2 at $X_t^{i,N}$ and $X_t^{j,N}$ almost surely, by the first point of Hypothesis 1. This will allow us to compute the Itô's decomposition of $\phi_D(X^{i,N})$ and $\phi_D(X^{j,N})$ at any time $t \in [s_{2n}, s_{2n+1}[$, using Itô's formula.

For all $n \geq 0$, we have

$$P\left(\exists t \in [s_{2n+1}, s_{2n+2}[,\ \sqrt{\phi_D(X_t^{i,N})^2 + \phi_D(X_t^{j,N})^2} \leq a\right) = 0,\ \forall a < a_0/2. \tag{4}$$

Fix $n \geq 0$ and let us now prove that there exists a constant $C > 0$ such that

$$P\left(\exists t \in [s_{2n}, s_{2n+1}[,\ \phi_D(X_t^{i,N}) \leq a \text{ and } \phi_D(X_t^{j,N}) \leq a\right) \leq \frac{C}{\log\left(\frac{\gamma}{a}\right)} E\left(s_{2n+1} - s_{2n}\right). \tag{5}$$

We define the positive semi-martingale Y^i by

$$Y_t^i = \begin{cases} \phi_D(X_{s_{2n}+t}^{i,N}) \text{ if } t < s_{2n+1} - s_{2n}, \\ a_0/2 + |W_t^i| \text{ if } t \geq s_{2n+1} - s_{2n}, \end{cases} \tag{6}$$

where W^i is a standard one dimensional Brownian motion independent of the rest of the process. The extension after time $s_{2n+1} - s_{2n}$ (we recall that n is fixed here) allows us to define Y_t^i at any time $t \in [0, +\infty[$. We define similarly the semi-martingale Y^j. The inequality (5) is proved using [14, Proposition 4.1] applied to the pair of semi-martingale Y^i, Y^j. In order to do so, we need the Itô's decompositions of Y^i and Y^j. Let us set

$$\pi_t^i = \begin{cases} f_i^N(s_{2n} + t, o_{s_{2n}+t}^{i,N}, X_{s_{2n}+t}^{i,N}), & \text{if } 0 \leq t < s_{2n+1} - s_{2n}, \\ 1, & \text{if } t \geq s_{2n+1} - s_{2n} \end{cases}$$

and

$$\rho_t^i = \begin{cases} g_i^N(s_{2n} + t, o_{s_{2n}+t}^{i,N}, X_{s_{2n}+t}^{i,N}), & \text{if } t < s_{2n+1} - s_{2n}, \\ 0, & \text{if } t \geq s_{2n+1} - s_{2n}, \end{cases}$$

where f_i^N and g_i^N are given by Hypothesis 1. By the Itô's formula applied to Y^i, we have

$$dY_t^i = dM_t^i + b_t^i dt + dK_t^i + Y_t^i - Y_{t-}^i,$$

where M^i is a local martingale such that

$$d\langle M^i \rangle_t = (\pi_t^i + \rho_t^i) dt;$$

b^i is the adapted process given, if $t < s_{2n+1} - s_{2n}$, by

$$b_t^i = \sum_{k=1}^{d'i} \frac{\partial \phi_i}{\partial x_k}(X_{t_{2n}+t}^{i,N})[\eta_i^N]_k(t_{2n} + t, o_{t_{2n}+t}^{i,N}, X_{t_{2n}+t}^{i,N})$$
$$+ \frac{1}{2} \sum_{k,l=1}^{d_i} \frac{\partial^2 \phi}{\partial x_k \partial x_l}(X_{t_{2n}+t}^{i,N})[\sigma_i^N \sigma_i^{N*}]_{kl}(t_{2n} + t, o_{t_{2n}+t}^{i,N}, X_{t_{2n}+t}^{i,N}),$$

and, if $t \geq s_{2n+1} - s_{2n}$, by $b_t^i = 0$; K^i is a non-decreasing process given by the local time of $|W_t|$ at 0 after time $s_{2n+1} - s_{2n}$. By the 3th point of Hypothesis 1, we have, for all $t \geq 0$,

$$c_0 \wedge 1 \leq \pi_t^i + \rho_t^i \leq C_0 \vee 1, \text{ and } |\rho_t^i| \leq k_g Y_t^i \tag{7}$$

