



HAL
open science

Simple asymptotic forms for Sommerfeld and Brillouin precursors

Bruno Macke, Bernard Ségard

► **To cite this version:**

Bruno Macke, Bernard Ségard. Simple asymptotic forms for Sommerfeld and Brillouin precursors. 2012. hal-00680959v1

HAL Id: hal-00680959

<https://hal.science/hal-00680959v1>

Preprint submitted on 20 Mar 2012 (v1), last revised 1 Jun 2012 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Simple asymptotic forms for Sommerfeld and Brillouin precursors

Bruno Macke and Bernard Ségard*

*Laboratoire de Physique des Lasers, Atomes et Molécules ,
CNRS et Université Lille 1, 59655 Villeneuve d'Ascq, France*

(Dated: March 20, 2012)

We examine from a physical viewpoint the classical problem of the propagation of a causal optical field in a dense Lorentz-medium when the propagation distance is such that the medium is opaque in a broad spectral region including the frequency of the optical carrier. The transmitted signal is then reduced to the celebrated precursors of Sommerfeld and Brillouin, well separated in time. In these conditions, we obtain explicit analytical expressions of the first (Sommerfeld) precursor, which only depend on the nature and the importance of the initial discontinuity of the incident field, and we show that the second (Brillouin) precursor has a Gaussian or Gaussian-derivative shape, depending whether the time-integral (algebraic area) of the incident field differs or not from zero. We demonstrate that the Brillouin precursor that has been actually observed in a Debye medium at decimetric wavelengths is also Gaussian. We complete these results by establishing a more general expression of the Brillouin precursor in the Lorentz medium, containing the previous Gaussian one and that obtained by Brillouin himself as particular cases. The propagation of pulses with a square or Gaussian envelope is also studied and we determine the pulse parameters optimizing the Brillouin precursor. Obtained by standard Laplace-Fourier procedures, our analytical results are explicit and contrast by their simplicity from those derived by the uniform asymptotic methods, the complexity of which frequently hides the basic behaviors evidenced in the present article.

PACS numbers: 42.25.Bs, 42.50.Md, 41.20.Jb

I. INTRODUCTION

More than one century ago, in a short communication [1] made at the 79th congress of the German physicists, Sommerfeld examined the apparent inconsistency between the theory of special relativity and the possibility of superluminal group velocity predicted by the classical wave theory. Considering an incident wave switched on at time $t = -T$ and off at time $t = T$ (square-wave modulation), he mathematically demonstrated that, regardless of the value of the group velocity at the frequency of the optical carrier, no signal can be transmitted by any linear dispersive-attenuative medium before the instant $t = -T + z/c$, where z is the propagation distance and c the velocity of light in vacuum. In the discussion following the Sommerfeld's communication, Voigt proposed a simple physical interpretation of this result. He remarked that the front of the wave encounters a medium that, due to its inertia, seems optically empty and, thus, that the propagation of the very first beginning of the signal will proceed undisturbed with the velocity of light in vacuum. In other words, local causality implies relativistic causality. The analysis of what happens after the arrival of the wavefront was subsequently conducted by Sommerfeld and Brillouin in the case of a step-wave modulation (field switched on at time $t = 0$), the medium being modeled as an ensemble of damped harmonic oscillators with the same resonance frequency ω_0 and the same damping rate γ (Lorentz medium) [2–5]. They found that, in suitable conditions, the transmitted signal consists in two successive transients (that they named “forerunners”) preceding the establishment of

the steady-state field at the frequency ω_c of the optical carrier (the “main field”). The first and second forerunners, now called the Sommerfeld and Brillouin precursors, were associated with the frequencies respectively high and low compared to the resonance frequency ω_0 of the medium. These results were obtained by means of a spectral approach involving the newly developed saddle-point method [3] but also classical complex analysis [2] and stationary phase method [4]. Following these pioneering works, precursors became a canonical problem in electromagnetism and optics [6, 7]. Results completing, improving and even correcting those of Sommerfeld and Brillouin were obtained by means of uniform asymptotic methods [8–11]. The problem was also studied by a purely temporal approach [12]. At the present time, the theoretical study of precursors continues to raise a considerable interest, due in part to their hoped application to the transmission of light in very absorbing media. An abundant bibliography can be found in the recent Oughstun's book [13]. Complementary studies on the effects of a finite turn-on time of the incident field on the precursors are reported in [14–17].

From an experimental point of view, the observation of Sommerfeld and Brillouin precursors *in the optical range* raises serious difficulties. Indeed the excitation of the Sommerfeld and Brillouin precursors requires the corresponding frequencies (respectively high and low compared to ω_0) be present at a significant level in the spectrum of the incident pulse. An experiment intended to observe the Brillouin precursor in water is reported in [18]. Using pulses at a wavelength of 700 nm with a bandwidth of 60 nm, the authors observed pulse breakup

in a linear regime as well as a sub-exponential attenuation with distance of the new peak. They attributed these features to the formation of a Brillouin precursor. This interpretation has been soundly disputed, in particular because the pulse bandwidth was in fact not broad enough to perform the excitation of precursors [19]. Alternative explanations of the observations have been proposed [19, 20] and, as far as we know, the direct observation of optical precursors as considered by Sommerfeld and Brillouin in a dense medium keeps an open experimental challenge. Anyway, the theoretical interest of the problem remains because of its connection with the theory of special relativity and its central importance in the theory of wave propagation in attenuative-dispersive media [13].

While well distinguishable Sommerfeld and Brillouin precursors occur when the medium is opaque in a broad spectral region, coherent transients of another kind are obtained in the opposite case where the width of the opacity region is very small compared to the resonance frequency ω_0 . They have been naturally named resonant precursors [21] but also Sommerfeld-Brillouin precursors [22]. Indeed they may be seen as resulting from the coalescence of the Sommerfeld and Brillouin precursors, originating a well-marked beat when the optical thickness of the medium is large enough [23]. The conditions required to achieve experimental evidence of these precursors are relatively easy to meet. They have been actually observed in various systems, in particular in a molecular gas [23], in a solid-state sample with a narrow exciton line [22] and in clouds of cold atoms [24, 25]. The theoretical study of these precursors has been conveniently carried out in the frame of the slowly varying envelope approximation.

In the present paper we come back to the classical problem of the propagation of a causally modulated wave in a dense Lorentz medium when the propagation distance is such that the medium is opaque in a broad spectral region including the frequency ω_c of the optical carrier. We remark that *these conditions are met for the parameters considered by Brillouin* [26] and often referred to in the literature. We then succeed in obtaining *simple and explicit analytical expressions* of both Sommerfeld and Brillouin precursors, having a clear physical interpretation. When it is necessary, we determine the range of validity of these analytical solutions by comparing them to exact numerical solutions obtained by fast Fourier transform (FFT). The arrangement of our paper is as follows. In Section II, we precisely define the problem under consideration and give some general results, useful for the following. Section III is devoted to the study of the Sommerfeld precursor. We establish the corresponding expression of the impulse response of the medium and apply it to determine the transmitted optical field when the incident field is turned on instantaneously, with a discontinuity of its n^{th} derivative

or ideally smoothly. We show in Section IV that, in a strict asymptotic limit, the impulse response associated with the Brillouin precursor is Gaussian and that the Brillouin precursor has itself a Gaussian or Gaussian-derivative shape, depending whether the time-integral (area) of the incident field differs or not from zero. Although this Gaussian shape holds in the conditions considered by Brillouin, it has been overlooked in the pioneering works of this author [3, 4] as well as in most subsequent papers. We demonstrate that it also holds for the Brillouin precursor that has been directly observed in a Debye medium. A more general expression of the Brillouin precursor in the Lorentz medium (simple asymptotic limit) is established in Section V, containing the previous one (dominant-attenuation limit) and that obtained by Brillouin (dominant-dispersion limit) as particular cases. The propagation in both media of pulses with a square or Gaussian envelope is finally examined in Section VI and we determine the pulse parameters optimizing the Brillouin precursor, strongly different from those proposed in the literature. We conclude in Section VII by summarizing and discussing our main results.

II. GENERAL ANALYSIS

We consider a one-dimensional optical wave propagating in a Lorentz medium in the z -direction, with an electric field linearly polarized in the x -direction (x, y, z : Cartesian coordinates). We denote $e(0, t)$ the algebraic amplitude of the field at time t for $z = 0$ (inside the medium) and $e(z, t)$ its value after a propagation distance z through the medium. The incident field $e(0, t)$ being given, the problem is to determine the transmitted field $e(z, t)$. We take for $e(0, t)$ the general form

$$e(0, t) = u(t) \cos(\omega_c t - \varphi), \quad (1)$$

including as particular cases the different forms considered in the literature. ω_c is the frequency of the optical carrier, φ is the phase (eventually time-dependent) and $u(t) \geq 0$ is the amplitude modulation or field envelope. On the other hand the medium is fully characterized in the frequency domain by its transfer function $H(z, \omega)$ relating the Fourier transform $E(z, \omega)$ of $e(z, t)$ to that $E(0, \omega)$ of $e(0, t)$ [27].

$$E(z, \omega) = H(z, \omega)E(0, \omega). \quad (2)$$

In all the following, we take for t a retarded time equal to the real time minus the luminal propagation time z/c (retarded-time picture). $H(z, \omega)$ then reads

$$H(z, \omega) = \exp \left\{ -i \frac{\omega z}{c} [\tilde{n}(\omega) - 1] \right\}. \quad (3)$$

Here $\tilde{n}(\omega)$ is the complex refractive index of the medium at the frequency ω , that is for the Lorentz medium

$$\tilde{n}(\omega) = \left(1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2 - 2i\gamma\omega} \right)^{1/2}, \quad (4)$$

where ω_0 is the resonance frequency, γ is the damping or relaxation rate and ω_p is the so-called plasma frequency whose square is proportional to the number density of absorbers. $\Re[\tilde{n}(\omega)]$ is the usual (real) refractive index $n(\omega)$ and the absorption coefficient $\alpha(\omega)$ for the amplitude is given by the relation $\alpha(\omega) = -(\omega/c)\Im[\tilde{n}(\omega)]$.

In the time domain, the medium will be characterized by its impulse response $h(z, t)$, inverse Fourier transform of $H(z, \omega)$, and the transmitted signal $e(z, t)$ is given by the convolution product [27]

$$e(z, t) = h(z, t) \otimes e(0, t). \quad (5)$$

Some general properties of $h(z, t)$ and $e(z, t)$ can be deduced from Eqs.(3-5). First $h(z, t)$ fulfills the condition of relativistic causality, namely $h(z, t) = 0$ for $t < 0$ [28]. Its area reads as $\int_{-\infty}^{+\infty} h(z, t) dt = H(z, 0) = 1$. It keeps thus constant and normalized to unity regardless of the propagation distance z . Consequently $E(z, 0) = E(0, 0)$, that is

$$\int_{-\infty}^{+\infty} e(z, t) dt = \int_{-\infty}^{+\infty} e(0, t) dt. \quad (6)$$

The area of the optical field (to distinguish from that of its envelope) is conserved during the propagation. Finally the fact that $H(z, \infty) = 1$ entails that $h(z, t)$ will start by a Dirac delta-function $\delta(t)$. This implies that the propagation of the very first beginning of any incident signal $e(0, t)$ will always proceed undisturbed at the velocity c , in agreement with the Voigt's remark on the Sommerfeld's communication [1].

The previous results are valid whatever the values of the parameters may be. Examine now in what conditions the medium is opaque in a broad spectral region. To be definite, we will consider that the medium is opaque at the frequency ω when its optical thickness $\alpha(\omega)z$ exceeds 20, the amplitude transmission $|H(z, \omega)| = \exp[-\alpha(\omega)z]$ being then about 2×10^{-9} . Assuming as Brillouin [26] that γ is relatively small compared to ω_0 (moderate damping) and that ω_p is of the order of ω_0 (dense medium), we easily get from Eq.(4) an approximate expression of the optical thickness on resonance, namely $\alpha(\omega_0)z \approx (z\omega_p/2c)\sqrt{\omega_0/\gamma} = (\xi/\omega_p)\sqrt{\omega_0/\gamma}$. Here $\xi = \omega_p^2 z/2c$ is the parameter, homogeneous to a frequency, introduced by Sommerfeld to study his eponymous precursor [2]. It will be used in the following to characterize the propagation distance ($\xi \propto z$). Far from resonance, the optical thickness falls down to $\alpha(\omega)z \approx 2\gamma\xi/\omega^2$ at high frequency ($\omega^2 \gg \omega_0^2$) and

to $\alpha(\omega)z \approx 2\gamma\xi\omega^2 / (\omega_0^4 \sqrt{1 + \omega_p^2/\omega_0^2})$ at low frequency ($\omega^2 \ll \omega_0^2$). Provided that $\gamma\xi/(10\omega_0^2) \gg 1$, the medium will thus be opaque in the broad spectral region comprised between ω_- and ω_+ , with $\omega_+/\omega_0 \approx \sqrt{\gamma\xi/(10\omega_0^2)}$ and $\omega_-/\omega_0 \approx (1 + \omega_p^2/\omega_0^2)^{1/4} \sqrt{10\omega_0^2/(\gamma\xi)}$. The inequality $\gamma\xi/(10\omega_0^2) \gg 1$ is over-satisfied (strict asymptotic limit) for the parameters values considered by Brillouin [26], namely $\omega_0 = 4 \times 10^{16} \text{s}^{-1}$, $\omega_p^2 = 1.24\omega_0^2$, $\gamma^2 = \omega_0^2/200$ and $z = 10^{-2} \text{m}$. Not to reduce our study to a particular system or region of the spectrum, we will refer all the frequencies (the times) to their natural unit ω_0 ($1/\omega_0$). We then get $\omega_p/\omega_0 \approx 1.11$, $\gamma/\omega_0 \approx 7.07 \times 10^{-2}$, $\xi/\omega_0 \approx 8.31 \times 10^5$ and $\gamma\xi/(10\omega_0^2) \approx 5.87 \times 10^3$.

