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Pierre-Vincent Koseleff, Daniel Pecker

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Harmonic knots

P. -V. Koseleff, D. Pecker

Université Pierre et Marie Curie
4, place Jussieu, F-75252 Paris Cedex 05
e-mail: {koseleff,pecker}@math.jussieu.fr

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Abstract

The harmonic knot $H(a, b, c)$ is parameterized as $K(t) = (T_a(t), T_b(t), T_c(t))$ where a, b and c are relatively coprime integers and T_n is the degree n Chebyshev polynomial of the first kind. We classify the harmonic knots $H(a, b, c)$ for $a \leq 4$. We show that the knot $H(2n - 1, 2n, 2n + 1)$ is isotopic to $H(4, 2n - 1, 2n + 1)$ (up to mirror symmetry). We study the knots $H(5, n, n + 1)$ and give a table of the simplest harmonic knots.

keywords: Polynomial curves, Chebyshev curves, rational knots, continued fractions
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Contents

| | | |
|----------|---|-----------|
| 1 | Introduction | 2 |
| 2 | Continued fractions and rational Chebyshev knots | 3 |
| 2.1 | Continued fractions | 4 |
| 2.2 | Continued fractions $[\pm 1, \pm 2, \dots, \pm 1, \pm 2]$ | 5 |
| 3 | The harmonic knots $H(a, b, c)$ | 7 |
| 3.1 | The harmonic knots $H(4, b, c)$ | 8 |
| 3.2 | Proof of theorem 3.6 | 10 |
| 4 | Some families with $a \geq 5$ | 14 |
| 4.1 | The harmonic knots $H(2n - 1, 2n, 2n + 1)$ | 14 |
| 4.2 | The harmonic knots $H(5, k, k + 1)$ | 17 |
| 4.3 | Some new findings on harmonic knots | 20 |

1 Introduction

A harmonic curve (or Chebyshev curve) is defined to be a curve which admits a parametrization $x = T_a(t)$, $y = T_b(t)$, $z = T_c(t)$ where a , b and c are relatively coprime integers and $T_n(t)$ are the classical Chebyshev polynomials defined by $T_n(\cos t) = \cos nt$. A harmonic knot is a non singular harmonic curve. In 1897 Comstock proved that a harmonic curve is a knot if and only if a, b, c are pairwise coprime integers ([Com, KP3, FF]).

We observed in [KP1] that the trefoil can be parametrized by Chebyshev polynomials: $x = T_3(t)$; $y = T_4(t)$; $z = T_5(t)$. This led us to study harmonic knots in [KP3].

Harmonic knots are polynomial analogues of the famous Lissajous knots ([BDHZ, BHJS, Cr, HZ, JP, La1, La2]). The symmetries of harmonic knots, obvious from the parity of Chebyshev polynomials, are different from those of Lissajous. For example, the figure-eight knot which is not a Lissajous knot, is the harmonic knot $H(3, 5, 7)$.

We proved that the harmonic knot $H(a, b, ab - a - b)$ is alternating, and deduced that there are infinitely many amphicheiral harmonic knots and infinitely many strongly invertible harmonic knots. We also proved in [KP3] that the torus knot $T(2, 2n + 1)$ is the harmonic knot $H(3, 3n + 2, 3n + 1)$.

The harmonic knots $H(3, b, c)$ are classified in [KP4]; they are two-bridge knots and their Schubert fractions $\frac{\alpha}{\beta}$ verify $\beta^2 \equiv \pm 1 \pmod{\alpha}$.

In this article, we give the classification of the harmonic knots $H(4, b, c)$ for b and c coprime odd integers. We also study some infinite families of harmonic knots for $a \geq 5$.

In section 2 we recall the Conway notation for two-bridge knots, and the computation of their Schubert fractions with continued fractions. We observe that the knots $H(4, b, c)$ are two-bridge knots, and their Schubert fractions are given by continued fractions of the form $[\pm 1, \pm 2, \dots, \pm 1, \pm 2]$. We show results on these continued fraction expansions. In section 3 we compute the Schubert fractions of the harmonic knots $H(4, b, c)$. We deduce the classification of these knots:

Theorem 3.7. *Let b and c be relatively prime odd integers, and let $K = H(4, b, c)$. There is a unique pair (b', c') such that (up to mirroring)*

$$K = H(4, b', c'), \quad b' < c' < 3b', \quad b' \not\equiv c' \pmod{4}.$$

K has a Schubert fraction $\frac{\alpha}{\beta}$ such that $\beta^2 \equiv \pm 2 \pmod{\alpha}$. Furthermore, there is an algorithm to find (b', c') , and the crossing number of K is $N = (3b' + c' - 2)/4$.

We notice that the trefoil is the only knot which is both of form $H(3, b, c)$ and $H(4, b, c)$. In section 4 we study some families of harmonic knots $H(a, b, c)$ with $a \geq 5$. In general the bridge number of these knots is greater than two, this is why the following result is surprising.