By Hypothesis 1, ϕ_D is of class C^2 on D^{a_0}, with uniformly bounded derivatives, and η_i^N, σ_i^N are uniformly bounded. This implies that there exists $b_\infty > 0$ (independent of i and N) such that, for all $t \geq 0$,

$$b_t^i \geq -b_\infty. \tag{8}$$

Similarly, we get the decomposition of Y^j, with π^j, ρ^j and b^j fulfilling inequalities (7) and (8) (without loss of generality, we keep the same constants c_0, C_0, k_g and b_∞).

Let us now compute the Itô's decompositions of π^i and π^j. We deduce from the Itô's formula that there exist a local martingale N^i and a finite variational process L^i such that, for all $t \geq 0$,

$$d\pi_t^i = dN_t^i + dL_t^i + \pi_t^i - \pi_{t-}^i,$$

where, for all $t \in [0, s_{2n+1} - s_{2n}[$,

$$L_t^i = \int_0^t \left(\sum_{k=1}^{d_0} \frac{\partial f_i^N}{\partial e^k}(s_{2n}+s, o_{s_{2n}+s}^i, X_{s_{2n}+s}^i) + \sum_{k=1}^{d_0'} \frac{\partial f_i^N}{\partial x^k}(s_{2n}+s, o_{s_{2n}+s}^i, X_{s_{2n}+s}^i) \right) ds.$$

By Hypothesis 1, the derivatives of f_i^N are uniformly bounded, so that there exists a constant C_L such that

$$E\left(|L^i|_{s_{2n+1}-s_{2n}}\right) \leq C_L E\left(s_{2n+1} - s_{2n}\right). \tag{9}$$

Let us set, for all $t < s_{2n+1} - s_{2n}$,

$$\begin{aligned}\xi_t^i = &\sum_{k=1,l}^{d_i} \frac{\partial f_i^N}{\partial e_k}(t, o_t^{i,N}, X_t^{i,N}) \frac{\partial f_i^N}{\partial e_l}(t, o_t^{i,N}, X_t^{i,N})[s_i^N s_i^{N*}]_{kl}(t, o_t^{i,N}, X_t^{i,N}) \\ &+ \sum_{k=1,l}^{d_i'} \frac{\partial f_i^N}{\partial x_k}(t, o_t^{i,N}, X_t^{i,N}) \frac{\partial f_i^N}{\partial x_l}(t, o_t^{i,N}, X_t^{i,N})[\sigma_i^N \sigma_i^{N*}]_{kl}(t, o_t^{i,N}, X_t^{i,N})\end{aligned}$$

and, for all $t \geq s_{2n+1} - s_{2n}$, $\xi_t^i = 0$. Then we have

$$\langle N^i \rangle_t = \xi_t^i dt.$$

Thanks to the regularity assumptions on f_i^N and the boundedness of s_i^N and σ_i^N, there exists $C_\xi > 0$ such that

$$\xi_t^i \leq C_\xi. \tag{10}$$

The same decomposition and inequalities hold for π^j, with the same constants C_L and C_ξ. We emphasize that these constants are chosen independently of i, j and N, since the bounds that we used are by assumption uniform in i, j, N.

