

Meromorphic functions of uniqueness

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Abstract: Let E be an algebraically closed field of characteristic 0 which is either \mathbb{C} or a complete ultrametric field K . We consider the composition of meromorphic functions $h \circ f$ where h is meromorphic in all E and f is meromorphic either in E or in an open disk of K . We then look for a condition on h in order that if 2 similar functions f, g satisfy $h \circ f(a_m) = h \circ g(a_m)$ where (a_m) is a bounded sequence satisfying certain condition, this implies $f = g$. Particularly we generalize to meromorphic functions previous results on polynomials of uniqueness. The condition on h involves the zeros (c_n) of h' and the values $h(c_n)$ but is weaker than this introduced by H.Fujimoto (injectivity on the set of zeros of h'). The main tool is the Nevanlinna Theory but also involves some specific p-adic properties and basic affine properties. Results concerning p-adic entire functions only suppose a property involving 2 zeros of h' . Polynomials of uniqueness for entire functions are characterized. Every polynomial P of prime degree $n \geq 3$ is a polynomial of uniqueness for p-adic entire functions, except if is of the form $A(x-a)^n + B$. A polynomial P such that P' has exactly two distinct zeros is a polynomial of uniqueness for meromorphic functions in K if and only if both zeros have a multiplicity order greater than 1. Results on p-adic functions have applications to rational functions in any field of characteristic 0.

Keywords: *p-adic and complex analytic functions, uniqueness, affinely rigid sets, values on zeros of the derivative*

1. Introduction and basic results

Throughout the paper, E is either an algebraically closed field K of characteristic zero complete with respect to an ultrametric absolute value or the field \mathbb{C} and L is an algebraically closed field of characteristic 0 without any assumption of absolute value. The paper is aimed at studying sufficient conditions assuring that if the composition of meromorphic functions of the form $h \circ f$ and $h \circ g$ are equal, then f and g are equal. This kind of problem follows many other problems of uniqueness studied in the past years, particularly on unique range sets with (or without) multiplicities and polynomials of uniqueness for analytic or meromorphic functions in the complex field and in an ultrametric field [1], [3], [13], [4], [6], [7], [8], [10], [21]. Polynomials of uniqueness were introduced and studied in \mathbb{C} by X.H Hua and C.C. Yang [19], H. Fujimoto [14], P. Li and C.C. Yang [25], H.H. Khoai and C.C. Yang, [16], E. Mayerhofer and the author [12] and were also studied in a p-adic field, particularly by T.T. H. An, H.H. Khoai, Julie Tzu-Yueh Wang, Pitmann Wong, C.C. Yang and the author, E. Mayerhofer [15], [1], [2] [22], [25], [11]. A polynomial P is called *polynomial of uniqueness* for a family of functions \mathcal{F} if for any two $f, g \in \mathcal{F}$ such that $P \circ f = P \circ g$ we have $f = g$. (Notice that P is called a *polynomial of strong uniqueness* for \mathcal{F} if for any two $f, g \in \mathcal{F}$ such that $P \circ f = \lambda P \circ g$ for certain λ in the ground field E , we have $f = g$).

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Here we mean to consider *meromorphic functions of uniqueness* for a family of functions \mathcal{F} defined in a subset of the field E : a meromorphic function h in the whole field E will be called a *function of uniqueness* for a family \mathcal{F} of functions defined in a suitable subset of E if given any two functions $f, g \in \mathcal{F}$ satisfying $h \circ f = h \circ g$, f and g are identical. Similarly, we shall consider the same question in the purely algebraic context. Let $h \in L(x)$ and let \mathcal{F} be a subset of $L(x)$. Then h will be called a *function of uniqueness* for \mathcal{F} if given any two functions $f, g \in \mathcal{F}$ satisfying $h \circ f = h \circ g$, f and g are identical.

First, we shall characterize polynomials of uniqueness for entire functions in K and similarly for polynomials in $L[x]$.

A subset S of L is said to be *affinely rigid* if there exists no affine mapping γ from L to L , other than the identity, such that $\gamma(S) = S$. Let P be the polynomial admitting S as the set of its zeros, all of order one. S is called an URS(CM) for a family \mathcal{F} of functions if, for any two functions $f, g \in \mathcal{F}$ such that $P(f)$ and $P(g)$ have the same zeros (counting multiplicities), then $f = g$. Actually, for functions in $\mathcal{A}(K)$, to say that $P(f)$ and $P(g)$ have the same zeros (counting multiplicities) is equivalent to say that $\frac{P(f)}{P(g)}$ is a constant.

Then by [6], [8] we have Theorem A:

Theorem A: *Let $P \in K[x]$ (resp. $P \in L[x]$) be of degree n and have all its zeros of order 1 and let S be the set of zeros of P . Then P is a polynomial of strong uniqueness for $\mathcal{A}(K)$ (resp. for $L[x]$) if and only if S is affinely rigid.*

Definition: We shall call *similarity* or *affine mapping* in the field L a mapping from L to L of the form $\gamma(x) = \alpha x + \beta$. If $\alpha \neq 1$, then $\gamma(x)$ is of the form $a + \alpha(x - a)$, the point a will be called *the center of γ* and γ will be called a *centered similarity*.

A subset S of L is said to be *affinely rigid* if there exists no similarity γ from L to L , other than the identity, such that $\gamma(S) = S$.

Proposition B: *If a finite subset S of L is preserved by a similarity γ , that γ is a centered similarity.*

First we shall characterize non-affinely rigid sets and next we'll characterize polynomials of uniqueness for $\mathcal{A}(K)$ and for $L[x]$.

Definition: Here a subset S of L will be said to be a *centered non-affinely rigid set* (resp. *non-centered non-affinely rigid analytic set*) if there exists a centered similarity γ from L to L , other than the identity, such that $\gamma(S) = S$ and such that the center of γ lies in S (resp. does not lie in S).

Theorem 1: *Let $P \in L[x]$ be of degree n and have all its zeros of order 1 and let S be the set of zeros of P . Then S is not affinely rigid if and only if there exists a centered similarity of center a : $\gamma(x) = a + \alpha(x - a)$, such that $P(\gamma(x)) = \alpha^n P(x)$, with $\alpha \neq 1$. Let S be non-affinely rigid and let γ be such a centered similarity $\gamma(x) = a + \alpha(x - a)$ preserves S , then putting $u = x - a$, $Q(u) = P(x)$, Q is of one the following two forms:*

i) $Q(u) = \sum_{k=0}^q a_{kd} u^{kd}$, with $a_0 \neq 0$, $\alpha^d = 1$ and $d \geq 2$ and then S is non-centered.

ii) $Q(u) = \sum_{k=0}^q a_{kd+1} u^{kd+1}$, with $a_1 \neq 0$, $\alpha^d = 1$ and $d \geq 2$ and then S is centered.

Moreover if two centered similarities preserve S , they have the same center. Further S is never both centered and non-centered.

Theorem 2: Let $P(x) = \sum_{k=0}^q a_{kd+1} x^{kd+1}$, with $a_1 \neq 0$, $d \geq 2$ and let Z be the set of the zeros of P' . For all $c \in K \setminus P(Z)$, $P - c$ admits $qd + 1$ distinct zeros and its set of zeros is affinely rigid.

Remark: Let δ be an affine mapping and let S be a non-affinely rigid set. Then $\delta(S)$ is a non-affinely rigid set. Moreover, if S is centered (resp. non-centered) so is $\delta(S)$.

Notation: We denote by $\mathcal{A}(E)$ the algebra of analytic functions in all E also called *entire functions* and by $\mathcal{M}(E)$ the field of meromorphic functions in E i.e. the field of fractions of $\mathcal{A}(E)$.

Theorem 3: Let $P \in K[x]$ (resp. $P \in L[x]$) have all its zeros of order 1 and let S be the set of zeros of P . Then P is not a function of uniqueness for $\mathcal{A}(K)$ (resp. for $L[x]$) if and only if S is a non-centered non-affinely rigid set.

Example: Let $P(x) = x + x^3 + x^5$. Then P is a function of uniqueness for $\mathcal{A}(K)$ and for $L[x]$.

Corollary 3.1: Let $P \in K[x]$ (resp. $P \in L[x]$) be of degree n , a prime number ≥ 3 . Then P is a function of uniqueness for $\mathcal{A}(K)$ (resp. for $P \in L[x]$) if and only if P is not of the form $A(x - a)^n + B$ ($A, B \in K$), (resp. $A, B \in L$).

Remark: In [2] T.T.H. An and J.T.Y. Wang give some sufficient conditions to assure that a polynomial is a polynomial of uniqueness for entire functions in a field of positive characteristic.

2. Generalities on Meromorphic Functions

In Paragraph 3 we mean to generalize results obtained by H. Fujimoto in \mathbb{C} and by T.T.H. An and H.H. Khoai in several ways: we shall consider a meromorphic function h instead of a polynomial P in \mathbb{C} as well as in K and we'll only assume that a few zeros c_1, \dots, c_k of h' satisfy $h(c_j) \neq h(d)$ for every other zero d of h' . We shall then examine the

situation in \mathbb{C} and four cases in K : f , g entire or meromorphic functions in the whole p -adic field K or in \mathbb{C} , or "unbounded" meromorphic function inside an "open" disk of K .

On the other hand, in each main claim, instead of assuming that the equality $h(f(x)) = h(g(x))$ holds in the whole set of definition, thanks to properties of analytic sets, we'll check that it is sufficient to have the equality on a bounded sequence having no cluster point at the poles of f and g . This is obvious in the complex context and easily proved in K by using properties of analytic elements [9].

In the field K as in \mathbb{C} , the composition of two meromorphic functions $h \circ f$ is not a meromorphic function, in the general case: a pole of f is currently narrowed by poles of $h \circ f$ coming from the poles of h . This is why we first have to study general and basic properties of such functions, particularly in the ultrametric case.

Notation: In K , given $a \in K$ and $r > 0$, we denote by $d(a, r^-)$ the disk $\{x \in K \mid |x - a| < r\}$ and by $d(a, r)$ the disk $\{x \in K \mid |x - a| \leq r\}$.

Throughout the paper, we consider a disk $d(a, R^-)$ in K and we denote by $\mathcal{A}(d(a, R^-))$ the algebra of the *analytic functions in $d(a, R^-)$* i.e. the set of power series in $x - a$ converging for $|x - a| < R$, we denote $\mathcal{A}_b(d(a, R^-))$ the K -subalgebra of $\mathcal{A}(d(a, R^-))$ consisting of the bounded functions $f \in \mathcal{A}(d(a, R^-))$ (i.e. the set of power series in $\sum_{n=0}^{\infty} a_n(x - a)^n$ such that $\sup_n |a_n|R^n < \infty$). And we put $\mathcal{A}_u(d(a, R^-)) = \mathcal{A}(d(a, R^-)) \setminus \mathcal{A}_b(d(a, R^-))$.

We denote by $\mathcal{M}(d(a, R^-))$ the field of meromorphic functions in $d(a, R^-)$ i.e. the field of fractions of $\mathcal{A}(d(a, R^-))$, by $\mathcal{M}_b(d(a, R^-))$ the field of fractions of $\mathcal{A}_b(d(a, R^-))$ and we put $\mathcal{M}_u(d(a, R^-)) = \mathcal{M}(d(a, R^-)) \setminus \mathcal{M}_b(d(a, R^-))$.

Let $h \in \mathcal{M}(E)$ (resp. $h \in \mathcal{M}(d(a, R^-))$). We shall denote by $\mathcal{P}(h)$ the set of poles of h and by $\mathcal{C}(h)$ the set of zeros of h . Let $f \in \mathcal{M}(E)$, (resp. let $f \in \mathcal{M}(d(a, R^-))$). We set $\mathcal{T}(f, h) = \{x \in E \mid f(x) \in \mathcal{P}(h)\}$ (resp. $\mathcal{T}(f, h) = \{x \in d(a, R^-) \mid f(x) \in \mathcal{P}(h)\}$) and we shall denote by $\mathcal{S}(f, h)$ the set $\mathcal{P}(f) \cup \mathcal{T}(f, h)$.

