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Itaï Ben Yaacov

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THE LINEAR ISOMETRY GROUP OF THE GURARIJ SPACE IS UNIVERSAL

ITAÏ BEN YAACOV

ABSTRACT. We give a positive answer to a question of Uspenskij, whether the linear isometry group of the Gurarij space \mathbf{G} is a universal Polish group.

In doing so, we also give a Katětov-style construction of the Gurarij space, via a generalisation of the Arens-Eells space to metric space with a distinguished normed subspace.

INTRODUCTION

Let $\text{Iso}(X)$ denote the isometry group of a metric space X , which we equip with the topology of point-wise convergence, rendering it, for complete separable X , a Polish group. We let $\text{Iso}_L(E)$ denote the linear isometry group of a normed space E (throughout this paper, over the reals). This is a closed subgroup of $\text{Iso}(E)$, and is therefore Polish as well when E is a separable Banach space.

It was shown by Uspenskij [Usp90] that the isometry group of the Urysohn space \mathbf{U} is a universal Polish group, following a construction of \mathbf{U} due to Katětov [Kat88]. The Gurarij space \mathbf{G} (see [Gur66, Lus76]) is, in *some ways*, the analogue of the Urysohn space in the category of Banach spaces. Either one is the unique separable, universal and approximately ultra-homogeneous object in its respective category, or equivalently, either one is the (necessarily unique) *Fraïssé limit* (see [Ben]) of the finitely generated objects in its respective category. It is therefore natural to compare properties of the two, and ask whether $\text{Iso}_L(\mathbf{G})$ is a universal Polish group as well. Before discussing this question, let us briefly recall Uspenskij's argument for the Urysohn space.

Let X be a metric space. We say that a real-valued function ξ on X is *Katětov* if $\xi(x) \leq \xi(y) + d(x, y)$ and $d(x, y) \leq \xi(x) + \xi(y)$ for all $x, y \in X$ (equivalently, if ξ is the distance function from points in X to some fixed point in a metric extension of X). Let $K(X)$ denote the set of Katětov functions on X . Equipped with the supremum distance, $K(X)$ is a complete metric space, endowed with a natural isometric embedding $X \hookrightarrow K(X)$, sending $x \mapsto d(x, \cdot)$. This construction is functorial, in the sense that an isometric embedding $Y \hookrightarrow X$ gives rise to a natural isometric embedding $K(Y) \hookrightarrow K(X)$, where ξ goes to $\hat{\xi}(x) = \inf_{y \in Y} d(x, y) + \xi(y)$, and everything composes and commutes as one would expect.

Even when X is separable, $K(X)$ need not be separable. More generally, if χ denotes the density character (which is always taken to be at least \aleph_0 , for convenience), we only have $\chi(K(X)) \leq |K(X)| \leq 2^{\chi(X)}$, and the bound can be attained. When $Y \subseteq X$, let us identify $K(Y)$ with its image in $K(X)$, and define

$$K_0(X) = \overline{\bigcup_{Y \subseteq X \text{ finite}} K(Y)}.$$

Then we have $X \hookrightarrow K_0(X)$ and this is functorial as for $K(X)$, and moreover $\chi(K_0(X)) = \chi(X)$. We now define $X_0 = X$, $X_{n+1} = K_0(X_n)$ and $X_\omega = \bigcup X_n$. If X is separable then so is X_ω , and the latter is a Urysohn space. Moreover, since everything was functorial, we obtain continuous embeddings of the isometry groups $\text{Iso}(X) \subseteq \text{Iso}(X_\omega) = \text{Iso}(\mathbf{U})$, whence the desired result. We refer the reader to [Usp08, Section 3] for more details.