We define the process

$$\Phi_t \stackrel{def}{=} -\frac{1}{2}\log\left(\frac{(Y_t^i)^2}{\pi_t^1}+\frac{(Y_t^j)^2}{\pi_t^2}\right),\ t\geq 0,$$

and we set, for all $\epsilon>0$, $T_\epsilon=\inf\{t\in[0,T],\ \Phi_t\geq\epsilon^{-1}\}$. By the previous Itô's decompositions, one can apply [14, Proposition 4.1] to the pair of semi-martingales Y^1,Y^2. Thus, for any stopping time θ, we have

$$P\left(T_\epsilon\leq\theta\right)\leq\frac{1}{\epsilon^{-1}-\Phi_0}C\left(E(|L^i|_\theta+|L^j|_\theta)+E(\theta)\right).$$

Applying this result to $\theta=s_{2n+1}-s_{2n}$ (which is a stopping time for the filtration of the process $(X^{i,N},X^{j,N})$ after time s_{2n}) and using (9), we deduce that there exists a constant $C'>0$, which only depend on the constants $b_\infty,k_g,c_0,C_0,C_\xi$, such that

$$P\left(T_\epsilon\in[0,s_{2n+1}-s_{2n}[\right)\leq\frac{1}{\epsilon^{-1}-\Phi_0}C'(2\,C_L+1)E\left(s_{2n+1}-s_{2n}\right). \tag{11}$$

By Hypothesis 1, we have

$$\Phi_0\leq-\log\left(\sqrt{\frac{(Y_0^1)^2}{C_0}+\frac{(Y_0^2)^2}{C_0}}\right).$$

If $s_{2n}=T$, then $s_{2n+1}-s_{2n}=0$, so that $Y_0^1=Y_0^2=a_0$. If $s_{2n}<T$, then we have $s_{2n}=S_\gamma^{i,N}$ or $s_{2n}>S_\gamma^{i,N}$. If $s_{2n}=S_\gamma^{i,N}$, then, by definition of $S_\gamma^{i,N}$ and by the right continuity of the process, we have $\phi_D(X_{s_{2n}}^{i,N})\geq\gamma$, that is $Y_0^i\geq\gamma$. If $s_{2n}>S_\gamma^{i,N}$, then, by definition of s_{2n}, we have

$$\sqrt{\phi_D(X_{s_{2n}\text{-}}^{i,N})^2+\phi_D(X_{s_{2n}\text{-}}^{j,N})^2}\geq a_0/2.$$

Since $\sqrt{\phi_D(X^{i,N})^2+\phi_D(X^{j,N})^2}$ can only have positive jumps, we deduce that

$$\sqrt{\phi_D(X_{s_{2n}}^{i,N})^2+\phi_D(X_{s_{2n}}^{j,N})^2}=a_0/2\geq\gamma,$$

that is $\sqrt{(Y_0^1)^2+(Y_0^2)^2}\geq a_0/2\geq\gamma$. Finally, in all cases, we have

$$\Phi_0\leq-\log\left(\frac{\gamma}{\sqrt{C_0}}\right).$$

Thus we deduce from (11) that, for $\epsilon>0$ small enough,

$$P\left(T_\epsilon\in[0,s_{2n+1}-s_{2n}]\right)\leq\frac{1}{\epsilon^{-1}+\log\left(\frac{\gamma}{\sqrt{C_0}}\right)}C(2\,C_L+1)E\left(s_{2n+1}-s_{2n}\right).$$

which implies that

$$P\left(\exists t\in[s_{2n},s_{2n+1}],\ \sqrt{\phi_D(X_t^{i,N})^2+\phi_D(X_t^{j,N})^2}\le\sqrt{c_0}e^{-\frac{1}{\epsilon}}\right)$$
$$\le\frac{1}{\epsilon^{-1}+\log\left(\frac{\gamma}{\sqrt{C_0}}\right)}C(2\,C_L+1)E\left(s_{2n+1}-s_{2n}\right).$$

Replacing ϵ^{-1} by $-\log(a/\sqrt{c_0})$, we deduce that

$$P\left(\exists t\in[s_{2n},s_{2n+1}],\ \sqrt{\phi_D(X_t^{i,N})^2+\phi_D(X_t^{j,N})^2}\le a\right)$$
$$\le\frac{C(2\,C_L+1)}{\log\left(\frac{\gamma}{a}\sqrt{\frac{c_0}{C_0}}\right)}E\left(s_{2n+1}-s_{2n}\right).$$