Proposition C is classical in \mathbb{C} as in the field K :

Proposition C: *Let $h \in \mathcal{M}(E)$ and let $f \in \mathcal{M}(E)$ (resp. let $f \in \mathcal{M}(d(a, R^-))$). Then $E \setminus (\mathcal{S}(f, h))$ is an open subset of E dense in E (resp. $d(a, R^-) \setminus (\mathcal{S}(f, h))$ is an open subset of $d(a, R^-)$ dense in $d(a, R^-)$). For each $\alpha \in E \setminus \mathcal{P}(f)$ (resp. $\alpha \in d(a, R^-) \setminus \mathcal{P}(f)$), $h \circ f(x)$ is equal to a Laurent series in $x - \alpha$ in any set of the form $d(\alpha, r) \setminus \{\alpha\}$ included in $E \setminus \mathcal{P}(\alpha)$ (resp. included in $d(a, R^-) \setminus (\mathcal{P}(f))$). If $\alpha \notin \mathcal{T}(f, h)$, the Laurent series in $x - \alpha$ of $h \circ f$ has no terms of negative index. If $\alpha \in \mathcal{T}(f, h)$, then the Laurent series of $h \circ f$ in $x - \alpha$ has finitely many terms of negative indices (i.e. $h \circ f$ is meromorphic in a disk of E of center α and has a pole at α).*

Definitions: Let $h \in \mathcal{M}(E)$ and let $f \in \mathcal{M}(E)$ (resp. let $f \in \mathcal{M}(d(a, R^-))$). Let $\alpha \in E \setminus \mathcal{P}(f)$ (resp. $\alpha \in d(a, R^-) \setminus \mathcal{P}(f)$). We shall call *Laurent series of $h \circ f$ at α* the Laurent series in $x - \alpha$ equal to $h \circ f(x)$ in a neighborhood of α . If the Laurent series of

$h \circ f$ at α is a power series, α is called a *regular point* for $h \circ f$. If the Laurent series at $\alpha \in \mathcal{T}(f, h)$ is of the form $\sum_{n=-q}^{\infty} a_n(x - \alpha)^n$, then α is called a *pole of order q* for $h \circ f$. A

point $\alpha \in E$ (resp. $\alpha \in d(a, R^-)$) which is not regular for $h \circ f$ will be called a *singular point* for $h \circ f$.

By Proposition C, a singular point for $h \circ f$ which is not a pole of $h \circ f$ belongs to $\mathcal{P}(f)$ and will be called a *point of high singularity* for $h \circ f$.

Remarks: By definition, a point of high singularity for $h \circ f$ is a pole of f . Conversely, a pole of f is just a pole for $h \circ f$ when h is a (non-constant) polynomial. And when h is a rational function tending to a finite limit at infinity, then a pole of f is a regular point for $h \circ f$.

Definition: Let $(a_n)_{n \in \mathbb{N}}$ be a bounded sequence in E . The number $\sup\{|a_n - a_m| \mid n, m \in \mathbb{N}\}$ will be called *the diameter of the sequence*.

Proposition D: Let $h \in \mathcal{M}(E)$ and let $f, g \in \mathcal{M}(E)$ (resp. let $f, g \in \mathcal{M}(d(a, R^-))$). Let $(a_m)_{m \in \mathbb{N}}$ be a bounded sequence of $E \setminus \mathcal{S}(f, h) \cup \mathcal{S}(g, h)$ (resp. a bounded sequence of diameter $< R$ in $d(a, R^-) \setminus \mathcal{S}(f, h) \cup \mathcal{S}(g, h)$) admitting no cluster point in $\mathcal{P}(f) \cup \mathcal{P}(g)$, satisfying further $h \circ f(a_m) = h \circ g(a_m) \forall m \in \mathbb{N}$. Then $h \circ f(x) = h \circ g(x) \forall x \in E \setminus (\mathcal{S}(f, h) \cup \mathcal{S}(g, h))$ (resp. $\forall x \in d(a, R^-)$).

Proposition E: Let $h \in \mathcal{M}(E) \setminus E$ and let $f, g \in \mathcal{M}(E)$ (resp. $f, g \in \mathcal{M}(d(a, R^-))$) satisfy $h \circ f(x) = h \circ g(x) \forall x \in E \setminus \mathcal{S}(f, h) \cup \mathcal{S}(g, h)$ (resp. $h \circ f(x) = h \circ g(x) \forall x \in d(a, R^-) \setminus \mathcal{S}(f, h) \cup \mathcal{S}(g, h)$). Moreover, if f, g do not belong to $\mathcal{A}(E)$ (resp. if f, g do not belong to $\mathcal{A}(d(a, R^-))$), we assume that $h \notin E(x) \setminus E[x]$. Then f, g satisfy $\mathcal{P}(f) = \mathcal{P}(g)$, $\mathcal{S}(f, h) = \mathcal{S}(g, h)$.

Remarks: In order to avoid the restriction : if f, g do not belong to $\mathcal{A}(E)$ (resp. if f, g do not belong to $\mathcal{A}(d(a, r^-))$), we assume that $h \notin E(x) \setminus E[x]$, we would like to show $\mathcal{P}(f) = \mathcal{P}(g)$ when h is a rational function. But it is hopeless as shows the following situation. Suppose $h \in E(x)$ is not a function of uniqueness for meromorphic functions and let $f, g \in \mathcal{M}(E)$ satisfy $h \circ f = h \circ g$ and $f(c) \neq g(c)$ for some $c \in E$ (resp. $c \in d(a, r^-)$). If c is a pole of f , this just shows $\mathcal{P}(f) \neq \mathcal{P}(g)$. Suppose c is not a pole for f and g . Let $b = f(c)$, let $\phi(x) = \frac{1}{f(x) - b}$, let $\psi(x) = \frac{1}{g(x) - b}$ and let $H(u) = h(b + \frac{1}{u})$. Then H belongs to $E(x)$ and we can check that $H \circ \psi = H \circ \phi$ and that c lies in $\mathcal{P}(\phi)$ but not in $\mathcal{P}(\psi)$.

Notation: Let $f \in \mathcal{A}(d(0, R^-))$. For each $r \in]0, R[$, the supremum of $|f(x)|$ in the disk $d(0, r)$ will be denoted by $|f|_r$.

In the proof of Theorem 4, we shall use the following Lemma F:

Lemma F: Let $h \in \mathcal{A}(K)$ and let $f \in \mathcal{A}(K) \setminus K$ (resp. $f \in \mathcal{A}_u(d(a, R^-))$). There exists $s > 0$ (resp. $s \in]0, R[$) such that $|h \circ f|_r = |h|_{(|f|_r)} \forall r \geq s$ (resp. $\forall r \in [s, R[$).

3. Main results and examples

Notation: Let $h \in \mathcal{M}(E) \setminus E$ (resp. $h \in L(x)$) and let $\Xi(h)$ be the set of zeros c of h' such that $h(c) \neq h(d)$ for every zero d of h' other than c . If $\Xi(h)$ is finite, we denote by $\Phi(h)$ its cardinal and if $\Xi(h)$ is not finite, we put $\Phi(h) = +\infty$.

Theorem 4: Let $h \in \mathcal{M}(K) \setminus K$, let $f, g \in \mathcal{A}(K) \setminus K$ and let $(a_m)_{m \in \mathbb{N}}$ be a bounded sequence of $K \setminus \mathcal{T}(f, h) \cup \mathcal{T}(g, h)$ satisfying $h \circ f(a_m) = h \circ g(a_m) \forall m \in \mathbb{N}$. Then $\mathcal{T}(f, h) = \mathcal{T}(g, h)$ and $h \circ f(x) = h \circ g(x) \forall x \in K \setminus \mathcal{T}(f, h)$. Moreover, if $\Phi(h) \geq 2$, then $f = g$.

Remark: Throughout Corollaries 4.1, 4.2, 7.1, we shall apply to polynomials (resp. rational functions) with coefficients in L results proved for analytic (resp. meromorphic) functions in K . Indeed, as it was often previously done, since L has characteristic 0, there exists a finite extension M of \mathbb{Q} containing all coefficients, zeros and poles of all functions involved and consequently we can consider M as a subfield of \mathbb{C}_p (w

Corollary 4.1: Let $h \in \mathcal{M}(K)$ (resp. $h \in L(x)$) satisfy $\Phi(h) \geq 2$. Then h is a function of uniqueness for $\mathcal{A}(K)$ (resp. for $L[x]$).

Remarks: Conversely, a polynomial of degree 2 is never a uniqueness function for any family of functions because through a suitable translation of the variable, it is possible to put it in the form of an even polynomial.

The condition $\Phi(h) \geq 2$ is not a necessary condition to assure that h is a function of uniqueness for entire or meromorphic functions: for instance, a linear fractional function has a derivative which has no zero, but obviously is a function of uniqueness for meromorphic functions in E or in $d(a, R^-)$.

Examples: 1) Let $h(x) = \frac{x(x-1)}{x-2}$. Hence $h'(x) = \frac{x^2 - 4x + 2}{(x-2)^2}$. Let $\sqrt{2}$ denote a square root of 2 in the field E . The zeros of h' are $c_1 = 2 + \sqrt{2}$, $c_2 = 2 - \sqrt{2}$. Thus $h(c_1) = 3 - 2\sqrt{2}$, $h(c_2) = 3 + \sqrt{2}$ hence h satisfies the hypothesis of Corollary 4.1. (whenever $E = K$).

2) Let $b \in E^*$ be a zero of the polynomial $Q(x) = \frac{x^5}{20} - \frac{x^3}{6} + \frac{x}{4} + \frac{2}{15}$ and let $h(x) = \frac{x^5}{5} - \frac{bx^4}{4} - \frac{x^3}{3} + \frac{bx^2}{2}$. Then $h'(x) = x(x-1)(x+1)(x-b)$. Now, we notice that $h(0) = 0$, $h(1) = \frac{2}{15} + \frac{b}{4}$, $h(-1) = -\frac{2}{15} + \frac{b}{4}$, $h(b) = -\frac{b^5}{20} + \frac{b^3}{6}$. Since $Q(b) = 0$ we

have $h(1) = h(b)$ and clearly $h(1) \neq h(0)$, $h(-1) \neq h(1)$, $h(0) \neq h(b)$, $h(-1) \neq h(b)$. Consequently, h' has 4 zeros $c_1 = 0$, $c_2 = -1$, $c_3 = 1$, $c_4 = b$ satisfying $h(c_j) \neq h(c_l) \forall j = 1, 2, l \neq j, 1 \leq l \leq 4$. Therefore h satisfies the hypothesis of Corollary 3.1 (whenever $E = K$). However, h does not satisfy Hypothesis (F) because $h(c_3) = h(c_4)$.

Corollary 4.2: *Let $h, f, g \in \mathcal{A}(\mathbb{C})$ have all coefficients in \mathbb{Q} and also lie in $\mathcal{A}(\mathbb{C}_p)$ for some prime p . Let $(a_m)_{m \in \mathbb{N}}$ be a bounded sequence of \mathbb{C} satisfying $h \circ f(a_m) = h \circ g(a_m) \forall m \in \mathbb{N}$. Then $h \circ f(x) = h \circ g(x) \forall x \in \mathbb{C}$. Moreover, if $\Phi(h) \geq 2$, then $f = g$.*

Proof: We know that the identity $h \circ f = h \circ g$ is obvious in $\mathcal{A}(\mathbb{C})$, which means the coefficients of the two functions are the same, hence the identity also holds in \mathbb{C}_p . Therefore, if $\Phi(h) \geq 2$, by Theorem 4 we have $f = g$ in $\mathcal{A}(\mathbb{C}_p)$, i.e. the coefficients of f, g are the same, hence this identity obviously holds in $\mathcal{A}(\mathbb{C})$.

Theorem 5: *Let $h \in \mathcal{M}(K) \setminus K$, let $f, g \in \mathcal{A}_u(d(a, R^-))$ and let $(a_m)_{m \in \mathbb{N}}$ be a bounded sequence of $d(a, R^-)$, of diameter $< R$, satisfying $h \circ f(a_m) = h \circ g(a_m) \forall m \in \mathbb{N}$. Then $\mathcal{S}(f, h) = \mathcal{S}(g, h)$ and $h \circ f(x) = h \circ g(x) \forall x \in d(a, R^-) \setminus \mathcal{S}(f, h)$. Moreover, if $\Phi(h) \geq 3$, then $f = g$.*

Example: Let $h(x) = \frac{x^3 - x^2 + x - 2}{x - 2}$. Hence $h'(x) = \frac{2x^3 - 7x^2 + 4x}{(x - 2)^2}$. Let $\sqrt{17}$ denote a square root of 17 in the field K . The zeros of h' are $c_1 = 0$, $c_2 = \frac{7 - \sqrt{17}}{4}$, $c_3 = \frac{7 + \sqrt{17}}{4}$. Thus $h(c_1) = 1$, $h(c_2) = \frac{73 - 17\sqrt{17}}{2}$, $h(c_3) = \frac{73 + 17\sqrt{17}}{2}$, hence h satisfies the hypothesis of Theorem 5.

Corollary 5.1: *Let $h \in \mathcal{A}(K)$, satisfy $\Phi(h) \geq 3$. Then h is a function of uniqueness for $\mathcal{A}_u(d(a, R^-))$*

Similarly to Corollary 4.2, we can state Corollary 5.2:

Corollary 5.2: *Let $h, f, g \in \mathcal{A}(\mathbb{C})$ have all coefficients in \mathbb{Q} and assume that h also lies in $\mathcal{A}(\mathbb{C}_p)$ for some prime p and f, g lie in $\mathcal{A}_u(d(a, R^-))$ (with respect to the field \mathbb{C}_p). Let $(a_m)_{m \in \mathbb{N}}$ be a bounded sequence of \mathbb{C} satisfying $h \circ f(a_m) = h \circ g(a_m) \forall m \in \mathbb{N}$. Then $h \circ f(x) = h \circ g(x) \forall x \in \mathbb{C}$. Moreover, if $\Phi(h) \geq 3$, then $f = g$.*

Remark: It was shown that if P is of the form $x^n - bx^{n-1} + t$, ($t \in K$), then it is not a function of uniqueness for $\mathcal{M}(K)$ [10], [26] and an immediate generalization shows that the same holds when P' has exactly 2 distinct zeros, one of them being of order 1. Assuming again that the set of zeros S of a polynomial P is affinely rigid and P satisfies Hypothesis (F), it is shown in [1] (Theorem 1) that if P' has exactly two distinct zeros c_j of order m_j ($j = 1, 2$), then it is a function of uniqueness for $\mathcal{M}(K)$ if and only if

$\min(m_1, m_2) \geq 2$. As in previous examples, the following Theorem 6 shows that the hypotheses "S affinely rigid" and "Hypothesis (F)" are not necessary to this characterization, concerning polynomials P such that P' has exactly two distinct zeros.