Theorem 4.5. *The harmonic knot $H(2n - 1, 2n, 2n + 1)$ is isotopic to the two-bridge harmonic knot $H(4, 2n - 1, 2n + 1)$, up to mirror symmetry.*

We also find an infinite family of two-bridge harmonic knots which are not of the form $H(a, b, c)$ for $a \leq 4$:

Theorem 4.6.

The knot $H(5, 5n + 1, 5n + 2)$ is the two-bridge knot of Conway form $C(2n + 1, 2n)$.

The knot $H(5, 5n + 3, 5n + 4)$ is the two-bridge knot of Conway form $C(2n + 1, 2n + 2)$.

Except for $H(5, 6, 7) = H(4, 5, 7)$ and $H(5, 3, 4)$, these knots are not of the form $H(a, b, c)$ with $a \leq 4$.

Then, we give an example of a composite harmonic knot, which disproves a conjecture of Freudentburg and Freudentburg [FF]. We show that the nonreversible knot 8_{17} is a harmonic knot.

Then, we identify the knots $H(a, b, c)$ for $(a - 1)(b - 1) \leq 30$. We conclude the paper with some questions and conjectures.

2 Continued fractions and rational Chebyshev knots

A two-bridge knot (or link) admits a diagram in Conway's normal form. This form, denoted by $C(a_1, a_2, \dots, a_n)$ where a_i are integers, is explained by the following picture (see [Con], [Mu] p. 187). The number of twists is denoted by the integer $|a_i|$, and the sign of a_i is

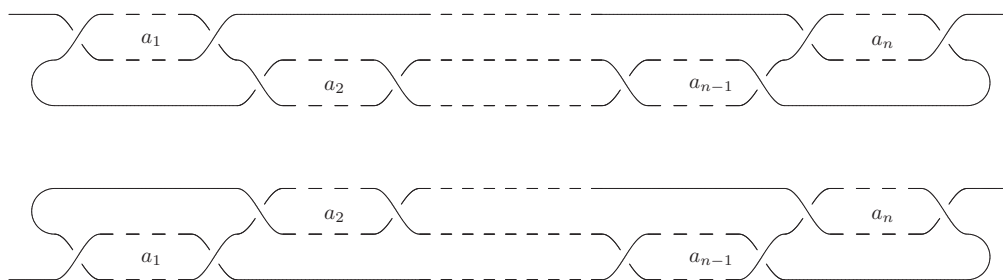


Figure 1: Conway normal forms for polynomial knots (n odd, and n even)

defined as follows: if i is odd, then the right twist is positive, if i is even, then the right twist is negative. In Figure 1 the a_i are positive (the a_1 first twists are right twists).

The two-bridge knots (or links) are classified by their Schubert fractions

$$\frac{\alpha}{\beta} = a_1 + \frac{1}{a_2 + \frac{1}{\dots + \frac{1}{a_n}}}, \quad \alpha > 0.$$

We shall denote $S(\frac{\alpha}{\beta})$ a two-bridge knot (or link) with Schubert fraction $\frac{\alpha}{\beta}$. The two-bridge knots (or links) $S(\frac{\alpha}{\beta})$ and $S(\frac{\alpha'}{\beta'})$ are equivalent if and only if $\alpha = \alpha'$ and $\beta' \equiv \beta^{\pm 1} \pmod{\alpha}$. If $K = S(\frac{\alpha}{\beta})$, its mirror image is $\overline{K} = S(\frac{\alpha}{-\beta})$.

We shall study knots with a diagram illustrated by figure 2. In this case, the a_i and the

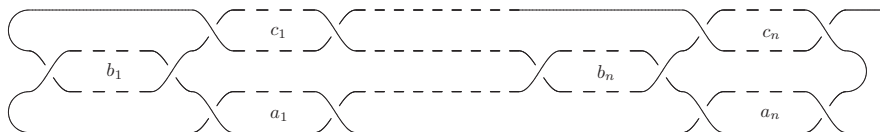


Figure 2: A knot isotopic to $C(b_1, a_1 + c_1, b_2, a_2 + c_2, \dots, b_n, a_n + c_n)$

c_i are positive if they are left twists, the b_i are positive if they are right twists (in our figure a_i, b_i, c_i are positive). Such a knot is equivalent to a knot with Conway's normal form $C(b_1, a_1 + c_1, b_2, a_2 + c_2, \dots, b_n, a_n + c_n)$ (see [Mu] p. 183-184). Many of our knots have a Chebyshev diagram $\mathcal{C}(4, k) : x = T_4(t), y = T_k(t)$. In this case we obtain diagrams of the form illustrated by Figure 2. Consequently, such a knot has a Schubert fraction of the form $[b_1, d_1, b_2, d_2, \dots, b_n, d_n]$ with $b_i = \pm 1, d_i = \pm 2$ or $d_i = 0$.