Summing over $n\ge 0$ and using equality (4), we deduce that

$$P\left(\exists t\in[S_\gamma,t_0],\ \sqrt{\phi_D(X_t^{i,N})^2+\phi_D(X_t^{j,N})^2}\le a\right)\ \le\frac{C(2\,C_L+1)t_0}{\log\left(\frac{\gamma}{a}\sqrt{\frac{c_0}{C_0}}\right)}.$$

This immediately leads to Lemma 3. □

Proof of Lemma 4. In order to prove Lemma 4, we build a coupling between $\phi_D(X^{i,N})$ and a time changed reflected Brownian motion with drift.

Let $(\theta_n)_{n\ge 0}$ be the sequence of stopping times defined by

$$\theta_0=\inf\{t\in[0,T],\ \phi_D(X_t^{i,N})\le a_0/2\}\wedge T$$

and, for all $n\ge 0$,

$$\theta_{2n+1}=\inf\{t\in[\theta_{2n},T],\ \phi_D(X_t^{i,N})\ge a_0\}\wedge T$$
$$\theta_{2n+2}=\inf\{t\in[\theta_{2n+1},T],\ \phi_D(X_t^{i,N})\le a_0/2\}\wedge T.$$

It is immediate that (θ_n) converges almost surely to T and that

$$\phi_D(X_t^{i,N})\ge\frac{a_0}{2},\ \forall t\in[0,\theta_0]\text{ and }\forall t\in\cup_{n=0}^{\infty}[\theta_{2n+1},\theta_{2n+2}[$$
$$\phi_0(X_t^{i,N})<a_0,\ \forall t\in\cup_{n=0}^{\infty}[\theta_{2n},\theta_{2n+1}[.$$

Let Γ be a 1-dimensional Brownian motion independent of the process $(.,\mathbb{O}^{(N)},\mathbb{X}^{(N)})$. We set

$$M_t=\Gamma_t,\ \text{for } t\in[0,\theta_0[,$$

and, for all $n \geq 0$,

$$M_t = M_{\theta_{2n}} + \int_{\theta_{2n}}^{t} \sum_{k=1}^{d_0'} \frac{\partial \phi_D}{\partial x_k}[\sigma_i]_{kl}(t, o_t^{i,N}, X_t^{i,N}) d[B_s^i]_l \text{ for } t \in [\theta_{2n}, \theta_{2n+1}[,$$
$$M_t = M_{\theta_{2n+1}} + (\Gamma_t - \Gamma_{\theta_{2n+1}}) \text{ for } t \in [\theta_{2n+1}, \theta_{2n+2}[,$$

Informally, M is a square-integrable martingale which is parallel to the martingale part of $\phi_D(X^{i,N})$ when this one is near 0 (at least strictly smaller than a_0), and equal to an independent Brownian motion when $\phi_D(X^{i,N})$ is sufficiently far from 0 (at least bigger than $a_0/2$). By [13, Theorem 1.9 (Knight)], M is a time changed Brownian motion. More precisely, there exists a 1-dimensional Brownian motion W such that, for all $t \geq 0$,

$$M_t = W_{\langle M \rangle_t}.$$

By Itô's formula, we have

$$\frac{\partial \langle M \rangle_t}{\partial t} = \begin{cases} f_i^N(t, o_t^{i,N}, X_t^{i,N}) + g_i^N(t, o_t^{i,N}, X_t^{i,N}) \text{ if } \exists n \geq 0 \text{ such that } t \in [\theta_{2n}, \theta_{2n+1}[, \\ 1 \text{ if } \exists n \geq 0 \text{ such that } t \in [\theta_{2n+1}, \theta_{2n+2}[. \end{cases}$$

By Hypothesis 1, we deduce that

$$c_0 \wedge 1 \leq \frac{\partial}{\partial t} \langle M \rangle_t \leq C_0 \vee 1. \tag{12}$$