Theorem 6: *Let $P \in K[x]$ (resp. $P \in L[x]$) be such that P' has exactly 2 distinct zeros: c_1 of order m_1 , c_2 of order m_2 . Then P is a function of uniqueness for $\mathcal{A}(K)$ (resp. for $P \in L[x]$). Moreover, P is a function of uniqueness for $\mathcal{M}(K)$ (resp. for $P \in L(x)$) if and only if $\min(m_1, m_2) \geq 2$.*

Theorem 7: *Let $h \in \mathcal{M}(K) \setminus (K \cup (K(x) \setminus K[x]))$, let $f, g \in \mathcal{M}(K) \setminus K$ and let $(a_m)_{m \in \mathbb{N}}$ be a bounded sequence of $K \setminus \mathcal{S}(f, h) \cup \mathcal{S}(g, h)$ of diameter $< R$ admitting no cluster point in $\mathcal{P}(f) \cup \mathcal{P}(g)$, satisfying further $h \circ f(a_m) = h \circ g(a_m) \forall m \in \mathbb{N}$. Then $\mathcal{S}(f, h) = \mathcal{S}(g, h)$ and $h \circ f(x) = h \circ g(x) \forall x \in K \setminus \mathcal{S}(f, h)$. Moreover, if $\Phi(h) \geq 3$, then $f = g$.*

Corollary 7.1: *Let $h \in \mathcal{A}(K)$ (resp. $h \in L[x]$) satisfy $\Phi(h) \geq 3$. Then h is a function of uniqueness for $\mathcal{M}(K)$ (resp. for $L(x)$).*

Corollary 7.2: *Let $h \in \mathcal{M}(K)$ satisfy $\Phi(h) \geq 3$. Then h is a function of uniqueness for $\mathcal{A}(K)$.*

Remark: In [1] it is shown that a polynomial $P \in K[x]$ satisfying (F), whose set of zeros is affinely rigid, is a function of uniqueness for $\mathcal{M}(K)$ if and only if either P' has at least 3 distinct zeros, or P' has just 2 distinct zeros, both of order ≥ 2 . By Theorems 7 we can find other polynomials of uniqueness for $\mathcal{M}(K)$ having sets of zeros which is not affinely rigid.

Example: Let $P(x) = x^4 - 4x$, let j be a cubic root of 1 different from 1 and let $a \in K$ be a cubic root of 4. Then P has 4 distinct zeros $\{0, a, ja, j^2a\}$. Thus the set of zeros of P is not affinely rigid (but is centered). Next, the zeros of P' is $\{1, j, j^2\}$ and we can check that P satisfies (F), hence $\Phi(P) = 3$, therefore P is a uniqueness function for $\mathcal{M}(K)$.

Remark: According to [10], (Lemma 3.2), given a polynomial $P(x) \in L[x]$ of degree 4 and the zeros c_1, c_2, c_3 of P' , the following 3 conditions are equivalent:

- (i) $P(c_i) = P(c_j)$ for some $i, j, i \neq j$
- (ii) P is of the form $[(x - a + l)(x - a - l)]^2 + A$ with $A \in L, l \in L^*$.
- (iii) There exists an affine change of variable transforming P into an even function.

Consequently, a polynomial P of degree 4 satisfies the hypotheses of Theorem 5 and Corollaries 7.1, 7.2 if and only if it is not of the form $[(x - a + l)(x - a - l)]^2 + A$ with $A \in L, l \in L^*$ (which means P is not an even function, up to some affine change of variable).

Example: $P(x) = (x^2 - a^2)^2 + x$ satisfies the hypotheses of Corollaries 5.1 and 7.1.

Now let P be a polynomial of degree 5 such that P' admits 4 distinct zeros c_1, c_2, c_3, c_4 . If $\Phi(P) > 0$, then $\Phi(P) \geq 2$. Indeed, suppose $\Phi(P) = 1$. We may assume that $P(c_1) = P(c_2) = P(c_3)$ and $P(c_1) \neq P(c_4)$. But then, $P - P(c_1)$ admits 3 zeros of order 2, a contradiction with $\deg(P) = 5$.

Similarly, if $\Phi(P) = 0$, then up to a good indexation we have $P(c_1) = P(c_2)$ and $P(c_3) = P(c_4)$.

Theorem 8: *Let $h \in \mathcal{M}(K) \setminus (K \cup (K(x) \setminus K[x]))$, let $f, g \in \mathcal{M}_u(d(a, R^-))$ and let $(a_m)_{m \in \mathbb{N}}$ be a bounded sequence of $d(a, R^-) \setminus (\mathcal{S}(f, h) \cup \mathcal{S}(g, h))$ admitting no cluster point in $\mathcal{P}(f) \cup \mathcal{P}(g)$, satisfying further $h \circ f(a_m) = h \circ g(a_m) \forall m \in \mathbb{N}$. Then $\mathcal{S}(f, h) = \mathcal{S}(g, h)$ and $h \circ f(x) = h \circ g(x) \forall x \in d(a, R^-) \setminus \mathcal{S}(f, h)$. Moreover, if $\Phi(h) \geq 4$, then $f = g$.*

Corollary 8.1: *Let $h \in \mathcal{A}(K)$ satisfy $\Phi(h) \geq 4$. Then h is a function of uniqueness for $\mathcal{M}_u(K)$.*

Theorem 9: *Let $h, f, g \in \mathcal{M}(\mathbb{C}) \setminus \mathbb{C}$ and let $(a_m)_{m \in \mathbb{N}}$ be a bounded sequence of $\mathbb{C} \setminus \mathcal{S}(f, h) \cup \mathcal{S}(g, h)$ admitting no cluster point in $\mathcal{P}(f) \cup \mathcal{P}(g)$, satisfying further $h \circ f(a_m) = h \circ g(a_m) \forall m \in \mathbb{N}$. Moreover, if $f, g \notin \mathcal{A}(\mathbb{C})$, we assume that $h \notin \mathbb{C}(x)$. Then $\mathcal{S}(f, h) = \mathcal{S}(g, h)$ and $h \circ f(x) = h \circ g(x) \forall x \in \mathbb{C} \setminus \mathcal{S}(f, h)$. Moreover, if $\Phi(h) \geq 4$, then $f = g$.*

Corollary 9.1: *Let $h \in \mathcal{A}(\mathbb{C})$ satisfy $\Phi(h) \geq 4$. Then h is a function of uniqueness for $\mathcal{M}(\mathbb{C})$.*

Corollary 9.2: *Let $h \in \mathcal{M}(\mathbb{C})$ satisfy $\Phi(h) \geq 4$. Then h is a function of uniqueness for $\mathcal{A}(\mathbb{C})$.*

Various examples and remarks: Let Ω be an algebraic closure of \mathbb{Q} .

1) Let $h(x) = \cos x + \frac{x}{2}$. The zeros of h' are the points $\frac{\pi}{6} + 2n\pi$ and $(2n+1)\pi - \frac{\pi}{6}$.

Thus we have:

$h(\frac{\pi}{6} + 2n\pi) = n\pi + \frac{\pi}{12} + \frac{\sqrt{3}}{2}$, hence $h(\frac{\pi}{6} + 2n\pi) \neq h(\frac{\pi}{6} + 2m\pi) \forall m \neq n$ and

$h(-\frac{\pi}{6} + (2n+1)\pi) = n\pi + \frac{\pi}{12} - \frac{\sqrt{3}}{2}$, hence $h(-\frac{\pi}{6} + (2n+1)\pi) \neq h((2m+1)\pi - \frac{\pi}{6}) \forall m \neq n$.

Moreover, since π is transcendental, we check that

$h(\frac{\pi}{6} + 2n\pi) \neq h((2m+1)\pi - \frac{\pi}{6}) \forall m, n \in \mathbb{Z}$. So, h satisfies the hypothesis of Theorem 9.

2) Let $R \in \mathbb{Q}(x)$, let $\mathcal{C}(R') = \{c_1, \dots, c_q\}$ and assume that $R(c_j) \neq R(c_n) \forall j = 1, \dots, k, \forall n \leq q$. Let $h(x) = e^{R(x)}$. Then $\mathcal{C}(h') = \{c_1, \dots, c_q\}$ and we shall check that $h(c_j) \neq h(c_n) \forall j = 1, \dots, k, \forall n \leq q$. Indeed, suppose that $h(c_j) = h(c_l)$ with $j \neq l$ and $j \leq k$. Then $R(c_j) - R(c_l)$ is of the form $2id\pi$ with $d \in \mathbb{Z}$, which is impossible because $R(c_j) - R(c_l)$ lies in Ω .

For instance, let α be a zero of the polynomial $D(x) = \frac{x^4}{42} - \frac{x^2}{4} + \frac{2}{3}$ and let

$$P(x) = x^7 - 7x^5 + \frac{28}{3}x^3 - \frac{7\alpha}{6}x^6 + \frac{35\alpha}{4}x^4 - 14\alpha x^2 + A \quad (\text{with } A \in \Omega).$$

We check that $\alpha \notin \mathbb{Q}$ (because it is a square root of a zero of $\frac{u^2}{42} - \frac{u}{4} + \frac{2}{3}$). Then

$$P'(x) = 7x^6 - 35x^4 + 28x^2 + 7\alpha x^5 + 35\alpha x^3 - 28\alpha x \text{ admits 6 distinct zeros:}$$

$c_1 = 1, c_2 = -1, c_3 = 2, c_4 = -2, c_5 = 0, c_6 = \alpha$. We notice that $P(c_5) = P(c_6) = A$. Next, for all $j, l, (1 \leq j < l \leq 6)$, $P(c_j) - P(c_l)$ is of the form $s + t\alpha$ with $s, t \in \mathbb{Q}$ and $t \neq 0$, except if $j = 5$ and $l = 6$. Consequently $P(c_j) \neq P(c_l)$ for all $j, l, 1 \leq j < l \leq 6$ such that $j < 5$. Therefore, P (playing the role of h) satisfies the hypothesis of Theorems 8 and 9. And the function $h(x) = e^{P(x)}$ satisfies the hypothesis of Theorem 9. However both do not satisfy Hypothesis (F).

3) Let $\alpha \in \Omega$ and let $P \in \Omega[x]$ be of degree 4, such that $\alpha P + P'$ has 4 distinct zeros. Let $h(x) = P(x)e^{\alpha x}$. The zeros of h' are the 4 zeros $c_j, j = 1, 2, 3, 4$ of $\alpha P + P'$ and are obviously algebraic over \mathbb{Q} . Suppose now that $h(c_j) = h(c_l)$ with $j \neq l$. Then $\frac{P(c_j)}{P(c_l)} = e^{\alpha(c_l - c_j)}$. Since c_j, c_l are algebraic, so are $\frac{P(c_j)}{P(c_l)}$ and $\alpha(c_l - c_j)$. But then, by Hermit-Lindeman's Theorem, $e^{\alpha(c_l - c_j)}$ is transcendental [24]. Consequently $h(c_j) \neq h(c_l)$ whenever $j \neq l$, hence h satisfies the hypothesis of Theorem 9.

For instance, consider $P(x) = x^4 - 10x^3 + 41x^2 - 88x + 88$. We check that $P(x) + P'(x) = x^4 - 6x^3 + 11x^2 - 6x = x(x-1)(x-2)(x-3)$, hence $P(x)e^x$ satisfies the hypothesis of Theorem 9.

4) In all Theorems and Corollaries above, the hypothesis:
let $(a_m)_{m \in \mathbb{N}}$ be a bounded sequence of $K \setminus \mathcal{S}(f, h) \cup \mathcal{S}(g, h)$ admitting no cluster point in $\mathcal{P}(f) \cup \mathcal{P}(g)$, satisfying further $h \circ f(a_m) = h \circ g(a_m) \forall m \in \mathbb{N}$
is obviously satisfied when the two considered functions f, g satisfy $h(f(x)) = h(g(x))$ inside a certain disk.

5) On the other hand, a typical example of complex entire function h such that $\Phi(h) = 0$ is given by $\sin(ax + b)$ with $a, b \in \mathbb{C}$.