Once again, the situation is best explained by typical examples. Figure 3 represents two knots with the same Chebyshev diagram $\mathcal{C}(4, 5) : x = T_4(t), y = T_5(t)$. A Schubert fraction of the first knot is $\frac{5}{2} = [1, 0, 1, 2]$; it is the figure-eight knot. A Schubert fraction of the second knot is $\frac{7}{-4} = [-1, -2, 1, 2]$; it is the twist knot 5_2 . By symmetry, the Chebyshev diagrams

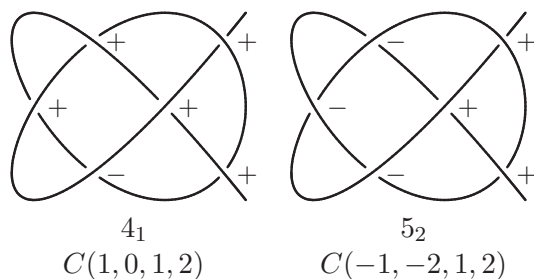


Figure 3: Knots with the Chebyshev diagram $\mathcal{C}(4, 5)$

of harmonic knots $H(4, b, c)$ are of Conway form $C(\pm 1, \pm 2, \dots, \pm 1, \pm 2)$. Consequently, the diagram $C(1, 0, 1, 2)$ is not the Chebyshev diagram of a harmonic knot $H(4, b, c)$.

2.1 Continued fractions

Let K be a two-bridge knot defined by a continued fraction expansion of its Schubert fraction $\frac{\alpha}{\beta} = [q_1, q_2, \dots, q_n]$, where the q_i are not necessarily positive. It is often possible to

obtain directly the crossing number of K .

Definition 2.1. Let $r > 0$ be a rational number, and $r = [q_1, \dots, q_n]$ be a continued fraction with $q_i > 0$. The crossing number of r is defined by $\text{cn}(r) = q_1 + \dots + q_n$.

The following result is proved in [KP4].

Proposition 2.2. Let $\frac{\alpha}{\beta} = [a_1, \dots, a_n]$ be a continued fraction such that $a_1 a_2 > 0$, $a_{n-1} a_n > 0$, and there is no two consecutive sign changes in the sequence a_1, a_2, \dots, a_n . Then its crossing number is

$$\text{cn}\left(\frac{\alpha}{\beta}\right) = \sum_{k=1}^n |a_k| - \#\{j, a_j a_{j+1} < 0\}. \quad (1)$$

2.2 Continued fractions $[\pm 1, \pm 2, \dots, \pm 1, \pm 2]$

We begin with a useful lemma:

Lemma 2.3. Let $r = [1, 2e_2, e_3, 2e_4, \dots, e_{2m-1}, 2e_{2m}]$, $e_i = \pm 1$. We suppose that there are no three consecutive sign changes in the sequence e_1, \dots, e_{2m} . Then $r > 0$, and $r > 1$ if and only if $e_2 = 1$.

Proof. By induction on m .

If $m = 1$, then $r = [1, 2] = \frac{3}{2}$ or $r = [1, -2] = \frac{1}{2}$, and the result is true.

Suppose the result true for $m - 1$, and let us prove it for m .

First, let us suppose $r = [1, 2, e_3, \dots, 2e_{2m}]$.

If $e_3 = 1$, then $r = [1, 2, y] = \frac{3y+1}{2y+1}$, where $y = [1, \pm 2, \dots]$. By induction we have $y > 0$, and then $r > 1$.

If $e_3 = -1$ and $e_4 = 1$. Then $e_5 = 1$ and $r = [1, 2, -1, 2, y] = y + 2$ with $y = [1, \pm 2, \dots]$. We have $y > 0$ by induction, and then $r > 2 > 1$.

If $e_3 = e_4 = -1$, then $r = [1, 2, -y] = \frac{3y-1}{2y-1} = \frac{3}{2} + \frac{1}{2(2y-1)}$ with $y = [1, 2, \pm 1, \dots]$. We have $y > 1$ by induction, and then $r > \frac{3}{2} > 1$.

Now, let us suppose $r = [1, -2, \dots]$.

If $r = [1, -2, -1, \dots]$. Then $r = [1, -2, -y] = \frac{y+1}{2y+1}$, with $y = [1, \pm 2, \dots]$. By induction, we have $y > 0$, and then $0 < r < 1$.

If $r = [1, -2, 1, \dots]$. Then $r = [1, -2, 1, 2, \dots] = [1, -2, y] = \frac{y-1}{2y-1}$ where $y = [1, 2, \pm 1, \dots]$. By induction we have $y > 1$, and then $0 < r < 1$.

This completes the proof. \square

Remark 2.4. Because of the identities $x = [1, -2, 1, -2, x]$ and $x = [2, -1, 2, -1, x]$, we see that the condition on the sign changes is necessary. It is also necessary in the following theorem.