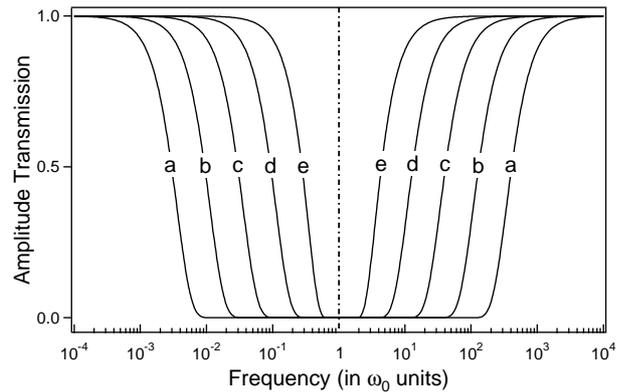


Figure 1: Amplitude transmission $|H(z, \omega)|$ of the medium as a function of the frequency modulus $|\omega|$ (logarithmic scale). Parameters (in ω_0 units) : $\omega_p = 1.11$, $\gamma = 0.0707$ and $\xi = 8.31 \times 10^5$ for the curve (a) corresponding to the Brillouin choice ($z = 10^{-2} \text{m}$). The curves (b), (c), (d) and (e) are obtained for propagation distances (and thus ξ) respectively 10, 100, 1 000 and 10 000 times smaller.

Figure 1 shows the profiles of the amplitude transmission $|H(z, \omega)| = \exp[-\alpha(\omega)z]$ as a function of the reduced frequency ω/ω_0 in the Brillouin conditions (curve a) and for propagation distances 10, 100, 1 000 and 10 000 times shorter (curves b to e). The corresponding opacity regions [$\alpha(\omega)z > 20$] respectively are (a) $0.016 < \omega/\omega_0 < 77$, (b) $0.05 < \omega/\omega_0 < 24$, (c) $0.16 < \omega/\omega_0 < 7.8$, (d) $0.43 < \omega/\omega_0 < 2.9$ and (e) $0.77 < \omega/\omega_0 < 1.64$. Though $\gamma\xi/(10\omega_0^2)$ falls down to 5.87 for the profile (d) the boundaries of the opacity region remain fairly close to their asymptotic values ω_- and ω_+ . The profile (e), given for reference, corresponds to a case where the width of the opacity region is of the order of ω_0 .

The medium being opaque for $\omega_- < \omega < \omega_+$, the transfer function may be written as

$$H(z, \omega) = H_S(z, \omega) + H_B(z, \omega), \quad (7)$$

with $H_S(z, \omega) \approx 0$ for $\omega < \omega_+$ and $H_B(z, \omega) \approx 0$ for $\omega > \omega_-$. H_S and H_B are respectively associated with the Sommerfeld and the Brillouin precursor. For $\omega = 0$,

$H_S(z, 0) \approx 0$ and $H_B(z, 0) \approx H(z, 0) = 1$. So long as ω_c lies in the opacity region, this implies that *the Sommerfeld precursor will have a zero area while the area of the Brillouin precursor will be equal to that of the incident field*.

The formation of the optical precursors is generally governed by combined effects of attenuation (considered above) and dispersion. The dispersion effects can be soundly characterized by the group delay $\tau_g(z, \omega) = -d\Phi/d\omega = z/v_g(\omega) - z/c$, where $\Phi(z, \omega)$ is the argument of $H(z, \omega)$ and $v_g(\omega)$ the group velocity [28]. Note that the group velocity is superluminal ($\tau_g < 0$) in a spectral region whose width is roughly equal to that of the band that would be forbidden in the absence of damping ($1 < \omega/\omega_0 < \sqrt{1 + \omega_p^2/\omega_0^2}$) and slightly broader than the anomalous-dispersion band where the refractive index $n(\omega)$ is a decreasing function of the frequency. For the Brillouin parameters, the superluminal and anomalous-dispersion bands are, respectively, $0.96 < \omega/\omega_0 < 1.43$ and $0.965 < \omega/\omega_0 < 1.38$. Both are well inside the opacity region and will not directly contribute to the formation of precursors. For the high and low frequencies respectively associated with the Sommerfeld and Brillouin precursors, we get the asymptotic forms $\tau_g \approx \xi/\omega^2$ [28] and $\tau_g \approx t_B + \omega^2/(\eta b^3)$ where

$$t_B = \frac{[n(0) - 1]z}{c} = \frac{2\xi}{\omega_p^2} \left[\left(1 + \frac{\omega_p^2}{\omega_0^2} \right)^{1/2} - 1 \right] \quad (8)$$

$$b = \omega_0 \left(3 \frac{\xi}{\omega_0} \right)^{-1/3} \left(1 + \frac{\omega_p^2}{\omega_0^2} \right)^{1/6} \quad (9)$$

$$\frac{1}{\eta} = 1 - \frac{4\gamma^2}{\omega_0^2} \left(1 + \frac{3\omega_p^2}{4\omega_0^2} \right) / \left(1 + \frac{\omega_p^2}{\omega_0^2} \right). \quad (10)$$

$t_B = \tau_g(z, 0) - \tau_g(z, \infty)$ is obviously indicative of the time delay of the Brillouin precursor (low frequency) with respect to the Sommerfeld precursor (high frequency). The two precursors will be fully separated when t_B is much larger than the damping time $1/\gamma$. Since $\gamma t_B = O(\gamma\xi/\omega_0^2)$, this condition is automatically fulfilled when the condition of broad opacity-region [$\gamma\xi/(10\omega_0^2) \gg 1$] holds. Another important point is that τ_g is minimum (stationary) for $\omega \rightarrow \infty$ and $\omega \rightarrow 0$. As pointed out by Brillouin [4], this ensures that the precursors will not be washed out by the group velocity dispersion.

III. SOMMERFELD PRECURSOR

In the limit considered here $\omega^2 \geq \omega_+^2 \gg \omega_0^2$ and $H_S(z, \omega)$ takes the following asymptotic form, accounting

for both dispersion (main contribution) and attenuation.

$$H_S(z, \omega) \approx \exp \left[-\frac{\xi}{i\omega + 2\gamma} \right]. \quad (11)$$

The corresponding impulse response $h_S(z, t)$ is easily determined by using standard results of Laplace transforms [29]. We get

$$h_S(z, t) = \delta(t) - \sqrt{\frac{\xi}{t}} J_1 \left(2\sqrt{\xi t} \right) e^{-2\gamma t} u_H(t), \quad (12)$$

where $J_n(s)$ and $u_H(t)$ respectively designate the first kind Bessel-function of index n and the Heaviside unit-step function. Except for their very first oscillation, the Bessel functions $J_n(s)$ are perfectly approximated by their asymptotic form

$$J_n(s) \approx \sqrt{\frac{2}{\pi s}} \cos \left(s - n\frac{\pi}{2} - \frac{\pi}{4} \right), \quad (13)$$

and the impulse response $h_S(z, t)$ can be characterized by an instantaneous frequency $\omega \approx d(2\sqrt{\xi t})/dt = \sqrt{\xi/t}$. Note that, in the asymptotic limit considered here, ξ is very large compared to ω_0 and extremely large compared to γ ($\xi/\omega_0 \approx 8.31 \times 10^5$ and $\xi/\gamma > 10^7$ in the Brillouin conditions). The range of validity of Eq.(12) may be estimated by determining the change $\delta H_S(z, \omega)$ of $H_S(z, \omega)$ due to the first term neglected in the asymptotic expansion of $\ln[H_S(z, \omega)]$ used to obtain Eq.(11). We find $\delta H_S(z, \omega)/H_S(z, \omega) = O(\xi\omega_0^2/\omega^3)$, negligible when $\omega^3 \gg \xi\omega_0^2$, i.e. when $\xi^{1/2} \gg \omega_0^2 t^{3/2}$. In fact, Eq.(12) fits very well the exact impulse response as soon as $\xi^{1/2}$ exceeds $\omega_0^2 t^{3/2}$ by a factor $\sqrt{10}$ (half an order of magnitude). This is achieved as long as $t \leq t_S$, with

$$\omega_0 t_S = \sqrt[3]{\frac{\xi}{10\omega_0}}. \quad (14)$$

In the strict asymptotic limit, t_S is so large that $\exp(-2\gamma t_S) \approx 0$ and, as expected, the entirety of the impulse response is reproduced by Eq.(12).

The Sommerfeld precursor $e_S(z, t)$ is obtained by convoluting $h_S(z, t)$ with the incident field $e(0, t) = u(t) \cos(\omega_c t - \varphi)$ introduced in the general analysis [Eq.(1)]. We are mainly interested here in the physical case where the incident field is causal [$e(0, t) = 0$ for $t < 0$], $u(t)$ being either a unit step $u_H(t)$ or a function *smoothly* and *monotonously* rising from 0 to 1 with a rate $r \lesssim \omega_c$ for $t > 0$ (step or step-like modulation). The convolution product of Eq.(5) takes the form:

$$e_S(z, t) = \int_{-\infty}^t h_S(z, \theta) e(0, t - \theta) d\theta, \quad (15)$$

that can be transformed by repeated integrations per parts to yield

$$e_S(z, t) = \sum_{n=0}^{\infty} d_n h_S^{(n+1)}(z, t). \quad (16)$$

Here d_n is the discontinuity of the n^{th} derivative of $e(0, t)$ at the initial time [30] and $f^{(n)}(t)$ is a short-hand notation for $\int_{-\infty}^t \int_{-\infty}^{t_1} \cdots \int_{-\infty}^{t_{n-1}} f(t_n) dt_n \cdots dt_2 dt_1$. In a frequency description, the previous result can be retrieved by expanding the Fourier transform $E(0, \omega)$ of $e(0, t)$ in powers of $1/i\omega$ and exploiting the equivalence between multiplication by $1/i\omega$ in the frequency domain and integration in the time domain [27]. Writing the impulse response under the form $h_S(z, t) = k_S(z, t) \exp(-2\gamma t)$, we easily show by means of standard Laplace procedures [29] that $k_S^{(n+1)}(z, t) = (t/\xi)^{n/2} J_n(2\sqrt{\xi t}) u_H(t)$. Insofar as $k_S(z, t)$ is very rapidly varying compared to $\exp(-2\gamma t)$, $h_S^{(n+1)}(z, t) \approx k_S^{(n+1)}(z, t) \exp(-2\gamma t)$ and we finally get

$$e_S(z, t) \approx \sum_{n=0}^{\infty} d_n \left(\frac{t}{\xi}\right)^{n/2} J_n(2\sqrt{\xi t}) \exp(-2\gamma t) u_H(t). \quad (17)$$

The n^{th} term of the series has a maximal amplitude $a_0 = |d_0|$ at $t = t_0 = 0$ for $n = 0$ and

$$a_n = \frac{1}{\sqrt{\pi}} |d_n| \left(\frac{2n-1}{8e}\right)^{(2n-1)/4} \left(\frac{\gamma}{\xi}\right)^{1/4} (\gamma\xi)^{-n/2}, \quad (18)$$

at $t \approx t_n = (2n-1)/8\gamma$ for $n > 0$. An essential point is that, in the (strict) asymptotic limit, a_n is (extremely) rapidly decreasing with n , so that a good (excellent) approximation of the exact result is obtained by keeping only the first term $n = p$ of the series for which $d_p \neq 0$. In the frequency description, this amounts to restrict the asymptotic expansion of $E(0, \omega)$ to its first non vanishing term [7]. We then get

$$e_S(z, t) \approx d_p \left(\frac{t}{\xi}\right)^{p/2} J_p(2\sqrt{\xi t}) \exp(-2\gamma t) u_H(t). \quad (19)$$

Eq.(19) will be a good approximation of the exact result if $\varepsilon = a_q/a_p \ll 1$, where q is the next integer following p for which $d_q \neq 0$. Its validity is in principle limited to the time domain $t \leq t_S$ [see Eq.(14)] but, as shown hereafter, Eq.(19) correctly reproduces the essential features of the precursor, in particular the amplitude of its maximum, even when the latter occurs at a time moderately exceeding t_S .

As a first illustration of the previous results, we consider the very instructive case where the incident field is instantaneously turned on (Fig.2). This is achieved with an incident field $e(0, t) = u_H(t) \cos(\omega_c t)$, such that $p = 0$ with $d_0 = 1$ [30] and $q = 2$ with $d_2 = -\omega_c^2$. Eq.(19) then reads as

$$e_S(z, t) \approx J_0(2\sqrt{\xi t}) \exp(-2\gamma t) u_H(t), \quad (20)$$

with $\varepsilon \approx 0.13\omega_c^2\gamma^{-3/4}\xi^{-5/4}$. The precursor does not depend on ω_c and the initial discontinuity of the incident field is integrally transmitted, in agreement with the general analysis. For $\omega_c < \omega_+ = \sqrt{\gamma\xi/10}$ (opacity condition), ε is always smaller than $0.013(\gamma/\xi)^{1/4}$,

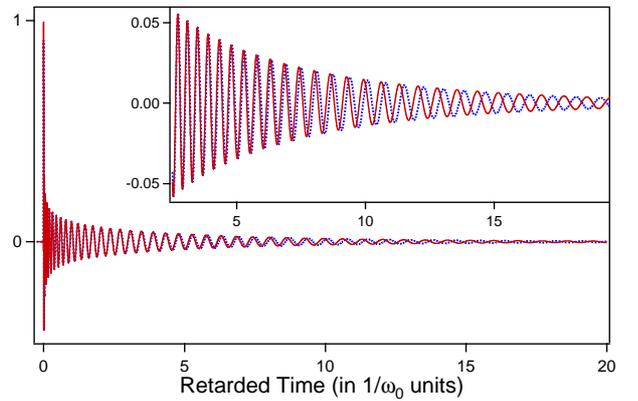


Figure 2: Sommerfeld precursor generated by the incident field $u_H(t) \cos(\omega_c t)$ in the simple asymptotic limit. The solid (dashed) line is the exact numerical solution (the approximate analytic solution). Parameters (in ω_0 units) : $\omega_c = 1$, $\omega_p = 1.11$, $\gamma = 0.0707$ and $\xi = 831$. Inset : enlargement of the tail of the precursor.

that is about 2.2×10^{-4} in the Brillouin conditions and 1.2×10^{-3} for a propagation distance 1 000 times smaller (simple asymptotic limit). In the first case, $\omega_0 t_S = 44$ and $\exp(-2\gamma t_S) \approx 2 \times 10^{-3}$. We are close to the strict asymptotic limit and the precursor is perfectly reproduced by its asymptotic form at any time where it has significant amplitude. This remark also holds for the cases $p = 1, 2, 3$ considered hereafter. In the simple asymptotic limit $\omega_0 t_S = 4.4$ and, as expected, Eq.(20) perfectly fits the exact solution for $\omega_0 t \leq 4.4$. For larger times, the fit remains very good except for a slight drift of the instantaneous frequency of the oscillations whose envelope is very well reproduced at any time (Fig.2).