By the uniform bounds assumptions of Hypothesis 1, there exists a positive constant $C_1 > 0$ such that, for all $t \in [\theta_{2n}, \theta_{2n+1}[$,

$$-C_1 \leq \frac{1}{2} \sum_{k,l=1}^{d_0} \frac{\partial^2 \phi_D}{\partial x_k \partial x_l}(X_t) \left[\sigma_i^N \left(\sigma_i^N\right)^*\right]_{kl} (t, o_t^{i,N}, X_t^{i,N}) + \sum_{k=1}^{d_0} \frac{\partial \phi_D}{\partial x_k}(X_t^{i,N}) \left[\eta_i^N\right]_k (t, o_t^{i,N}, X_t^{i,N}), \tag{13}$$

which is the drift part of the semi-martingale $\phi_D(X^{i,N})$. Let U be the diffusion process reflected on 0 and a, defined by

$$dU_t = dW_t - \frac{C_1}{c_0 \wedge 1} dt + dL_t^0 - dL_t^a, \; U_0 = 0,$$

where L^0 (resp. L^a) is the local time of U on 0 (resp. a). In particular, we have

$$dU_{\langle M \rangle_t} = dM_t - \frac{C_1}{c_0 \wedge 1} \frac{\partial}{\partial t} \langle M \rangle_t dt + dL^0_{\langle M \rangle_t} - dL^a_{\langle M \rangle_t},$$

where, by the third point of Hypothesis 1 and inequalities (12) and (13),

$$-\frac{C_1}{c_0\wedge 1}\frac{\partial}{\partial t}\langle M\rangle_t \leq \frac{1}{2}\sum_{k,l=1}^{d_0}\frac{\partial^2\phi_D}{\partial x_k\partial x_l}(X_t)\left[\sigma_i^N\left(\sigma_i^N\right)^*\right]_{kl}(t,o_t^{i,N},X_t^{i,N})$$
$$+\sum_{k=1}^{d_0}\frac{\partial\phi_D}{\partial x_k}(X_t^{i,N})\left[\eta_i^N\right]_k(t,o_t^{i,N},X_t^{i,N}),$$

which is the drift part of the semi-martingale $\phi_D(X_t^{i,N})$. Informally, $U_{\langle M\rangle_t}$ evolves as $\phi_D(X_t^{i,N})$ but with a stronger drift toward 0, $U_{\langle M\rangle_t}$ is reflected on 0 while $\phi_D(X_t^{i,N})$ makes positive jumps when it hits 0 and $U_{\langle M\rangle_t}$ is reflected on a while $\phi_D(X_t^{i,N})$ can become greater than a. As a consequence (see [15, Proposition 2.2] for a rigorous and detailed argumentation of this fact), we have

$$0\leq U_{\langle M\rangle_t}\leq \phi_D(X_t^{i,N}),\ \forall t\in[0,T].$$

Then, for all $\gamma>0$,

$$\left\{\phi_D(X_t^{i,N})\geq\gamma\right\}\supset\left\{U_{\langle M\rangle_t}\geq\gamma\right\},$$

where $\langle M\rangle_t\in[\frac{t}{C_0\vee 1},\frac{t}{c_0\wedge 1}]$ by inequality (12). It yields that

$$\left\{\exists t\in[0,T]|\phi_D(X_t^{i,N})\geq\gamma\right\}\supset\left\{\exists t\in[0,\frac{T}{c_0\wedge 1}]\text{ such that }U_t\geq\gamma,\ \right\},$$

which implies that

$$P(S_\gamma^{i,N}\leq T)\geq P\left(\exists t\in[0,\frac{T}{c_0\wedge 1}]\text{ such that }U_t\geq\gamma\right).$$