Theorem 2.5. Let $r = \frac{\alpha}{\beta} > 0$ be a fraction with α odd and β even. There is a unique continued fraction expansion $r = [1, \pm 2, \dots, \pm 1, \pm 2]$ without three consecutive sign changes.

Proof. The existence of this continued fraction expansion is given in [KPR]. Its uniqueness is a direct consequence of lemma 2.3. \square

The next result will be useful to describe the continued fractions of harmonic knots $H(4, b, c)$.

Proposition 2.6. Let $r = \frac{\alpha}{\beta}$ be a rational number given by a continued fraction of the form $r = [e_1, 2e_2, e_3, 2e_4, \dots, e_{2m-1}, 2e_{2m}]$, $e_1 = 1$, $e_i = \pm 1$. We suppose that the sequence of sign changes is palindromic, that is $e_k e_{k+1} = e_{2m-k} e_{2m-k+1}$ for $k = 1, \dots, 2m - 1$.

Then we have $\beta^2 \equiv \pm 2 \pmod{\alpha}$.

Proof. We shall use the Möbius transformations

$$A(x) = [1, x] = \frac{x+1}{x+0}, \quad B(x) = [2, x] = \frac{2x+1}{x+0}, \quad S(x) = -x$$

and their matrix notations

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad AB = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \quad ASB = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.$$

We shall consider the mapping (analogous to matrix transposition)

$$\tau : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & \frac{c}{2} \\ 2b & d \end{bmatrix}.$$

We have $\tau(XY) = \tau(Y)\tau(X)$, $\tau(AB) = AB$, $\tau(ASB) = ASB$ and $\tau(S) = S$.

Let G be the Möbius transformation defined by $G(z) = [1, 2e_2, e_3, 2e_4, \dots, e_{2m-1}, 2e_{2m}, z]$, we have $\frac{\alpha}{\beta} = G(\infty)$. Let us write $G = X_1 \cdots X_n$ where $X_i = A, B$ or S , $X_1 = A$ and $X_n = B$. One can suppose that G contains no subsequence of the form AA, ASA, BB, SS and BSB . Moreover, the palindromic condition means that if $X_i = S$, then $X_{n+1-i} = S$.

Let us show that if $P = X_1 \cdots X_n$ is a product of terms A, B, S having these properties, then $\tau(P) = P$. By induction on $s = \#\{i, X_i = S\}$.

If $s = 0$ then $P = (AB)^m$, and $\tau(P) = (\tau(AB))^m = (AB)^m = P$.

Let $k = \min\{i, X_i = S\}$.

If $k = 2q+1$ then $q \geq 1$ and $P = (AB)^q S P' S (AB)^q$. By induction we have $\tau(P') = P'$, and then $\tau(P) = \tau((AB)^q) \tau(S) \tau(P') \tau(S) \tau((AB)^q) = P$.

If $k = 2q$ then $P = (AB)^{q-1} (ASB) P' (ASB) (AB)^{q-1}$. By induction we have $\tau(P') = P'$, and then $\tau(P) = P$. This concludes our induction proof.

Consequently we have $\tau(G) = G$. Since $G \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, we see that $G = \begin{bmatrix} \alpha & \gamma \\ \beta & \lambda \end{bmatrix}$, with $\beta = 2\gamma$. Since $\det(G) = \pm 1$, we obtain $\beta^2 \equiv \pm 2 \pmod{\alpha}$. \square

3 The harmonic knots $H(a, b, c)$

We shall first show some properties of the plane Chebyshev curves $x = T_a(t)$, $y = T_b(t)$. We shall need the following result proved in [KP3]:

Proposition 3.1. *Let a and b be coprime integers. The $\frac{1}{2}(a-1)(b-1)$ double points of the Chebyshev curve $x = T_a(t)$, $y = T_b(t)$ are obtained for the parameter pairs*

$$t = \cos\left(\frac{k}{a} + \frac{h}{b}\right)\pi, \quad s = \cos\left(\frac{k}{a} - \frac{h}{b}\right)\pi,$$

where h, k are positive integers such that $\frac{k}{a} + \frac{h}{b} < 1$.

Using the symmetries of Chebyshev polynomials, we see that this set of parameters is symmetrical about the origin. Let us define a right twist as in Figure 4(a) and a left twist as in Figure 4(b); this notion depends on the choice of the coordinate axes.

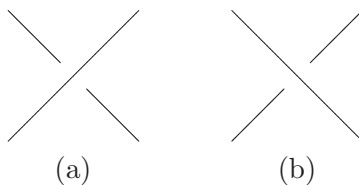


Figure 4: A right twist (a) and a left twist (b).

We will write $x \sim y$ when $\text{sign}(x) = \text{sign}(y)$. We shall need the following result proved in [KP3, KPR].