It turns out that the same strategy works for the Gurarij space as well. In Section 1 we identify the space of isomorphism types of one point extensions of a Banach space E (in model-theoretic terminology we would speak of *quantifier-free 1-types* over E) as the space $K_C(E)$ of *convex* Katětov functions. The passage from K to K_0 carries over essentially unchanged, in that $K_{C,0}(E) = K_C(E) \cap K_0(E)$ will do. The main technical difficulty lies in the fact that, while $K(X)$ is a metric space extension of X , $K_C(E)$ is *not* a Banach space extension of E . In Section 2 we show that $K_C(E)$ embeds canonically in a Banach

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space extension of E . In fact we prove a stronger result, generalising the Arens-Eells construction from pointed metric spaces to metric spaces over a normed space. We conclude in Section 3, constructing, for a separable Banach space E , a chain of separable Banach space extensions as above, with limit E_ω , and show that $E_\omega \cong \mathbf{G}$, with only marginally more complexity than for pure metric spaces.

1. CONVEX KATĚTOV FUNCTIONS

Let E be a normed space, $X \subseteq E$ convex, and define $K_C(X) \subseteq K(X)$ to consist of all convex Katětov functions. We observe that the inclusion $K(X) \subseteq K(E)$ restricts to $K_C(X) \subseteq K_C(E)$, for indeed, if $\xi \in K_C(X) \subseteq K(E)$ and $x, y \in E$ then

$$\frac{\xi(x) + \xi(y)}{2} = \sup_{x', y' \in X} \frac{\|x - x'\| + \xi(x') + \|y - y'\| + \xi(y')}{2} \geq \sup_{x', y' \in X} \left\| \frac{x+y}{2} - \frac{x'+y'}{2} \right\| + \xi\left(\frac{x'+y'}{2}\right) \geq \xi\left(\frac{x+y}{2}\right).$$

For $\xi \in K(X)$ we define the generated convex function $\xi^C: X \rightarrow \mathbf{R}$ to be the greatest convex function lying below ξ , namely

$$\xi^C(a) = \inf_{n; \bar{b} \in A(a, n)} \sum \xi(b_k)/n, \quad \text{where } A(a, n) = \{\bar{b} \in X^n : a = \sum b_i/n\}.$$

We observe that ξ^C is Katětov as well, so $\xi^C \in K_C(X)$, and the map $\xi \mapsto \xi^C$ is 1-Lipschitz. Moreover, if $X \subseteq Y \subseteq E$ are all convex then passing from ξ to ξ^C (on X) and then extending to Y gives the same result as first extending ξ to Y and then passing to ξ^C , so we may refer to $\xi^C \in K_C(Y)$ without ambiguity.

For a convex set X (most often this will be the entire Banach space), we define $K_{C,0}(X) = K_0(X) \cap K_C(X)$. We claim that this intersection is not too small, and in particular, that it contains ξ^C for every $\xi \in K_0(X)$. More precisely, we claim that

$$K_{C,0}(X) = \overline{\bigcup_{Y \subseteq X \text{ convex compact}} K_C(Y)} = \{\xi^C : \xi \in K_0(X)\}.$$

Indeed, for the first inclusion \supseteq it will suffice to observe that $K_0(X)$ contains $K(Y)$ for every compact $Y \subseteq X$. The second holds since $\xi \mapsto \xi^C$ is continuous and the convex hull of a finite set is compact, and $\{\xi^C : \xi \in K_0(X)\} \supseteq K_{C,0}(X)$ trivially.

Lemma 1.1. *Given $\xi \in K_C(E)$, let $E(x) = E \oplus \mathbf{R}x$, and for $\alpha x - a \in E(x)$ define*

$$\|\alpha x - a\|^\xi = \begin{cases} |\alpha| \xi(a/\alpha) & \alpha \neq 0 \\ \|a\| & \alpha = 0. \end{cases}$$

The map $\xi \mapsto \|\cdot\|^\xi$ is a bijection between $K_C(E)$ and semi-norms on $E(x)$ extending $\|\cdot\|_E$, namely with the space of quantifier-free 1-types over E (see [BH]), whose inverse sends $\|\cdot\|$ to $\|x - \cdot\|$.

