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# Long time behaviour of a stochastic nano particle

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## Abstract

In this article, we are interested in the behaviour of a single ferromagnetic mono-domain particle submitted to an external field with a stochastic perturbation. This model is the first step toward the mathematical understanding of thermal effects on a ferromagnet. In a first part, we present the stochastic model and prove that the associated stochastic differential equation is well defined. The second part is dedicated to the study of the long time behaviour of the magnetic moment and in the third part we prove that the stochastic perturbation induces a non reversibility phenomenon. Last, we illustrate these results through numerical simulations of our stochastic model. The main results presented in this article are the rate of convergence of the magnetization toward the unique stable equilibrium of the deterministic model. The second result is a sharp estimate of the hysteresis phenomenon induced by the stochastic perturbation (remember that with no perturbation, the magnetic moment remains constant).

**Keywords:** convergence rate, stochastic dynamical systems, long time study, magnetism, hysteresis.

**AMS Classification:** 60F10, 60F15, 65Z05

## 1 Introduction

Thermal effects in ferromagnetic materials are essential in order to understand their behaviour at ambient temperature or, more critically, in electronic devices where the joule effect induces high heat fluxes. This effect is commonly modeled by the introduction of a noise at micro-scopic scale on the magnetic moment direction and at the meso-scopic scale by a transition of behaviour. In ferromagnetic materials, the transition between the non-linear behaviour and the linear behaviour is managed by the struggle between the Heisenberg interaction and this disorder induced by the heating. This model explains the critical temperatures such as the Curie temperature for ferromagnetic materials. In this context, it is essential to understand the impact of introducing stochastic perturbations in deterministic models of ferromagnetic materials such as the micromagnetism (see Brown

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(1962, 1963)).

The understanding of this phenomena is a key point in order to simulate realistic ferromagnetic devices such as micro electronic circuits. Furthermore, heating has a real effect on the microstructure dynamics in magnets; then, efficiently controlled, the dynamic of microstructures could accelerate processes such as the magnetization switching, which is the basics of magnetic recording techniques.

During the last decade, several studies have been initialized in several articles by physicists (e.g. Mercer et al. (2011); Zheng et al. (2003); Raikher et al. (2004); Atkinson et al. (2003); Raikher and Stepanov (2007); Scholz et al. (2001); Martinez et al. (2007); Smith (2001)), but no mathematical models justifying this kind of effects have been developed yet. In this article, our goal is to improve the understanding of thermal effects in ferromagnets. To achieve this goal, we focus this first study on the dynamic of a single magnetic moment submitted to a stochastic perturbation. Our main aim is to characterize precisely the dynamic of the moment, giving estimations and general behaviour in long time. In particular, we will exhibit an hysteresis behaviour of the magnetization in our model.

The model we are studying mimics the behaviour of a single magnetic moment  $\mu(t)$  (function from  $\mathbb{R}$  into  $S(\mathbb{R}^3) = \{u \in \mathbb{R}^3; |\mu| = 1\}$ ) submitted to an external field  $b$ . The dynamic of such a system is, at the micro-scale, described by the Larmor precession equation

$$\frac{d\mu}{dt} = -\mu \wedge b.$$

Nevertheless, this equation is non dissipative and, in order to make the theoretical study easier, we introduce a dissipative part using the Landau-Lifchitz equation

$$\frac{d\mu}{dt} = -\mu \wedge b - \alpha \mu \wedge (\mu \wedge b),$$

where  $\alpha$  is a positive real constant and we set the initial condition  $\mu(0) = \mu_0 \in S(\mathbb{R}^3)$ . We point out two major properties of this system

- i.*  $\forall t \in \mathbb{R}, |\mu(t)| = 1,$
- ii.*  $\forall t \in \mathbb{R}, \frac{d}{dt}(\mu(t) \cdot b) \geq 0.$

The first property, which is definitely essential, will have to be preserved by the stochastic system and the second property is the energy decreasing induced by the introduction of the dissipation term. The dynamic of this deterministic system is classical; in fact, one knows that  $\lim_{t \rightarrow \infty} \mu(t) = \frac{1}{|b|}b$  provided that  $\mu(0) \neq -b$ . In this work, we will develop such a result for the stochastic system. In order to build this stochastic system, the first question is how to introduce the stochastic perturbation in the deterministic system. We want to model the thermal effects which are external perturbations of the magnetic moment. In fact, this perturbation could be modeled has an external perturbation field. In the sequel, we will choose to build a stochastic system by perturbing the external field with a Brownian motion. We will write down

$$\begin{cases} dY_t &= -\mu_t \wedge (b dt + \varepsilon dW_t) - \alpha \mu_t \wedge \mu_t \wedge (b dt + \varepsilon dW_t) \\ Y_0 &= y \in \mathcal{S}(\mathbb{R}^3). \end{cases}$$

where  $\varepsilon$  is a strictly positive real number and  $W$  a standard Brownian motion with values in  $\mathbb{R}^3$ . But, an easy computation of  $d(|Y_t|^2)$  using Itô's formula shows that the process  $Y$  will not stay in  $S(\mathbb{R}^3)$ , then, in order to preserve this essential behaviour, we have to renormalise the previous equation and set

$$\mu_t = \frac{Y_t}{|Y_t|}.$$

Given this system, we prove the following results

*i.*  $\mu_t \cdot b \xrightarrow[t \rightarrow \infty]{} |b|$ , a.s.,

*ii.* For all  $\beta < 1/2$ ,  $t^\beta(|b| - \mu_t \cdot b)$  tends to 0 in probability.

*iii.* The family  $\{\sqrt{t}(|b| - \mu_t \cdot b); t \geq 0\}$  is tight in  $\mathbb{R}$ .

Note that these results are only valid for  $\alpha > 0$ , as for  $\alpha = 0$ , it is easy to show that the function  $e(t) = \mathbb{E}(\mu_t \cdot b)$  satisfies the following ordinary differential equation  $e'(t) = -e(t)\frac{h'(t)}{h(t)}$ . Hence,  $e(t) = \frac{e(0)}{h(t)} \xrightarrow[t \rightarrow \infty]{} 0$ . This contradicts the a.s. convergence of  $\mu_t$  to  $\frac{b}{|b|}$ .

*iv.* When  $\mu$  is submitted to a time varying external field, an hysteresis phenomenon appears. If we consider  $\mathbf{b} \in \mathcal{S}(\mathbb{R}^3)$  and let  $b$  linearly vary between  $+\mathbf{b}$  and  $-\mathbf{b}$  over the time interval  $[0, T]$ , then  $\mathbb{E}(\mu_t \cdot \mathbf{b})$  is bounded from below by  $\frac{1}{\sqrt{1+ct}}$  for  $t \leq T/2$  where  $c$  is a constant depending only on  $\varepsilon$  and  $\alpha$ .

First, we make precise the derivation of the stochastic model. Then, we lead a detailed study of its asymptotic behaviour and in particular we point out an hysteresis phenomenon. This phenomenon is obtained by slow variations of the external field such that the dynamic of relaxation of the magnetization toward this field becomes instantaneous when the speed ratio of the external excitation goes to zero. The results shown in this article are finally illustrated by numerical simulations.

## 2 Model and notations

Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a probability space. We consider a standard Brownian motion  $W$  defined on this space with values in  $\mathbb{R}^3$  and denote by  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  its natural filtration augmented with  $\mathbb{P}$ -null sets.

Let  $b \in \mathbb{R}^3$  be the magnetic field. We model the  $\mathcal{S}(\mathbb{R}^3)$ -valued magnetic moment process  $\mu = (\mu_t)_{t \geq 0}$  by the following coupled stochastic differential equation (SDE in short)

$$\begin{cases} dY_t &= -\mu_t \wedge (b dt + \varepsilon dW_t) - \alpha \mu_t \wedge \mu_t \wedge (b dt + \varepsilon dW_t) \\ \mu_t &= \frac{Y_t}{|Y_t|} \\ Y_0 &= y \in \mathcal{S}(\mathbb{R}^3), \end{cases} \quad (2.1)$$

where  $\alpha > 0$  is the magnitude of the damping term and  $\varepsilon > 0$  is the magnitude of the noise term.