Theorem 10: 1) Let $(b_n)_{n \in \mathbb{N}}$ be a sequence of K satisfying $b_0 = b_1 = 0, b_2 = 1, |2b_4| < |9(b_3)^2|, |3b_3b_5| < |4(b_4)^2|, |4b_4| \geq |5b_5| \left| \frac{b_4}{b_5} \right|, |4b_4| > |nb_n| \left| \frac{b_4}{b_5} \right|^{n-4} \forall n > 5$ and such that the sequence $\left| \frac{b_n}{b_{n+1}} \right|_{n \geq 2}$ is strictly increasing, of limit $+\infty$. Let $h(x) = \sum_{n=0}^{\infty} b_n x^n$ and let $(c_n)_{n \in \mathbb{N}^*}$ be the sequence of zeros of h' ordered in such a way that $|c_n| \leq |c_{n+1}|$. Then h belongs to $\mathcal{A}(K)$ and satisfies $h(c_i) \neq h(c_n) \forall i = 1, 2, 3 \forall n \neq i$.

2) Let $(b_n)_{n \in \mathbb{N}}$ be a sequence of K satisfying $b_0 = b_1 = 0, b_2 = 1, |2b_4| < |9(b_3)^2|, |3b_3b_5| < |4(b_4)^2|, |4b_4| \geq |5b_5| \left| \frac{b_4}{b_5} \right|, |4b_4b_6| < |5(b_5)^2|, |5b_5| \geq |6b_6| \left| \frac{b_5}{b_6} \right|, |5b_5| > |nb_n| \left| \frac{b_5}{b_6} \right|^{n-5}$

$\forall n > 6$ and be such that the sequence $\left| \frac{b_n}{b_{n+1}} \right|_{n \geq 2}$ is strictly increasing, of limit $+\infty$. Let

$h(x) = \sum_{n=0}^{\infty} b_n x^n$ and let $(c_n)_{n \in \mathbb{N}^*}$ be the sequence of zeros of h' ordered in such a way that $|c_n| \leq |c_{n+1}|$. Then h belongs to $\mathcal{A}(d(0, R^-))$ and satisfies $h(c_i) \neq h(c_n) \forall i = 1, 2, 3, \forall n \neq i$.

Example of a function $h \in \mathcal{A}(\mathbb{C})$ also lying in $\mathcal{A}(\mathbb{C}_p)$, satisfying the hypotheses of Theorem 10:

Let $|\cdot|_{\infty}$ be the Archimedean absolute value on \mathbb{C} and let $|\cdot|_p$ be the p -adic absolute value on \mathbb{C}_p . Let p be a prime integer ≥ 3 and let q be an integer ≥ 2 . Consider a power

series of the form $h(x) = x^2 + \sum_{n=3}^{\infty} \frac{p^{n^q}}{t_n} x^n$ where t_n is an integer prime to p , satisfying

$|t_n|_{\infty} > p^{n^{q+1}}$. Then $\lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{p^{n^q}}{t_n} \right|_p} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{p^{n^q}}{t_n} \right|_{\infty}} = 0$, hence h belongs to both $\mathcal{A}(\mathbb{C})$

and $\mathcal{A}(\mathbb{C}_p)$. Moreover we can check that h satisfies the hypothesis 2) of Theorem 10 hence $\Phi(h) \geq 3$ in $\mathcal{A}(\mathbb{C}_p)$.

4. The proofs

Proof of Proposition B: Since L has characteristic 0, if a finite set S is preserved by an affine mapping γ , that γ is necessarily a centered similarity because if γ is not a centered similarity, it is of the form $\gamma(x) = x + b$, so the sequence $(\gamma^n(x))_{n \in \mathbb{N}}$ is of the form $(x + nb)_{n \in \mathbb{N}}$ and therefore is injective, a contradiction to $\gamma(S) = S$.

Proof of Theorem 1: Considering the coefficients of degree n , it is obvious that S is not affinely rigid if and only if there exists a rotation of center a : $\gamma(x) = a + \alpha(x - a)$, such that $P(\gamma(x)) = \alpha^n P(x)$, with $\alpha \neq 1$.

If i) or ii) are satisfied, S is obviously preserved by the similarity γ of center a . Moreover, in Case i), a does not lie in S , hence S is non-centered. In Case ii), a lies in S hence S is centered.

Now, Suppose S is not affinely rigid and let $\gamma(x) = a + \alpha(x - a)$ be a similarity preserving S . Then $P(\gamma(x)) = P(a + \alpha u) = Q(\alpha u)$ and $\alpha^n P(x) = \alpha^n Q(u)$, hence $Q(\alpha u) = \alpha^n Q(u)$. Let $Q(u) = \sum_{j=0}^n a_j u^j$. We have $\alpha^j a_j = \alpha^n a_j \forall j = 0, \dots, n$. Consequently

$$(1) a_j (\alpha^{n-j} - 1) = 0 \forall j = 0, \dots, n.$$

Suppose first $a_0 \neq 0$. Clearly we have $\alpha^n = 1$. Let d be the order of α as a n -th root of 1. Since γ is not the identity, $\alpha \neq 1$. Consequently, by (1) we notice that $d \geq 2$ because $\alpha^{n-1} \neq 1$. Then n is of the form qd ($q \in \mathbb{N}^*$) and by (1) we have $a_j = 0$ for

every j which is not multiple of d . Consequently, we have obtained $Q(u) = \sum_{k=0}^q a_{kd} u^{kd}$,

with $\alpha^d = 1$ and $d \geq 2$, hence S is non-centered.

Suppose now $a_0 = 0$. Since the zeros of P are not multiple, neither are those of Q . Consequently, $a_1 \neq 0$, hence by (1) we have $\alpha^{n-1} = 1$. Let d be the order of α as a $(n-1)$ -th root of 1. Since $\alpha \neq 1$, by (1) we have $d \geq 2$ because $\alpha^{n-2} \neq 1$. Then n is of the form $qd+1$ ($q \in \mathbb{N}^*$) and by (1) we have $a_j = 0$ for every j which is not of the form

$kd+1$. Consequently, we have obtained $Q(u) = \sum_{k=0}^q a_{kd+1} u^{kd+1}$, with $\alpha^d = 1$ and $d \geq 2$,

hence S is centered.

Suppose S is a non-affinely rigid set which is both centered and non-centered and let it be the set of zeros of P be defined as in i). As a centered non-affinely rigid set, it admits a non-identical similarity δ of center b preserving S . Without loss of generality we may assume that $b = 0$ and hence, similarly to the case ii), we can show that $P(x)$

is of the form $P(x) = \sum_{k=0}^s b_{kt+1} x^{kt+1}$, with $\alpha^t = 1$ and $t \geq 2$. Now, since S is also

non-centered, there exists a similarity φ of the form $\varphi(x) = \lambda x + \mu$ preserving S such that the center of φ does not lie in S , hence is different from 0. Then $P(\varphi(x)) = \lambda^n P(x)$.

So, we have $\sum_{k=0}^s b_{kt+1} (\lambda x + \mu)^{kt+1} = (\lambda)^{st+1} \sum_{k=0}^s b_{kt+1} x^{kt+1}$. Examining terms of degree

st , we can check that $b_{st+1}(st+1)\lambda^{st}\mu = 0$ because $t \geq 2$. Consequently, $\mu = 0$ because $\deg(P) = st+1$. Therefore the center of φ is 0, a contradiction. This ends the proof of Theorem 1.

Proof of Theorem 2: Let $c \in K \setminus P(Z)$. It is obvious that $P - c$ admits $qd+1$ distinct zeros because none of them is a zero of P' . Let S be its set of zeros and suppose that S is not affinely rigid. Let $\gamma(x) = a + \alpha(x - a)$ be a similarity preserving S . Thus, $P(\gamma(x)) = \alpha^{qd+1} P(x)$, $\alpha^{qd+1} = 1$. Now, examining terms of degree qd as in the proof of Theorem 1, we see that $a_{qd+1}(qd+1)\alpha^{qd}a = 0$ because $d \geq 2$. Consequently, $a = 0$. Therefore the center of γ is 0. But since $c \neq 0$, we see that P is neither of the form i) nor of the form ii) in Theorem 1, a contradiction to the hypothesis: S is not affinely rigid.

Proof of Theorem 3: Suppose first that S is a non-centered non-affinely rigid set, hence

the polynomial Q associated to P is defined by i): $P(x) = Q(u) = \sum_{k=0}^q a_{kd} u^{kd}$, with $\alpha^d = 1$

and $d \geq 2$. Then we have $Q(f) = Q(\alpha f)$ and therefore P is not a function of uniqueness for $\mathcal{A}(K)$.

Now, suppose that S is not a non-centered non-affinely rigid set and that P is not a function of uniqueness for $\mathcal{A}(K)$. If S is affinely rigid, by Theorem D P is polynomial of strong uniqueness for $\mathcal{A}(K)$. So, it only remains to suppose that it is a centered non-affinely

rigid set, hence the polynomial Q associated to P is of the form ii): $Q(u) = \sum_{k=0}^q a_{kd+1} u^{kd+1}$,

with $\alpha^d = 1$ and $d \geq 2$. Since P is not a function of uniqueness for $\mathcal{A}(K)$, neither is Q nor $Q + c \forall c \in K$, hence the set of zeros of $Q + c$ is not affinely rigid whenever $c \in K$, a contradiction to Theorem 2.

Notation: Throughout the section, we shall denote by \log a real logarithm function of base > 1 .

Given a subset A of E and positive numbers $t, r \in]0, t[$, we set $\mathcal{D}(t, r, A) = d(0, t) \setminus \bigcup_{\alpha \in A} d(\alpha, r^-)$.

Let D be a closed bounded subset of K . We denote by $R(D)$ the K -algebra of the rational functions $h(x) \in K(x)$ without pole in D and we denote by $H(D)$ the completion of $R(D)$ with respect to the norm of uniform convergence on D .

Let $f \in \mathcal{M}(E)$ (resp. $f \in \mathcal{M}(d(0, R^-))$) and let $\alpha \in E$ (resp. $d(0, R^-)$). If f has a zero (resp. a pole) of order n at α , we put $\omega_\alpha(f) = n$ (resp. $\omega_\alpha(f) = -n$). If $f(\alpha) \neq 0$ and ∞ , we put $\omega_\alpha(f) = 0$.

Let $f \in \mathcal{M}(E)$ (resp. $f \in \mathcal{M}(d(0, R^-))$), with $f(0) \neq 0, \infty$. We denote by $Z(r, f)$ the counting function of zeros of f in E (resp. $d(0, R^-)$)

$$Z(r, f) = \sum_{\omega_\alpha(f) > 0, |\alpha| \leq r} \omega_\alpha(f) \log \frac{r}{|\alpha|}.$$

Next, we put

$$\bar{Z}(r, f) = \sum_{\omega_\alpha(f) > 0, \alpha \in d(0, r^-)} \log \frac{r}{|\alpha|}.$$

We shall also consider the counting functions of poles of f in E (resp. in $d(0, R^-)$):

$$N(r, f) = Z(r, \frac{1}{f}) \quad \text{and} \quad \bar{N}(r, f) = \bar{Z}(r, \frac{1}{f}).$$

Moreover, we will consider counting functions under certain conditions in that way. Consider a subset F of E (resp. of $d(0, R^-)$). We put

$$Z(r, f \mid x \in F) = \sum_{\omega_\alpha(f) > 0, |\alpha| \leq r, \alpha \in F} \omega_\alpha(f) \log \frac{r}{|\alpha|}.$$

The Nevanlinna function $T(r, f)$ is defined by

$$T(r, f) = \max[Z(r, f) + \log |f(0)|, N(r, f)].$$

Given functions φ, ψ from $]0, +\infty[$ to $]0, +\infty[$ we shall write $\varphi(r) \leq \psi(r) + o(\varphi)$ if there exists a subset J of $]0, +\infty[$ of finite Lebesgue measure such that $\limsup_{r \rightarrow +\infty, r \notin J} \frac{\psi(r) - \varphi(r)}{\varphi(r)} = 0$.

Given functions φ, ψ from $]0, +\infty[$ (resp. from $]0, R[$) to $]0, +\infty[$ we shall write $\varphi(r) \leq \psi(r) + O(1)$ if there exists a constant C such that $\limsup_{r \rightarrow +\infty} \psi(r) - \varphi(r) \leq C$, (resp. if $\limsup_{r \rightarrow R} \psi(r) - \varphi(r) \leq C$).

Remark: By definition, we have $\overline{Z}(r, f) \leq Z(r, f) \leq T(r, f) + O(1)$, $\overline{N}(r, f) \leq N(r, f) \leq T(r, f)$ in $]0, +\infty[$ whenever $f \in \mathcal{M}(E)$ (resp. in $]0, R[$ whenever $f \in \mathcal{M}(d(0, R^-))$).

We must recall classical lemmas on the functions $T(r, f)$.