Proof. All there is to show is that if $\xi \in K_C(E)$ then $\|\cdot\|^\xi$ is indeed a semi-norm, and for this it will suffice to show that $\|\alpha x - a\|^\xi + \|\beta x - b\|^\xi \geq \|(\alpha + \beta)x - (a + b)\|^\xi$. We consider several cases:

- (i) If both α and β are zero then there is nothing to show.
- (ii) If α and β are non zero with equal sign, say $0 > \alpha, \beta$ and $\alpha + \beta = 1$, then we use convexity: $\alpha \xi(a/\alpha) + \beta \xi(b/\beta) \geq \xi(\alpha a/\alpha + \beta b/\beta) = \xi(a + b)$.
- (iii) If $\alpha = -\beta$, say $\alpha = 1$, then this is $\xi(a) + \xi(-b) \geq \|a + b\| = d(a, -b)$, which follows from ξ being Katětov.
- (iv) The last case is when α and β have distinct signs and absolute values, say $\beta \leq 0 < \alpha$ and $\alpha + \beta = 1$. Then indeed, using the fact that ξ is Katětov,

$$\begin{aligned} \alpha \xi(a/\alpha) - \beta \xi(b/\beta) &= \xi(a/\alpha) - \beta [\xi(a/\alpha) + \xi(b/\beta)] \\ &\geq \xi(a/\alpha) - \beta \|a/\alpha - b/\beta\| \\ &= \xi(a/\alpha) + \|a/\alpha - (a + b)\| \geq \xi(a + b). \end{aligned} \quad \blacksquare_{1.1}$$

Given two (isometric) extensions $E \subseteq F_i$, $i = 0, 1$, there is a greatest (compatible) norm on $F_0 \oplus_E F_1$, namely $\|a - b\| = \inf_{c \in E} \|a - c\| + \|c - b\|$. With $F_0 = (E(x), \|\cdot\|^\xi)$ for some $\xi \in K_C(E)$ and $F_1 = F$ fixed, we obtain a canonical extension of $\|\cdot\|^\xi$ to $F(x) = E(x) \oplus_E F$, compatible with the embedding $K_C(E) \subseteq K_C(F)$.

Lemma 1.2. *Let E be a normed space and $\xi_i \in K_C(E)$, $i = 0, 1$. Then for every $d(\xi_0, \xi_1) = r_0 \leq r \leq r_1 = \inf_{a \in E} \xi_0(a) + \xi_1(a)$ there exists a semi-norm $\|\cdot\|_r$ on $E(x_0, x_1) = E \oplus \mathbf{R}x_0 \oplus \mathbf{R}x_1$ whose restriction to $E(x_i)$ is $\|\cdot\|^{\xi_i}$, such that $\|x_0 - x_1\|_r = r$.*

Proof. When $r = r_0$, this is a special case of an amalgamation result of Henson, see [BH, Proposition 1.7]. Indeed, for $a \in \alpha x_0 + \beta x_1 \in E(x_0, x_1)$ define

$$\|a + \alpha x_0 + \beta x_1\|' = \inf_{b \in E, \gamma \in \mathbf{R}} \|b + (\alpha + \gamma)x_0\|^{\xi_0} + \|a - b + (\beta - \gamma)x_1\|^{\xi_1} + |\gamma|r_0.$$

This is clearly a semi-norm, and $\|x_0 - x_1\|' \leq r_0$. For all $a + \alpha x_0 \in E(x_0)$ and b, γ , we have, by choice of r_0 ,

$$\|a + \alpha x_0\|^{\xi_0} \leq \|b + (\alpha + \gamma)x_0\|^{\xi_0} + \|a - b - \gamma x_1\|^{\xi_1} + |\gamma|r_0,$$

so $\|\cdot\| = \|\cdot\|^{\xi_0}$ on $E(x_0)$, and similarly $\|\cdot\| = \|\cdot\|^{\xi_1}$ on $E(x_1)$. It follows that $\|x_0 - x_1\|' = r_0$, as desired.