The term  $\mu_t \wedge dW_t$  in (2.1) is naturally defined by introducing the antisymmetric operator  $L : \mathbb{R}^3 \mapsto \mathbb{R}^{3 \times 3}$  associated to the vector product in  $\mathbb{R}^3$

$$L(x) = \begin{pmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & -x^1 \\ -x^2 & x^1 & 0 \end{pmatrix}.$$

Hence, for a 3-dimensional  $\mathbb{F}$ -adapted process  $H$  satisfying  $\int_0^t |H_s|^2 ds < \infty$  a.s. for all  $t > 0$ , the process  $\int_0^t H_s \wedge dW_s$  is defined by  $\int_0^t L(H_s) dW_s$ , which is a standard multi-dimensional Itô stochastic integral. When dealing with the differential expression, we will either write  $H_t \wedge dW_t$  or  $L(H_t) dW_t$ .

### Notations:

- For  $a$  and  $b$  in  $\mathbb{R}^3$  we denote by  $a \cdot b$  their scalar product,  $a \cdot b = \sum_{i=1}^3 a^i b^i$ .
- For  $a$  in  $\mathbb{R}^3$ , we denote by  $|a| = \sqrt{a \cdot a}$  the Euclidean norm of  $a$ .
- We like to encode elements of  $\mathbb{R}^3$  as column vectors. For  $x \in \mathbb{R}^3$ ,  $x^*$  is a row vector. Similarly, we use the star notation “ $*$ ” to denote the transpose of matrices.
- If  $H = (H_t)_{t \geq 0}$  is a 3-dimensional  $\mathbb{F}$ -adapted process satisfying  $\int_0^t |H_u|^2 du < \infty$  a.s. for all  $t$ , we may write  $\int_0^t H_u \cdot dW_u$  for  $\sum_{i=1}^3 \int_0^t H_u^i dW_u^i$  and use the differential form  $H_t \cdot dW_t$  for  $\sum_{i=1}^3 H_t^i dW_t^i$ .

## 3 Main results: long time behaviour

### 3.1 First properties of the magnetic moment $\mu$

**Proposition 1.** *Let  $(Y, \mu)$  be a pair of processes satisfying (2.1), then*

$$d|Y_t|^2 = 2\varepsilon^2(\alpha^2 + 1)dt$$

and therefore  $|Y_t| = \sqrt{2\varepsilon^2(\alpha^2 + 1)t + 1}$  is non random.

**Remark 2.** *The fact that  $|Y_t|$  is non random is definitely essential in all the following computations. In particular, we deduce from this result that  $|Y_t|$  has finite variation.*

*Proof.* Using Itô’s lemma we have

$$d|Y_t|^2 = 2Y_t \cdot dY_t + \sum_{i=1}^3 d\langle Y^i, Y^i \rangle_t = \sum_{i=1}^3 d\langle Y^i, Y^i \rangle_t,$$

where we have used the fact that  $Y_t$  and  $dY_t$  are orthogonal. But, using the identity  $a \wedge (b \wedge c) = (a \cdot c)b - (a \cdot b)c$ , we have

$$\begin{aligned} dY_t &= \varepsilon \left[ -(\mu_t \wedge dW_t) - \alpha((\mu_t \cdot dW_t)\mu_t - (\mu_t \cdot \mu_t)dW_t) \right] + \dots dt \\ &= \varepsilon A(\mu_t) dW_t + \dots dt, \end{aligned}$$

where we have set  $A(\mu) = \alpha I - \alpha(\mu\mu^*) - L(\mu)$  and used  $|\mu_t| = 1$ . Thus,

$$\begin{aligned} d\langle Y, Y \rangle_t &= \varepsilon^2 AA^*(\mu_t) dt \\ &= \varepsilon^2 \left[ \alpha^2 I - \alpha^2(\mu_t \mu_t^*) + \alpha L(\mu_t) - \alpha^2(\mu_t \mu_t^*) + \alpha^2(\mu_t \mu_t^*)(\mu_t \mu_t^*) - \alpha(\mu_t \mu_t^*)L(\mu_t) \right. \\ &\quad \left. - \alpha L(\mu_t) + L(\mu_t)(\mu_t \mu_t^*) - L(\mu_t)L(\mu_t) \right] dt \\ &= \varepsilon^2 \left[ \alpha^2 I - \alpha^2(\mu_t \mu_t^*) + L(\mu_t)L^*(\mu_t) \right] dt, \end{aligned}$$

where we have used  $L(\mu)\mu = 0$ ,  $L^*(\mu) = -L(\mu)$  and again  $\mu_t^* \mu_t = 1$ . Thus, for each  $1 \leq i \leq 3$  we have

$$d\langle Y^i, Y^i \rangle_t = \varepsilon^2 [\alpha^2 - \alpha^2 (\mu_t^i)^2 + \sum_{k=1}^3 (L_{ik}(\mu_t))^2].$$

Then, summing over  $i$ , we get

$$\begin{aligned} d|Y_t|^2 &= \varepsilon^2 [3\alpha^2 - \alpha^2 |\mu_t|^2 + \sum_{i,j=1}^3 (L_{ij}(\mu_t))^2] dt \\ &= \varepsilon^2 [3\alpha^2 - \alpha^2 |\mu_t|^2 + 2|\mu_t|^2] dt, \end{aligned}$$

The result ensues by remembering that  $|\mu_t|^2 = 1$ . ■

With the help of Proposition 1, we can establish the SDE satisfied by the one dimensional process  $(\mu_t \cdot b)_t$ . We introduce the function

$$h(t) = |Y(t)| = \sqrt{2\varepsilon^2(\alpha^2 + 1)t + 1}. \quad (3.1)$$

Since  $|Y_t|$  is non random, we deduce from Equation (2.1) that

$$\begin{aligned} d\mu_t &= -\frac{\mu_t h'(t)}{h(t)} dt + \frac{dY_t}{h(t)} \\ &= -\frac{\mu_t h'(t)}{h(t)} dt - \frac{1}{h(t)} (\mu_t \wedge (b dt + \varepsilon dW_t) + \alpha \mu_t \wedge \mu_t \wedge (b dt + \varepsilon dW_t)) \\ d\mu_t &= -\frac{\mu_t h'(t) + \mu_t \wedge b + \alpha(\mu_t(\mu_t \cdot b) - b)}{h(t)} dt - \frac{\varepsilon}{h(t)} (L(\mu_t) + \alpha(\mu_t \mu_t^* - I)) dW_t \end{aligned} \quad (3.2)$$

By taking the scalar product with  $b$ , we get

$$\begin{aligned} d(\mu_t \cdot b) &= -(\mu_t \cdot b) \frac{h'(t)}{h(t)} dt - \frac{\alpha}{h(t)} ((\mu_t \cdot b)^2 - |b|^2) dt \\ &\quad - \frac{\varepsilon}{h(t)} ((\mu_t \wedge dW_t) \cdot b + \alpha(\mu_t \cdot b)(\mu_t \cdot dW_t) - \alpha(b \cdot dW_t)) \\ d(\mu_t \cdot b) &= -(\mu_t \cdot b) \frac{h'(t)}{h(t)} dt - \frac{\alpha}{h(t)} ((\mu_t \cdot b)^2 - |b|^2) dt \\ &\quad - \frac{\varepsilon}{h(t)} (-L(\mu_t)b + \alpha((\mu_t \cdot b)\mu_t - b)) \cdot dW_t \end{aligned} \quad (3.3)$$

We may call this equation the SDE satisfied by  $\mu_t \cdot b$  whereas it is not an SDE properly speaking since the r.h.s member actually depends on all the components of  $\mu_t$  and not only  $\mu_t \cdot b$ . Nonetheless, we may use this abuse of terminology throughout the paper.