Lemma 3.2. *Let $H(a, b, c)$ be a harmonic knot.*

A crossing point of parameter $t = \cos\left(\frac{k}{a} + \frac{h}{b}\right)\pi$ is a right twist if and only if

$$D = \left(z(t) - z(s)\right)x'(t)y'(t) > 0$$

where

$$z(t) - z(s) \sim -\sin\left(\frac{ch}{b}\pi\right)\sin\left(\frac{ck}{a}\pi\right) \text{ and } x'(t)y'(t) \sim (-1)^{h+k}\sin\left(\frac{ah}{b}\pi\right)\sin\left(\frac{bk}{a}\pi\right).$$

Remark 3.3. The sign of $\sin(r\pi)$ is simply given by $(-1)^{\lfloor r \rfloor}$.

From this lemma we immediately deduce

Corollary 3.4. *Let a, b, c be coprime integers. Suppose that the integer c' verifies $c' \equiv c \pmod{2a}$ and $c' \equiv -c \pmod{2b}$. Then the knot $H(a, b, c')$ is the mirror image of $H(a, b, c)$.*

Proof. At each crossing point we have $T_{c'}(t) - T_{c'}(s) = -(T_c(t) - T_c(s))$. \square

Corollary 3.5. *Let a, b, c be coprime integers. Suppose that the integer c is of the form $c = \lambda a + \mu b$ with $\lambda, \mu > 0$. Then there exists $c' < c$ such that $H(a, b, c) = \overline{H(a, b, c')}$*

Proof. Let $c' = |\lambda a - \mu b|$. The result follows immediately from corollary 3.4 \square

This corollary is often used to reduce the degree of a harmonic knot. In a recent paper, G. and J. Freudenburg proved the following stronger result. *There is a polynomial automorphism Φ of \mathbf{R}^3 such that $\Phi(H(a, b, c)) = H(a, b, c')$.* They also conjectured that for any coprime integers $a < b$, the knots $H(a, b, c)$, $a < b < c$, $c \neq \lambda a + \mu b$, $\lambda, \mu > 0$ are different knots ([FF], Conjecture 6.2).

In [KP4] we obtained the Schubert fractions of the harmonic knots $H(3, b, c)$. We deduced their classification, which provides a proof of Freudenburg's conjecture for $a = 3$. We shall follow the same strategy to study the harmonic knots $H(4, b, c)$.

3.1 The harmonic knots $H(4, b, c)$.

The following result will allow us to classify the harmonic knots of the form $H(4, b, c)$.

Theorem 3.6. *Let b, c be odd integers such that $b \not\equiv c \pmod{4}$. The Schubert fraction of the knot $K = H(4, b, c)$ is given by the continued fraction*

$$\frac{\alpha}{\beta} = [e_1, 2e_2, e_3, 2e_4, \dots, e_{b-2}, 2e_{b-1}], \text{ where } e_j = \text{sign}(\sin(j-b)\theta), \theta = \frac{3b-c}{4b}\pi.$$

We have $\beta^2 \equiv \pm 2 \pmod{\alpha}$. If $b < c < 3b$, then the crossing number of K is $N = (3b+c-2)/4$.

The proof will be given in section 3.2, p. 10.

We are now able to classify the harmonic knots of the form $H(4, b, c)$.

Theorem 3.7. *Let b and c be relatively prime odd integers, and let $K = H(4, b, c)$. There is a unique pair (b', c') such that (up to mirroring)*

$$K = H(4, b', c'), \quad b' < c' < 3b', \quad b' \not\equiv c' \pmod{4}.$$

K has a Schubert fraction $\frac{\alpha}{\beta}$ such that $\beta^2 \equiv \pm 2 \pmod{\alpha}$. Furthermore, there is an algorithm to find (b', c') , and the crossing number of K is $N = (3b' + c' - 2)/4$.

Proof. Let us first prove the uniqueness of this pair. Let $K = H(4, b, c)$ with $b < c < 3b$, $c \not\equiv b \pmod{4}$. We thus have $c = 3b \pmod{4}$, so $\lambda = \frac{1}{4}(3b - c)$ is an integer and by Theorem 3.6, K admits the Schubert fraction $\frac{\alpha}{\beta} = [e_1, 2e_2, e_3, 2e_4, \dots, e_{b-2}, 2e_{b-1}]$ where $e_j = \text{sign}(\sin(j - b)\theta) = (-1)^\lambda \text{sign}(\sin j\theta)$.

As $0 < \theta < \frac{\pi}{2}$, we deduce that e_1 and e_2 have the same signs. Furthermore $|\beta| < \alpha$ by Lemma 2.3, $\beta^2 = \pm 2 \pmod{\alpha}$ by Theorem 3.6 and β is even by Theorem 2.5. We then have $\alpha \neq 5$.

Suppose that $\frac{\alpha}{\beta'}$ is another Schubert fraction of K with $|\beta'| \leq \alpha$, $\beta'^2 = \pm 2 \pmod{\alpha}$ and β' even. We then must have $\beta\beta' \equiv 1 \pmod{\alpha}$ so $\pm 4 \equiv 1 \pmod{\alpha}$. We thus deduce that $\frac{\alpha}{\beta} = \frac{3}{2}$, and then K is a trefoil.