Following Sommerfeld and Brillouin, most authors have considered an incident field of the *canonical form* $e(0, t) = u_H(t) \sin(\omega_c t)$ for which $p = 1$ with $d_1 = \omega_c$ and $q = 3$ with $d_3 = -\omega_c^3$. We then get

$$e_S(z, t) \approx \omega_c \sqrt{\frac{t}{\xi}} J_1(2\sqrt{\xi t}) \exp(-2\gamma t) u_H(t), \quad (21)$$

with $\varepsilon \approx 0.34(\omega_c^2/\gamma\xi)$. We incidentally remark that, insofar as $\exp(-2\gamma t)$ is slowly varying compared to $\sqrt{t/\xi} J_1(2\sqrt{\xi t})$, the Sommerfeld precursor obtained when the field is instantaneously turned on [Eq.(20)] is simply the time derivative of that given Eq.(21), divided by ω_c . A derivation of Eq.(5) with respect to t shows that this relation holds for the entirety of the transmitted field $e(z, t)$, without any approximation or restriction. The result given Eq.(21) differs from that originally obtained by Sommerfeld [2] by the presence of the damping term $\exp(-2\gamma t)$. Though the formation of the Sommerfeld precursor is mainly governed by the medium dispersion, the presence of this term (associated with the absorption) is obviously necessary to avoid that $e_S(z, t)$ diverges with time. The precursor attains its

maximum at $t \approx t_1 = 1/(8\gamma)$ ($\omega_0 t_1 = 1.77$) and its amplitude $a_S = a_1 \approx 0.26 \omega_c \gamma^{-1/4} \xi^{-3/4}$ is proportional to ω_c . For $\omega_c = \omega_0$, $a_S \approx 1.8 \times 10^{-5}$ with $\varepsilon \approx 5.8 \times 10^{-6}$ in the Brillouin conditions whereas $a_S \approx 3.25 \times 10^{-3}$ with $\varepsilon \approx 5.8 \times 10^{-3}$ in the simple asymptotic limit. In the latter case, Fig.3 shows that Eq.(21) actually fits very well the exact result for $t \leq t_S$, again with a slight drift of the instantaneous frequency of the oscillations for $t > t_S$. In order to check the proportionality of the precursor to ω_c , we have compared the exact forms of $(\omega_0/\omega_c) e_S(z, t)$ obtained when ω_c lies at the boundaries ω_- or ω_+ of the opacity region to that obtained when $\omega_c = \omega_0$. As expected we have found that the three results are nearly undistinguishable, except for an amplitude 1.3% larger for $\omega_c = \omega_+$ (below the corresponding value of ε , namely $\varepsilon = 0.034$). For this value of ω_c , the amplitude of the precursor is $a_S \approx 0.082 (\gamma/\xi)^{1/4}$, that is 1.4×10^{-3} in the Brillouin conditions and 7.9×10^{-3} in the simple asymptotic limit.

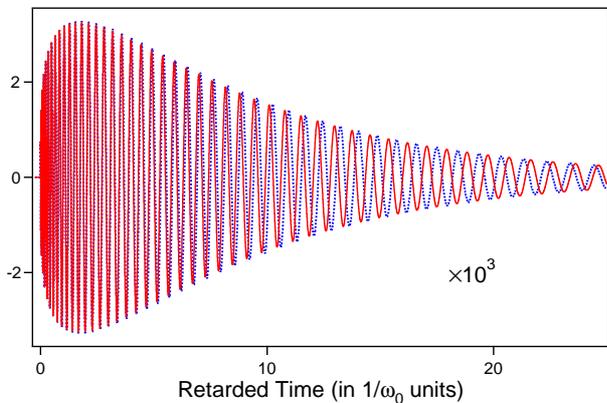


Figure 3: Sommerfeld precursor generated by the canonical incident field $\sin(\omega_c t) u_H(t)$ in the simple asymptotic limit. The solid (dashed) line is the exact numerical solution (the approximate analytic solution). Parameters as in Fig.2.

A gradual turning on of the incident field is expected to reduce the amplitude of the Sommerfeld precursor. To study this so-called rise-time effect, Ciarkowski [14, 17] has considered an incident field of the form $e(0, t) = u(t) \sin(\omega_c t)$ with a field envelope $u(t) = u_2(t) = \tanh(rt) u_H(t)$. We have then $p = 2$ with $d_2 = 2r\omega_c$, $q = 4$ with $d_4 = -4\omega_c r (2r^2 + \omega_c^2)$ and the asymptotic form of the precursor reads as

$$e_S(z, t) \approx 2\omega_c r \left(\frac{t}{\xi} \right) J_2(2\sqrt{\xi t}) \exp(-2\gamma t) u_H(t), \quad (22)$$

with $\varepsilon \approx 1.21 (2r^2 + \omega_c^2) / \gamma \xi$. The precursor attains its maximum at $t \approx t_2 = 3/(8\gamma)$ ($\omega_0 t_2 \approx 5.3$) with an amplitude $a_S = a_2 \approx 0.26 r \omega_c \gamma^{-3/4} \xi^{-5/4}$. Compared to the precursor obtained with the canonical incident field [Eq.(21)], the maximum is shifted to larger time ($t_2 = 3t_1$) and its amplitude is reduced by a factor $\rho \approx \sqrt{\gamma \xi} / r$. Fig.4, obtained in the simple asymptotic

limit, shows that Eq.(22) fits quite satisfactorily the exact precursor though its maximum now lies at a time slightly larger than t_S . To check that the precursor is mainly determined by the lowest order initial discontinuity of the incident field regardless of its subsequent evolution, we have compared the precursor obtained when the envelope $\tanh(rt) u_H(t)$ is replaced by $(1 - e^{-rt}) u_H(t)$, the two incident fields having the same initial discontinuity $d_2 = 2r\omega_c$. Though q is different (3 instead of 4) and the 10 – 90% rise time T_r is significantly larger ($2.20/r$ instead of $1.37/r$), we find that the two precursors are actually very close, with a deviation that does not exceed 5% of their maximum amplitude.

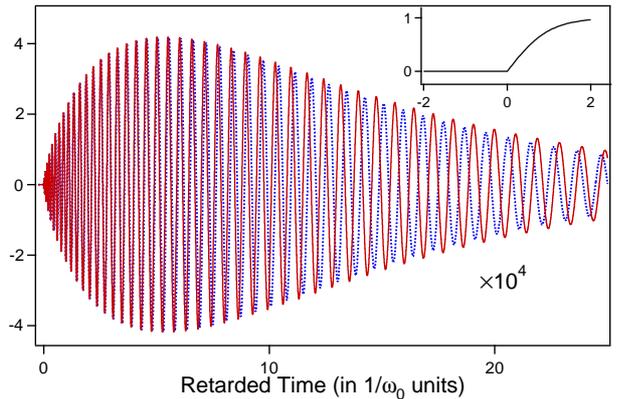


Figure 4: Sommerfeld precursor generated by the incident field $u(t) \sin(\omega_c t)$ with $u(t) = u_2(t) = \tanh(rt) u_H(t)$. The solid (dashed) line is the exact numerical solution (the approximate analytic solution) obtained for $r = \omega_0$. Other parameters as in Fig.2. Inset : $u(t)$ as a function of rt .

The reduction of the amplitude of the precursor becomes more and more important when the order p of the discontinuity increases. Oughstun [31] has considered an incident field $\sin(\omega_c t)$ modulated by $[1 - \cos(\beta_r t)] [u_H(t) - u_H(t - \pi/\beta_r)]$. The corresponding incident field has a discontinuity of order $p = 3$ at $t = 0$ but the second discontinuity at $t = \pi/\beta_r$ also generates a Sommerfeld precursor that interferes with the first one and partly jams it. We consider instead the simpler modulation $u_3(t) = [1 - \exp(-r^2 t^2)] u_H(t)$. The field $e(0, t)$ has a single discontinuity $p = 3$ at $t = 0$, with $d_3 = 6\omega_c r^2$ and $q = 5$, with $d_5 = -20\omega_c r^2 (3r^2 + \omega_c^2)$. We then get

$$e_S(z, t) \approx 6\omega_c r^2 \left(\frac{t}{\xi} \right)^{3/2} J_3(2\sqrt{\xi t}) \exp(-2\gamma t) u_H(t), \quad (23)$$

with $\varepsilon = 2.89 (3r^2 + \omega_c^2) / \gamma \xi$. The maximum of the precursor is now shifted to $t \approx t_3 = 5/(8\gamma)$ ($\omega_0 t_3 \approx 8.8$), with an amplitude $a_S = a_3 \approx 0.54 \omega_c r^2 \gamma^{-5/4} \xi^{-7/4}$, smaller than that obtained with the canonical incident field by the factor $\rho \approx 0.48 \gamma \xi / r^2$. For the parameters previously considered (Figs.2-4), $t_3 \approx 2t_S$ and $\varepsilon \approx 20\%$.

Despite that, the agreement between Eq.(23) and the exact result is not too bad (Fig.5). It becomes very good when the propagation distance, still keeping very short compared to that considered by Brillouin, is increased by a factor 4 (we have then $\omega_0 t_S \approx 7.0$ and $\varepsilon \approx 5\%$). Again, the agreement is nearly perfect in the Brillouin conditions.

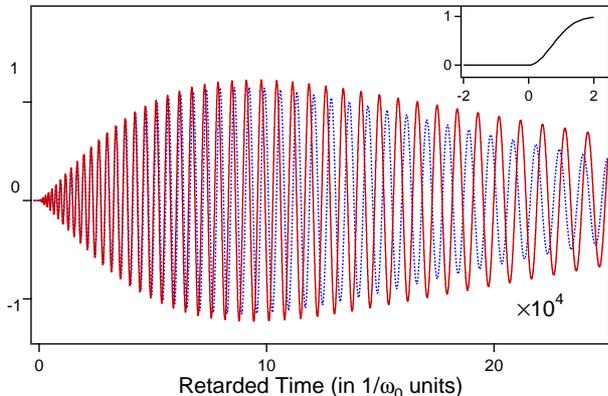


Figure 5: Same as Fig.4 for $u(t) = u_3(t) = [1 - \exp(-r^2 t^2)] u_H(t)$.

To summarize, Eq.(19) relating the Sommerfeld precursors generated by a *causal* incident field to the lowest-order initial discontinuity of the latter is exact in a strict asymptotic limit ($z \rightarrow \infty$). It has been applied to the reference cases where the order p of this discontinuity is equal to 0, 1, 2 and 3 to obtain the equations 20 to 23. These equations have been shown to provide a good approximation of the exact numerical solutions for propagation distances much shorter than that considered by Brillouin (simple asymptotic limit). A remarkable consequence of these results is that *the global shape of the precursor is practically independent of the carrier frequency* (below, inside or beyond the anomalous dispersion band), at least when this frequency lies inside the opacity band. When the incident field is turned on progressively ($p > 1$), the amplitude of the precursor is smaller than that obtained with the canonical incident field ($p = 1$) by a factor ρ scaling as $(\sqrt{\gamma\xi}/r)^{p-1}$. The rise time effects will thus be moderate if the rise rate r is not too small compared to $\sqrt{\gamma\xi}$ and not to $\gamma \ll \sqrt{\gamma\xi}$, as sometimes considered [31]. Note however that, when the order p of discontinuity increases, Eq.(19) only holds for larger and larger propagation distances, such that $t_p \leq t_S \propto z^{1/3}$.

At the light of the previous results, the rise time effects are expected to become dramatic when the incident field is analytic with continuous derivatives in every point. Such fields were considered in the literature [13, 16, 31] though they are not causal and, in principle, not physically realizable (in the sense of the linear systems theory). We considered in [16] an incident field

$\sin(\omega_c t)$ modulated by the ideally smooth analytic function $u_{an}(t) = [1 + \text{erf}(rt)]/2$ where $\text{erf}(s)$ designates the error function. Extrapolating the results obtained in the causal case, the Sommerfeld precursor is expected to be strongly delayed with respect to the precursor generated by the canonical incident field, with a reduction ρ of its amplitude larger than in the causal case but, as in this case, mainly depending on $\sqrt{\gamma\xi}/r$. We empirically find that ρ very roughly scales as $\exp(\sqrt{\gamma\xi}/r)$.

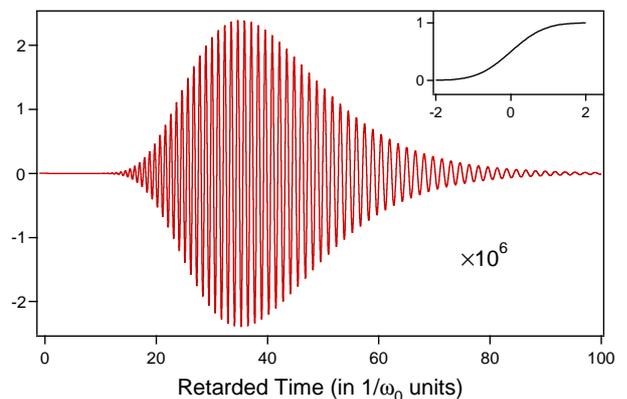


Figure 6: Same as Fig.4 for $u(t) = u_{an}(t) = [1 + \text{erf}(rt)]/2$.

Fig.6 shows the Sommerfeld precursor obtained in the simple asymptotic limit with $u(t) = u_{an}(t)$, $\omega_c = \omega_0$ and $r/\omega_0 = 1$. Compared to the precursor obtained with the canonical incident field $\sin(\omega_c t) u_H(t)$ (see Fig.3), the maximum occurs at a time more than 20 times larger and its amplitude is dramatically smaller (2.4×10^{-6} instead of 3.25×10^{-3}). Quite generally, the amplitude of the precursor rapidly decreases with the propagation distance and this phenomenon becomes really catastrophic when the rise of the amplitude of the incident field is ideally smooth [16]. Conversely, the propagation distance being given, extremely short rise times T_r of $u_{an}(t)$ are required to obtain precursors whose amplitude is not negligible, say larger than 10^{-8} . Using our empirical law $\rho \propto \exp(\sqrt{\gamma\xi}/r)$, we find that, in the Brillouin conditions, it would be necessary that $T_r < T_0/100$, where $T_0 = 2\pi/\omega_0$ is the resonance period. A less delirious value is obtained with the realistic (causal) incident field $u_3(t) \sin(\omega_c t)$. Using our analytical results, we find then the condition $T_r < T_0/14$ (confirmed by numerical simulations), which is less severe but seems impossible to satisfy in a real experiment. It appears thus that the observation of the Sommerfeld precursor in the conditions considered by Brillouin is quite improbable. As shown in the following sections, the constraints for the observation of the Brillouin precursor in the same conditions are much less severe.