**Remark 3** (Remark on the existence and uniqueness of solutions to Equation (2.1)). *Let us consider the following coupled SDE*

$$dY_t = -\mu_t \wedge (b dt + \varepsilon dW_t) - \alpha \mu_t \wedge \mu_t \wedge (b dt + \varepsilon dW_t) \quad (3.4a)$$

$$d\mu_t = -\frac{\mu_t h'(t) + \mu_t \wedge b + \alpha(\mu_t(\mu_t \cdot b) - b)}{h(t)} dt - \frac{\varepsilon}{h(t)} (L(\mu_t) + \alpha(\mu_t \mu_t^* - I)) dW_t \quad (3.4b)$$

$$Y_0 = \mu_0 \in \mathcal{S}(\mathbb{R}^3)$$

This system is actually decoupled as the SDE on  $\mu$  is autonomous (this has only been possible because  $|Y_t|$  is non random). The existence and uniqueness of a solution to Equation (3.4) boil down to the ones of Equation (3.4b). By computing  $d(|\mu_t|^2)$ , we deduce that if there exists a solution  $\mu$  to Equation (3.4b),  $|\mu_t|^2 = 1$  a.s. for all  $t$ . Hence, it is sufficient to check the standard global Lipschitz behaviour of the coefficients on  $\mathbb{R}_+ \times \mathcal{S}(\mathbb{R}^3)$  to prove existence and uniqueness of a strong solution to Equation (3.4b).

We have already seen above that if a pair  $(Y, \mu)$  is solution of Equation (2.1), it also solves Equation (3.4).

Conversely, if  $(Y, \mu)$  is the unique strong solution of Equation (3.4), it is clear that  $|Y_t| = h(t)$  by following the proof of Proposition 1 and moreover the computation of  $d(Y_t/|Y_t|)$  shows that the process  $(Y_t/|Y_t|)_t$  solves the same SDE as  $\mu$ , hence for all  $t$   $\mu_t = \frac{Y_t}{|Y_t|}$  a.s. This last argument proves that Equations (2.1) and (3.4) have the same solutions. Therefore, we deduce that Equation (2.1) admits a unique strong solution denoted by  $(Y, \mu)$  in the sequel.

### 3.2 Almost sure convergence

In this part, we prove the almost sure convergence of  $\mu_t$  to  $b/|b|$  when  $t$  goes to infinity. This is achieved by studying the pathwise behaviour of the process  $(\mu_t \cdot b)_t$ . First, we prove that any northern polar cap is recurrent (see Proposition 4); this especially implies that  $\mu$  cannot get trapped in  $-b$  which is yet an equilibrium position of the deterministic system.

We are interested in characterising the hitting time of any polar caps. Let  $0 < \eta < |b|$ , we introduce the family of hitting times for  $s > 0$ ,

$$\tilde{\tau}_s = \inf\{t \geq 0 : \mu_{t+s} \cdot b \geq \eta\} \quad (3.5)$$

Note that for every  $s > 0$ ,  $s + \tilde{\tau}_s$  is an  $(\mathcal{F}_t)$ -stopping time.

**Proposition 4.** *For all  $s > 0$  we have  $\mathbb{P}(\tilde{\tau}_s < \infty) = 1$ .*

*Proof.* We have

$$\mathbb{P}(\tilde{\tau}_s < \infty) = \mathbb{P}(\tilde{\tau}_s < \infty, \mu_s \cdot b < \eta) + \mathbb{P}(\tilde{\tau}_s < \infty, \mu_s \cdot b \geq \eta).$$

As  $\mathbb{P}(\tilde{\tau}_s < \infty, \mu_s \cdot b \geq \eta) = \mathbb{P}(\mu_s \cdot b \geq \eta)$  it suffices to show that  $\mathbb{P}(\tilde{\tau}_s < \infty, \mu_s \cdot b < \eta) = \mathbb{P}(\mu_s \cdot b < \eta)$ . Let  $A_s = \{\mu_s \cdot b < \eta\}$ . Note that  $A_s \in \mathcal{F}_s$ .

Integrating (3.3), and applying the stopping theorem, we get

$$\mathbb{E}(\mu_{t \wedge (s + \tilde{\tau}_s)} \cdot b \mathbf{1}_{\{A_s\}}) - \mathbb{E}(\mu_s \cdot b \mathbf{1}_{\{A_s\}}) = \mathbb{E} \left[ \int_s^t \left( -(\mu_u \cdot b) \frac{h'(u)}{h(u)} + \frac{\alpha}{h(u)} |\mu_u \wedge b|^2 \right) \mathbf{1}_{\{u \leq s + \tilde{\tau}_s\}} du \mathbf{1}_{\{A_s\}} \right].$$

We always have  $|\mu_u \cdot b| \leq |b|$ . Let  $u \leq s + \tilde{\tau}_s$ . On the set  $\{0 < \mu_u \cdot b < \eta\}$  we have  $|\mu_u \wedge b|^2 \geq |b|^2 - \eta^2$ . Thus,

$$-(\mu_u \cdot b) \frac{h'(u)}{h(u)} + \frac{\alpha}{h(u)} |\mu_u \wedge b|^2 \geq -|b| \frac{h'(u)}{h(u)} + \alpha \frac{|b|^2 - \eta^2}{h(u)}.$$

There exists  $U > s$  (non random) such that for all  $u \geq U$ ,

$$-|b| \frac{h'(u)}{h(u)} + \alpha \frac{|b|^2 - \eta^2}{h(u)} \geq \alpha \frac{|b|^2 - \eta^2}{2h(u)}.$$

On the set  $\{\mu_u \cdot b \leq 0\}$  we have

$$\begin{aligned} -(\mu_u \cdot b) \frac{h'(u)}{h(u)} + \frac{\alpha}{h(u)} |\mu_u \wedge b|^2 &\geq (-(\mu_u \cdot b) + \alpha |\mu_u \wedge b|^2) \frac{h'(u)}{h(u)} \\ &\geq \min(|b|, \alpha |b|^2) \frac{1}{2} \frac{h'(u)}{h(u)}. \end{aligned}$$

The last inequality comes from the fact that if  $\pi/2 \leq x \leq 3\pi/2$ , we have either  $-\cos(x) \geq \sqrt{2}/2$  or  $|\sin(x)| \geq \sqrt{2}/2$ .

Therefore, there exists  $\bar{U} \geq U$  (non random), such that for all  $u \geq \bar{U}$ , on the event  $A_s \cap \{u \leq s + \tilde{\tau}_s\}$ ,  $-(\mu_u \cdot b) \frac{h'(u)}{h(u)} + \frac{\alpha}{h(u)} |\mu_u \wedge b|^2$  is bounded from below by  $c \frac{h'(u)}{h(u)}$  which is non integrable and where  $c$  is a positive real constant depending on  $\eta$ .

$$\begin{aligned} \mathbb{E}(\mu_{t \wedge \tilde{\tau}_s} \cdot b \mathbf{1}_{\{A_s\}}) - \mathbb{E}(\mu_s \cdot b \mathbf{1}_{\{A_s\}}) &\geq \mathbb{E} \left( \int_s^{\bar{U}} \left( -(\mu_u \cdot b) \frac{h'(u)}{h(u)} + \alpha \frac{|\mu_u \wedge b|^2}{h(u)} \right) \mathbf{1}_{\{u \leq s + \tilde{\tau}_s, A_s\}} du \right) \\ &\quad + \int_{\bar{U}}^t c \frac{h'(u)}{h(u)} \mathbb{P}(u \leq \tilde{s} + \tau_s, A_s) du \end{aligned}$$

As  $c \frac{h'(u)}{h(u)}$  is a non integrable and positive function,  $\mathbb{P}(u \leq s + \tilde{\tau}_s, A_s)$  must converge to zero as  $u$  goes to infinity, otherwise  $\mathbb{E}(\mu_{t \wedge \tilde{\tau}_s} \cdot b \mathbf{1}_{\{A_s\}})$  would tend to infinity with  $t$  which would contradict the boundedness of the process  $\mu$ .

Thus,  $\lim_{u \rightarrow \infty} \mathbb{P}(u > s + \tilde{\tau}_s, A_s) = \mathbb{P}(A_s)$ . Moreover  $\lim_{u \rightarrow \infty} \mathbb{P}(u > s + \tilde{\tau}_s, A_s) = \mathbb{P}(s + \tilde{\tau}_s < \infty, A_s)$ , which yields the expected result.  $\blacksquare$

From Proposition 3.5, we know that any polar caps is visited within a finite time. Hence we can define the exit time of any polar caps. Let  $0 < \delta < \eta$ , we define

$$\tau_s = \inf\{t \geq 0; \mu_{s+\tilde{\tau}_s+t} \cdot b \leq \eta - \delta\} \quad (3.6)$$

For every  $s > 0$ ,  $s + \tilde{\tau}_s + \tau_s$  is an  $(\mathcal{F}_t)$ -stopping time.