Lemma 1 [5]: *Let $f, g \in \mathcal{M}(K)$ (resp. $f, g \in \mathcal{M}(d(0, R^-))$). Then $T(r, f + g) \leq T(r, f) + T(r, g) + O(1)$. Moreover, if $f, g \in \mathcal{A}(K)$ (resp. $f, g \in \mathcal{A}(d(0, R^-))$), then $T(r, f + g) \leq \max(T(r, f), T(r, g)) + O(1)$. Further, $T(r, f \cdot g) \leq T(r, f) + T(r, g) + O(1)$.*

Lemma 2: *Let $h \in \mathcal{M}(K)$, let $f \in \mathcal{M}(K)$ (resp. let $f \in \mathcal{M}(d(a, R^-))$). Then for every $t > 0$ (resp. $t \in]0, R[$) and $r \in]0, t[$ both $\mathcal{P}(f) \cap d(0, t)$ and $\mathcal{T}(f, h) \cap \mathcal{D}(t, r, \mathcal{P}(f))$ are finite, any cluster point of $\mathcal{S}(f, h)$ is a pole of f , the number of holes of $\mathcal{D}(t, r, \mathcal{S}(f, h))$ is finite and $h \circ f$ belongs to $H(\mathcal{D}(t, r, \mathcal{S}(f, h)))$.*

Proof: We assume $a = 0$. Let $t > 0$ (resp. $t \in]0, R[$) be fixed and let $r \in]0, t[$. Let $\{\alpha_1, \dots, \alpha_q\}$ be the finite set of all poles of f and g in $d(0, t)$. We notice that any cluster point of $\mathcal{S}(f, h)$ in $d(0, t)$ is necessarily one of the α_j . Indeed, let $\alpha \in \mathcal{S}(f, h) \cap d(0, t)$ with $\alpha \neq \alpha_j \forall j = 1, \dots, q$. Then $f(\alpha)$ is a pole of h , hence there exists a disk $d(f(\alpha), s)$ such that h has no singularity in $d(f(\alpha), s)$ but $f(\alpha)$, i.e. h is of the form $\frac{g}{(x - f(\alpha))^u}$ with $g \in H(d(f(\alpha), s))$. And of course there exists a disk $d(\alpha, \rho)$ such that $f(d(\alpha, \rho)) \subset d(f(\alpha), s)$, which shows that $h \circ f$ has no singularity but α in $d(\alpha, \rho)$, hence α is not a cluster point of $\mathcal{S}(f, h)$.

Now, since $\mathcal{D}(t, r, \mathcal{P}(f))$ has finitely many holes, it is a closed bounded set with no T-filter, hence for every $\lambda \in K$, $f - \lambda$ is quasi-invertible and therefore has finitely many zeros [9]. Thus, the set $\mathcal{T}(f, h) \cap \mathcal{D}(t, r, \mathcal{P}(f))$ is finite, hence $\mathcal{D}(t, r, \mathcal{S}(f, h))$ has finitely many holes and therefore it has no T-filter. Consequently, by [9], Corollary 38.15, we know that $h \circ f$ belongs to $H(\mathcal{D}(t, r, \mathcal{S}(f, h)))$.

Notation: Let $f \in \mathcal{A}(d(0, R^-))$. For each $r \in]0, R[$, the supremum of $|f(x)|$ in the disk $d(0, r)$ will be denoted by $|f|_r$.

Proof of Lemma F: Let $b = f(0)$. By classical results, $f(d(0, r))$ is a disk $d(b, t)$ [9]. Suppose f belongs to $\mathcal{A}(K) \setminus K$, hence it admits a zero $\alpha \in K$. Let $s = |\alpha|$ and let us take $r \geq s$. Then $f(d(0, r))$ is a disk $d(b, t)$ equal to $d(0, t)$, hence $t = |f|_r$. Consequently, $|h \circ f|_r = \sup\{|h(f(x))| \mid x \in d(0, r)\} = \sup\{|h(u)| \mid u \in d(0, |f|_r)\}$.

Similarly, suppose f belongs to $\mathcal{A}_u(d(a, R^-)) \setminus K$. Since f is unbounded, it admits a zero $\alpha \in d(a, R^-)$. By putting again $s = |\alpha|$ we can go on as in the previous case.

As a corollary of Lemma F, we note Lemma 3:

Lemma 3 : *Let $h \in \mathcal{A}(K)$ and let $f \in \mathcal{A}(K)$ (resp. $f \in \mathcal{A}(d(a, R^-))$) satisfy $h(f(0))h(0) \neq 0$. Then there exists $s > 0$ (resp. $r \in]0, R[$) such that $Z(r, h \circ f) - Z(|f|_r, h)$ is a constant for all $r \geq s$.*

By classical results [20] we have Lemma 4:

Lemma 4: *Let $h \in \mathcal{M}(K)$. There exists $\phi, \psi \in \mathcal{A}(K)$ with no common zeros such that $h = \frac{\phi}{\psi}$.*

Lemma 5: *Let $h \in \mathcal{M}(K)$ and let $f \in \mathcal{A}(K)$ (resp. $f \in \mathcal{A}(d(a, R^-))$) satisfy $h(f(0))h(0) \neq 0$. Then $T(r, h \circ f) = T(|f|_r, h) + O(1)$.*

Proof: Let $\phi, \psi \in \mathcal{A}(K)$ with no common zeros such that $h = \frac{\phi}{\psi}$. Then $Z(r, \phi \circ f) = Z(r, h \circ f)$ and $Z(r, \psi \circ f) = N(r, h \circ f)$. On the other hand, by Lemma 3 we have $Z(r, \phi \circ f) = Z(|f|_r, \phi) + O(1)$, $Z(r, \psi \circ f) = Z(|f|_r, \psi) + O(1)$. Consequently $T(r, h \circ f) = \max(Z(|f|_r, \phi), Z(|f|_r, \psi)) + O(1)$. But now, $\max(Z(|f|_r, \phi), Z(|f|_r, \psi)) = T(|f|_r, h) + O(1)$ which ends the proof.

Lemma 6: *Let $h \in \mathcal{A}(K)$ satisfy $h(0) \neq 0$, $s > 0$ (resp. $s \in]0, R[$) and let θ, τ be increasing continuous functions from $]0, +\infty[$ (resp. from $[0, R[$ to $]0, +\infty[$) satisfying $\lim_{r \rightarrow +\infty} \theta(r) = \lim_{r \rightarrow +\infty} \tau(r) = +\infty$ (resp. $\lim_{r \rightarrow R} \theta(r) = \lim_{r \rightarrow R} \tau(r) = +\infty$) and $Z(\theta(r), h) = Z(\tau(r), h)$ whenever $r \geq s$ (resp. whenever $r \in [s, R[$). Then $\log(\theta(r)) - \log(\tau(r))$ is bounded in $[s, +\infty[$ (resp. in $[s, R[$). Moreover, if h is not a polynomial, (resp. if h belongs to $\mathcal{A}_u(d(a, R^-))$) then $\log(\theta(r)) - \log(\tau(r))$ tends to 0 when r tends to $+\infty$ (resp. to R).*

Proof: Since $Z(\theta(r), h) = \log(|h|_{\theta(r)}) - \log(|h(0)|)$ and $Z(\tau(r), h) = \log(|h|_{\tau(r)}) - \log(|h(0)|)$, we can see that $\log(|h|_{\theta(r)}) - \log(|h|_{\tau(r)})$ is a constant C in $[s, +\infty[$ (resp. in $[s, R[$). For each $\rho \in [s, +\infty[$ (resp. for each $\rho \in [s, R[$), let $\nu(\rho, h)$ be the number of zeros of h in the disk $d(0, \rho)$. Then by classical results [9] we know that

$$\begin{aligned} |\log(|h|_{\theta(r)}) - \log(|h|_{\tau(r)})|_\infty &\geq \min(\nu(\tau(r), h), \nu(\theta(r), h)) |\log(\theta(r)) - \log(\tau(r))|_\infty \\ &\geq \min(\nu(\tau(r), h), \nu(\theta(r), h)) C. \end{aligned}$$

Suppose first $h \in \mathcal{A}(K)$. If h is not a polynomial, we have $\lim_{r \rightarrow +\infty} \nu(\theta(r), h) = \lim_{r \rightarrow +\infty} \nu(\tau(r), h) = +\infty$, hence $\lim_{r \rightarrow +\infty} \log(\theta(r)) - \log(\tau(r)) = 0$. And if h is a polynomial of degree q then when r is big enough, we have $\nu(\theta(r)) = \nu(\tau(r)) = q$, hence $\log(\theta(r)) - \log(\tau(r))$ is constant.

Now, if h belongs to $\mathcal{A}_u(d(a, R^-))$ then $\lim_{r \rightarrow R} \nu(\theta(r), h) = \lim_{r \rightarrow R} \nu(\tau(r), h) = +\infty$, hence $\lim_{r \rightarrow R} \log(\theta(r)) - \log(\tau(r)) = 0$ again.

Lemma 7: *Let $h \in \mathcal{M}(K)$ and let $f, g \in \mathcal{A}(K)$ (resp. $f, g \in \mathcal{A}(d(a, R^-))$) satisfy $h(f(0))h(g(0))h(0)f(0)g(0) \neq 0$ and $h \circ f = h \circ g$. Then $T(r, f) - T(r, g)$ is bounded in $]0, +\infty[$ (resp. in $]0, R[$).*

Proof: Let $\phi, \psi \in \mathcal{A}(K)$ with no common zeros such that $h = \frac{\phi}{\psi}$. Then $Z(r, \phi \circ f) = Z(r, h \circ f)$ and $Z(r, \phi \circ g) = Z(r, h \circ g)$. On the other hand, by Lemma 3 there exists $s', s'' > 0$ (resp. $s', s'' \in]0, R[$) such that $Z(r, \phi \circ f)$ is of the form $Z(|f|_r, \phi) + C'$ with $C' \in \mathbb{R}$ whenever $r \geq s'$ (resp. whenever $r \in [s', R[$) and similarly $Z(r, \phi \circ g)$ is of the form $Z(|g|_r, \phi) + C''$ with $C'' \in \mathbb{R}$, whenever $r \geq s''$ (resp. whenever $r \in [s'', R[$). Consequently, putting $s = \max(s', s'')$, we have $Z(|f|_r, \phi) + C' = Z(|g|_r, \phi) + C'' \forall r \geq s$ hence $Z(|f|_r, \phi) - Z(|g|_r, \phi)$ is a constant C whenever $r \geq s$. Now, since the functions $|f|_r, |g|_r$ are continuous strictly increasing functions of r , tending to $+\infty$ when r tends to $+\infty$ (resp. when r tends to R), then by Lemma 6, $\log(|f|_r) - \log(|g|_r)$ is bounded in $[0, +\infty[$ (resp. in $]0, R[$) hence so is $T(r, f) - T(r, g)$.

Lemma 8 is classical:

Lemma 8: *Let $h \in \mathcal{A}(E)$ and let $f \in \mathcal{M}(E)$ (resp. $f \in \mathcal{M}(d(a, R^-))$). Let α be a pole of f in E (resp. in $\mathcal{M}(d(a, R^-))$). Then α is a singular point for $h \circ f$.*

Let $h \in \mathcal{M}(E) \setminus E(x)$ and let $f \in \mathcal{M}(E)$ (resp. $f \in \mathcal{A}(d(a, R^-))$). Let α be a pole of f in E (resp. in $\mathcal{M}(d(a, R^-))$). Then α is a point of high singularity for $h \circ f$.

Proof of Proposition D: Suppose first that $E = K$. We assume $a = 0$. Suppose that $h \circ f$ and $h \circ g$ are two distinct functions. Let $\rho = \sup_{m \in \mathbb{N}}(|a_m|)$. If f belongs to $\mathcal{M}(d(a, R^-))$, since the diameter of the sequence (a_m) is $< R$, we have $\rho < R$. Let $t > \rho$ (resp. $t \in]\rho, R[$) be fixed and let $r \in]0, t[$. Let $\{\alpha_1, \dots, \alpha_q\}$ be the finite set of all poles of f and g in $d(0, t)$. By Lemma 2 $\mathcal{D}(t, r, \mathcal{S}(f, h))$ has finitely many holes and $h \circ f$ belongs to $H(\mathcal{D}(t, r, \mathcal{S}(f, h)))$. Similarly, $\mathcal{D}(t, r, \mathcal{S}(f, h))$ has finitely many holes and $h \circ f$ belongs to $H(\mathcal{D}(t, r, \mathcal{S}(f, h)))$. Let $D = \mathcal{D}(t, r, \mathcal{S}(f, h)) \cap \mathcal{D}(t, r, \mathcal{S}(g, h)) = \mathcal{D}(t, r, \mathcal{S}(f, h) \cup \mathcal{S}(g, h))$. Then D has finitely many holes hence it has no T-filter. As it is obviously open, by [9], Theorem 38.9, every element of $H(D)$ is quasi-invertible or identically zero, hence so is $h \circ g - h \circ f$. Consequently, if $h \circ g(x) - h \circ f(x)$ is identically zero in an open subset of $K \setminus (\mathcal{S}(f, h) \cup \mathcal{S}(g, h))$ (resp. of $d(a, R^-) \setminus (\mathcal{S}(f, h) \cup \mathcal{S}(g, h))$), it is identically zero in all $K \setminus (\mathcal{S}(f, h) \cup \mathcal{S}(g, h))$ (resp. in all $d(a, R^-) \setminus (\mathcal{S}(f, h) \cup \mathcal{S}(g, h))$).