IV. BRILLOUIN PRECURSOR IN THE STRICT ASYMPTOTIC LIMIT

In the limit considered now $\omega^2 \leq \omega_-^2 \ll \omega_0^2$ and $H_B(z, \omega)$ is conveniently developed under the form

$$H_B(z, \omega) = \exp \left(\sum_{n=1}^{\infty} \frac{(-i\omega)^n}{n!} k_n(z) \right). \quad (24)$$

Here $k_n(z)$ are the so-called cumulants, generally introduced in probability theory [29], but also quite useful to study deterministic signals [32, 33]. The cumulants $k_1(z)$, $k_2(z)$ and $k_3(z)$ have remarkable properties. $k_1(z)$ and $k_2^{1/2}(z)$ respectively are the center-of-mass and the root-mean-square duration of the impulse response $h_B(z, t)$, inverse Fourier transform of $H_B(z, \omega)$, whereas $\kappa(z) = k_3(z)/k_2^{3/2}(z)$ is its normalized asymmetry or skewness [29]. From Eqs.(3,4), we easily get $k_1 = t_B$ (as expected), $k_2 = 4\gamma/(3b^3)$, $k_3 = -2/(\eta b^3)$ and $\kappa = -(1/4\eta)(3b/\gamma)^{3/2}$, where t_B , b and η are defined by Eqs.(8-10). When $z \rightarrow \infty$ (strict asymptotic limit), $\kappa \propto b^{3/2} \propto z^{-1/2} \rightarrow 0$ and the expansion of Eq.(24) may be limited to the term $n = 2$. Taking a new origin of time at $t = t_B$, the transfer function then reads as

$$H_B(z, \omega) \approx \exp \left(-\frac{\omega^2}{4\beta^2} \right), \quad (25)$$

where $\beta = \sqrt{3b^3/8\gamma} \propto 1/\sqrt{z}$ is very small compared to ω_0 . This Gaussian form is that of the normal distribution derived by means of the central limit theorem in probability theory. This theorem can also be used to obtain an approximate evaluation of the convolution of n deterministic functions [27]. It can be applied to our case by splitting the medium into n cascaded sections, $h_B(z, t)$ being the convolution of the impulses responses of each section. By calculating the inverse Fourier transform of $H_B(z, \omega)$, we get

$$h_B(z, t) = \frac{\beta}{\sqrt{\pi}} \exp(-\beta^2 t'^2), \quad (26)$$

where $t' = t - t_B$. The impulse response has a duration (amplitude) proportional (inversely proportional) to \sqrt{z} , with an area constantly equal to 1 (in agreement with the general analysis). We remark that the approximation leading to Eq.(25) and Eq.(26), valid in the strict asymptotic limit, amounts to neglect the effects of the group delay dispersion, the formation of the Brillouin precursor being then governed by the frequency dependence of the medium attenuation (dominant-attenuation limit). In the Brillouin conditions, $|\kappa| = 5.2\%$ and, as shown later, the impulse response is perfectly approximated by Eq.(26). The approximation remains very good for a propagation distance 10 times smaller for which $|\kappa|$ is $\sqrt{10}$ times larger ($|\kappa| = 16\%$). Due to the negative value of κ , the trailing edge of $h_B(z, t)$ is then very

slightly steeper than its leading edge and its maximum is insignificantly delayed with respect to t_B .

The Gaussian forms of Eq.(25) and Eq.(26) are not specific to the Lorentz medium but have some generality [34]. They hold for the Debye medium [35], for some random media [36] and, more generally, whenever the transfer function of the medium can be expanded in cumulants and the propagation distance is such that $|\kappa| \ll 1$. Stoudt et al. [35] showed in particular that the results of their experiments on water (Debye medium) at decimetric wavelengths can be numerically reproduced by neglecting the group delay dispersion, as it has been made to obtain Eq.(25). See also [37–40]. Using a purely temporal approach, Karlsson and Ritke [12] early remarked that the impulse response of the Debye medium is very close to a normalized Gaussian. This property is obviously a consequence of the previous analysis. The complex refractive index now reads as $\tilde{n}(\omega) = [1 + (n_0^2 - 1)/(1 + i\omega\tau)]^{1/2}$ where n_0 is the refractive index for $\omega \rightarrow 0$ and τ is the relaxation time for the orientation of the polar molecules [38]. Including $\tilde{n}(\omega)$ in Eq.(3) and following the procedure used for the Lorentz medium, we easily get $\beta = [2(n_0^2 - 1)\tau z/cn_0]^{1/2}$ and, taking into account that $n_0^2 \gg 1$, $\kappa \approx 2.25\sqrt{c\tau/n_0z}$. Note that β and κ depends on z as $1/\sqrt{z}$ (as in the Lorentz medium). The normalized Gaussian of Eq.(26) will thus also be obtained for sufficient propagation distances. Using the parameters of water [38], namely $n_0 = \sqrt{79}$ and $\tau = 8.5 \times 10^{-12}$ s, we find that the skewness of 5.2%, obtained in a Lorentz medium for a propagation distance larger by more of four orders of magnitude than the optical wavelengths considered, is now attained for a propagation distance $z \approx 0.55$ m comparable to the wavelengths involved in the experiments reported in [35]. Despite strongly different scales, Brillouin precursors in the Lorentz medium in the strict asymptotic limit and in the Debye medium pertain to the same physics, namely that of the dominant-attenuation limit, and will be described by the same laws. On the other hand, the Debye medium is fully opaque at high frequency and Sommerfeld precursors cannot be generated in this medium.

The Brillouin precursor generated by an arbitrary incident field $e(0, t)$ is obtained by convoluting the latter with $h_B(z, t)$ or by multiplying its Fourier transform $E(0, \omega)$ by $H_B(z, \omega)$ and determining the inverse Fourier transform of the product. We consider first the case where $e(0, t)$ is rapidly varying compared to $h_B(z, t)$. This requires in particular that $\omega_c \gg \beta$. Compared to $E(0, \omega)$, $H_B(z, \omega)$ then appears as a narrow peak centered on $\omega = 0$ and, provided that $E(0, 0) \neq 0$, $E_B(z, \omega) \approx E(0, 0) H_B(z, \omega)$. Remembering that $E(0, 0)$ is the algebraic area \mathcal{A} of the incident field (see Sec. II), we finally get:

$$e_B(z, t) \approx \mathcal{A} h_B(z, t) = \frac{\mathcal{A}\beta}{\sqrt{\pi}} \exp(-\beta^2 t'^2). \quad (27)$$

For the canonical incident field $\sin(\omega_c t) u_H(t)$, $E(0,0) = 1/\omega_c$ and the precursor has an amplitude $a_B = \beta/(\omega_c\sqrt{\pi})$ inversely proportional to ω_c (no matter its value provided that $\omega_c \gg \beta$) and to \sqrt{z} . Note that the law $a_B \propto 1/\sqrt{z}$, sometimes considered as general, is only valid in the strict asymptotic limit considered here (for which $|\kappa| \ll 1$).

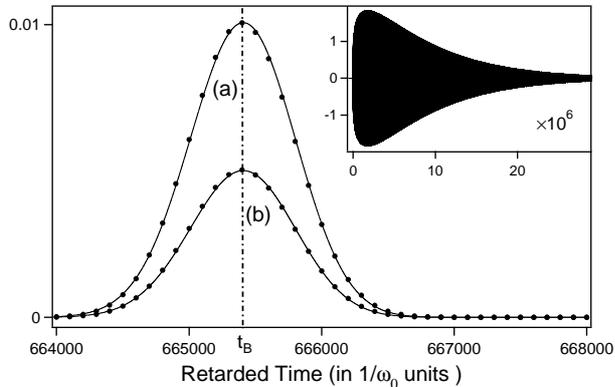


Figure 7: Brillouin precursor obtained in the Brillouin conditions, namely for $\omega_c = 0.1$, $\omega_p = 1.11$, $\gamma = 0.0707$ and $\xi = 8.31 \times 10^5$ (in ω_0 units). For these parameters, $\omega_0 t_B \approx 6.654 \times 10^5$ and $\beta \approx 1.78 \times 10^{-3} \omega_0 = 1.78 \times 10^{-2} \omega_c$. Solid lines (bullets \bullet) are the exact numerical solutions (the analytic solutions). Curve (a) is the precursor obtained with the canonical incident field $\sin(\omega_c t) u_H(t)$. The precursor of curve (b) is generated by the incident field $e(0,t) = \sin(\omega_c t) [1 + \text{erf}(rt)]/2$ for $r = \omega_c/2\sqrt{2}$. Inset: Sommerfeld precursor obtained in the conditions of curve (a). It fully vanishes in the conditions of curve (b).

Fig.7 shows that the precursor obtained in *all* the Brillouin conditions [curve (a)] is perfectly fitted by the Gaussian form of Eq.(27). We incidentally note that, for the carrier frequency retained by Brillouin ($\omega_c = \omega_0/10$), the medium is fully opaque at this frequency [$\alpha(\omega_c)z \approx 800$], in contradiction with his artist's view showing a "main field" (at ω_c) larger than the precursors. On the other hand, the condition $\omega_c \gg \beta$ is well satisfied. The inset in Fig.7 shows the Sommerfeld precursor obtained in the same conditions. As already mentioned, it is perfectly fitted by the analytical expression of Eq.(21). Note however that its amplitude is about four orders of magnitude smaller than that of the Brillouin precursor. Eq.(27) also holds when the envelope of the incident field rises in a finite time provided that the rate r , as ω_c , is large compared to β . Curve (b) of Fig.7 shows the Brillouin precursor generated by the incident field $e(z,t) = \sin(\omega_c t) u_{an}(t)$ already considered in Sec. III. We have then $E(0,0) = (1/\omega_c) \exp(-\omega_c^2/4r^2)$ and the area of the incident pulse, equal to $1/\omega_c$ for $r \rightarrow \infty$, falls to $1/2\omega_c$ for $r = \omega_c/2\sqrt{\ln(2)}$ ($r \approx 0.60\omega_c$). As expected, the Brillouin precursor is identical to the previous one with amplitude reduced by half. The same result is obtained when $u_{an}(t)$ is replaced by $u_2(t)$ or $u_3(t)$, also

introduced in Sec. III. The area of the incident field and thus the amplitude of the Brillouin precursor are then reduced by half for $r \approx 0.72\omega_c$ and for $r \approx 0.905\omega_c$, respectively. In every case, the corresponding Sommerfeld precursor completely vanishes.

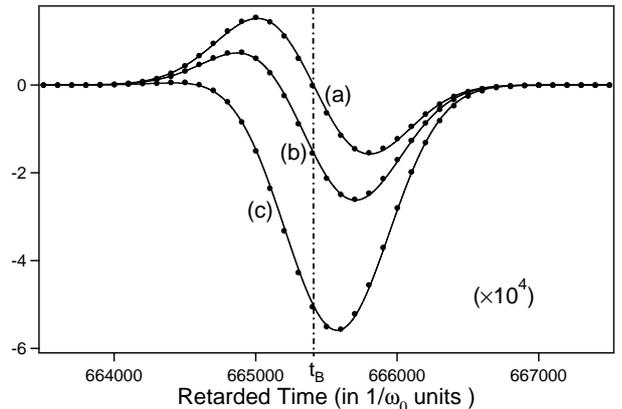


Figure 8: Brillouin precursor obtained with the incident fields (a) $\cos(\omega_c t) u_H(t)$, (b) and (c) $(1 - e^{-rt}) \cos(\omega_c t) u_H(t)$ with $r = 65\omega_c$ and $r = 20\omega_c$ respectively (solid lines). Other parameters as in Fig.7. The bullets \bullet correspond to the analytical solutions given by Eq.(28) or by the combination of this equation with Eq.(27).

Even if $\omega_c, r \gg \beta$, Eq.(27) obviously fails when $\mathcal{A} = E(0,0) = 0$. This occurs in particular in the extreme case where the incident field is instantaneously turned on, with $e(0,t) = \cos(\omega_c t) u_H(t)$. It is then necessary to consider the next term in the expansion of $E(0,\omega)$ in powers of $i\omega$. We get in this case $E(0,\omega) \approx i\omega/\omega_c^2$ and $E_B(z,\omega) \approx i\omega H_B(z,\omega)/\omega_c^2$. Using the correspondence $i\omega \leftrightarrow d/dt$ between frequency and time descriptions [27] and denoting by a dot the time derivative, we finally get:

$$e_B(z,t) \approx \frac{1}{\omega_c^2} \dot{h}_B(z,\omega t) = -\frac{2\beta^2}{\omega_c^2 \sqrt{\pi}} \beta t' \exp(-\beta^2 t'^2). \quad (28)$$

This result may also be seen as a consequence of the property mentioned in Sec. III that the field $e(z,t)$ transmitted for $e(0,t) = \cos(\omega_c t) u_H(t)$ is the derivative of that transmitted for $e(0,t) = \sin(\omega_c t) u_H(t)$, divided by ω_c . Insofar as Sommerfeld and Brillouin precursors are well separated, this property holds for both precursors. As shown Fig.8 [curve (a)], the analytical expression of Eq.(28) perfectly fits the exact numerical results obtained by FFT. The precursor is a Gaussian derivative with a peak amplitude $a_B = [2/(\pi e)]^{1/2} (\beta/\omega_c)^2$, smaller than that attained with the canonical incident field by a factor $\omega_c \sqrt{e}/(\beta\sqrt{2})$ (≈ 65 in *all* the Brillouin conditions) and decreasing much more rapidly with the propagation distance (as $1/z$ instead of as $1/\sqrt{z}$). We however remark that the case considered here is quasi pathologic and that the precursor so obtained is not robust. Indeed it suffices that the incident field suffers a short rise time to retrieve

a precursor mainly governed by the area law of Eq.(27). To illustrate this point, we have again considered an incident field of the form $(1 - e^{-rt}) \cos(\omega_c t) u_H(t)$ that tends to $\cos(\omega_c t) u_H(t)$ for $r \rightarrow \infty$. For $r \gg \omega_c$ (very short rise time), $E(0, \omega) \approx -1/r + i\omega/\omega_c^2$. The incident field has gained a (negative) area $\mathcal{A} = -1/r$. The precursor is then the sum of two contributions, respectively given by Eq.(27) with $\mathcal{A} = -1/r$ and by Eq.(28). Curve (b) of Fig.8 shows the result obtained when the two contributions have the same amplitude, that is when $r/\omega_c = \omega_c \sqrt{e}/(\beta\sqrt{2}) \approx 65$. When r decreases by remaining large compared to ω_c , the Gaussian part of the precursor rapidly prevails on the Gaussian-derivative part and, as shows Fig.8 [curve (c)], the precursor becomes nearly Gaussian (downwards) for r as large as $20\omega_c$. The cancellation of the area is not specific to the pathologic incident field $\cos(\omega_c t) u_H(t)$. It also occurs, e.g., with the incident field $\sin(\omega_c t) u_3(t)$ already considered. Its area cancels when $s = \omega_c/2r$ is such that $s\sqrt{\pi}\text{erfi}(s) e^{-s^2} = 1$, that is for $r = 0.541 \omega_c$ [erfi(s) designates here the imaginary error function]. Numerical simulations show that it suffices that r deviates by a few percents from this value to retrieve a dominantly Gaussian precursor. This confirms that the Gaussian-derivative form is not robust in the sense that it does not resist to a slight change of parameters (at least in the strict asymptotic limit).