When $r = r_1$, just take the greatest norm on $E(x_0) \oplus_E E(x_1)$ compatible with $\|\cdot\|^{\xi_i}$ as above. Intermediate values can be obtained as convex combinations of the two extremal cases. $\blacksquare_{1.2}$

2. THE ARENS-EELLS SPACE OVER A NORMED SPACE

Notation 2.1. Given two pointed metric spaces $(X, 0)$ and $(Y, 0)$, we let $\text{Lip}_0(X, Y)$ denote the space of all Lipschitz functions $\theta: X \rightarrow Y$ which send $0 \mapsto 0$, and for $\theta \in \text{Lip}_0(X, Y)$ (or for that matter, for any Lipschitz function θ) we let $L(\theta)$ denote its Lipschitz constant. If $Y = \mathbf{R}$, we omit it.

We recall the following facts from Weaver [Wea99, Chapter 2.2]:

Fact 2.2. *Let $(X, 0)$ be a pointed metric space. Then there exists a Banach space $\text{AE}(X)$, together with an isometric embedding $X \subseteq \text{AE}(X)$ sending $0 \mapsto 0$, called the Arens-Eells space of X , having the following universal property: every $\theta \in \text{Lip}_0(X, F)$, where F is a Banach space, admits a unique continuous linear extension $\theta': \text{AE}(X) \rightarrow F$, and this unique extension satisfies $\|\theta'\| \leq L(\theta)$.*

This universal property characterises the Arens-Eells space up to a unique isometric isomorphism. Its dual Banach space $\text{AE}(X)^$ is canonically isometrically isomorphic to $\text{Lip}_0(X)$, the isomorphism consisting of sending a linear functional to its restriction to X .*

We shall generalise this as follows:

Definition 2.3. Let E be a fixed normed space. By a *metric space over E* we mean a pair (X, φ) where X is a metric space, $\varphi: E \rightarrow X$ is isometric, and for each $x \in X$ the function $a \mapsto d(\varphi a, x)$ is convex. Most of the time φ will just be an inclusion map, in which case it is replaced with E or simply omitted.

For a normed space F we also define $\text{Lip}_E(X, F)$ to consist of all Lipschitz functions $\theta: X \rightarrow F$ which are linear on E , and if $F = \mathbf{R}$ then we omit it (of course we could extend this definition to Lipschitz functions between metric spaces over respective Banach spaces, but no such notion will be needed here).

For the trivial normed space 0 , a metric space over 0 is the same thing as a pointed metric space. Also, every isometric inclusion of normed spaces $E \subseteq F$ renders F metric over E .

Theorem 2.4. *Let X be a metric space over a normed space E . Then there exists a Banach space $\text{AE}(X, E)$, together with an isometric embedding $X \subseteq \text{AE}(X, E)$ linear on E , having the following universal property: every $\theta \in \text{Lip}_E(X, F)$, where F is a Banach space, admits a unique continuous linear extension $\theta': \text{AE}(X, E) \rightarrow F$, and this unique extension satisfies $\|\theta'\| \leq L(\theta)$.*

This universal property characterises $\text{AE}(X, E)$ up to a unique isometric isomorphism, and we shall call it the Arens-Eells space of X over E . Its dual $\text{AE}(X, E)^$ is isometrically isomorphic to $\text{Lip}_E(X)$ via restriction to X .*

Proof. Let $\iota: E \rightarrow \text{AE}(X)$ denote the inclusion map, so that we may distinguish between algebraic operations in E and in $\text{AE}(X)$. Let $F_0 \subseteq \text{AE}(X)$ be the closed subspace generated by all expressions of the form $\iota(a + b) - \iota a - \iota b$ for $a, b \in E$, observing that it also contains $\iota \alpha a - \alpha \iota a$ for $a \in E$ and $\alpha \in \mathbf{R}$. Define $\text{AE}(X, E)$ as the completion of the quotient $\text{AE}(X)/F_0$, and let $\psi: X \rightarrow \text{AE}(X, E)$ be the natural map. Then ψ is linear and 1-Lipschitz, and both the universal property and characterisation of the dual hold, so all we need to show is that ψ is isometric.