**Proposition 5.** *For all  $t > 0$ ,  $\lim_{s \rightarrow \infty} \mathbb{P}(\tau_s \leq t) = 0$ , ie. the family  $(\tau_s)_s$  tends to infinity in probability.*

*Proof.* We know from Proposition 4, that  $\tilde{\tau}_s$  is a.s. finite, hence the pathwise continuity of  $\mu$  proves that  $\mu_{s+\tilde{\tau}_s} \cdot b \geq \eta$  a.s. and moreover we have the following equality  $\{\mu_{(s+\tilde{\tau}_s+\tau_s) \wedge (s+\tilde{\tau}_s+t)} \cdot b \leq \eta - \delta\} = \{\tau_s \leq t\}$ . Therefore, we deduce

$$\begin{aligned} \mathbb{P}(\tau_s \leq t) &= \mathbb{P}(\mu_{(s+\tilde{\tau}_s+\tau_s) \wedge (s+\tilde{\tau}_s+t)} \cdot b \leq \eta - \delta) \\ &\leq \mathbb{P} \left( -\delta \geq \int_{s+\tilde{\tau}_s}^{(s+\tilde{\tau}_s+\tau_s) \wedge (s+\tilde{\tau}_s+t)} -(\mu_u \cdot b) \frac{h'(u)}{h(u)} + \alpha \frac{|\mu_u \wedge b|^2}{h(u)} du \right. \\ &\quad \left. + \varepsilon \int_{s+\tilde{\tau}_s}^{(s+\tilde{\tau}_s+\tau_s) \wedge (s+\tilde{\tau}_s+t)} \frac{\mu_u \wedge b - \alpha(\mu_u \cdot b)\mu_u + \alpha b}{h(u)} \cdot dW_u \right) \end{aligned}$$

We have

$$\begin{aligned} \mathbb{P}(\tau_s \leq t) &\leq \mathbb{P} \left( \delta \leq \int_{s+\tilde{\tau}_s}^{(s+\tilde{\tau}_s+\tau_s) \wedge (s+\tilde{\tau}_s+t)} |b| \frac{h'(u)}{h(u)} du \right. \\ &\quad \left. - \varepsilon \int_{s+\tilde{\tau}_s}^{(s+\tilde{\tau}_s+\tau_s) \wedge (s+\tilde{\tau}_s+t)} \frac{\mu_u \wedge b - \alpha(\mu_u \cdot b)\mu_u + \alpha b}{h(u)} \cdot dW_u \right) \end{aligned}$$



Using Markov's inequality we find,

$$\begin{aligned}
\mathbb{P}(\tau_s \leq t) &\leq \frac{2}{\delta^2} \mathbb{E} \left( \left( \int_{s+\tilde{\tau}_s}^{(s+\tilde{\tau}_s+\tau_s) \wedge (s+\tilde{\tau}_s+t)} |b| \frac{h'(u)}{h(u)} du \right)^2 \right. \\
&\quad \left. + \varepsilon^2 \left( \int_{s+\tilde{\tau}_s}^{(s+\tilde{\tau}_s+\tau_s) \wedge (s+\tilde{\tau}_s+t)} \frac{\mu_u \wedge b - \alpha(\mu_u \cdot b)\mu_u + \alpha b}{h(u)} \cdot dW_u \right)^2 \right) \\
&\leq \frac{2}{\delta^2} |b|^2 \mathbb{E} \left( \log \frac{h(s+\tilde{\tau}_s+t)}{h(s+\tilde{\tau}_s)} \right)^2 + \frac{2\varepsilon^2}{\delta^2} \int_0^t \frac{1}{h^2(s+u)} du \\
&\leq \frac{2}{\delta^2} |b|^2 \left( \log \frac{h(s+t)}{h(s)} \right)^2 + \frac{2|b|^2(1+2\alpha)^2\varepsilon^2}{\delta^2} \frac{t}{h^2(s)}
\end{aligned}$$

where the stochastic integral term has been treated by applying Lemma 10 after having noticed that the integrand is bounded by  $|b|(1+2\alpha)/h(u)$ . Hence, we deduce that  $\lim_{s \rightarrow \infty} \mathbb{P}(\tau_s \leq t) = 0$ .  $\blacksquare$

**Theorem 6.** *We have  $\mu_t \cdot b \rightarrow |b|$  a.s.*

*Proof.* From Proposition 5, we know that  $\tau_s$  converges in probability to infinity when  $s$  goes to infinity, which means that we can extract a subsequence  $(\tau_{s_k})_k$  converging a.s. to infinity, i.e.  $\lim_{k \rightarrow \infty} \mathbb{P}(\tau_{s_k} = +\infty) = 1$ . Let us define  $m_t = \inf_{u \geq t} (\mu_u \cdot b)$ . The process  $(m_t)_t$  is decreasing and bounded from below by  $-|b|$ , hence it converges a.s. to a random variable  $m$ . If we use the equality

$$\{\tau_{s_k} = \infty\} = \{\forall t \geq s_k + \tilde{\tau}_{s_k}; \mu_t \cdot b > \eta - \delta\} = \left\{ \inf_{t \geq s_k + \tilde{\tau}_{s_k}} \mu_t \cdot b > \eta - \delta \right\},$$

we get  $\lim_{k \rightarrow \infty} \mathbb{P}(m_{s_k + \tilde{\tau}_{s_k}} > \eta - \delta) = 1$ . As we also know that  $m_t$  converges a.s. to  $m$ , we obtain that  $\mathbb{P}(m > \eta - \delta) = 1$ . As this result, is true for all  $0 < \delta < \eta < |b|$ , it proves that  $m = |b|$  a.s., which leads in turn to the a.s. convergence of  $\mu_t \cdot b$  to  $|b|$  since for all  $t$ , we have the inequalities  $|b| \geq \mu_t \cdot b \geq m_t$  a.s.  $\blacksquare$

### 3.3 Convergence rate

Let  $0 < \eta < 2|b|$  and  $0 < \delta < 1$  be fixed. For any fixed  $t_0 > 0$ , we introduce the family of sets  $(A_t^{t_0})_{t \geq t_0}$  defined by

$$A_t^{t_0} = \left\{ \sup_{t_0 \leq u \leq t} |b| - \mu_u \cdot b < \eta \right\}. \quad (3.7)$$

This family is decreasing : for  $t \geq s$ ,  $A_t^{t_0} \subset A_s^{t_0}$ . Since the process  $(\mu_u)_u$  is continuous and the supremum over the interval  $[t_0, t]$  is an increasing function of  $t$ , it is easy to show that the process  $(I_t = \mathbf{1}_{\{A_t^{t_0}\}})_{t \geq t_0}$  is càdlàg and decreasing with at most one jump and hence has finite variations. From Theorem 6, we know that we can choose  $t_0$ , such that for all  $t \geq t_0$ ,  $\mathbb{P}(A_t) \geq 1 - \delta$ . From now on, we assume that  $t_0$  satisfies this condition.

We are interested in the asymptotic behaviour of  $f(t) = \mathbb{E}(h(t)(|b| - \mu_t \cdot b)I_t)$ . Since the process  $(I_t)_t$  is right continuous, it ensues from the bounded convergence theorem, that the function  $f$  is also right continuous.