The process $(U_t)_{t\geq 0}$ is a reflected Brownian motion with bounded drift, whose law doesn't depend on i,N. As a consequence, there exists $\gamma_\epsilon>0$ independent of i,N such that $P\left(\exists t\in[0,\frac{T}{c_0\wedge 1}]\text{ such that }U_t\geq\gamma_\epsilon\right)\geq 1-\epsilon/3$. This allows us to conclude the proof of Lemma 4. □

4 Uniform tightness for the conditional distribution of time-inhomogeneous diffusions

In this section, we use Theorem 2 in order to prove the uniform tightness of the family of conditional distributions of time-inhomogeneous environment-dependent diffusion processes. More precisely, let E be an open subset of $\mathbb{R}^d$ $(d\geq 0)$ and D a bounded open subset of $\mathbb{R}^{d'}$, with $(d'\geq 1)$. We consider the diffusion process $\mathcal{Z}$ with values in $[0,+\infty[\times E\times D$ and denoted by $\mathcal{Z}_t=(t,e_t,Z_t)$ at time $t\geq 0$, which fulfils the stochastic differential system

$$\begin{aligned}de_t&=s(t,e_t,Z_t)d\beta_t+m(t,e_t,Z_t)dt\\dZ_t&=\sigma(t,e_t,Z_t)dB_t+\eta(t,e_t,Z_t)dt.\end{aligned}\tag{14}$$

Here (β, B) is a $d+d'$ standard Brownian motion and s, m, σ, η are measurable functions. We also assume that the process is subject to *hard killing* at ∂D and to *soft killing* with rate $\kappa(t, e_t, Z_t) \geq 0$, where κ is a non-negative measurable function. We denote by τ_∂ the killing time of $\mathcal{Z}$, defined by

$$\tau_\partial = \inf\{t \geq 0,\ \mathcal{Z}_t \text{ is killed at time } t\}.$$

Our first assumption ensures that the above differential system has a unique solution, and that this solution is strongly Markov.

Hypothesis 3. *We assume that s, m, σ and η are continuous uniformly Lipschitz in e, z, uniformly in t. This means that there exists a constant $k_l > 0$ such that*

$$\begin{aligned}&\|s(t,e,z) - s(t,e',z')\| + \|m(t,e,z) - m(t,e',z')\| \\ &\qquad + \|\sigma(t,e,z) - \sigma(t,e',z')\| + |\eta(t,e,z) - \eta(t,e',z')| \leq k_l \left(|z - z'| + |e - e'|\right).\end{aligned}$$

Under this hypothesis, the system (14) has a solution (see [6, Theorem 3.10, Chapter 5]), which is pathwise unique and Markov up to time $\tau_\partial = \inf\{t \geq 0,\ Z_t \notin D\}$ (see [6, Theorem 3.7, Chapter 5]).

Our second assumption ensures that a Fleming-Viot type particle system with particles evolving as $\mathcal{Z}$ between the jumps is well defined at any time $t \geq 0$.

Hypothesis 4. *We assume that*

1. *ϕ_D is of class C^2 on the boundary's neighbourhood D^{a_0}, for a given $a_0 > 0$ (we recall that D^{a_0} is defined above Hypothesis 1),*
2. *κ is uniformly bounded over $[0, +\infty[\times E \times D$ and s, m, σ and η are uniformly bounded over $[0, +\infty[\times E \times D^{a_0}$,*
3. *there exist two measurable functions $f : [0, +\infty[\times E \times D^{a_0} \to \mathbb{R}_+$ and $g : [0, +\infty[\times E \times D^{a_0} \to \mathbb{R}$ such that $\forall (t, e, z) \in [0, +\infty[\times E \times D^{a_0}$,*

$$\sum_{k,l} \frac{\partial \phi_D}{\partial x_k}(z) \frac{\partial \phi_D}{\partial x_l}(z) [\sigma\sigma^*]_{kl}(t,e,z) = f(t,e,z) + g(t,e,z),$$

and such that

   (a) *f is of class C^1 in time and of class C^2 in environment/space, and the derivatives of f are uniformly bounded,*

   (b) *there exists a positive constant $k_g > 0$ such that, for all $(t, e, z) \in [0, +\infty[\times E \times D^{a_0}$,*

$$|g(t,e,z)| \leq k_g \phi_D(z),$$

We are now able to state the main result of this section, which concerns the uniform tightness of the family of conditional distributions of $\mathcal{Z}$.