Indeed, let $x_i \in X$, $i = 0, 1$, and let $\xi_i = d(\cdot, x_i) \in K_C(E)$. Let $r_0 \leq r_1$ be as in Lemma 1.2, and observe that by the triangle inequality $r_0 \leq d(x_0, x_1) \leq r_1$. Therefore, by Lemma 1.2 there exists a semi-norm $\|\cdot\|$ on $E(x_0, x_1)$ which induces the same distance on $E \cup \{x_0, x_1\}$ as X does. By the Hahn-Banach Theorem, there exists a linear functional $\lambda \in E(x_0, x_1)^*$ which norms $x_0 - x_1$: $\|\lambda\| = 1$ and $\lambda(x_0 - x_1) = d(x_0, x_1)$. The restriction of λ to $E \cup \{x_0, x_1\}$ is 1-Lipschitz, and by [Wea99, Theorem 1.5.6], it extends to a 1-Lipschitz function $\lambda': X \rightarrow \mathbf{R}$. Then $\lambda' \in \text{Lip}_E(X) = \text{AE}(X, E)^* \subseteq \text{Lip}_0(X)$ and $\|\lambda'\| = 1$, witnessing that $\|\psi x_0 - \psi x_1\| \geq d(x_0, x_1)$, completing the proof. $\blacksquare_{2.4}$

Similarly, it is shown in [Wea99] that if $Y \subseteq X$ then $\text{AE}(Y)$ is naturally identified with the subspace of $\text{AE}(X)$ generated by Y , and the same can be deduced for metric spaces over a normed space E .

Corollary 2.5. *Let E be a normed space and $E \subseteq X \subseteq K_C(E)$ (here we identify E with its image in $K_C(E)$). Then there exists a Banach space $E[X]$ together with an isometric embedding $X \subseteq E[X]$ such that $E \subseteq E[X]$ is a Banach space extension, and the following universal property holds: every Lipschitz map $\theta: X \rightarrow F$, where F is a Banach space, which is linear on E , admits a unique continuous linear extension $\theta': E[X] \rightarrow F$ with $\|\theta'\| \leq L(\theta)$.*

In particular, $E[X]$ is uniquely determined by this universal property, and is generated as a Banach space by X .

Proof. Just take $E[X] = \text{AE}(X, E)$. ■_{2.5}

3. MAIN THEOREM

Definition 3.1. For a Banach space E we define

- $E' = E[K_{C,0}(E)]$ as per Corollary 2.5,
- $E_0 = E$, $E_{n+1} = E'_n$, $E_\omega = \bigcup E_n$.

In order to show that E_ω is a Gurarij space, it will be easiest to use the following criterion, which may simply be taken as the definition.

Fact 3.2. *A separable Banach space E is a Gurarij space if and only if for every $\xi \in K_C(E)$, every finitely many points $a_0, \dots, a_{n-1} \in E$ and every $\varepsilon > 0$ there exists $b \in E$ such that $|\|b - a_i\| - \xi(a_i)| < \varepsilon$ for all $i < n$. Moreover, we may restrict the a_i to some dense subset of E .*

Proof. The main assertion is a restatement of [BH, Lemma 2.2], and the moreover part follows. ■_{3.2}

Fact 3.3. *The Gurarij space \mathbf{G} exists and is unique.*

Proof. Existence (which also follows from Lemma 3.6 below) is due to Gurarij [Gur66], as is almost isometric uniqueness. Isometric uniqueness is due to Lusky [Lus76], with a recent plethora of more “elementary” (that is to say, in the author’s view, more model-theoretic) proofs [KS, BH, Ben]. ■_{3.3}

We recall from Uspenskij [Usp08],

Definition 3.4. An isometric embedding of metric spaces $X \subseteq Y$ is said to be a *g-embedding* if there exists a continuous homomorphism $\Theta: \text{Iso}(X) \rightarrow \text{Iso}(Y)$ such that $\Theta\varphi$ extends φ for each $\varphi \in \text{Iso}(X)$.