In any case $\frac{\alpha}{\beta}$ is the unique Schubert fraction of K which satisfies $|\beta| < \alpha$, $\beta^2 = \pm 2 \pmod{\alpha}$ and β even. The integer $b - 1$ is then the length of the continued fraction expansion of $\frac{\alpha}{\beta} = [e_1, 2e_2, e_3, 2e_4, \dots, e_{b-2}, 2e_{b-1}]$. Since we also have $3b + c - 2 = 4 \text{cn}(K)$, we conclude that (b, c) is uniquely determined by K .

Now, let us prove the existence of the pair (b', c') . Let $K = H(4, b, c)$, $b < c$. We have only to show that if the pair (b, c) does not satisfy the condition of the theorem, then it is possible to reduce it.

If $c \equiv b \pmod{4}$, then $c = b + 4\mu$, $\mu > 0$, and we can reduce the pair (b, c) by corollary 3.5.

If $c \not\equiv b \pmod{4}$ and $c > 3b$, we have $c = 3b + 4\mu$, $\mu > 0$, and we can reduce (b, c) by 3.5. \square

Remark 3.8. Our theorem gives a positive answer to the Freudentburg conjecture ([FF, 6.2]) for $a = 4$: *the knots $H(4, b, c)$, $4 < b < c$, $c \neq 4\lambda + \mu b$, $\lambda, \mu > 0$ are different knots.*

We also see that the only knot belonging to the two families of knots $H(3, b, c)$ and $H(4, b, c)$ is the trefoil $H(3, 4, 5) = \overline{H}(4, 3, 5)$.

Corollary 3.9. *The harmonic knot $H(4, 2k - 1, 2k + 1)$ is the two-bridge knot of Conway form $C(3, 2, \dots, 2)$ and crossing number $2k - 1$.*

Proof. By Theorem 3.6, the knot $H_k = H(4, 2k - 1, 2k + 1)$ has crossing number $2k - 1$ and Conway form $C(e_1, 2e_2, \dots, e_{2k-3}, 2e_{2k-2})$, where $e_j = \text{sign}(\sin(j - b)\theta)$, $\theta = \frac{\pi}{2}(1 - \frac{1}{2k-1})$.

Since the knots $C(a_1, \dots, a_{2m})$ and $C(-a_{2m}, \dots, -a_1)$ are isotopic, we deduce that H_k is isotopic to the knot $C(2\varepsilon_1, \varepsilon_2, \dots, 2\varepsilon_{2k-3}, \varepsilon_{2k-2})$ where $\varepsilon_i = \text{sign}(\sin i\theta) = (-1)^{\lfloor \frac{i-1}{2} \rfloor}$.

We deduce that the rational number $r_k = [2, 1, -2, -1, \dots, (-1)^{k-2}2, (-1)^{k-2}]$ (length $2k - 2$) is a Schubert fraction of H_k . We have $r_2 = 3$, and $r_k = [2, 1, -r_{k-1}]$. Using the identity $[2, 1, x] = [3, x - 1]$, by an easy induction we obtain $r_k = [3, 2, \dots, 2]$. \square

Example 3.10 (The Twist knots). The Twist knots $\mathcal{T}_n = C(n, 2)$ are not harmonic knots $H(4, b, c)$ for $n > 3$. They are not harmonic knots $H(3, b, c)$ for $n > 2$.

Proof. The Schubert fractions of $\mathcal{T}_n = S(n + \frac{1}{2})$ (or $\overline{\mathcal{T}}_n$) with an even denominator are $\frac{2n+1}{2}$, and $\frac{2n+1}{-n}$ or $\frac{2n+1}{n+1}$ according to the parity of n . The only such fractions verifying $\beta^2 \equiv \pm 2 \pmod{\alpha}$ are $\frac{3}{2}, \frac{7}{4}, \frac{9}{4}$. The first two are the Schubert fractions of the trefoil and the $\overline{5}_2$ knot, which are harmonic for $a = 4$. We have only to study the case of $6_1 = S(\frac{9}{4})$. We have $\frac{9}{4} = [1, 2, -1, 2, 1, -2, 1, 2]$. Since this continued fraction expansion has two consecutive sign changes, we see that 6_1 is not of the form $H(4, b, c)$. \square

But there also exist infinitely many rational knots whose Schubert fractions $\frac{\alpha}{\beta}$ satisfy $\beta^2 \equiv -2 \pmod{\alpha}$ that are not harmonic knots for $a = 4$.