By hypothesis, the sequence $(a_m)_{m \in \mathbb{N}}$ may not admit one of the α_j ($1 \leq j \leq q$) as a cluster point. We will show that it does not admit any $\beta \in \mathcal{T}(f, h)$ as a cluster point, either. Indeed, suppose that a subsequence of the sequence $(a_m)_{m \in \mathbb{N}}$ converges to $\beta \in \mathcal{T}(f, h)$. Since β doesn't lie in $\mathcal{S}(f, h) \cup \mathcal{S}(g, h)$, we have a disk $d(\beta, r)$ such that, for $0 < |x - \beta| \leq r$,

both $f(x)$ and $g(x)$ are equal to a Laurent series $\sum_{n=q}^{\infty} \lambda_n(x - \beta)^n$, with $\lambda_q \neq 0$. If $q < 0$,

then $\lim_{x \rightarrow \beta} |h(f(x)) - h(g(x))| = +\infty$, which excludes a sequence of zeros converging to β .

And if $q \geq 0$, then $f(x) - g(x)$ is analytic in $d(\beta, r)$ which excludes a sequence of zeros converging to β , except if $h(f(x)) - h(g(x))$ is identically zero in $d(\beta, r)$, but then it is identically zero in all $K \setminus \mathcal{S}(f, h) \cup \mathcal{S}(g, h)$ (resp. in all $d(a, R^-) \setminus \mathcal{S}(f, h) \cup \mathcal{S}(g, h)$), our conclusion. Consequently, the sequence $(a_m)_{m \in \mathbb{N}}$ may not admit any cluster point in $\mathcal{S}(f, h) \cup \mathcal{S}(g, h)$.

We will show that $\inf\{|a_m - x| \mid m \in \mathbb{N}, x \in \mathcal{S}(f, h)\} > 0$. Indeed, let β be a cluster point of $\mathcal{S}(f, h)$ in $d(0, t)$. Suppose β is not a pole of f . By Lemma 2 $h \circ f(x)$ is equal to a Laurent series in a set $d(\beta, \rho) \setminus \{\beta\}$, hence, by Lemma 2 again, there is no sequence of $\mathcal{T}(f, h)$ converging to β and therefore there is no sequence of $\mathcal{S}(f, h)$ converging to β because $\mathcal{P}(f)$ is discrete. Consequently, β lies in $\mathcal{P}(f)$. Suppose that $\inf\{|a_m - x| \mid m \in \mathbb{N}, x \in \mathcal{S}(f, h)\} = 0$. There must exist a sequence of the form $(a_{\tau(n)}, \beta_n)_{n \in \mathbb{N}}$ with $\beta_n \in \mathcal{T}(f, h)$, satisfying $\lim_{n \rightarrow \infty} |a_{\tau(n)} - \beta_n| = 0$. By Lemma 2, the β_n are in finite number in $\mathcal{D}(t, r, \mathcal{P}(f))$, hence at least one of the holes $d(\alpha_j, r^-)$ contains an infinity of them. This is true for all $r > 0$, so we can extract a subsequence $(a_{\theta(n)}, \beta_{\nu(n)})_{n \in \mathbb{N}}$ where the subsequence $(\beta_{\nu(n)})_{n \in \mathbb{N}}$ converges to one of the α_j and therefore so does the sequence $(a_{\theta(\tau(n))})_{n \in \mathbb{N}}$ because $\lim_{n \rightarrow \infty} |a_{\tau(n)} - \beta_n| = 0$. This contradicts the hypothesis: "the sequence (a_m) has no cluster point in $\mathcal{S}(f, h)$ ". Thus we have proved that $\inf\{|a_m - x| \mid m \in \mathbb{N}, x \in \mathcal{S}(f, h)\} > 0$. Similarly, $\inf\{|a_m - x| \mid m \in \mathbb{N}, x \in \mathcal{S}(g, h)\} > 0$.

Let $r \in]0, \inf\{|a_m - x| \mid m \in \mathbb{N}, x \in \mathcal{S}(f, h) \cup \mathcal{S}(g, h)\}[$. Thanks to the choice of t and r we can see that all the a_m lie in $\mathcal{D}(t, r, (\mathcal{P}(f) \cup \mathcal{P}(g))) = D$, hence $h \circ f - h \circ g$ has an infinity of zeros in D . Consequently, it is not quasi-invertible and therefore it must be identically zero.

We now suppose that $E = \mathbb{C}$. The proof is similar but easier to this when $E = K$. The function $h \circ f(x) - h \circ g(x)$ is holomorphic in the set $B = \mathbb{C} \setminus (\mathcal{S}(f, h) \cup \mathcal{S}(g, h))$ and admits a bounded sequence of zeros without cluster points in $\setminus(\mathcal{P}(f) \cup \mathcal{S}(g))$. But it does admit a cluster point β in \mathbb{C} and β may not be a regular point, hence $\beta \in \mathcal{T}(f) \cup \mathcal{T}(g)$. Then β is either a pole or a regular point for $h \circ f$ and similarly for g . Consequently, β is either a pole or a regular point for $h \circ f - h \circ g$. In both cases there exists no sequence of zeros of $h \circ f - h \circ g$ converging to β .

Proof of Proposition E: First we shall show that $\mathcal{P}(f) = \mathcal{P}(g)$. Suppose α is a pole of f . If $h \in \mathcal{A}(E)$ (resp. if $h \in \mathcal{A}(d(a, R^-))$), then by Lemma 8 α is a singular point for $h \circ f$, hence for $h \circ g$, hence α is a singular point for g and hence it is a pole for g . Now suppose that $h \notin \mathcal{A}(E)$, hence $h \in \mathcal{M}(E) \setminus E(x)$ (resp. $h \notin \mathcal{A}(d(a, R^-))$), hence $h \in \mathcal{M}(d(a, R^-)) \setminus E(x)$. Since h does not lie in $E(x)$, by Lemma 8 α is a high singularity for $h \circ f$, hence for $h \circ g$. But if α is not a pole for g , it is a regular point for g , hence it is either a regular point or a pole for $h \circ g$, a contradiction. Consequently, α is a pole for g and therefore, since f and g play the same role, we have $\mathcal{P}(f) = \mathcal{P}(g)$. Now, suppose

$\alpha \in \mathcal{T}(f, h)$, hence $h \circ f$ has a pole at α and so does $h \circ g$, thereby $\alpha \in \mathcal{T}(g, h)$. Consequently $\mathcal{T}(f, h) \subset \mathcal{T}(g, h)$ hence $\mathcal{T}(f, h) = \mathcal{T}(g, h)$ and therefore $\mathcal{S}(f, h) = \mathcal{S}(g, h)$ which completes the proof.

We must now recall the classical Nevanlinna Second Main Theorem in \mathbb{C} [23], [13], [17] and in K [5], [18].

Theorem N: *Let $f \in \mathcal{M}(\mathbb{C})$ be non-constant. Let $q \in \mathbb{N} \setminus \{0, 1\}$ and let $a_1, \dots, a_q \in \mathbb{C}$. Let $A = \{a_1, \dots, a_q\}$. Suppose that $f(0) \neq 0$, $f(0) \neq \infty$ and $f(0) \neq a_i$ for every $i = 1, \dots, q$. Then we have:*

$$(q-1)T(r, f) \leq \overline{N}(r, f) + \sum_{i=1}^q \overline{Z}(r, f - a_i) - Z(r, f' \mid f(x) \neq a_i, 1 \leq i \leq q) + o(T(r, f)).$$

Theorem N': *Let $f \in \mathcal{M}(K)$ (resp. $f \in \mathcal{M}(d(a, R^-))$) be non-constant. Let $q \in \mathbb{N} \setminus \{0, 1\}$ and let $a_1, \dots, a_q \in K$ be such that $|a_i - a_j| \geq \delta$ for $1 \leq i \neq j \leq q$. Let $A = \{a_1, \dots, a_q\}$. Suppose that $f(0) \neq 0$, $f(0) \neq \infty$ and $f(0) \neq a_i$ for every $i = 1, \dots, q$. Then we have:*

$$(q-1)T(r, f) \leq T(r, f) + \overline{N}(r, f) + \sum_{i=1}^q \overline{Z}(r, f - a_i) - Z(r, f' \mid f(x) \neq a_i, 1 \leq i \leq q) - \log r + O(1).$$

Lemma 9: *Let $h \in \mathcal{M}(E) \setminus E$ and let $f, g \in \mathcal{M}(E)$ (resp. $f, g \in \mathcal{M}(d(a, r^-))$) satisfy $h \circ f(x) = h \circ g(x) \forall x \in E \setminus \mathcal{S}(f, h) \cup \mathcal{S}(g, h)$ (resp. $h \circ f(x) = h \circ g(x) \forall x \in d(a, R^-) \setminus \mathcal{S}(f, h) \cup \mathcal{S}(g, h)$). Moreover, if f, g do not belong to $\mathcal{A}(E)$ (resp. if f, g do not belong to $\mathcal{A}(d(a, r^-))$), we assume that $h \notin E(x) \setminus E[x]$. Then f, g satisfy $\mathcal{P}(f) = \mathcal{P}(g)$, $\mathcal{S}(f, h) = \mathcal{S}(g, h)$.*

Let $\mathcal{C}(h') = \{c_1, \dots, c_n, \dots\}$ and for each $j = 1, \dots, k$ let $q_j = \omega_{c_j}(h')$. We assume that $h(c_j) \neq h(c_n) \forall j = 1, \dots, k, \forall n \neq j$. Then f, g satisfy

$$\overline{N}(r, f) + \sum_{j=1}^k \overline{Z}(r, f - c_j) \leq \overline{Z}(r, \frac{1}{f} - \frac{1}{g}) + \sum_{j=1}^k \frac{1}{q_j} Z(r, g' \mid f(x) = c_j, g(x) \notin \mathcal{C}(h')).$$

Furthermore, if $f, g \in \mathcal{A}(E)$, (resp. if $f, g \in \mathcal{A}(d(a, r^-))$) then

$$\sum_{j=1}^k \overline{Z}(r, f - c_j) \leq \overline{Z}(r, f - g) + \sum_{j=1}^k \frac{1}{q_j} Z(r, g' \mid f(x) = c_j, g(x) \notin \mathcal{C}(h')).$$

Proof: First, we shall show that $\mathcal{P}(f) = \mathcal{P}(g)$. Suppose α is a pole of f . If $h \in \mathcal{A}(E)$ (resp. if $h \in \mathcal{A}(d(a, R^-))$), then by Lemma 8 α is a singular point for $h \circ f$, hence for $h \circ g$, hence α is a singular point for g , hence it is a pole for g . Now suppose that $h \notin \mathcal{A}(E)$, hence $h \in \mathcal{M}(E) \setminus E(x)$ (resp. $h \notin \mathcal{A}(d(a, R^-))$), hence $h \in \mathcal{M}(d(a, R^-)) \setminus E(x)$. Since h does not lie in $E(x)$, by Lemma 8 α is a high singularity point for $h \circ f$, hence for $h \circ g$. But if α is not a pole for g , it is a regular point for g , hence it is either a regular point or a pole for $h \circ g$, a contradiction. Consequently, α is a pole for g and therefore, since f and g play the same role, we have $\mathcal{P}(f) = \mathcal{P}(g)$. Now, suppose $\alpha \in \mathcal{T}(f, h)$, hence $h \circ f$ has a pole at α and so does $h \circ g$, thereby $\alpha \in \mathcal{T}(g, h)$. Consequently $\mathcal{T}(f, h) \subset \mathcal{T}(g, h)$ hence $\mathcal{T}(f, h) = \mathcal{T}(g, h)$ and therefore $\mathcal{S}(f, h) = \mathcal{S}(g, h)$.

We now assume that $h(c_j) \neq h(c_n) \forall j = 1, \dots, k, \forall n \neq j$. Without loss of generality, we may assume that $0 \notin \mathcal{C}(h')$. Indeed, if $0 \in \mathcal{C}(h')$ we can find $\gamma \in E$ such that $c_n + \gamma \neq 0 \forall n \in \mathbb{N}^*$. Set $\bar{f}(x) = f(x) - \gamma$, $\bar{g}(x) = g(x) - \gamma$ and $\bar{h}(z) = h(z + \gamma)$. Then we have $\bar{h}(\bar{f}(x)) = \bar{h}(\bar{g}(x)) \forall x \in E \setminus \mathcal{S}(\bar{f}, \bar{h})$, (resp. $\bar{h}(\bar{f}(x)) = \bar{h}(\bar{g}(x)) \forall x \in d(a, R^-) \setminus \mathcal{S}(\bar{f}, \bar{h})$), thereby we may process in the same way with \bar{f} , \bar{g} , \bar{h} . Moreover, we notice that if f, g lie in $\mathcal{A}(E)$ (resp. in $\mathcal{A}(d(a, r^-))$), then so do \bar{f}, \bar{g} . Consequently, in order to simplify a deduction, we will assume that $c_n \neq 0 \forall n \in \mathbb{N}^*$.