The previous results are valid for the Lorentz medium in the strict asymptotic limit also as in the Debye medium provided that $\omega_c \gg \beta$. Fortunately enough, the simplicity of the Gaussian impulse response enables us to obtain exact expressions of the transmitted field for arbitrary values of the ratio ω_c/β . This occurs in the Lorentz medium when ω_c resides below the opacity region and *direct* observations of the field transmitted in such conditions have been performed by Stoudt *et al.* in a Debye medium [35]. The transmitted field $e(z, t)$ is calculated directly in the time domain by convoluting $h_B(z, t)$ given Eq.(26) with the incident field. For the canonical incident field, the convolution product can be written as:

$$e(z, t) = \frac{\beta}{\sqrt{\pi}} \int_{-\infty}^{t'} e^{-\beta^2 \theta^2} \sin[\omega_c (t' - \theta)] d\theta. \quad (29)$$

After some simple transformations, we finally get

$$e(z, t) = \frac{1}{2} e^{-\omega_c^2/4\beta^2} \Im \left\{ \left[1 + \text{erf} \left(\beta t' + \frac{i\omega}{2\beta} \right) \right] e^{i\omega_c t'} \right\}, \quad (30)$$

where $e^{-\omega_c^2/4\beta^2} \approx e^{-\alpha(\omega_c)z}$ and, as previously, $t' = t - t_B$. For $t' \rightarrow \infty$, $e(z, t)$ tends to $e^{-\omega_c^2/4\beta^2} \sin(\omega_c t')$ which is nothing but that the steady state or main field, not negligible when ω_c and β are comparable. If we take t_B ($1/\beta$) as time origin (time unit), the transmitted field only depends on the ratio ω_c/β , *regardless of the particular system considered*. When $\omega_c \gg \beta$, it tends to $\beta/(\omega_c \sqrt{\pi}) \exp(-\beta^2 t'^2)$ in agreement with Eq.(27), the

main field being then negligible. When $\omega_c \geq 4\beta$, Eq.(30) is well approximated by the expression:

$$e(z, t) \approx \frac{1 + \text{erf}(\beta t')}{2} \sin(\omega_c t') e^{-\alpha(\omega_c)z} + \frac{\beta'}{\omega_c \sqrt{\pi}} e^{-\beta'^2 t'^2}, \quad (31)$$

where $\beta' = \beta(1 + 2\beta^2/\omega_c^2) \rightarrow \beta$ for $\omega_c \gg \beta$. The first (second) term of Eq.(31) obviously corresponds to the main field (the Brillouin precursor). Figure 9 shows the transmitted field as a function of $\beta t' = \beta(t - t_B)$ for $\omega_c \approx 3.84\beta$ and $\omega_c \approx 7.67\beta$ (inset). In the study on water (Debye medium) at decimetric wavelengths [35], these values are obtained with $\omega_c = 2\pi \times 10^9 \text{s}^{-1}$, for $z = 0.75 \text{m}$ and $z = 3 \text{m}$ respectively. As expected Eq.(30) perfectly fits the exact numerical result in both cases. Eq.(31) provides a good approximation for $\omega_c \approx 3.84\beta$, excellent for $\omega_c \approx 7.67\beta$. In the latter case, the Brillouin precursor prevails over the main field whose relative amplitude is negligible. The signals shown Fig.9 are in good agreement with those directly observed in the experiments reported in [35].

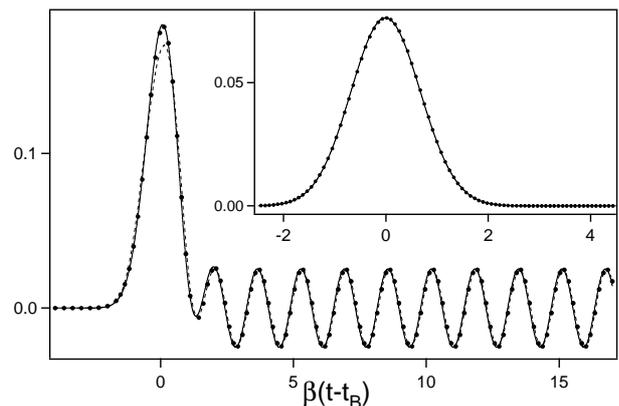


Figure 9: Brillouin precursor and main field obtained for $\omega_c \approx 3.84\beta$ as a function of $\beta(t - t_B)$. The solid line, the bullets \bullet and the dashed line are respectively the exact numerical solution, the analytical solution given Eq.(30) and its approximate form given Eq.(31). Inset: Brillouin precursor obtained for $\omega_c \approx 7.67\beta$. The two analytical solutions are undistinguishable in this case and the amplitude of the main field is negligible.

V. EXTENDED EXPRESSION OF THE BRILLOUIN PRECURSOR

We come back in this section to the Brillouin precursor in the Lorentz medium. Numerical simulations show that the solutions obtained in the strict asymptotic or dominant-attenuation limit continue to provide good (not too bad) approximations of the exact solutions when

the propagation distance is 10 times (100 times) shorter than that considered by Brillouin [26], though the skewness κ then rises up to 16% (52%). For shorter distances, it is obviously necessary to take into account the effects of the group-delay dispersion neglected in the strict asymptotic approximation. The transfer function then reads as

$$H(z, \omega) \approx H_B(z, \omega) \approx \exp \left[-i\omega t_B - \frac{i}{3\eta b^3} (\omega^3 - 2i\eta\gamma\omega^2) \right], \quad (32)$$

where t_B , b and η are defined by Eqs.(8-10), with $2\gamma/(3b^3) = 1/4\beta^2$. Remarking that $(\omega^3 - 2i\eta\gamma\omega^2)$ is the beginning of $(\omega - 2i\eta\gamma/3)^3$ and taking a new origin of time at $t_B + 4\eta\gamma^2/9b^3$, we get:

$$H_B(z, \omega) \approx \exp \left[-\frac{i}{3\eta b^3} \left(\omega - \frac{2}{3}i\eta\gamma \right)^3 - \frac{\eta^2}{3} \left(\frac{2\gamma}{3b} \right)^3 \right]. \quad (33)$$

By means of an inverse Fourier transform, we finally find:

$$h_B(z, t) \approx B \text{Ai} \left(-\eta^{1/3} b t'' \right) \exp(-2\eta\gamma t''/3). \quad (34)$$

Here $B = \eta^{1/3} b \exp \left[-(\eta^2/3) (2\gamma/3b)^3 \right]$, $t'' = t - t_B - 4\eta\gamma^2/9b^3$ and $\text{Ai}(s)$ designates the Airy function. The range of validity of Eq.(34) can be roughly estimated by means of a strategy similar to that used for the Sommerfeld precursor. By taking account of the cumulants k_4 (correction of the attenuation) and k_5 (correction of the dispersion), the transfer function associated with the Brillouin precursor approximately reads as $H_B(z, \omega) \times (1 - a_4\omega^4 - ia_5\omega^5)$ where $a_4 = -k_4/24 > 0$ and $a_5 = k_5/120 > 0$. $H_B(z, \omega)$ will be a good approximation if $a_4\omega^4$ and $a_5\omega^5$ are small compared to 1 (say $\leq 1/\sqrt{10}$). For sake of simplicity, we take for the ratios ω_p/ω_0 and γ/ω_0 the values retained by Brillouin, representative of a dense Lorentz medium with moderate damping. We get then $\eta \approx 1.018 \approx 1$. Besides, in a cavalier manner, we assimilate ω to the instantaneous frequency derived from the asymptotic form $\text{Ai}(-s) \approx \pi^{-1/2} s^{-1/4} \sin(2s^{3/2}/3 + \pi/4)$ that provides a good approximation of $\text{Ai}(-s)$ when $s > 1$. We get so $\omega \approx \sqrt{b^3 t''}$. With all these hypotheses, we finally find that the corrections due to the cumulants k_4 and k_5 will be small if $\omega t'' \leq 2(\omega_0/b)^{3/2}$ and $\omega t'' \leq (\omega_0/b)^{9/5}$, respectively. Despite the roughness of the procedure leading to these conditions, it will appear below that they are realistic and even too severe.

When $h_B(z, t)$ is slowly varying compared to $e(0, t)$, the Brillouin precursor generated by the canonical incident field $\sin(\omega_c t) u_H(t)$ takes again the simple form $e_B(z, t) = \mathcal{A} h_B(z, t)$, that is

$$e_B(z, t) \approx \frac{b}{\omega_c} \text{Ai} \left(-\eta^{1/3} b t'' \right) \exp(-2\eta\gamma t''/3). \quad (35)$$

It is assumed by writing Eq.(35) that the instantaneous frequency $\sqrt{b^3 t''}$ is small compared to ω_c (say $\sqrt{b^3 t''} \leq \omega_c/\sqrt{10}$) and that the conditions of validity of $h_B(z, t)$ are met. All these restrictions are summarized by the inequality

$$\omega_0 t'' \leq \min \left[2(\omega_0/b)^{3/2}, (\omega_0/b)^{9/5}, \omega_0 \omega_c^2 / 10b^3 \right]. \quad (36)$$

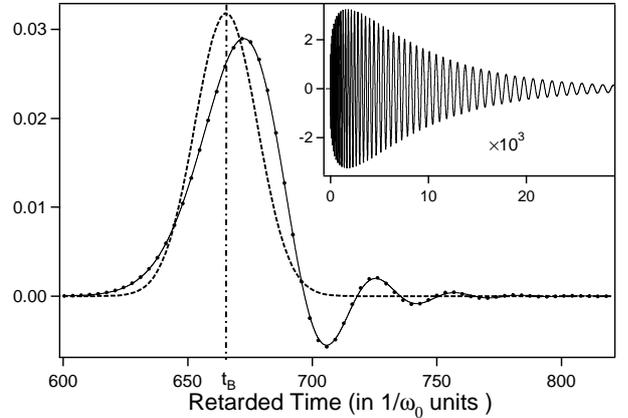


Figure 10: Brillouin precursor obtained in the simple asymptotic limit with the canonical incident field $\sin(\omega_c t) u_H(t)$. Parameters (in ω_0 units): $\omega_c = 1$, $\omega_p = 1.11$, $\gamma = 0.0707$ and $\xi = 831$, leading to $\omega_0 t_B \approx 665.4$, $b \approx 8.44 \times 10^{-2} \omega_0$ and $\beta \approx 5.64 \times 10^{-2} \omega_0$. The solid line, the bullets \bullet and the dashed line are respectively the exact numerical solution, the analytical solution given Eq.(35) and the Gaussian given Eq.(27). The conditions are those of Fig.3. The corresponding Sommerfeld precursor is given in inset for reference.

Fig.10 shows the Brillouin precursor obtained in the simple asymptotic limit considered in the study of the Sommerfeld precursor (Fig.3). The inequality of Eq.(36) then leads to $\omega_0 t \leq \min(750, 760, 840)$. Insofar as the amplitude of the precursor is negligible for $\omega_0 t = 750$, the analytical expression of Eq.(35) perfectly fits the exact numerical result. The Gaussian that would be obtained in the strict asymptotic or dominant-attenuation limit [Eq.(27)] is also given Fig.10 for reference (dashed line). We see that it significantly departs from the exact result but continues to give a satisfactory order of magnitude of the precursor amplitude.

To check the limit of validity of Eq.(35), we have considered what happens when the propagation distance is 10 times shorter than the previous one (Fig.11). The width of opacity region is then of the order of ω_0 and far from being large in the sense given to this expression in the present paper. Surprisingly enough, Eq.(35) provides a not too bad approximation of the exact result and the entirety of the first oscillation of the Brillouin precursor is even very well reproduced. For these parameters, the inequality of Eq.(36) leads to $\omega_0 t \leq \min [92, 89, 83.5]$. We see Fig.11 that $\omega_0 t \approx 85$ actually limits the time domain where the Brillouin precursor is well approximated by

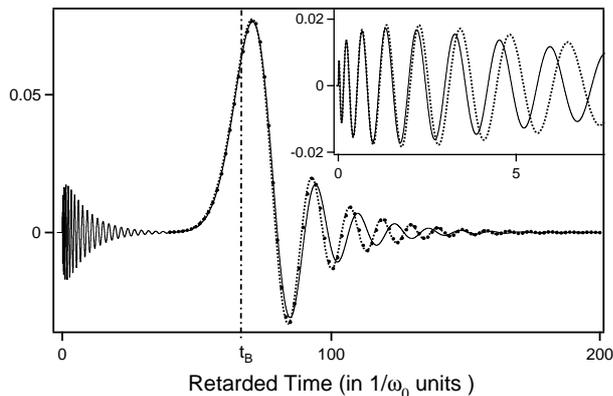


Figure 11: Comparison of the Brillouin precursor obtained outside the asymptotic limit (solid line) with the analytical forms given Eq.(35) (●) and Eq.(37) (dashed line). Parameters (in ω_0 units): $\omega_c = 1$, $\omega_p = 1.11$, $\gamma = 0.0707$ and $\xi = 83.1$, leading to $\omega_0 t_B \approx 66.54$, $b \approx 0.182\omega_0$ and $\beta \approx 0.178\omega_0$. Inset: corresponding Sommerfeld precursor (solid line) compared to the analytic form given Eq.(21) (dashed line).