Proposition 3.11. *The knots $S(n + \frac{1}{2n})$ are not harmonic knots $H(4, b, c)$ for $n > 1$. Their crossing number is $3n$ and their Schubert fractions $\frac{\alpha}{\beta} = \frac{2n^2 + 1}{2n}$ satisfy $\beta^2 \equiv -2 \pmod{\alpha}$.*

Proof. We shall use the Möbius transformations $F(x) = [1, 2, x] = \frac{3x+1}{2x+1}$,
 $C(x) = [1, 2, -1, 2, x] = x+2$, $D(x) = [1, -2, 1, 2, x] = \frac{x}{4x+1}$, and $D^k(x) = \frac{x}{4kx+1}$.
 If $n = 2k$, we have $C^k(x) = 2k+x$ and $D^k(\infty) = \frac{1}{4k}$, and then

$$n + \frac{1}{2n} = C^k D^k(\infty).$$

If $n = 2k+1$, we have $FD^k(\infty) = F(\frac{1}{4k}) = \frac{2n+1}{2n}$, and then

$$n + \frac{1}{2n} = n - 1 + \frac{2n+1}{2n} = C^k FD^k(\infty).$$

These continued fractions are such that $\beta^2 \equiv -2 \pmod{\alpha}$. Nevertheless, for $n > 1$ these continued fractions have two consecutive sign changes, and therefore they do not correspond to harmonic knots $H(4, b, c)$. \square

3.2 Proof of theorem 3.6

The crossing points of the plane projection of $H = H(4, b, c)$ are obtained for parameter pairs (t, s) where $t = \cos(\frac{m}{4b}\pi)$, $s = \cos(\frac{m'}{4b}\pi)$. We shall denote $\lambda = \frac{3b-c}{4}$ (or $c = 3b - 4\lambda$), and $\theta = \frac{\lambda}{b}\pi$. We will consider the two following cases.

The case $b = 4n + 1$.

For $k = 0, \dots, n - 1$, let us consider the following crossing points

- A_k corresponding to $m = 4k + 1$, $m' = 2b - m$,
- B_k corresponding to $m = 4k + 2$, $m' = 4b - m$,
- C_k corresponding to $m = 4k + 3$, $m' = 2b + m$,
- D_k corresponding to $m = 2b - 4(k + 1)$, $m' = 4b - m$.

Then we have

- $x(A_k) = \cos\left(\frac{4k + 1}{b}\pi\right)$, $y(A_k) = (-1)^k \cos\frac{\pi}{4} \neq 0$,
- $x(B_k) = \cos\left(\frac{4k + 2}{b}\pi\right)$, $y(B_k) = 0$,
- $x(C_k) = \cos\left(\frac{4k + 3}{b}\pi\right)$, $y(C_k) = (-1)^k \cos\frac{3\pi}{4} \neq 0$,
- $x(D_k) = \cos\left(\frac{4k + 4}{b}\pi\right)$, $y(D_k) = 0$.

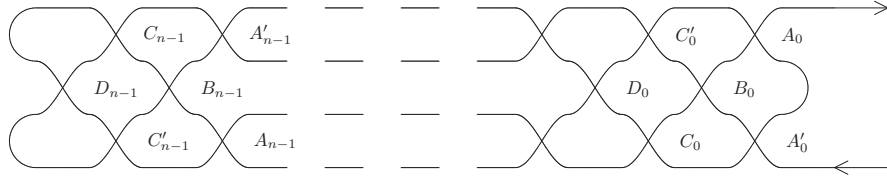


Figure 5: $H(4, 4n + 1, c)$

Hence our $4n$ points satisfy

$$x(A_0) > x(B_0) > x(C_0) > x(D_0) > \dots > x(A_{n-1}) > x(B_{n-1}) > x(C_{n-1}) > x(D_{n-1}).$$

Let A'_k (respectively C'_k) be the reflection of A_k (respectively C_k) in the x -axis. The crossings of our diagram are the points $A_k, A'_k, B_k, C_k, C'_k$, and D_k .

The Conway sign of a crossing point M is $s(M) = \text{sign}(D(M))$ if $y(M) = 0$, and $s(M) = -\text{sign}(D(M))$ if $y(M) \neq 0$.

By symmetry, we have $s(A'_k) = s(A_k)$ and $s(C'_k) = s(C_k)$ because symmetric points correspond to opposite parameters. The Conway form of H is then (see paragraph 2) :

$$C\left(s(D_{n-1}), 2s(C_{n-1}), s(B_{n-1}), 2s(A_{n-1}), \dots, s(B_0), 2s(A_0)\right).$$

Using the identity $T'_a(\cos \tau) = a \frac{\sin a\tau}{\sin \tau}$, we get $x'(t)y'(t) \sim \sin\left(\frac{m}{b}\pi\right) \sin\left(\frac{m}{4}\pi\right)$. Consequently,

- For A_k we have $x'(t)y'(t) \sim \sin\left(\frac{4k + 1}{b}\pi\right) \sin\left(\frac{4k + 1}{4}\pi\right) \sim (-1)^k$.