Let $\phi = \frac{1}{f} - \frac{1}{g}$. Since $\mathcal{P}(f) = \mathcal{P}(g)$, for each pole α of f , we have $\phi(\alpha) = 0$, therefore

$$(1) \quad \bar{N}(r, f) \leq \bar{Z}(r, \phi \mid x \in \mathcal{P}(f)).$$

Let us fix $j \in \{1, \dots, k\}$ and let $\alpha \in E$ (resp. $\alpha \in d(a, R^-)$) satisfy $f(\alpha) = c_j$. Suppose first that $g(\alpha)$ lies in $\mathcal{C}(h')$. Thanks to the hypothesis $h(c_n) \neq h(c_j) \forall n \neq j$, if $g(\alpha) \neq c_j$ then $h(g(\alpha)) \neq h(c_j)$, a contradiction to $h(g(\alpha)) = h(f(\alpha))$. So we have $g(\alpha) = f(\alpha) = c_j$ and since $c_j \neq 0$, then $\phi(\alpha) = 0$. Consequently,

$$(2) \quad \bar{Z}(r, f - c_j \mid g(x) \in \mathcal{C}(h')) \leq \bar{Z}(r, \phi \mid f(x) = c_j)$$

and similarly if $f, g \in \mathcal{A}(E)$ or if $f, g \in \mathcal{A}_u(d(a, R^-))$

$$(3) \quad \bar{Z}(r, f - c_j \mid g(x) \in \mathcal{C}(h')) \leq \bar{Z}(r, f - g \mid f(x) = c_j).$$

Consequently, since $\mathcal{P}(f) \cap \mathcal{C}(h') = \emptyset$, in the general case, by (1) and (2) we derive

$$(4) \quad \bar{N}(r, f) + \sum_{j=1}^k \bar{Z}(r, f - c_j \mid g(x) \in \mathcal{C}(h')) \leq \bar{Z}(r, \phi).$$

Similarly, if $f, g \in \mathcal{A}(E)$, (resp. if $f, g \in \mathcal{A}(d(a, r^-))$) then by (3) we have

$$(5) \quad \sum_{j=1}^k \bar{Z}(r, f - c_j \mid g(x) \in \mathcal{C}(h')) \leq \bar{Z}(r, f - g).$$

In order to complete the proof, we shall show

$$(6) \quad \bar{Z}(r, f - c_j \mid g(x) \notin \mathcal{C}(h')) \leq \frac{1}{q_j} Z(r, g' \mid f(x) = c_j, g(x) \notin \mathcal{C}(h')).$$

Indeed, consider α such that $g(\alpha) \notin \mathcal{C}(h')$. Since $h'(f(\alpha)) = h'(c_j) = 0$, we notice that $f'(\alpha)h'(f(\alpha)) = g'(\alpha)h'(g(\alpha)) = 0$. But since $g(\alpha) \notin \mathcal{C}(h')$, we have $h'(g(\alpha)) \neq 0$, hence $g'(\alpha) = 0$. Consequently, we obtain

$$(7) \quad \overline{Z}(r, f - c_j \mid g(x) \notin \mathcal{C}(h')) \leq \overline{Z}(r, g' \mid f(x) = c_j, g(x) \notin \mathcal{C}(h')).$$

On the other hand, since $f(\alpha) = c_j$, we see that $\omega_\alpha(f'(x)h'(f(x))) \geq q_j$ hence

$$(8) \quad \omega_\alpha(g'(x)h'(g(x))) \geq q_j.$$

But since $g(\alpha) \notin \mathcal{C}(h')$, we have $h'(g(\alpha)) \neq 0$, hence by (8), $\omega_\alpha(g') \geq q_j$, and consequently

$$(9) \quad \overline{Z}(r, g' \mid f(x) = c_j, g(x) \notin \mathcal{C}(h')) \leq \frac{1}{q_j} Z(r, g' \mid f(x) = c_j, g(x) \notin \mathcal{C}(h'))$$

Thus, by (7) and (9) we obtain (6) which, by (4) proves

$$\overline{N}(r, f) + \sum_{j=1}^k \overline{Z}(r, f - c_j) \leq \overline{Z}(r, \frac{1}{f} - \frac{1}{g}) + \sum_{j=1}^k \frac{1}{q_j} Z(r, g' \mid f(x) = c_j, g(x) \notin \mathcal{C}(h')).$$

Similarly, by (5) and (6) we have

$$\sum_{j=1}^k \overline{Z}(r, f - c_j) \leq \overline{Z}(r, f - g) + \sum_{j=1}^k \frac{1}{q_j} Z(r, g' \mid f(x) = c_j, g(x) \notin \mathcal{C}(h')).$$

Proof of Theorems 4, 5, 7, 8, 9: By Proposition D when $E = K$ we have $h \circ f(x) = h \circ g(x) \forall x \in K \setminus (\mathcal{S}(f, h) \cup \mathcal{S}(g, h))$ in Theorems 4, 5 and $h \circ f(x) = h \circ g(x) \forall x \in d(a, R^-) \setminus (\mathcal{S}(f, h) \cup \mathcal{S}(g, h))$ in Theorems 7, 8. In Theorem 8, by Proposition E we also have $h \circ f(x) = h \circ g(x) \forall x \in \mathbb{C} \setminus (\mathcal{S}(f, h) \cup \mathcal{S}(g, h))$. Suppose that f and g are not identical.

Then by Lemma 9 we have $\mathcal{P}(f) = \mathcal{P}(g)$ and $\mathcal{S}(f, h) = \mathcal{S}(g, h)$. Without loss of generality we can obviously assume that $f(0) \neq 0, \infty$, $g(0) \neq 0, \infty$. Since c_1, \dots, c_k lie in $\mathcal{C}(h')$, clearly by applying Theorem N', we obtain in Theorems 4, 7:

$$(1) \quad (k-1)T(r, f) \leq \sum_{j=1}^k \overline{Z}(r, f - c_j) + \overline{N}(r, f) - Z(r, f' \mid f(x) \notin \mathcal{C}(h')) - \log r + O(1) \quad (r > 0)$$

$$(2) \quad (k-1)T(r, g) \leq \sum_{j=1}^k \overline{Z}(r, g - c_j) + \overline{N}(r, g) - Z(r, g' \mid g(x) \notin \mathcal{C}(h')) - \log r + O(1) \quad (r > 0),$$

in Theorems 5, 8 we have:

$$(1\text{bis}) \quad (k-1)T(r, f) \leq \sum_{j=1}^k \overline{Z}(r, f - c_j) + \overline{N}(r, f) - Z(r, f' \mid f(x) \notin \mathcal{C}(h')) + O(1) \quad (r \in]0, R[),$$

$$(2\text{bis}) \quad (k-1)T(r, g) \leq \sum_{j=1}^k \overline{Z}(r, g - c_j) + \overline{N}(r, g) - Z(r, g' \mid g(x) \notin \mathcal{C}(h')) + O(1) \quad (r \in]0, R[),$$

and by applying Theorem N, in Theorem 9 we have:

$$(1\text{ter}) \quad (k-1)T(r, f) \leq \sum_{j=1}^k \bar{Z}(r, f - c_j) + \bar{N}(r, f) - Z(r, f' \mid f(x) \notin \mathcal{C}(h')) + o(T(r, f)),$$

$$(2\text{ter}) \quad (k-1)T(r, g) \leq \sum_{j=1}^k \bar{Z}(r, g - c_j) + \bar{N}(r, g) - Z(r, g' \mid g(x) \notin \mathcal{C}(h')) + o(T(r, g)),$$

Now, let $\phi = \frac{1}{f} - \frac{1}{g}$ and for each $j = 1, \dots, k$, let $q_j = \omega_{c_j}(h')$. By Lemma 9, in Theorem 4, 7 we obtain

$$(3) \quad (k-1)T(r, f) \leq \bar{Z}(r, \phi) + \sum_{j=1}^k \frac{1}{q_j} Z(r, g' \mid f(x) = c_j, g(x) \notin \mathcal{C}(h')) - Z(r, f' \mid f(x) \notin \mathcal{C}(h')) - \log r + O(1),$$

and similarly:

$$(4) \quad (k-1)T(r, g) \leq \bar{Z}(r, \phi) + \sum_{j=1}^k \frac{1}{q_j} Z(r, f' \mid g(x) = c_j, f(x) \notin \mathcal{C}(h')) - Z(r, g' \mid g(x) \notin \mathcal{C}(h')) - \log r + O(1),$$

in Theorems 5, 8 we obtain:

$$(3\text{bis}) \quad (k-1)T(r, f) \leq \bar{Z}(r, \phi) + \sum_{j=1}^k \frac{1}{q_j} Z(r, g' \mid f(x) = c_j, g(x) \notin \mathcal{C}(h')) - Z(r, f' \mid f(x) \notin \mathcal{C}(h')) + O(1),$$

$$(4\text{bis}) \quad (k-1)T(r, g) \leq \bar{Z}(r, \phi) + \sum_{j=1}^k \frac{1}{q_j} Z(r, f' \mid g(x) = c_j, f(x) \notin \mathcal{C}(h')) - Z(r, g' \mid g(x) \notin \mathcal{C}(h')) + O(1),$$

and in Theorem 9 we obtain:

$$(3\text{ter}) \quad (k-1)T(r, f) \leq \bar{Z}(r, \phi) + \sum_{j=1}^k \frac{1}{q_j} Z(r, g' \mid f(x) = c_j, g(x) \notin \mathcal{C}(h')) - Z(r, f' \mid f(x) \notin \mathcal{C}(h')) + \theta(r),$$

$$(4\text{ter}) \quad (k-1)T(r, g) \leq \bar{Z}(r, \phi) + \sum_{j=1}^k \frac{1}{q_j} Z(r, f' \mid g(x) = c_j, f(x) \notin \mathcal{C}(h')) - Z(r, g' \mid g(x) \notin \mathcal{C}(h')) + \tau(r).$$

By adding in each case the two inequalities we have respectively obtained, in Theorems 4 and 7, by (3) and (4) we obtain:

$$(5) \quad (k-1)(T(r, f) + T(r, g)) \leq 2\bar{Z}(r, \phi) +$$

$$\begin{aligned}
& + \sum_{j=1}^k \frac{1}{q_j} [Z(r, f' \mid g(x) = c_j, f(x) \notin \mathcal{C}(h')) + Z(r, g' \mid f(x) = c_j, g(x) \notin \mathcal{C}(h'))] - \\
& - Z(r, f' \mid f(x) \notin \mathcal{C}(h')) - Z(r, g' \mid g(x) \notin \mathcal{C}(h')) - 2 \log r + O(1),
\end{aligned}$$

in Theorems 5 and 8 by (3bis) and (4bis) we have

$$\begin{aligned}
(5\text{bis}) \quad & (k-1)(T(r, f) + T(r, g)) \leq 2\bar{Z}(r, \phi) + \\
& + \sum_{j=1}^k \frac{1}{q_j} [Z(r, f' \mid g(x) = c_j, f(x) \notin \mathcal{C}(h')) + Z(r, g' \mid f(x) = c_j, g(x) \notin \mathcal{C}(h'))] - \\
& - Z(r, f' \mid f(x) \notin \mathcal{C}(h')) - Z(r, g' \mid g(x) \notin \mathcal{C}(h')) + O(1),
\end{aligned}$$

and in Theorem 9, by (3ter) and (4ter) we have:

$$\begin{aligned}
(5\text{ter}) \quad & (k-1)(T(r, f) + T(r, g)) \leq 2\bar{Z}(r, \phi) + \\
& + \sum_{j=1}^k \frac{1}{q_j} [Z(r, f' \mid g(x) = c_j, f(x) \notin \mathcal{C}(h')) + Z(r, g' \mid f(x) = c_j, g(x) \notin \mathcal{C}(h'))] - \\
& - Z(r, f' \mid f(x) \notin \mathcal{C}(h')) - Z(r, g' \mid g(x) \notin \mathcal{C}(h')) + \theta(r) + \tau(r).
\end{aligned}$$

Now, in each inequality, we notice that in the left side member we have the term

$$\sum_{j=1}^k \frac{1}{q_j} [Z(r, f' \mid g(x) = c_j, f(x) \notin \mathcal{C}(h'))] - Z(r, f' \mid f(x) \notin \mathcal{C}(h'))$$

or equal to zero and similarly

$$\sum_{j=1}^k \frac{1}{q_j} [Z(r, g' \mid f(x) = c_j, g(x) \notin \mathcal{C}(h'))] - Z(r, g' \mid g(x) \notin \mathcal{C}(h')) \leq 0.$$

Consequently, in Theorems 4, 7 we obtain

$$(6) \quad (k-1)(T(r, f) + T(r, g)) \leq 2\bar{Z}(r, \phi) - 2 \log r + O(1),$$

in Theorems 5, 8 we have:

$$(6\text{bis}) \quad (k-1)(T(r, f) + T(r, g)) \leq 2\bar{Z}(r, \phi) + O(1)$$

and in Theorem 9 we have:

$$(6\text{ter}) \quad (k-1)(T(r, f) + T(r, g)) \leq 2\bar{Z}(r, \phi) + \theta(r) + \tau(r).$$

Now, by Lemma 1 in theorems 7 and 8 we have $Z(r, \phi) \leq T(r, f) + T(r, g) + O(1)$

Consequently, in Theorem 7 we have $k \leq 2$ and in Theorem 8 we have $k \leq 3$.

In Theorem 9, by classical results in complex analysis [22], we have $Z(r, \phi) \leq T(r, \phi) + \gamma(r)$ with $\gamma \in B(\phi)$. And $T(r, \phi) \leq T(r, f) + T(r, g) + o(T(r, f) + T(r, g))$, hence $Z(r, \phi) \leq T(r, f) + T(r, g) + o(T(r, f) + T(r, g))$.