**Theorem 7.** *We have the following results*

$$\begin{aligned}\mathbb{E}(h(t)(|b| - \mu_t \cdot b)I_t) &\leq \frac{\varepsilon^2(1 + \alpha^2)}{\alpha(2 - \eta/|b|)} \quad \forall t \geq t_0, \\ \frac{\varepsilon^2(1 + \alpha^2)(1 - \delta)}{2\alpha} &\leq \liminf_{t \rightarrow \infty} \mathbb{E}(h(t)(|b| - \mu_t \cdot b)I_t).\end{aligned}$$

As a corollary of Theorem 7, we can prove the following result

**Theorem 8.** *The following results hold*

1. For all  $0 < \beta < 1/2$  and  $\bar{\eta} > 0$ ,  $\mathbb{P}(t^\beta(|b| - \mu_t \cdot b) \geq \bar{\eta}) \rightarrow 0$ .
2. The family  $\{h(t)(|b| - \mu_t \cdot b); t \geq 0\}$  is tight in  $\mathbb{R}$ .
3. For all  $t \geq t_0$  and all  $0 < \bar{\eta} < \eta$ , we have

$$\mathbb{P}(|b| - \mu_t \cdot b \geq \bar{\eta}; \sup_{t_0 \leq u \leq t} |b| - \mu_u \cdot b < \eta) \leq \frac{\varepsilon^2(1 + \alpha^2)}{\alpha(2 - \eta/|b|)\bar{\eta}} \frac{1}{h(t)}.$$

*Proof.* **Statement 1.** For all  $\bar{\eta} > 0$ , we have

$$\begin{aligned}\mathbb{P}(t^\beta(|b| - \mu_t \cdot b) \geq \bar{\eta}) &= \mathbb{P}(t^\beta(|b| - \mu_t \cdot b)I_t \geq \bar{\eta}) + \mathbb{P}(t^\beta(|b| - \mu_t \cdot b)(1 - I_t) \geq \bar{\eta}) \\ &\leq \frac{1}{\bar{\eta}} \mathbb{E}(t^\beta(|b| - \mu_t \cdot b)I_t) + \mathbb{P}(I_t = 0) \\ &\leq t^{\beta-1/2} \frac{1}{\bar{\eta} \sqrt{2\varepsilon^2(\alpha^2 + 1)}} \mathbb{E}(h(t)(|b| - \mu_t \cdot b)I_t) + \mathbb{P}(\bar{A}_t^{t_0})\end{aligned}$$

We know from Theorem 7 that the first term on the r.h.s can be made smaller than  $\delta$  where  $\delta$  was fixed at the beginning of Section 3.3. The parameter  $t_0$  has been chosen to ensure that for all  $t \geq t_0$ ,  $\mathbb{P}(\bar{A}_t^{t_0}) \leq \delta$ . Hence, for  $t$  large enough,  $\mathbb{P}(t^\beta(|b| - \mu_t \cdot b) \geq \bar{\eta}) \leq \delta$ . As the parameter  $\delta$  can be chosen arbitrarily small, the first result is proved.

**Statement 2.** Since the process  $(h(t)(|b| - \mu_t \cdot b))_t$  is continuous, it is sufficient to prove the tightness of  $\{h(t)(|b| - \mu_t \cdot b); t \geq t_0\}$ . Following the proof of the previous item, it is easy to show that for all  $M$  large enough,  $\mathbb{P}(h(t)(|b| - \mu_t \cdot b) \geq M) \leq 2\delta$ .

**Statement 3.** It is sufficient to notice that  $\mathbb{P}(|b| - \mu_t \cdot b \geq \bar{\eta}; \sup_{t_0 \leq u \leq t} |b| - \mu_u \cdot b < \eta) = \mathbb{P}((|b| - \mu_t \cdot b) \mathbf{1}_{\{\sup_{t_0 \leq u \leq t} |b| - \mu_u \cdot b < \eta\}} \geq \bar{\eta})$  and then to apply Markov's inequality.  $\blacksquare$

*Proof of Theorem 7.* Applying Itô's formula for jump processes, we obtain

$$f(t) - f(t_0) = \mathbb{E} \left( \int_{t_0}^t [b|h'(u) - \alpha(|b|^2 - (\mu_u \cdot b)^2)] I_u du \right) + \mathbb{E} \left( \sum_{t_0 \leq u \leq t} h(u)(|b| - \mu_u \cdot b) \Delta(I_u) \right).$$

Let us focus on the jump term. We define, the stopping time

$$\tau = \inf\{t \geq t_0 : I_t = 0\}.$$

Note that  $A_t = \{t < \tau\}$ , hence we can rewrite the definition of  $f$  as  $f(t) = \mathbb{E}(h(t \wedge \tau)(|b| - \mu_{t \wedge \tau} \cdot b)I_{t \wedge \tau})$  and the Itô formula yields

$$\begin{aligned} f(t) - f(t_0) &= \mathbb{E} \left( \int_{t_0}^t [|b|h'(u) - \alpha(|b|^2 - (\mu_u \cdot b)^2)] I_u du \right) \\ &\quad + \mathbb{E} \left( \sum_{t_0 \leq u \leq t \wedge \tau} h(u)(|b| - \mu_u \cdot b) \Delta(I_u) \right). \end{aligned} \quad (3.8)$$

The jump term writes  $\sum_{t_0 \leq u \leq t \wedge \tau} h(u)(|b| - \mu_u \cdot b) \Delta(I_u) = -h(t)(|b| - \mu_t \cdot b) \mathbf{1}_{\{t=\tau\}}$ . Therefore, Equation (3.8) becomes,

$$\begin{aligned} f(t) - f(t_0) &= \mathbb{E} \left( \int_{t_0}^t [|b|h'(u) - \alpha(|b|^2 - (\mu_u \cdot b)^2)] I_u du \right) + \mathbb{E} \left( h(t)(|b| - \mu_t \cdot b) \mathbf{1}_{\{t=\tau\}} \right) \\ &= \mathbb{E} \left( \int_{t_0}^t [|b|h'(u) - \alpha(|b|^2 - (\mu_u \cdot b)^2)] I_u du \right) + h(t)\eta\mathbb{P}(t = \tau). \end{aligned}$$

Since  $f$  and the expectation on the right hand side are both right continuous, we deduce that the function  $t \mapsto \mathbb{P}(\tau = t)$  is also right continuous, which implies from Lemma 11 that  $\mathbb{P}(\tau = t) = 0$  for all  $t \geq t_0$ . Hence,

$$\begin{aligned} f(t) - f(t_0) &= |b| \int_{t_0}^t h'(u) \mathbb{P}(A_u) du - \alpha \int_{t_0}^t \mathbb{E} \left( (|b|^2 - (\mu_u \cdot b)^2) I_u \right) du \\ f'(t) &= |b|h'(t)\mathbb{P}(A_t) - \alpha \mathbb{E} \left( (|b|^2 - (\mu_t \cdot b)^2) I_t \right). \end{aligned} \quad (3.9)$$

**Step 1:**

$$(|b|^2 - (\mu_t \cdot b)^2) I_t = (|b| - (\mu_t \cdot b))(|b| + (\mu_t \cdot b)) I_t \geq (2|b| - \eta)(|b| - (\mu_t \cdot b)) I_t.$$

Combining this upper bound with Equation (3.9) leads to

$$\begin{aligned} f'(t) &\leq |b|h'(t) - \alpha(2|b| - \eta) \frac{f(t)}{h(t)} \\ f'(t) + \alpha(2|b| - \eta) \frac{f(t)}{h(t)} &\leq |b|h'(t). \end{aligned}$$

Note that  $\frac{1}{h(t)} = \frac{h'(t)}{\varepsilon^2(1+\alpha^2)}$ , hence  $\int_{t_0}^t \frac{1}{h(u)} du = \frac{h(t) - h(t_0)}{\varepsilon^2(1+\alpha^2)}$ .

$$\begin{aligned} \left( f(t) e^{\frac{\alpha(2|b|-\eta)}{\varepsilon^2(1+\alpha^2)}(h(t)-h(t_0))} \right)' &\leq |b|h'(t) e^{\frac{\alpha(2|b|-\eta)}{\varepsilon^2(1+\alpha^2)}(h(t)-h(t_0))} \\ f(t) e^{\frac{\alpha(2|b|-\eta)}{\varepsilon^2(1+\alpha^2)}(h(t)-h(t_0))} - f(t_0) &\leq |b| \int_{t_0}^t h'(u) e^{\frac{\alpha(2|b|-\eta)}{\varepsilon^2(1+\alpha^2)}(h(u)-h(t_0))} du. \end{aligned}$$

The change of variable  $v = h(u)$  leads to

$$\begin{aligned} f(t) - f(t_0) e^{-\frac{\alpha(2|b|-\eta)}{\varepsilon^2(1+\alpha^2)}(h(t)-h(t_0))} &\leq e^{-\frac{\alpha(2|b|-\eta)}{\varepsilon^2(1+\alpha^2)}(h(t)-h(t_0))} |b| \int_{h(t_0)}^{h(t)} \frac{\alpha(2|b|-\eta)}{\varepsilon^2(1+\alpha^2)}(v-h(t_0)) dv \\ &\leq \frac{|b|\varepsilon^2(1+\alpha^2)}{\alpha(2|b|-\eta)}. \end{aligned}$$