Eq.(35). The corresponding Sommerfeld precursor (inset) is itself well reproduced by Eq.(21) in the region $\omega_0 t \leq \sqrt[3]{\xi/10\omega_0} \approx 2$ [see Eq.(14)], which comprises in particular the maximum of its envelope.

The expression of the Brillouin precursor given by Eq.(35) obviously includes as particular case the Gaussian obtained in the dominant-attenuation limit. In fact, retrieving the Gaussian precursor directly from Eq.(35) requires long and tedious calculations and this probably explains why the Gaussian solution has been generally overlooked. An other particular form of Eq.(35), also of special importance, is that obtained when the damping is very small, so that the formation of the Brillouin precursor is mainly governed by the group delay dispersion (dominant-dispersion limit). This requires in particular that $\gamma \ll b$. We then get $t'' \approx t - t_B$, $B \approx b$ and

$$e_B(z, t) \approx \frac{b}{\omega_c} Ai[-b(t - t_B)] \exp\left[-\frac{2}{3}\gamma(t - t_B)\right]. \quad (37)$$

Except for the exponential damping term, this result was established by Brillouin himself by means of the method of stationary phase [4, 41]. In the conditions of Fig.11, $\gamma/b \approx 0.39$ and that suffices for the signal given by Eq.(37) to be very close to that obtained with the more general Eq.(35). When the group-delay dispersion is fully dominant (say when $\gamma/b < 1/100$), the precursor has a well marked oscillatory behavior with a very weak damping and its maximum practically coincides with the first maximum of $Ai[-b(t - t_B)]$, attained for $t - t_B \approx 1, 02/b$ [16]. The corresponding amplitude is $a_B \approx 0.536(b/\omega_c)$ that scales as $z^{-1/3}$, instead of as $z^{-1/2}$ in the strict or dominant-attenuation limit.

Fig.12 shows an example of Brillouin precursor obtained in such conditions ($\gamma/b \approx 3.9 \times 10^{-3}$). It is

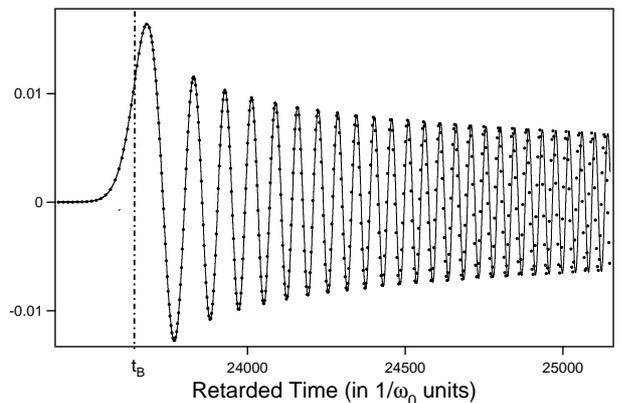


Figure 12: Brillouin precursor in the dispersion dominant limit. The solid line (bullets ●) is the exact numerical solution (the analytical solution). Parameters (in ω_0 units): $\omega_c = 0.836$, $\omega_p = 1.11$, $\gamma = 10^{-4}$ and $\xi = 2.95 \times 10^4$, leading to $\omega_0 t_B \approx 2.3641 \times 10^4$, $b \approx 0.0257\omega_0$ and $\beta \approx 0.252\omega_0$.

worth emphasizing that, since $b \propto z^{-1/3}$, the condition $\gamma/b \ll 1$ requires that the propagation distance is not too large. On the other hand, it should be large enough for the inequality of Eq.(36) to be satisfied for a time larger or at least comparable to the half-maximum duration of the precursor. In fact, the most severe restriction originates in the condition $\omega_0(t - t_B) \leq (\omega_0/b)^{9/5}$ associated with the dispersion correction. When $\gamma \ll b$, we easily deduce from the asymptotic form of the Airy function that the half-maximum of the precursor will be attained for $\omega_0(t - t_B) \approx 20(\omega_0/b)$. The precursor will thus be well reproduced by the expression $e_B(z, t) \approx (b/\omega_c) Ai[-b(t - t_B)]$ beyond its half-maximum amplitude if $\gamma \ll b$ and if $(\omega_0/b)^{4/5} > 20$, that is if $b/\omega_0 < 0.024$. The latter condition is approximately met Fig.12 for which $b/\omega_0 = 0.026$. As expected, the maximum amplitude of the precursor is $a_B \approx 0.536(b/\omega_c) \approx 0.0165$, with $\exp(-2\gamma(t - t_B)/3) \approx 0.997$ at the corresponding time.

VI. PROPAGATION OF PULSES WITH A SQUARE OR GAUSSIAN ENVELOPE

Up to now, in the spirit of the pioneering work of Sommerfeld and Brillouin, we have considered incident fields of infinite duration. In actual or even numerical experiments, this duration is naturally finite. As a matter of fact the simulations made to corroborate our previous analytical calculations were made by using a square-wave modulation (eventually suitably filtered) and choosing a square duration long enough to avoid that the precursors generated by the rise and the fall of the square overlap. On the contrary, we consider in this section the case where the duration of the incident field is small compared to the time-delay t_B separating the Brillouin precursor

from the Sommerfeld precursor and does not exceed few periods of the carrier. We will restrict the analysis to the Brillouin precursor. Indeed the Sommerfeld precursor, if it exists, is generally much smaller and will be often filtered out by rise-time effects, to which the Brillouin precursor is much less sensitive.

We consider first a square-modulated incident field $[u_H(t) - u_H(t - T)] \sin(\omega_c t)$. Of particular interest is the case where the square duration is an integer n of half-periods of the carrier, that is $T = nT_c/2 = n\pi/\omega_c$. The incident field can then be rewritten as $e(0, t) = u_H(t) \sin(\omega_c t) - (-1)^n u_H(t - T) \sin[\omega_c(t - T)]$ and the transmitted field reads as $e'(z, t) = e(z, t) - (-1)^n e(z, t - T)$ where $e(z, t)$ designates the transmitted field when only the incident field $u_H(t) \sin(\omega_c t)$ is on. This equation applies to the whole field and in particular to the Brillouin precursor to yield:

$$e'_B(z, t) = e_B(z, t) - (-1)^n e_B(z, t - T), \quad (38)$$

where $e_B(z, t)$ is given by Eq.(27) or Eq.(35), depending on the system and the parameters considered. The two components of e'_B are of opposite (same) sign when n is even (odd) and are well separated when it is large enough, so that T significantly exceeds the duration of the elementary precursor. On the other hand, $e_B(z, t)$ evolving slowly at the scale of T_c , the two components overlap and interfere if n is small. When $n = 2$ ($T = T_c$) as considered in [37, 42], the two components interfere nearly destructively to give a precursor $e'_B(z, t) \approx T_c \dot{e}_B(z, t - T_c/2)$. The case where n is odd and, in particular, where $n = 1$ ($T = T_c/2$) is much more favorable. Indeed the two precursors then interfere constructively to yield a precursor $e'_B(z, t) \approx 2e_B(z, t - T_c/4)$ whose amplitude is twice that obtained with a step modulation. This result is not really a surprise since the pulse area is itself twice that of $u_H(t) \sin(\omega_c t)$. On the contrary the pulse area equals zero when n is even. The previous results are illustrated Fig.13 that shows the Brillouin precursors obtained for $n = 1, 2$ for a Lorentz medium when attenuation and dispersion comparably contribute to the formation of the Brillouin precursor (simple asymptotic limit).

When the detection of the Brillouin precursor is not time-resolved an important parameter is the integrated "energy" $W_B(z) = \int_{-\infty}^{+\infty} |e'_B(z, t)|^2 dt$ [18, 43]. Thanks to the Parseval-Plancherel theorem [27], it can be written as

$$W_B(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |H_B(z, \omega)|^2 |E(0, \omega)|^2 d\omega. \quad (39)$$

In this expression all phases are eliminated and $|H_B(z, \omega)|^2$ is reduced to $\exp(-4\gamma\omega^2/3b^3) = \exp(-\omega^2/2\beta^2)$ in both strict and simple asymptotic cases. For $T = T_c/2$, $|H_B(z, \omega)E(0, \omega)|^2 \approx (4/\omega_c^2) \exp(-\omega^2/2\beta^2)$ and we get an energy

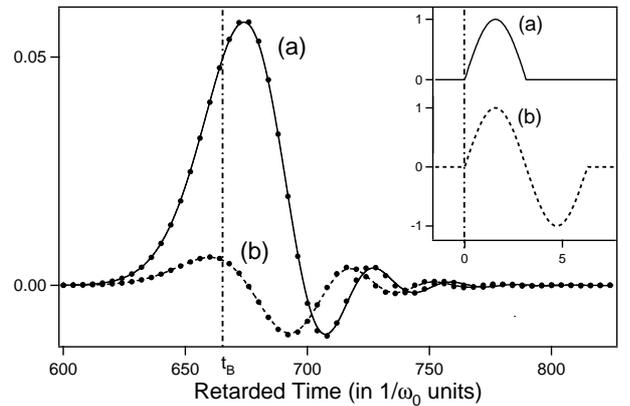


Figure 13: Comparison of the Brillouin precursors $e'_B(z, t)$ generated by an incident square-modulated field of duration (a) $T = T_c/2$ and (b) $T = T_c$. The parameters are those of Fig.10 (simple asymptotic limit). The solid and dashed lines are the exact numerical solutions, indiscernible from the analytical solutions given by Eq.(38). The bullets are the approximate solutions (a) $2e_B(z, t - T_c/4)$ and (b) $T_c \dot{e}_B(z, t - T_c/2)$. As expected the precursor amplitude for $T = T_c/2$ is twice that attained with a step-modulated field (see Fig.10) whereas that attained for $T = T_c$ is much smaller. Inset: corresponding incident fields.

$W_B(z) = 2^{3/2} \pi^{-1/2} \beta / \omega_c^2$ which slowly decays with the propagation distance (as $1/\sqrt{z}$). On the other hand, for $T = T_c$, $|H_B(z, \omega)E(0, \omega)|^2 \approx (2\pi\omega/\omega_c^2)^2 \exp(-\omega^2/2\beta^2)$ and $W_B(z) = (2\pi)^{3/2} \beta^3 / \omega_c^4$. As expected, $W_B(z)$ then decays very rapidly with the propagation distance (as $z^{-3/2}$). As already mentioned, the previous expressions of the energy are valid regardless of the relative contributions of the absorption and the dispersion to the formation of the precursor. For the Debye medium and the Lorentz medium in the dominant-attenuation limit, it is besides possible to derive from Eq.(38) and Eq.(27) explicit expressions of the maximum amplitude $a'_B(z)$ of the precursor $e'_B(z, t)$. We find that this amplitude, equal to $2\beta / (\omega_c \sqrt{\pi}) \approx 1.1 (\beta/\omega_c) \propto 1/\sqrt{z}$ when $T = T_c/2$, falls down to $2\sqrt{2\pi/e} (\beta/\omega_c)^2 \approx 3.0 (\beta/\omega_c)^2 \propto 1/z$ when $T = T_c$.

We examine now the case where the incident pulse has a Gaussian envelope and is eventually linearly chirped. The theoretical interest of such pulses is that it is possible to obtain (relatively) simple analytic expressions of the Brillouin precursor in both strict and simple asymptotic limits. Non-chirped incident fields of the form $e^{-t^2/T^2} \cos(\omega_c t)$ and $e^{-t^2/T^2} \sin(\omega_c t)$ have been respectively considered by Oughstun and Balictsis in [44] and by Ni and Alfano in [45]. When they are linearly chirped, it is convenient to consider them as the real and imaginary part of $\tilde{e}(0, t) = \exp(i\omega_c t - t^2/T^2 + i\chi^2 t^2)$ where χ^2 is the chirping parameter. The Fourier transform of $\tilde{e}(0, t)$ and of the corresponding transmitted field $\tilde{e}_B(z, t)$ simply read as $\tilde{E}(0, \omega) = \tilde{T} \sqrt{\pi} \exp[-(\omega - \omega_c)^2 \tilde{T}^2/4]$

and

$$\tilde{E}_B(z, \omega) = \tilde{\mathcal{A}} H_B(z, \omega) \exp\left(-\omega^2 \tilde{T}^2/4 + \omega \omega_c \tilde{T}^2/2\right). \quad (40)$$

In these expressions $\tilde{T} = T/\sqrt{1 - i\chi^2 T^2}$ and $\tilde{\mathcal{A}} = \tilde{T}\sqrt{\pi} \exp\left(-\omega_c^2 \tilde{T}^2/4\right)$ may be respectively seen as the (complex) duration and area of the pulse $\tilde{e}(0, t)$. In the strict asymptotic limit [see Eq.(25)], we get

$$\tilde{E}_B(z, \omega) = \tilde{\mathcal{A}} \exp\left[-\frac{\omega^2}{4} \left(\frac{1}{\beta^2} + \tilde{T}^2\right) + \omega \frac{\omega_c \tilde{T}^2}{2}\right], \quad (41)$$

and $\tilde{e}_B(z, t)$, inverse Fourier transform of $\tilde{E}_B(z, \omega)$, reads as

$$\tilde{e}_B(z, t) = \frac{\tilde{\mathcal{A}}\beta}{\sqrt{\pi(1 + \beta^2 \tilde{T}^2)}} \times \exp\left[-\frac{\beta^2(t' - i\omega_c \tilde{T}^2/2)}{1 + \beta^2 \tilde{T}^2}\right] \quad (42)$$

where $t' = t - t_B$. In the simple asymptotic limit (see Sec. V), Eq.(32) and Eq.(40) yield

$$\tilde{E}_B(z, \omega) = \tilde{\mathcal{A}} \exp\left[-i\omega\left(t_B + \frac{i\omega_c \tilde{T}^2}{2}\right)\right] \times \exp\left[-\omega^2\left(\frac{2\gamma}{3b^3} + \frac{\tilde{T}^2}{4}\right) - i\omega^3\left(\frac{1}{3\eta b^3}\right)\right] \quad (43)$$