Consequently, by (6ter) we obtain in Theorem 9:

$(k-1)(T(r, f) + T(r, g)) \leq 2(T(r, f) + T(r, g)) + o((T(r, f) + T(r, g)))$,
hence $k \leq 3$.

Now assume the hypotheses of Theorems 4, 5. By Lemma 9 we can replace $Z(r, \phi)$ by $Z(r, f - g)$ and by Lemma 7 we have
 $T(r, f - g) \leq T(r, f) + O(1)$, $T(r, f - g) \leq T(r, g) + O(1)$, hence
 $T(r, f - g) \leq \frac{1}{2}(T(r, f) + T(r, g)) + O(1)$.

Consequently in place of (6), in Theorem 4 we can obtain

$$(7) \quad (k-1)(T(r, f) + T(r, g)) \leq 2\bar{Z}(r, f - g) - 2 \log r + O(1) \leq T(r, f) + T(r, g) - 2 \log r + O(1)$$

and in place of (6bis), in Theorem 5 we have

$$(7\text{bis}) \quad (k-1)(T(r, f) + T(r, g)) \leq 2\bar{Z}(r, f - g) - 2 \log r + O(1) \leq T(r, f) + T(r, g) + O(1).$$

Thus we can conclude that $k \leq 1$ in Theorem 4 and $k \leq 2$ in Theorem 5.

Lemma 10: *$P \in L[x]$ be such that P' admits exactly 2 distinct zeros c_1, c_2 . Then $P(c_1) \neq P(c_2)$. Assume that all zeros of P are distinct, of order 1. If the set of zeros S of P is not affinely rigid, then P' is of the form $A((x - c_1)(x - c_2))^m$, $A \in K$ and the unique affine mapping preserving S , other than the identity, is the mapping $\gamma(x) = -x + c_1 + c_2$.*

Proof : Let $P'(x) = A(x - c_1)^{m_1}(x - c_2)^{m_2}$. Of course, $\deg(P) = m_1 + m_2 + 1$. Without loss of generality, we may obviously assume $A = 1$. Suppose first that $P(c_1) = P(c_2)$. Then $P - P(c_1)$ admits each c_j as a zero of order $m_j + 1$ ($j = 1, 2$) and therefore $\deg(P) = m_1 + m_2 + 2$, a contradiction. Hence $P(c_1) \neq P(c_2)$.

Suppose that S is not affinely rigid and let $\gamma(x) = ax + b$ be an affine mapping on L preserving S , other than the identity. Since γ preserves S and since all zeros of P are of order 1, $P \circ \gamma$ is a polynomial of same degree as P , admitting the same zeros, all of order 1 and therefore $P \circ \gamma$ is of the form λP with $\lambda \in K^*$. Consequently, $P'(x) = \lambda a(ax + b - c_1)^{m_1}(ax + b - c_2)^{m_2}$.

Suppose first that $m_1 \neq m_2$. Then we can identify c_1 with $\frac{c_1 - b}{a}$ and c_2 with $\frac{c_2 - b}{a}$. Consequently, $b = 0$, $a = 1$, a contradiction since γ is not the identity.

Thus we are led to assume that $m_1 = m_2$. Put $m = m_1 = m_2$. Thus, $P'(x) = [(x - c_1)(x - c_2)]^m$. We may now write $P'(x) = a[(ax + b - c_1)(ax + b - c_2)]^m$ and we see that

$$\begin{aligned} &\text{either } \frac{c_1 - b}{a} = c_1, \quad \frac{c_2 - b}{a} = c_2 \text{ which yields } a = 1, b = 0, \text{ again} \\ &\text{or } \frac{c_1 - b}{a} = c_2, \quad \frac{c_2 - b}{a} = c_1. \end{aligned}$$

And since γ is not the identity, the second conclusion is the only possible. Then we can see that $a = -1$ and $b = c_1 + c_2$.

Proof of Theorem 6: By lemma 10, we have $P(c_1) \neq P(c_2)$ hence by Theorem 4 P is a function of uniqueness for $\mathcal{A}(K)$. Let S be the set of zeros of P . Without loss of generality, through an affine change of variable, we may assume that $c_1 + c_2 = 0$. On the other hand, changing $P(0)$ does not change the property of being a function of uniqueness for $\mathcal{M}(K)$. Next, the set of constant $C \in K$ such that $P + C$ admits some multiple zero is the finite set $\{P(c_1), P(c_2)\}$. Thus, without loss of generality, we may assume that all zeros of P are of order 1 and that $P(0) \neq 0$. If $m_1 = 1$, then without loss of generality, through an affine change of variable, we may assume that P is of the form $A(x^{m+1} + x^m - t)$ (with $t \in K$) and we know that such polynomials are not functions of uniqueness. Indeed, by the proof of Theorem 2 in [10], given any $h \in \mathcal{M}(K)$, putting $g = \frac{h^{n-1}}{h^n}$ and $f = gh$, we have $P(f) = P(g)$. Now, assume that $\min(m_1, m_2) \geq 2$. If S is affinely rigid, since by Lemma 10 P satisfies Condition (F), then by Theorem 1 in [1] we know that P is a function of uniqueness for $\mathcal{M}(K)$. Thus, we are led to examine the situation when S is not affinely rigid. By Lemma 10 we know that P' is of the form $((x - c)(x + c))^m$ and the unique affine mapping preserving S , other than the identity, is the mapping $\gamma(x) = -x$. But since P' is an even polynomial, clearly $P - P(0)$ is an odd polynomial. Let $a \in S$. Then $P(-a) - P(0) = -(P(a) - P(0))$ hence $P(-a) = -P(a) + 2P(0)$. By hypothesis, $P(a) = 0$ and $P(0) \neq 0$, a contradiction. This shows that S is affinely rigid. But since $\gamma(S) = S$, both $a, -a$ lie in S , hence $P(a) = P(-a) = 0$, a contradiction. This completes the proof.

The following Lemma 11 is useful when proving properties of examples of p-adic functions of uniqueness:

Lemma 11: Let $h(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathcal{A}(K)$ satisfy $\left| \frac{a_n}{a_{n+1}} \right| < \left| \frac{a_{n+1}}{a_{n+2}} \right| \forall n \geq t$ and $|a_t| \left| \frac{a_t}{a_{t+1}} \right|^t \geq |a_n| \left| \frac{a_t}{a_{t+1}} \right|^n \forall n < t$. Then h admits t zeros in $d(0, \left| \frac{a_t}{a_{t+1}} \right|)$ (taking multiplicities into account), admits a unique zero of order 1 in each circle $C(0, \left| \frac{a_m}{a_{m+1}} \right|)$ for each $m > t$ and admits no other zero in K .

Proof of Theorem 10: We shall use the notation introduced in [20] and [9] concerning the valuation function $v(f, \mu)$ of a meromorphic function and more generally a Laurent series f , together with the indexes $N^+(f, \mu)$, $N^-(f, \mu)$.

By construction, we can see that h has a zero of order 2 at 0. For each $n \geq 2$, we set $r_n = \left| \frac{b_n}{b_{n+1}} \right|$. By hypothesis the sequence $\left(\left| \frac{b_n}{b_{n+1}} \right| \right)_{n \geq 2}$ is strictly increasing. Hence by Lemma 11 h has a unique zero b_n in the circle $C(0, r_n)$ and this zero is of order 1. This is true for each $n \geq 1$ and h does not admit any other zero in K , except 0 which is of order 2.

Now, let $\lambda \in K$ and $r > 0$ be such that $|\lambda| < |h|_r$. We know that $v(h - \lambda, \mu) = v(h, \mu) \forall \mu \leq -\log r$ and $N^+(h - \lambda, \mu) = N^+(h, \mu)$, $N^-(h - \lambda, \mu) = N^-(h, \mu) \forall \mu \leq -\log r$, hence $h - \lambda$ admits a unique zero in $C(0, r_n)$ for each n such that $r_n \geq r$, this zero being of order 1. And then $h - \lambda$ does not admit any other zeros in $K \setminus d(0, r^-)$.

Next, since $|2b_4| < |9(b_3)^2|$, we see that $\left| \frac{2}{3b_3} \right| < \left| \frac{b_3}{4b_4} \right| r_2$, hence $|c_2| < r_3$. Similarly, since $|3b_3b_5| < |4(b_4)^2|$, we see that $\left| \frac{3b_3}{4b_4} \right| < \left| \frac{4b_4}{5b_5} \right| r_4$, hence $|c_3| < r_4$.

Now, suppose that there exist $m \in \mathbb{N}$ and $i \in \{1, 2, 3\}$ such that $h(c_m) = h(c_i)$ with $m \neq i$. We first notice that $h(c_m) \neq 0 \forall m \neq 1$, because if $h(c_m) = 0$, then c_m is a zero of order 2 of h , hence $c_m = 0$. Thus, we have $m \geq 3$. Suppose $c_m \geq r_3$. Since $|c_2| < r_3$, we have $|h(c_2)| < |h|_{r_3}$, hence as it was seen, $h - h(c_2)$ admits a unique zero of order 1 in each circle $C(0, r_n)$ for each $n \geq 3$ and has no other zero in $K \setminus d(0, r_3^-)$, as does h . But if $|c_m| \geq r_3$, then it is a zero of order 2 for $h - h(c_m)$, a contradiction showing that $|c_m| < r_3$. Let $\rho = \max(|c_2|, |c_m|)$. So, $\rho < r_3$ and $h - h(c_2)$ must admit at least 2 multiple zeros in $d(0, \rho)$. But since $|b - 2| = r_3$, we know that $N^+(h, \mu) = N^-(h, \mu) = 3 \forall \mu \in] -\log(r_3), -\log(r_2)[$. On the other hand, when $\mu < -\log(\rho)$, we have seen that $N^+(h, \mu) = N^+(h - h(c_3), \mu)$, $N^-(h, \mu) = N^-(h - h(c_3), \mu)$, hence $N^+(h - h(c_3), \mu) = 3 \forall \mu \in] -\log(r_3), -\log(\rho)[$. Consequently, $h - h(c_3)$ admits at most 3 zeros in $d(0, \rho)$, taking multiplicities into account. This is a contradiction to the assumption $c_m \in d(0, r_3^-)$ and finishes showing that $h(c_m) \neq h(c_2) \forall m \neq 2$. Thus we have shown that $h(c_m) \neq h(c_j)$ for $j = 1, 2$ and $m \neq j$.

We now suppose that there exists $m \neq 3$ such that $h(c_m) = h(c_3)$. Clearly, as for $h(c_2)$ we have $h(c_3) \neq 0$ and by the above, $h(c_3) \neq h(c_2)$ hence $m > 3$. Since $|c_3| < r_4$, we have $|h(c_3)| < |h|_{r_4}$ hence $N^+(h - h(c_2), \mu) - N^-(h - h(c_2), \mu) \leq 1 \forall \mu \leq -\log r_4$, which shows that $c_m \in d(0, r_4^-)$. Thus, in $d(0, r_4^-)$, h' admits at least 4 zeros: c_1, c_2, c_3, c_m . But by the hypothesis $|4b_4| \geq |5b_5| \left| \frac{b_4}{b_5} \right|$ and $|4b_4| > |nb_n| \left| \frac{b_4}{b_5} \right|^{n-4} \forall n > 5$, we can see that $|4b_4|r_4^3 \geq |5b_5|r_4$ and $|4b_4|r_4^3 > |nb_n|r_4^{n-1} \forall n > 5$. Therefore we have $N^+(h', -\log r_4) \leq 4$ and $N^+(h', -\log r) \leq 3 \forall r < r_4$. Consequently, h' admits at most 3 zeros in $d(0, r_4^-)$, a contradiction to the existence of a $c_m \in d(0, r_4^-)$ such that $h(c_m) = h(c_3)$ with $m \geq 4$. Thus the first conclusion is now established.

We now assume further that $|5b_5| \geq |6b_6| \left| \frac{b_4}{b_5} \right|$, $|4b_4b_6| < |5(b_5)^2|$ and that $|5b_5| > |nb_n| \left| \frac{b_4}{b_5} \right|^{n-5} \forall n > 6$. Suppose that there exists $m \neq 4$ such that $h(c_m) = h(c_4)$. By what precedes, we have $m > 4$. Thanks to the hypothesis $|4b_4b_6| < |5(b_5)^2|$ we see that $|c_4| < r_5$. Consequently, $|h(c_m)| < |h|_{r_5}$ hence c_m lies in $d(0, r_5^-)$. Hence, $d(0, r_5^-)$ contains 5 zeros of h' : c_1, c_2, c_3, c_4, c_m . But thanks to the hypotheses $|5b_5| \geq |6b_6| \left| \frac{b_4}{b_5} \right|$,

$|5b_5| > |nb_n| \left| \frac{b_5}{b_6} \right|^{n-5} \forall n > 6$, we can see that $N^+(h - h(c_4), -\log(r)) - N^-(h, -\log(r)) \leq 4 \forall r < r_5$, hence $h - h(c_4)$ admits at most 4 zeros in $d(0, r_5^-)$, a contradiction to the assumption $h(c_4) = h(c_m)$ for some $m \neq 4$. This ends the proof of Theorem 10.

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