Notice that since the restriction map $\text{img } \Theta \rightarrow \text{Iso}(X)$, $\varphi \mapsto \varphi \upharpoonright_X$, is continuous as well, such a map Θ is necessarily a homeomorphic embedding.

Fact 3.5 (Mazur-Ulam Theorem). *For a normed space E , $\text{Iso}(E)$ is the group of affine isometries. Stated equivalently, $\text{Iso}_L(E) = \{\varphi \in \text{Iso}(E): \varphi 0 = 0\}$.*

Proof. [FJ03, Theorem 1.3.5]. ■_{3.5}

It follows that an isometric embedding of Banach spaces $E \subseteq F$ is a *g-embedding* if and only if there exists a continuous homomorphism $\Theta: \text{Iso}_L(E) \rightarrow \text{Iso}_L(F)$ such that $\Theta\varphi$ extends φ for all $\varphi \in \text{Iso}_L(E)$. Indeed, given $\Theta: \text{Iso}(E) \rightarrow \text{Iso}(F)$ we have $(\Theta\varphi)0 = \varphi 0 = 0$, so Θ restricts to $\Theta': \text{Iso}_L(E) \rightarrow \text{Iso}_L(F)$, and conversely, given $\Theta: \text{Iso}_L(E) \rightarrow \text{Iso}_L(F)$ we can define $\Theta': \text{Iso}(E) \rightarrow \text{Iso}(F)$ by $\Theta'\varphi = \Theta(\varphi - \varphi 0) + \varphi 0$.

Lemma 3.6. *Let E be a normed space. Then E' and E_ω are Banach space extensions of E with $\chi(E) = \chi(E') = \chi(E_\omega)$, and the embeddings $E \subseteq E' \subseteq E_\omega$ are *g-embeddings*.*

Moreover, if E is separable then E_ω is the Gurarij space.

Proof. Since $E[X]$ is generated by the image of X , we have $\chi(E[X]) = \chi(E) + \chi(X)$, whence the first assertion. For the second, every $\varphi \in \text{Iso}_L(E)$ of E extends to an isometry $\hat{\varphi}: \xi \mapsto \xi \circ \varphi^{-1}$ of $K_{C,0}(E)$ and by the universal property of $E[X]$, to an isometry $\varphi' \in \text{Iso}_L(E')$. Proceeding by induction we obtain an extension $\varphi_\omega \in \text{Iso}(E_\omega)$. We leave it to the reader to check that the maps $\varphi \mapsto \varphi'$ and therefore $\varphi \mapsto \varphi_\omega$ are continuous homomorphisms.

For the moreover part, it is clear that E_ω satisfies the criterion of Fact 3.2, with the a_i restricted to the dense set $\bigcup E_n$. ■_{3.6}

Theorem 3.7. *The group $\text{Iso}_L(\mathbf{G})$ of linear isometries of the Gurarij space is a universal Polish group. Moreover, every separable normed space is g -embeddable in \mathbf{G} .*

Proof. The moreover part follows immediately from Lemma 3.6. Now let H be a Polish group. Then it admits a compatible left-invariant distance d , which we may assume is bounded by 1. Adjoining a distinguished point $*$ to H with $d(*, h) = 1$ (or even $\frac{1}{2}$) for all $h \in H$ we obtain a pointed metric space H^* . The left action of $h \in H$ on H^* , fixing $*$, is an isometry, which extends by the universal property to a linear isometry $h' \in \text{Iso}_L(\text{AE}(H^*, *))$, and the map $h \mapsto h'$ is a continuous embedding. We conclude by the moreover part. ■_{3.7}

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ITAÏ BEN YAACOV, UNIVERSITÉ CLAUDE BERNARD – LYON 1, INSTITUT CAMILLE JORDAN, CNRS UMR 5208, 43 BOULEVARD DU 11 NOVEMBRE 1918, 69622 VILLEURBANNE CEDEX, FRANCE
 URL: <http://math.univ-lyon1.fr/~begnac/>