Since  $f(t_0) \geq 0$ , we get for all  $t \geq t_0$

$$f(t) \leq \frac{|b| \varepsilon^2 (1 + \alpha^2)}{\alpha (2|b| - \eta)}.$$

**Step 2:** If we go back to Equation (3.9), we can write

$$\begin{aligned} f'(t) &\geq |b| h'(t)(1 - \delta) - 2\alpha |b| \mathbb{E}\left((|b| - \mu_t \cdot b) I_t\right) \\ f'(t) &\geq |b| h'(t)(1 - \delta) - 2\alpha |b| \frac{f(t)}{h(t)}. \end{aligned}$$

We can reproduce the calculations of *Step 1* (with  $\eta = 0$  and the reverse inequality sign) to find

$$\begin{aligned} f(t) - f(t_0) e^{-\frac{2\alpha|b|}{\varepsilon^2(1+\alpha^2)(1-\delta)}(h(t)-h(t_0))} &\geq \frac{|b| \varepsilon^2 (1 + \alpha^2)(1 - \delta)}{2\alpha |b|} = \frac{\varepsilon^2 (1 + \alpha^2)}{2\alpha} \\ \liminf_{t \rightarrow \infty} f(t) &\geq \frac{\varepsilon^2 (1 + \alpha^2)(1 - \delta)}{2\alpha}. \end{aligned}$$

■

## 4 Hysteresis phenomena

In this section, we want to study the impact of the stochastic perturbation on the reversibility of the system; we are wondering whether the stochastic part may induce an hysteresis phenomenon. In order to observe this, the particle is submitted to an external field linearly varying from  $+\mathbf{b}$  to  $-\mathbf{b}$  where  $\mathbf{b} \in \mathcal{S}(\mathbb{R}^3)$  and with constant direction and bounded modulus. We have seen in Section 3 that when the external field is fixed, the magnetic moment  $\mu$  asymptotically stabilizes along this field. If the external field varies sufficiently slowly compared to the stabilization rate of  $\mu$ , we expect that  $\mu$  will take different back and forth paths when the external field switches from  $+\mathbf{b}$  to  $-\mathbf{b}$  and then from  $-\mathbf{b}$  to  $+\mathbf{b}$ : this is the non reversibility property of the system.

In order to highlight this property, we will study the evolution of a suitably rescaled system on the time interval  $[0, 1]$  and show that the average back and forth paths of  $\mu_t \cdot b$  can not cross at the point  $t = 1/2$ .

We consider a two time scale model: a slower scale for the variations of the external field and a faster scale for the Landau Lifshitz evolution of the magnetic moment.

Let  $\eta > 0$  be a fixed time scale and  $\mathbf{b} \in \mathcal{S}(\mathbb{R}^3)$  the direction of the external field. We define the external field  $b^\eta$  linearly varying between  $+\mathbf{b}$  and  $-\mathbf{b}$  on the interval  $[0, 1/\eta]$  by

$$b^\eta(t) = (1 - 2t \eta) \mathbf{b} \quad \text{for } t \in [0, 1/\eta]$$

We assume that the magnetic moment  $\mu^\eta$  is affected by  $b^\eta(t)$  according to the following equation for  $t \in [0, 1/\eta]$

$$\begin{cases} dY_t^\eta &= -\mu_t^\eta \wedge (b^\eta(t) dt + \varepsilon dW_t) - \alpha \mu_t^\eta \wedge \mu_t^\eta \wedge (b^\eta(t) dt + \varepsilon dW_t) \\ \mu_t^\eta &= \frac{Y_t^\eta}{|Y_t^\eta|} \\ Y_0^\eta &= \mathbf{b} \end{cases}$$

In order to work on the interval  $[0, 1]$ , we introduce rescaled versions of both the external field and the magnetic moment defined for  $t \in [0, 1]$ .

$$b(t) = b^\eta(t/\eta), \quad Z_t = Y_{t/\eta}^\eta, \quad \lambda_t = \mu_{t/\eta}^\eta.$$

Using the time scale property of the stochastic integral, we can write

$$dZ_t = -\lambda_t \wedge \left( b(t) \frac{1}{\eta} dt + \varepsilon dW_{t/\eta} \right) - \alpha \lambda_t \wedge \lambda_t \wedge \left( b(t) \frac{1}{\eta} dt + \varepsilon dW_{t/\eta} \right)$$

From the scaling property of the Brownian motion, we know that  $(\sqrt{\eta}W_{t/\eta})$  is still a Brownian motion. So we get

$$dZ_t = -\lambda_t \wedge \left( b(t) \frac{1}{\eta} dt + \varepsilon \frac{1}{\sqrt{\eta}} dW_t \right) - \alpha \lambda_t \wedge \lambda_t \wedge \left( b(t) \frac{1}{\eta} dt + \varepsilon \frac{1}{\sqrt{\eta}} dW_t \right)$$

It is important to notice that the factor  $\eta$  acts as a time scale parameter for the deterministic part, but that the corresponding scaling parameter for the stochastic part is  $\sqrt{\eta}$ .

Following the proof of Proposition 1, it is obvious to show that  $d|Z_t|^2 = 2(1 + \alpha^2)\varepsilon^2/\eta dt$ . Then, we introduce for  $t \in [0, 1]$

$$h^\eta(t) = \sqrt{2(1 + \alpha^2)\varepsilon^2 t/\eta + 1}.$$

The Ito formula applied to  $(\lambda_t \cdot \mathbf{b})_t$  yields

$$\begin{aligned} d(\lambda_t \cdot \mathbf{b}) &= -(\lambda_t \cdot \mathbf{b}) \frac{h^{\eta'}(t)}{h^\eta(t)} dt + \alpha \frac{1 - 2t}{\eta h^\eta(t)} |\lambda_t \wedge \mathbf{b}|^2 dt \\ &\quad - \frac{\varepsilon}{\sqrt{\eta} h^\eta(t)} ((\lambda_t \wedge dW_t) \cdot \mathbf{b} + \alpha(\lambda_t \cdot \mathbf{b})(\lambda_t \cdot dW_t) - \alpha(\mathbf{b} \cdot dW_t)) \end{aligned}$$

The stochastic part vanishes through when taking expectation as in the previous section to find

$$\mathbb{E}(\lambda_t \cdot \mathbf{b}) - \lambda_0 \cdot \mathbf{b} = \int_0^t -\mathbb{E}(\lambda_u \cdot \mathbf{b}) \frac{h^{\eta'}(u)}{h^\eta(u)} + \alpha \frac{1 - 2u}{\eta h^\eta(u)} \mathbb{E}|\lambda_u \wedge \mathbf{b}|^2 du \quad (4.1)$$

**Proposition 9.** For all  $t \in [0, 1/2]$ ,

$$\mathbb{E}(\lambda_t \cdot \mathbf{b}) \geq \frac{1}{h^\eta(t)} \geq \frac{1}{\sqrt{1 + \frac{(1 + \alpha^2)\varepsilon^2}{\eta}}}.$$

*Proof.* Since the process  $\lambda$  is pathwise continuous and bounded, it is easy to show that the deterministic function  $e(t) : t \mapsto \mathbb{E}(\lambda_t \cdot \mathbf{b})$  is of class  $C^1$ . Hence, we can differentiate Equation (4.1)

$$e'(t) = -e(t) \frac{h^{\eta'}(t)}{h^\eta(t)} + \alpha \frac{1 - 2t}{\eta h^\eta(t)} \mathbb{E}|\lambda_t \wedge \mathbf{b}|^2$$

As  $t \leq 1/2$ , the second term on the r.h.s is non-negative, hence

$$e'(t) \geq -e(t) \frac{h^{\eta'}(t)}{h^\eta(t)}$$

From this inequality, we deduce that  $(e(t)h^\eta(t))' \geq 0$ , which leads to the following lower bound

$$e(t) \geq \frac{1}{h^\eta(t)} \quad \text{for } t \leq 1/2$$

■

## 5 Numerical experiments

In this section, we want to illustrate the theoretical results obtained in Sections 3 and 4 using some numerical simulations.