Theorem 5. *Fix $t_0 > 0$ and assume that Hypotheses 3 and 4 hold. Then, for any $\epsilon > 0$, there exists $a_\epsilon > 0$ such that, for any initial distribution η of $\mathcal{Z}$,*

$$\mathbb{P}_\eta\left(\mathcal{Z}_t = (t, e_t, Z_t) \in [0, +\infty[\times E \times D^{a_\epsilon} | t < \tau_\partial\right) \leq \epsilon, \ \forall t \geq t_0.$$

In particular, letting $(\eta_n)_{n\geq 0}$ be any sequence of initial distributions for $\mathcal{Z}$ and $(t_n)_{n\geq 0}$ be any sequence of positive times such that $\inf_{n\geq 0} t_n > 0$, the family of conditional probability measures on D indexed by $n \geq 0$ and defined by

$$\mathbb{P}_{\eta_n}\left(\mathcal{Z}_{t_n} = (t_n, e_{t_n}, Z_{t_n}) \in [0, +\infty[\times E \times \cdot \mid t_n < \tau_\partial\right), \ \forall n \geq 0,$$

is uniformly tight.

Informally, Theorem 5 tells us that the conditional distribution of diffusion process doesn't degenerate to the boundary, even if its initial distribution does. We emphasize that the result still applies for random initial distributions.

Proof of Theorem 5. For any $N \geq 2$, we define the particle system $(t, \mathbb{O}_t^N, \mathbb{X}_t^N)_{t\geq 0}$ as in Section 2 with $\mathcal{Z}^N = \mathcal{Z}$ and the following values of $\mathcal{H}^N$ and $\mathcal{S}^N$, which correspond to the Fleming-Viot type system introduced by Burdzy *et al.* [3]. Assuming that the ith particle is killed at time t, the jump measures are given by

$$\mathcal{H}^N(t, \mathbb{O}_t^N, \mathbb{X}_t^N) = \frac{1}{N-1} \sum_{j=1, j\neq i}^{N} \delta_{(o_t^j, X_t^j)} \text{ for a hard killing}$$

and by

$$\mathcal{S}^N(t, \mathbb{O}_t^N, \mathbb{X}_t^N) = \frac{1}{N-1} \sum_{j=1, j\neq i}^{N} \delta_{(o_t^j, X_t^j)} \text{ for a soft killing.}$$

On the one hand, Hypothesis 4, clearly implies that Hypothesis 1 is fulfilled. On the other hand, $\mathcal{H}^N$ fulfils Hypothesis 2 with $h^N(u) = u$ and $p_0^N = 1$. Thus, by Theorem 1, we deduce that the particle system $(\cdot, \mathbb{O}^N, \mathbb{X}^N)$ is well defined at any time, for any $N \geq 2$. As a consequence, one can apply [14, Theorem 2.1], which states that this implies, for any $t \geq 0$,

$$\frac{1}{N} \sum_{i=1}^{N} \delta_{t, o_t^{i,N}, Z_t^{i,N}}(\cdot) \xrightarrow[N\to\infty]{} \mathbb{P}_\eta\left((t, e_t, Z_t) \in \cdot \times \cdot \times \cdot\right),$$

where the initial distributions m^N of the interacting particle systems are chosen so that

$$\frac{1}{N} \sum_{i=1}^{N} \delta_{0, o_0^{i,N}, Z_0^{i,N}}(\cdot) \xrightarrow[N\to\infty]{} \eta(\cdot).$$