This equation is easily transformed in an equation similar to Eq.(33). By this way, we find

$$\tilde{e}_B(z, t) = \tilde{\mathcal{A}} \tilde{B} \text{Ai}\left(-\eta^{1/3} b \tilde{t}\right) \exp\left(-\frac{2}{3}\eta \tilde{\gamma} \tilde{t}\right), \quad (44)$$

where $\tilde{\gamma} = \gamma + 3b^3 \tilde{T}^2/8$, $\tilde{B} = \eta^{1/3} b \exp\left[-(\eta^2/3)(2\tilde{\gamma}/3b)^3\right]$ and $\tilde{t} = t - t_B - 4\eta \tilde{\gamma}^2/9b^3 - i\omega_c \tilde{T}^2/2$. Finally the precursors generated by the incident fields $e^{-t^2/T^2} \cos(\omega_c t + \chi^2 t^2)$ and $e^{-t^2/T^2} \sin(\omega_c t + \chi^2 t^2)$ respectively read as $e_{cos}(z, t) = \Re[\tilde{e}_B(z, t)]$ and $e_{sin}(z, t) = \Im[\tilde{e}_B(z, t)]$. Eq.(44) and the derived expressions of $e_{cos}(z, t)$ and $e_{sin}(z, t)$ hold whatever the duration of the incident pulse may be. However, as shown below, the amplitude of the Brillouin precursor will be only significant when this duration does not exceed a few periods of the carrier. In the Fourier transform $H_B(z, \omega)\tilde{E}(0, \omega)$ of the transmitted field, $H_B(z, \omega)$ is then again much narrower than $\tilde{E}(0, \omega)$, which may be approximated by its first order expansion in powers of ω . We get so $\tilde{E}_B(z, \omega) \approx \tilde{\mathcal{A}}\left(1 + \omega \omega_c \tilde{T}^2/2\right) H_B(z, \omega)$ and finally

$$\tilde{e}_B(z, t) \approx \tilde{\mathcal{A}} \left[h_B(z, t) - \left(i\omega_c \tilde{T}^2/2\right) \dot{h}_B(z, t) \right]. \quad (45)$$

When there is no chirping, \tilde{T} and $\tilde{\mathcal{A}}$ are real, with $\tilde{T} = T$ and $\tilde{\mathcal{A}} = \mathcal{A} = T\sqrt{\pi} \exp\left[-\omega_c^2 T^2/4\right]$. Eq.(45) then leads to

$$e_{cos}(z, t) \approx \mathcal{A} h_B(z, t) = T\sqrt{\pi} \exp\left[-\frac{\omega_c^2 T^2}{4}\right] h_B(z, t) \quad (46)$$

$$e_{sin}(z, t) \approx -\frac{\mathcal{A}\omega_c T^2}{2} \dot{h}_B(z, t) = -\frac{\omega_c T^2}{2} \dot{e}_{cos}(z, t). \quad (47)$$

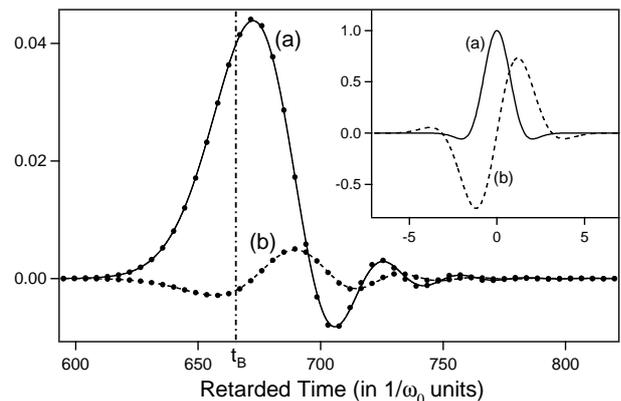


Figure 14: Brillouin precursors generated by the incident fields of Gaussian envelope (a) $e^{-(t/T)^2} \cos(\omega_c t)$ with $T = \sqrt{2}/\omega_c$ and (b) $e^{-(t/T)^2} \sin(\omega_c t)$ with $T = \sqrt{6}/\omega_c$. The parameters are those of Fig.10 (simple asymptotic limit). In both cases, the pulse duration has been chosen in order to maximize the precursor amplitude (see text). The solid and dashed lines are the exact numerical solutions whereas the bullets are the analytical solutions obtained in the short pulse approximation [Eq.(46) and Eq.(47)], indiscernible from those obtained without approximation [Eq.(44)]. Inset: corresponding incident fields. Numerical calculations shows that the Sommerfeld precursors generated by these fields have negligible amplitudes, respectively (a) 5.6×10^{-7} and (b) 1.15×10^{-10} .

As illustrated Fig.14, obtained in the simple asymptotic limit, these approximate analytic solutions perfectly fit the exact numerical solution. It is easily deduced from Eq.(46) [Eq.(47)] that the amplitude of the precursor $e_{cos}(z, t)$ [$e_{sin}(z, t)$] is maximum for a pulse duration $T = T_m = \sqrt{2}/\omega_c$ [$\sqrt{6}/\omega_c$]. Simple analytical expressions of the corresponding amplitude of the precursors can only be obtained in the strict asymptotic limit. By substituting the previous values of T_m in Eq.(46) and Eq.(47), we get $a_{Bm} \approx (2/e)^{1/2} (\beta/\omega_c) \propto 1/\sqrt{z}$ for $e_{cos}(z, t)$ and $a_{Bm} \approx (6\sqrt{3}/e^2) (\beta/\omega_c)^2 \propto 1/z$ for $e_{sin}(z, t)$. On the other hand, general expressions of the energy of the Brillouin precursors, valid both in the strict and simple asymptotic limit, can be obtained by the method already used in the case of a square modulation. We get so $W_B \approx (\pi/2)^{1/2} \left(\beta T^2 e^{-\omega_c^2 T^2/2}\right) \propto$

$1/\sqrt{z}$ for $e(0,t) = e^{-t^2/T^2} \cos(\omega_c t)$ and $W_B \approx (\pi/32)^{1/2} (\beta^3 \omega_c^2 T^6 e^{-\omega_c^2 T^2/2}) \propto z^{-3/2}$ for $e(0,t) = e^{-t^2/T^2} \sin(\omega_c t)$. The previous scaling laws in z are identical to those obtained with a square modulation. In fact they hold for every short and non-chirped incident pulse. In all cases, the transmitted pulse is indeed proportional to $h_B(z,t)$ when $E(0,0) = \mathcal{A} \neq 0$ or to $\dot{h}_B(z,t)$ when $\mathcal{A} = 0$, the proportionality coefficient depending only on the characteristics of the incident pulse and not on the propagation distance. For Gaussian incident pulses and, more generally, for smooth pulses, the amplitude and the energy of the Brillouin precursor rapidly decreases with the pulse duration. For example, the amplitude of the Brillouin precursor generated by the incident field $e^{-t^2/T^2} \cos(\omega_c t)$ is reduced by a factor exceeding 400 when T is taken four times larger than its optimum value $\sqrt{2}/\omega_c$ [see Eq.(46)]. This reduction of amplitude can however be compensated by using chirped pulses. When the pulse duration remains small enough, Eq.(45) holds and the Brillouin precursor generated by the incident field $e^{-t^2/T^2} \cos(\omega_c t + \chi^2 T^2)$ reads as

$$e_B(z,t) \approx h_B(z,t) \Re(\tilde{\mathcal{A}}) - \dot{h}_B(z,t) \Re(i\omega_c \tilde{\mathcal{A}} \tilde{T}^2/2). \quad (48)$$

Anticipating that the second term of this equation is small compared to the first one, we easily get the approximate expression

$$e_B(z,t) \approx \mathcal{A} h_B \left[z, t'' - \Re(i\omega_c \tilde{\mathcal{A}} \tilde{T}^2/2\mathcal{A}) \right], \quad (49)$$

where $\mathcal{A} = \Re(\tilde{\mathcal{A}})$ is the area of the incident pulse. This result differs from that obtained without chirping [see Eq.(46)] by a extra time-delay $\Re(i\omega_c \tilde{\mathcal{A}} \tilde{T}^2/2\mathcal{A})$ and, moreover, by the expression of the pulse area that reads as

$$\mathcal{A} = \Re \left\{ \frac{T\sqrt{\pi}}{\sqrt{1 - i\chi^2 T^2}} \exp \left[-\frac{\omega_c^2 T^2}{4(1 - i\chi^2 T^2)} \right] \right\}. \quad (50)$$

For a given duration T , the modulus of the (real) area \mathcal{A} may be considerably larger than that attained when the pulse is not chirped.

Fig.15 shows the result obtained for a pulse duration $T = 4\sqrt{2}/\omega_c$. In order to maximize the precursor amplitude, we have chosen for the chirping the value $\chi = \omega_c/4$ for which the function $\mathcal{A}(\chi)$ reaches its first extremum (negative minimum). For these parameters, $\Re(i\mathcal{A}\tilde{T}^2/2\mathcal{A})$ is also negative (time advancement). We remark that, despite the numerous approximations having led to Eq.(49), it provides a very good approximation of the exact result.

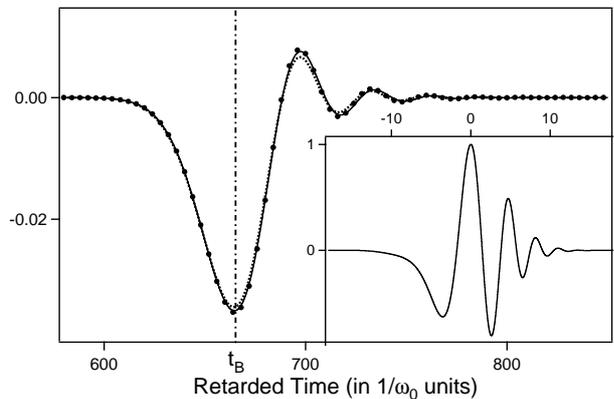


Figure 15: Brillouin precursor generated by a chirped incident pulse $e^{-(t/T)^2} \cos(\omega_c t + \chi^2 t^2)$, with $T = 4\sqrt{2}/\omega_c$ and $\chi = \omega_c/4$. The other parameters are as those of Fig.10 and Fig.14 (simple asymptotic limit). The solid line, the bullets and the dashed line respectively are the exact numerical solution, the analytic solution derived from Eq.(44) and the approximate analytic solution of Eq.(49), obtained in the short pulse approximation. Inset: incident pulse. The corresponding Sommerfeld precursor has fully negligible amplitude (9×10^{-11} !).

VII. SUMMARY AND DISCUSSION

We have studied the transmission of light pulses through a dense Lorentz medium whose thickness is such that the medium is opaque in a frequency range large compared to its resonance frequency ω_0 . As shown Fig.1, this condition is over-satisfied for the parameters considered by Brillouin [26], often referred to in the literature, and reasonably holds for much smaller thickness of the medium. When, as generally assumed in this paper, the frequency ω_c of the pulse carrier lies in the opacity region, the transmitted field is reduced to the Sommerfeld and Brillouin precursors which are well separated, in both frequency and time domains. Quite generally, we have shown that the very first beginning of the Sommerfeld precursor is identical to that of the incident field, delayed by the propagation time in vacuum, that its total area is equal to 0, and that the Brillouin precursor propagates with an area equal to that of the incident field. Explicit calculation of the precursors are made in two stages by calculating first the impulse response, intrinsic to the medium and independent of the incident field, and, second, by convoluting these two quantities.

When the opacity region is extremely large (strict asymptotic limit), we have obtained a simple analytic expression of the Sommerfeld precursors generated by causal incident fields (step or step-like modulation), evidencing that they are entirely determined by the order and the importance of the initial discontinuity of the incident field [Eq.(19)]. When it is the field itself that is discontinuous (0th order discontinuity), the amplitude of

the precursor is equal to the discontinuity and its shape is independent of the carrier frequency [Eq. 20]. For a 1st order discontinuity (discontinuity of the 1st derivative) as that of the canonical incident field $\sin(\omega_c t) u_H(t)$ considered by Sommerfeld and Brillouin and most authors, the amplitude (the shape) of the precursor is proportional to (independent of) the carrier frequency [Eq.(21)]. The precursors generated by incident fields with a higher order discontinuity, considered to determine the decay of the precursor amplitude with the discontinuity order, have the same properties [Eqs.(22, 23)]. We emphasize that *all these results are valid regardless of the carrier frequency* (below, inside or beyond the anomalous dispersion region) as long as it lies in the opacity region. In principle, they only apply in the strict asymptotic limit that is practically attained for the medium thickness considered by Brillouin. However we have found that they satisfactorily fit the exact numerical solutions for thicknesses that may be 1 000 times smaller (simple asymptotic limit). See Figs.(2-5).

The formation of both precursors is governed by interrelated effects of dispersion and attenuation but these effects are different on each precursor. Whereas the Sommerfeld precursor mainly originates from dispersion effects at high frequency, the attenuation may be of special importance in the formation of the Brillouin precursor. We have shown that in the strict asymptotic limit the Brillouin precursor is determined by the frequency-dependent attenuation of the medium at low frequency (dominant-attenuation limit) and does not depend on the group-delay dispersion. The impulse response $h_B(z, t)$ associated with the Brillouin precursor is then a normalized Gaussian of width (amplitude) proportional (inversely proportional) to \sqrt{z} (z : propagation distance). See Eq.(26). When ω_c lies in the opacity region, the precursor generated by an incident field whose area \mathcal{A} differs from zero is simply $\mathcal{A}h_B(z, t)$ and is thus also Gaussian. See Eq.(27) and Fig.7. For the canonical incident field $\sin(\omega_c t) u_H(t)$, $\mathcal{A} = 1/\omega_c$ and, consequently, *the amplitude of the precursor is inversely proportional to ω_c and to \sqrt{z}* . If $\mathcal{A} = 0$, the precursor is proportional to the time derivative of the Gaussian $h_B(z, t)$. Its amplitude is much lower, inversely proportional to ω_c^2 and to z . See Eq.(28) and Fig.8. We have also obtained an exact form of the transmitted field when ω_c lies below the opacity region [Eq.(30)]. We get then a field consisting in a superposition of the precursor with a residual main field at ω_c , in good agreement with the exact numerical result (Fig.9) and with the signals observed at decimetric wavelengths on a Debye medium [35].