**Long time behaviour.** We consider the stochastic system (2.1) and discretize it with the help of an Euler scheme  $(\bar{Y}, \bar{\mu})$ , on a time grid with step size  $\delta t > 0$ .

Figure 1 shows the long time behaviour of  $(\bar{\mu}_t \cdot b)_{t \geq 0}$ , for one path of the scheme  $(\bar{Y}, \bar{\mu})$ , with time step size  $\delta t = 0.01$  and for different values of the damping parameter  $\alpha$ . The parameter  $\varepsilon$  is fixed to 0.1, we have taken  $|b| = 1$ , and set  $\mu_0 = -b$ . The almost sure convergence of  $\mu_t \cdot b$  to  $|b|$ , as stated by Theorem 6, is well illustrated by Figure 1 and one can also see how the parameter  $\alpha$  impacts the characteristic time of the system, ie. the time needed to stabilize around the limit.

Now, we wish to compare the rates of convergence studied in Subsection 3.3 to numerical observations. Theorem 7 suggests that the limit of  $\frac{2\alpha\sqrt{2}}{\varepsilon\sqrt{1+\alpha^2}}\sqrt{t} \mathbb{E}(|b| - \mu_t \cdot b)$  when  $t \rightarrow \infty$  should be equal to one. This is illustrated by Figure 2 for different values of  $\alpha$ . This figure confirms that decreasing the parameter  $\alpha$  leads to a decrease of the convergence rate of  $\mathbb{E}(|b| - \mu_t \cdot b)$ .

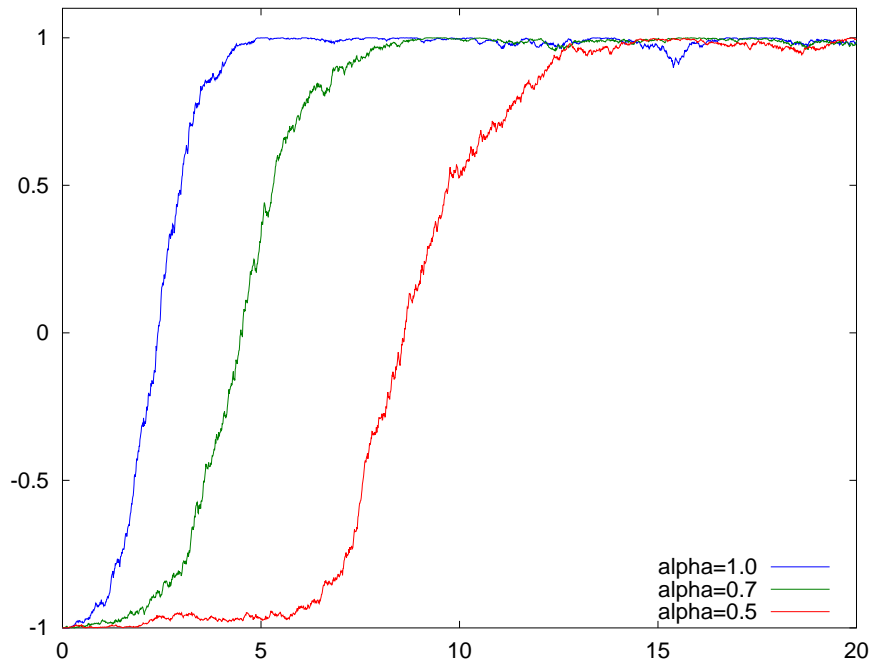


Figure 1: Almost sure convergence of  $\mu_t \cdot b$  with  $\mu_0 = -b$ ,  $|b| = 1$ ,  $\varepsilon = 0.1$ .

**Hysteresis phenomena.** On Figure 3, we can observe a typical pathwise hysteresis phenomenon, which not only illustrates Proposition 9 but also suggests that the result of this Proposition could well improved by proving a almost sure lower bound (probably for sufficiently small values  $\eta$ ). On Figure 3, the forward path (red curve) is almost stuck to the

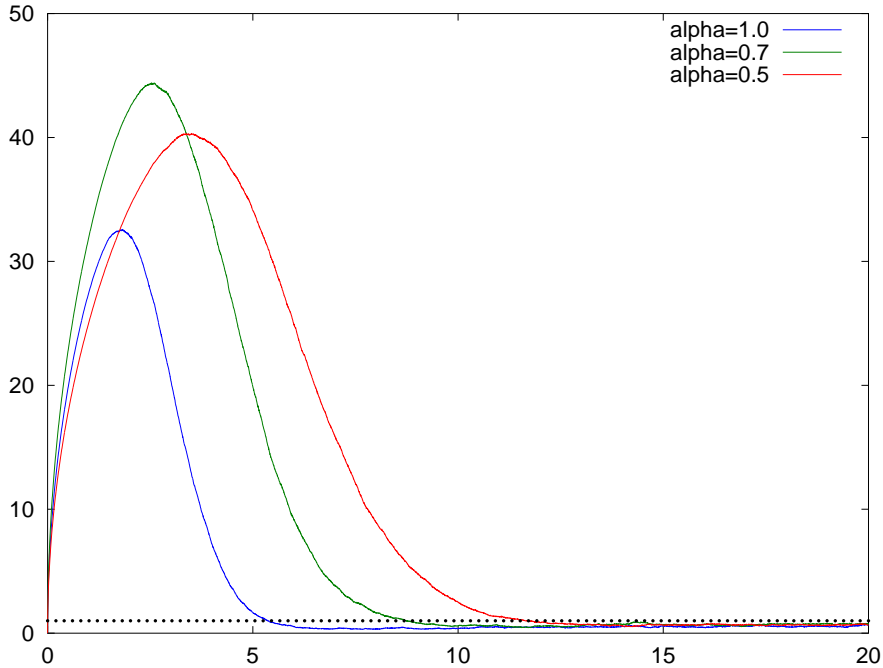


Figure 2: Convergence of  $\frac{2\alpha\sqrt{2}}{\varepsilon\sqrt{1+\alpha^2}}\sqrt{t} \mathbb{E}(|b| - \mu_t.b)$  with  $\mu_0 = -b$ ,  $|b| = 1$  and  $\varepsilon = 0.1$ . The horizontal dashed line is at level one. The expectation is computed using a Monte–Carlo method with 100 samples.

value 1 on the interval  $[0, 1]$ , we could then be tempted to think that the lower bound of Proposition 9 lacks some accuracy.

On the contrary, when  $\eta$  becomes small, which corresponds to a slower scale for the variations of the external field, Figures 4 and 5 show that the lower bound  $1/h^\eta(t)$  becomes nearly optimal for  $t$  lower but close to  $1/2$ .

## 6 Conclusion

In this article, we analyzed the long time behaviour of a non linear SDE modeling the evolution of a magnet submitted to a perturbed external field. The rate of convergence of the magnetic moment is particularly interesting and holds for any dissipation coefficient  $\alpha > 0$ . This result has been obtained by combining the ODE technique for dynamic systems with localisation techniques. The second results concerns the hysteresis behaviour of the system induced by the stochastic perturbation. The combination of these two results illustrated the dissipative effects of the stochastic perturbation on the system by giving a first glimpse of thermal effects on a ferromagnet can be modeled in the framework of micromagnetism.