In particular, one has, for any $t \geq 0$,

$$\mu_t^N(\cdot) = \frac{1}{N}\sum_{i=1}^{N} \delta_{Z_t^{i,N}}(\cdot) \xrightarrow[N\to\infty]{} \mathbb{P}_\eta\left((t, e_t, Z_t) \in [0, +\infty[\times E \times \cdot\right).$$

By Theorem 2, for any $\epsilon > 0$, there exists $a_\epsilon > 0$ and $N_\epsilon \geq 2$ such that

$$\mathbb{E}\left(\mu_t^N(D^{a_\epsilon})\right) \leq \epsilon,\ \forall N \geq N_\epsilon,\ \forall t \geq t_0,$$

independently of the initial distribution of the particle system. Thus we have

$$\mathbb{P}_\eta\left((t, e_t, Z_t) \in [0, +\infty[\times E \times D^{a_\epsilon}\right) \leq \epsilon,$$

for any initial distribution η. These concludes the proof of the first part of Theorem 5 and immediately implies its second part.

□

Acknowledgements This work has been partly written during mly PhD thesis and benefited from the support of the "Chaire Modélisation Mathématique et Biodiversité of Veolia Environnement-École Polytechnique-Museum National d'Histoire Naturelle-Fondation X". I am extremely grateful to my PhD advisor Sylvie Méléard for her support and to Steven N. Evans, reviewer of my PhD thesis, for his comments on the first version of this paper.

References

[1] M. Bieniek, K. Burdzy, and S. Finch. Non-extinction of a fleming-viot particle model. *Probability Theory and Related Fields*, pages 1–40, 2011.

[2] M. Bieniek, K. Burdzy, and S. Pal. Extinction of Fleming-Viot-type particle systems with strong drift. *ArXiv e-prints*, Oct. 2011.

[3] K. Burdzy, R. Holyst, D. Ingerman, and P. March. Configurational transition in a fleming-viot-type model and probabilistic interpretation of laplacian eigenfunctions. *J. Phys. A*, 29(29):2633–2642, 1996.

[4] K. Burdzy, R. Hołyst, and P. March. A Fleming-Viot particle representation of the Dirichlet Laplacian. *Comm. Math. Phys.*, 214(3):679–703, 2000.

[5] P. Del-Moral and P. Lezaud. Branching and interacting particle interpretations of rare event probabilities. *Stochastic hybrid systems*, pages 277–323, 2006.

[6] S. N. Ethier and T. G. Kurtz. *Markov processes.* Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1986. Characterization and convergence.

[7] I. Grigorescu and M. Kang. Hydrodynamic limit for a Fleming-Viot type system. *Stochastic Process. Appl.*, 110(1):111–143, 2004.

[8] I. Grigorescu and M. Kang. Immortal particle for a catalytic branching process. *Probab. Theory Related Fields*, pages 1–29, 2011. 10.1007/s00440-011-0347-6.

[9] A. Jakubowski. Tightness criteria for random measures with application to the principle of conditioning in Hilbert spaces. *Probab. Math. Statist.*, 9(1):95–114, 1988.

[10] M. Kolb and D. Steinsaltz. Quasilimiting behavior for one-dimensional diffusions with killing. *To appear in Annals of Probability.*

[11] J.-U. Löbus. A stationary Fleming-Viot type Brownian particle system. *Math. Z.*, 263(3):541–581, 2009.

[12] S. Méléard and D. Villemonais. Quasi-stationary distributions and population processes. *ArXiv e-prints*, Dec. 2011.

[13] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[14] D. Villemonais. Interacting particle processes and approximation of Markov processes conditioned to not be killed. *ArXiv e-prints*, June 2011.

[15] D. Villemonais. Interacting particle systems and yaglom limit approximation of diffusions with unbounded drift. *Electronic Journal of Probability*, 16:1663–1692, 2011.

[16] W. Zhen and X. Hua. Multi-dimensional reflected backward stochastic differential equations and the comparison theorem. *Acta Mathematica Scientia*, 30(5):1819 – 1836, 2010.