In the simple asymptotic limit, the effects of attenuation and group-delay dispersion in the formation of the Brillouin precursor are of the same order of magnitude and it is necessary to take account of the latter to determine the impulse response $h_B(z, t)$ of the medium. We have obtained in this case a reasonably simple expression

of $h_B(z, t)$ in terms of damped Airy function [Eq.(34)], from which we have deduced that of the precursor generated by the canonical incident field [Eq.35]. Despite the rough approximations made to yield the latter result, it fits remarkably well the exact numerical result (Fig.10). It shows that the precursor amplitude, determined by the area of the incident pulse, remains inversely proportional to ω_c but there is no exact simple law fixing how it depends on the propagation distance. Such a law is found again in the limit where the formation of the precursor is dominated by the dispersion of the group delay (dominant-dispersion limit). See Eq.(37) and Fig.12. We retrieve in this case the precursor obtained by Brillouin himself by means of the stationary phase method. Its amplitude scales as $z^{-1/3}$. It should however be remarked that this scaling law only holds for a restricted class of media and in a restricted range of propagation distance. It does not hold in particular in the Brillouin conditions [4], whatever the propagation distance may be.

We have finally examined what becomes the Brillouin precursor when the incident field consists in a short pulse whose duration does not exceed a few periods of the carrier, with a square or Gaussian envelope. The Brillouin precursor generated by a square modulated pulse may obviously be seen as resulting from the precursors generated by the rise and the fall of the pulse. An essential point is that these two components may be of the same sign for suitable pulse duration [Eq.(38)]. When this duration is half the carrier period, they interfere constructively to give a precursor whose amplitude is twice that of the precursor generated by the canonical incident field. On the contrary the two components interfere nearly destructively when the pulse duration is equal to one carrier period, as considered in [37, 42]. The resulting precursor is then much smaller (Fig.13). In the case of pulses of Gaussian envelope, eventually chirped, we have obtained exact expressions of the Brillouin precursor in both the strict and simple asymptotic limits [Eq.(42) and Eq.(44)]. They take simple forms when the pulse is short enough, a condition necessary to yield precursor of significant amplitude, this amplitude being maximum with a pulse duration $\sqrt{2}/\omega_c$ ($\sqrt{6}/\omega_c$) for an amplitude-modulated cosine (sine). See Eq.(46), Eq.(47) and Fig.14. For larger pulse durations, the precursor amplitude dramatically decreases but this effect can be compensated by using chirped pulses. We have obtained in this case an approximate expression of the Brillouin precursor that fits fairly well the exact numerical solution. See Eqs.(48, 49) and Fig.15.

As already mentioned, simple scaling laws for the dependence in z of the amplitude of the Brillouin precursor are only obtained in the dominant-absorption and dominant-dispersion limits. On the other hand, insofar as the phases are then irrelevant, a scaling law can be established in the general case for the energy $W_B(z)$ of the precursor [Eq.(39)]. We have found that $W_B(z)$ decays

as $1/\sqrt{z}$ when the area of the incident pulse differs from zero but much more rapidly (as $z^{-3/2}$) in the contrary case. This result has been explicitly demonstrated in the case of a square-modulated incident field but it holds for a wide class of incident fields, including in particular the canonical one.

Most of our results are analytic and explicit, the numerical simulations being only made to verify their validity. They have been obtained by standard Laplace-Fourier procedures whose simplicity strongly contrasts with the complexity of the asymptotic methods generally used [8–11, 13–15, 31, 37, 40, 42, 44]. This complexity results in particular from the fact that, depending on the time, saddle points can go to infinity, coalesce or go near a pole singularity. The elementary saddle point method as used by Brillouin [3] then partially fails and the calculations require the use of more complex uniform asymptotic methods. This complexity is naturally retrieved in the results of these calculations where it is sometimes difficult, even impossible, to recognize the simple asymptotic laws established in the present paper. On the other hand, our results, contrary to the previous ones, only hold in a time domain the more restricted the shorter is the propagation distance. See Eq.(14) and Eq.(36). We however remark that they provide a surprisingly good description of the Sommerfeld and Brillouin precursors even when the width of the opacity region is of the order of the resonance frequency. See Fig.11.

Insofar as the Sommerfeld and the Brillouin precursors are respectively associated with the high and low frequency behaviors of the transfer function, the previous results apply to any system having transfer function of comparable form in these regions. Nine years after the publication of the 1960 book by Brillouin [5], experimental evidence of both precursors was realized by means of microwave waveguide systems mimicking the Lorentz medium [46]. However, as far as we know, the sole direct observation of precursors in a material medium is that of the Gaussian Brillouin precursor obtained by Stoudt *et al.* in their study of the transmission of water at decimetric wavelengths [35]. In the foreword of his 1960 book, Brillouin commented the 1914 papers by Sommerfeld and himself on precursors, writing that “the subject was a fascinating one, but it had, at that time, only academic importance” [5]. Seeing the continuous flow of theoretical papers on precursors, the fascination indisputably remains one century later. On the other hand, at least in the optical domain, the observation of precursors and, consequently, their applications continue to encounter serious difficulties. Some experiments were conducted by using smooth femtosecond laser pulses [18, 47, 48]. The amplitude of the Sommerfeld precursor is then completely negligible and the attention was focused on the Brillouin precursor. Contrary to the claim by Choi *et al.* [18], it has been shown that a subexponential decay of the energy of the transmitted pulse does

not prove that a Brillouin precursor is generated and is not at odds with the Lambert-Beer law that holds in every case [43, 47, 48]. In fact, as already noted by Alfano *et al.* [19] in their comment on the experiments by Choi *et al.*, the bandwidth of the pulses used in all these experiments is too narrow to excite the Brillouin precursor. As shown in Sec. VI of the present paper, the generation of a Brillouin precursor of significant amplitude in a very absorbing medium would require incident pulses of full duration $2T$ smaller than the carrier half-period $T_c/2$ when the pulse is not chirped, going to about $2T_c$ with a suitable chirping (Fig.15). Clearly the production of the required chirped incident pulses (see inset of Fig. 15) is at the extreme limit of what could be realized at optical wavelengths. As discussed in [45], the time scale (picosecond instead of femtosecond) would be more favorable at Terahertz (THz) frequencies (submillimetric wavelengths). The authors indicate that there exist in these domains materials whose complex refractive index $\tilde{n}(\omega)$ is very close to that of the Lorentz medium and, most important, that both the amplitude and the phase of the transmitted field could be measured by taking advantage of the recent progress in THz technology. They suggest using a ZnTe semiconductor crystal for generation, propagation and detection of THz pulses. They however remark that the ZnTe samples used up to now were not thick enough to evidence the Brillouin precursor and their proposal was not followed by an actual experiment.

From a theoretical viewpoint, most of the complexity of the uniform saddle point methods originates from the square root in the expression of $\tilde{n}(\omega)$, resulting in a branch cut in the complex plan. As noted in [16], this difficulty disappears when the medium is dilute enough, so that the 2nd term in $\tilde{n}(\omega)$ is small compared to 1. See Eq.(4). The equation giving the saddle points is then of 4th degree (instead of 8th degree) and, by means of reasonable approximations, can be reduced to a biquadratic equation, with solutions analogous to those obtained in the study of quasi-resonant precursors in transparent media [49]. Combining the Laplace-Fourier procedures used in the present paper and the *elementary* saddle point method, it seems possible to obtain analytic expressions for the Sommerfeld and Brillouin precursors, valid at any time. This work is in progress.

* Electronic address: bernard.segard@univ-lille-1.fr

- [1] A. Sommerfeld, *Physikalische Zeitschrift* **23**, 841 (1907) [in German].
- [2] A. Sommerfeld, *Ann. Phys. (Leipzig)* **44**, 177 (1914) [in German].
- [3] L. Brillouin, *Ann. Phys. (Leipzig)* **44**, 203 (1914) [in German].
- [4] L. Brillouin, in *Comptes Rendus du Congrès Interna-*

- tional d'Electricité, Paris 1932* (Gauthier-Villars 1933), Vol.2, pp 739-788 [in French].
- [5] L. Brillouin, *Wave Propagation and Group Velocity* (Academic Press, New York 1960), Authorized translations in English of [2-4] can be found in this book, respectively in Ch. II, Ch. III, Ch. IV and Ch. V.
- [6] J.A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York 1941).
- [7] J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York 1975).
- [8] R.A. Handelsman and N. Bleistein, Arch. Rat. Mech. Anal. **35**, 267 (1969).
- [9] K.E. Oughstun and G.C. Sherman, *Proceedings of the URSI Symposium on Electromagnetic Wave Theory* (Stanford University, 1977), pp 34-36.
- [10] V.A. Vasilev, M.Y. Kelbert, I.A. Sazonov and I.A. Chaban, Opt. Spectrosc. **64**, 862 (1988) [Opt. Spectrosc. **64**, 513 (1988)].
- [11] K.E. Oughstun and G.C. Sherman, J. Opt. Soc. Am. A **6**, 1394 (1989).
- [12] A. Karlsson and S. Rike, J. Opt. Soc. Am. A **15**, 487 (1998).
- [13] K.E. Oughstun, *Electromagnetic and Optical Pulse Propagation 2 : Temporal Pulse Dynamics in Dispersive Attenuative Media* (Springer, New York, 2009).
- [14] A. Ciarkowski, J. Tech. Phys. **43**, 187 (2002).
- [15] A. Ciarkowski, J. Tech. Phys. **44**, 181 (2003).
- [16] B. Macke and B. Ségard, J. Opt. Soc. Am. B **28**, 450 (2011).
- [17] A. Ciarkowski, International Journal of Electronics and Telecommunications (JET) **57**, 251 (2011).
- [18] S.H. Choi and U. Österberg, Phys. Rev. Lett. **92**, 193903 (2004).
- [19] R.R. Alfano, J.L. Birman, X. Ni, M. Alrubaiee, and B.B. Das, Phys. Rev. Lett. **94**, 239401 (2005).
- [20] T.M. Roberts, Phys. Rev. Lett. **93**, 269401 (2004).
- [21] E. Varoquaux, G. A. Williams, and O. Avenel, Phys. Rev. B **34**, 7617(1986).
- [22] J. Aaviksoo, J. Kuhl, and K. Ploog, Phys. Rev. A **44**, R5353 (1991).
- [23] B. Ségard, J. Zemmouri, and B. Macke, Europhys. Lett. **4**, 47 (1987). See Fig.2 in this reference. Note that the experiment was performed at a millimeter wavelength instead of in the optical domain and was not analyzed in terms of precursors.
- [24] H. Jeong, A. M. C. Dawes, and D. J. Gauthier, Phys. Rev. Lett., **96**, 143901 (2006). In this experiment, the optical thickness of the medium for the amplitude was only 0.5, not sufficient to observe the characteristic beat between Sommerfeld and Brillouin precursors. See: B. Macke and B. Ségard, e-print arXiv:physics/065039
- [25] Dong Wei, J.F. Chen, M.M.T. Loy, G.K.L. Wong, and S. Du, Phys. Rev. Lett. **103**, 093602 (2009).
- [26] See [5], pp. 55-57 and 127-128.
- [27] We use the definitions, sign conventions and results of the linear system theory. See for example A.Papoulis, *The Fourier Integral and its Applications* (Mc Graw Hill, New York 1987).
- [28] Keep in mind that we use a retarded time picture where the time t is the real time minus the luminal propagation time z/c .
- [29] *Handbook of Mathematical functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972).
- [30] As usual, we consider that the zero-order derivative of a function is the function itself.
- [31] K.E Oughstun, J. Opt. Soc. Am. A **12**, 1715 (1995).
- [32] N. S. Bukhman, Quantum Electron. **34**, 299 (2004).
- [33] B. Macke and B. Ségard, Phys. Rev. A **73**, 043802 (2006).
- [34] J.R. Klauder, IEE Proc. Rad. Sonar Navig. **152**, 23 (2005) and e-print arXiv:math/0402410.
- [35] D. C. Stoudt, F. E. Peterkin, and B. J. Hankla, *Transient RF and Microwave Pulse Propagation in a Debye Medium (Water)*, NSWC Report JPOSTC-CRF-005-03 (Dalgren, 2001).
- [36] J. Garnier and K. Solna, Waves in Random and Complex Media **20**, 122 (2010).
- [37] K. E. Oughstun, IEEE Trans. Antennas Propag. **53**, 1582 (2005).
- [38] M. Pieraccini, A. Bicci, D. Mecatti, G. Macaluso, and C. Atzeni, IEEE Trans. Antennas Propag. **57**, 3612 (2009).
- [39] M. Dawood, H. R. Mohammed, and A. V. Alejos, Electron. Lett. **46**, 1645 (2010).
- [40] N. Cartwright, IEEE Trans. Antennas Propag. **59**, 1571 (2011).
- [41] Note that the Airy function $\mathcal{A}(x)$ used by Brillouin differs from that currently used $\text{Ai}(x)$, with $2\pi\text{Ai}(-x) = 3^{1/3}\mathcal{A}(3^{1/3}x)$. Peak amplitudes of $\mathcal{A}(x)$ and $\text{Ai}(-x)$ are respectively 2.33 and 0.536.
- [42] K. E. Oughstun and G. C. Sherman, Phys. Rev. A **41**, 6090 (1990).
- [43] D. Lukofsky, J. Bessette, H. Jeong, E. Garmire, and U. Österberg, J. Mod. Opt. **56**, 1083 (2009).
- [44] K. E. Oughstun and C. M. Balictsis, Phys. Rev. Lett. **77**, 2210 (1996).
- [45] X. Ni and R. R. Alfano, Optics Express **14**, 4188 (2006).
- [46] P. Pleshko and I. Palócz, Phys. Rev. Lett. **22**, 1201 (1969).
- [47] L.M. Naveira, B.D. Strycker, J. Wang, G.O. Ariunbold, A.V. Sokolov, and G.W. Kattawar, Appl. Opt. **48**, 1828 (2009).
- [48] M.M. Springer, W. Yang, A.A. Kolomenski, H.A. Schuessler, J. Strohaber, G.W. Kattawar, and A.V. Sokolov, Phys. Rev. A **83**, 043817 (2011).
- [49] B. Macke and B. Ségard, Phys. Rev. A **80**, 011803(R) (2009).