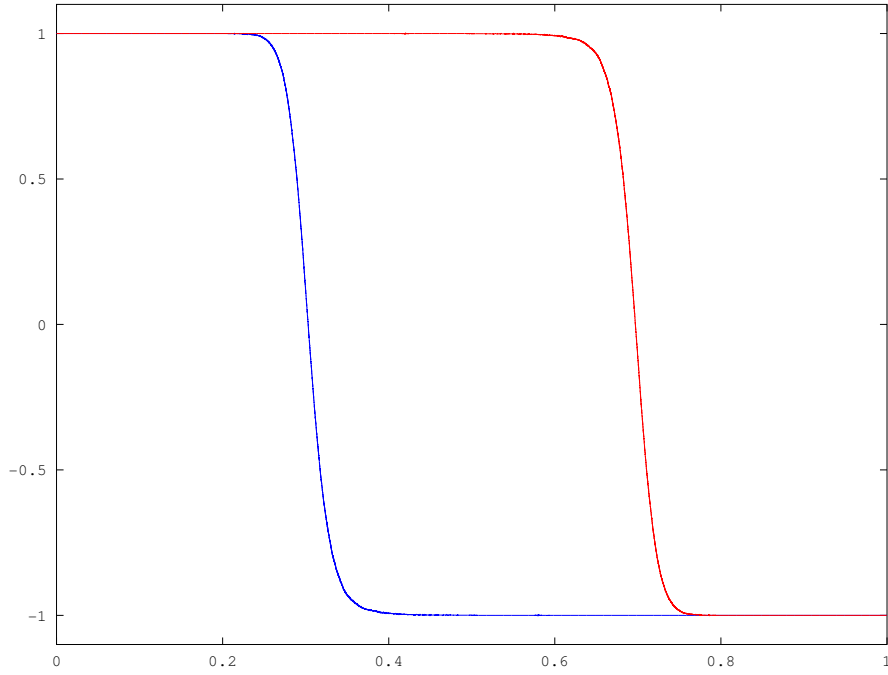


Figure 3: Pathwise hysteresis phenomena with  $\alpha = 1$ ,  $\varepsilon = 0.005$  and  $\eta = 0.01$ . The red curve is the forward path whereas the blue curve is the backward path.

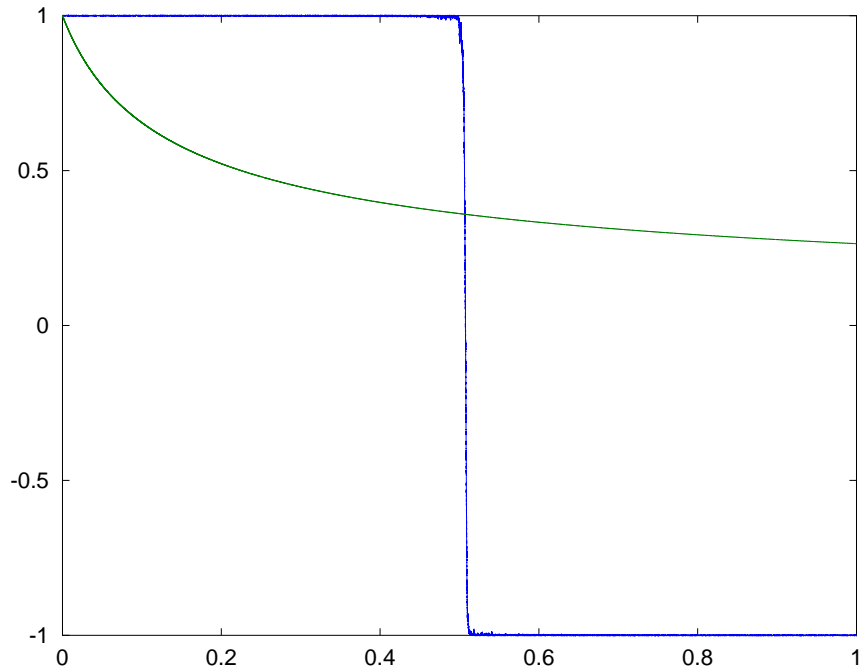


Figure 4: Pathwise hysteresis phenomena with  $\alpha = 1$ ,  $\varepsilon = 0.01$  and  $\eta = 3.1E - 5$ . The blue curve is the evolution of  $\mu_t \cdot \mathbf{b}$  and the green curve is  $1/h^\eta(t)$ .



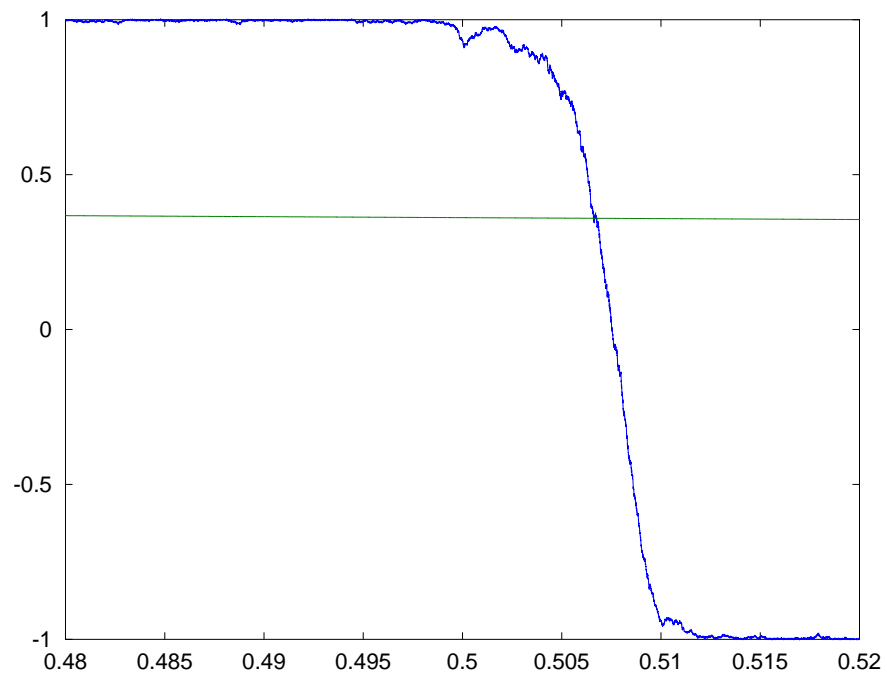


Figure 5: Zoom of Figure 4 around  $t = 1/2$ . The blue curve is the evolution of  $\mu_t \cdot \mathbf{b}$  and the green curve is  $1/h^\eta(t)$ .

## A Some auxiliary results

**Lemma 10.** *Let  $W$  be a standard Brownian motion defined on a filtered probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$  and  $H$  a continuous process adapted to the filtration  $(\mathcal{F}_t)_t$  and bounded by 1. Let  $S$  and  $T$  be two positive random variables such that  $S$  and  $S + T$  are two  $(\mathcal{F}_t)_t$ -stopping times. Then,*

$$E \left[ \left( \int_S^{(S+T) \wedge (S+t)} H_u dW_u \right)^2 \right] \leq t$$

*Proof.* Let us introduce  $W^{(S)}$  be defined by  $W_u^{(S)} = W_{S+u} - W_S$  for  $u \geq 0$ . The process  $W^{(S)}$  is Brownian motion adapted to the shifted filtration  $(\mathcal{F}'_u = \mathcal{F}_{S+u})_u$ . We can perform a shift of time in the stochastic integral to get

$$\int_S^{(S+T) \wedge (S+t)} H_u dW_u = \int_0^{T \wedge t} H_{u+S} dW_u^{(S)}$$

Note that  $T$  is actually a  $\mathcal{F}'$  stopping time. Moreover, as  $H$  is bounded, the stopped process  $(\int_0^{T \wedge t} H_{u+S} dW_u^{(S)})_t$  is martingale bounded in  $\mathbb{L}^2$  and Itô's isometry yields that

$$\mathbb{E} \left[ \left( \int_0^{T \wedge t} H_{u+S} dW_u^{(S)} \right)^2 \right] = \mathbb{E} \int_0^{T \wedge t} H_{u+S}^2 du \leq t. \quad \blacksquare$$

**Lemma 11.** *Let  $X$  be a real valued random variable such that the function  $x \mapsto \mathbb{P}(X = x)$  is right continuous, then for all  $x$ ,  $\mathbb{P}(X = x) = 0$ .*

*Proof.* Assume there exist  $x_0$  s.t.  $\mathbb{P}(X = x_0) \geq 2\varepsilon > 0$ . The right continuity ensures the existence of a right neighbourhood  $[x_0, x_1]$  with  $x_1 > x_0$  such that for all  $x \in [x_0, x_1]$ ,  $\mathbb{P}(X = x) \geq \varepsilon$ . Let  $n$  be an integer larger than  $1/\varepsilon$ , we consider  $\{y_1, \dots, y_n\} \subset [x_0, x_1]$ . Then,  $\mathbb{P}(X \in [x_0, x_1]) \geq \mathbb{P}(X \in \{y_1, \dots, y_n\}) = \sum_{i=1}^n \mathbb{P}(X = y_i) \geq \frac{n}{\varepsilon} > 1$ , which contradicts the existence of  $x_0$  such that  $\mathbb{P}(X = x_0) > 0$ .  $\blacksquare$

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