- Similarly, for B_k , C_k and D_k we obtain $x'(t)y'(t) \sim (-1)^k$.

On the other hand, at the crossing points we have

$$z(t) - z(s) = 2 \sin\left(\frac{c}{8b}(m' - m)\pi\right) \sin\left(\frac{c}{8b}(m + m')\pi\right).$$

We obtain the signs of our crossing points, with $c = 3b - 4\lambda$, $\theta = \frac{\lambda}{b}\pi$.

- For A_k we get: $z(t) - z(s) = 2 \sin\frac{c}{b}(n - k)\pi \sin c\frac{\pi}{4}$.
We have $\sin c\frac{\pi}{4} = \sin\frac{12n + 3 - 4\lambda}{4}\pi = (-1)^{n+\lambda} \sin\frac{3\pi}{4} \sim (-1)^{n+\lambda}$
and also $\sin\left(\frac{c}{b}(n - k)\pi\right) = \sin\left(\left(3 - \frac{4\lambda}{b}\right)(n - k)\pi\right)$
 $= (-1)^{n+k} \sin\left(\frac{4k - 4n}{b}\lambda\pi\right) = (-1)^{n+k+\lambda} \sin(4k + 1)\theta$

Consequently, the sign of A_k is $s(A_k) = -\text{sign}(\sin(4k + 1)\theta)$.

- For B_k , we have: $z(t) - z(s) = 2 \sin\left(\frac{c}{b}(2n - k)\pi\right) \sin c\frac{\pi}{2} = -2 \sin\left(\frac{c}{b}(2n - k)\pi\right)$
 $= 2 \sin\left(\left(3 - \frac{4\lambda}{b}\right)(k - 2n)\pi\right)$
 $= 2(-1)^{k+1} \sin\left(\frac{\lambda}{b}(4k - 8n)\pi\right) = (-1)^{k+1} \sin(4k + 2)\theta$.

Therefore the Conway sign of B_k is $s(B_k) = -\text{sign}(\sin(4k + 2)\theta)$.

- For C_k : $z(t) - z(s) = 2 \sin\left(\frac{c}{4}\pi\right) \sin\left(\frac{c}{b}(n + k + 1)\pi\right)$.

We know that $\sin\frac{c\pi}{4} \sim (-1)^{n+\lambda}$. Let us compute the second factor

$$\begin{aligned} \sin\left(\left(3 - \frac{4\lambda}{b}\right)(n + k + 1)\pi\right) &= (-1)^{n+k} \sin\left(\frac{\lambda}{b}(4n + 4k + 4)\pi\right) \\ &= (-1)^{n+k} \sin\left(\frac{\lambda}{b}(b + 4k + 3)\pi\right) = (-1)^{n+k+\lambda} \sin(4k + 3)\theta. \end{aligned}$$

Hence the sign of C_k is $s(C_k) = -\text{sign}(\sin(4k + 3)\theta)$.

- For D_k : $z(t) - z(s) = 2 \sin\left(\frac{c}{b}(k + 1)\pi\right) \sin\left(c\frac{\pi}{2}\right)$
 $= -2 \sin\left(\left(3 - \frac{4\lambda}{b}\right)(k + 1)\pi\right) = (-1)^{k+1} \sin(4k + 4)\theta$.

We conclude that $s(D_k) = -\text{sign}(\sin(4k + 4)\theta)$.

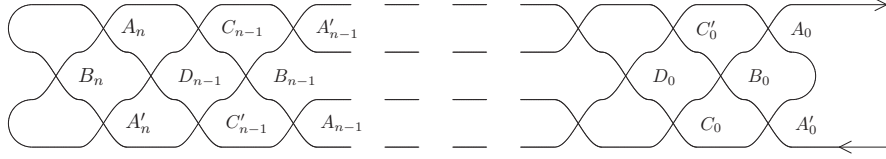
This completes the computation of our Conway normal form of H in this first case.

The case $b = 4n + 3$.

Here, the diagram is different. Let us consider the following $4n + 2$ crossing points.

For $k = 0, \dots, n$

- A_k corresponding to $m = 4k + 1$, $m' = 2b + m$,
- B_k corresponding to $m = 4k + 2$, $m' = 4b - m$.

Figure 6: $H(4, 4n + 3, c)$

For $k = 0, \dots, n - 1$

- C_k corresponding to $m = 4k + 3$, $m' = 2b - m$,
- D_k corresponding to $m = 2b + 4(k + 1)$, $m' = 4b - m$.

These points are such that

$$x(A_0) > x(B_0) > x(C_0) > x(D_0) > \dots > x(C_{n-1}) > x(D_{n-1}) > x(A_n) > x(B_n),$$

and we have $\text{sign}(x'(t)y'(t)) = (-1)^k$.

- For A_k we have $z(t) - z(s) = 2 \sin(c \frac{\pi}{4}) \sin(\frac{c}{b}(n + k + 1)\pi)$.

We easily get $\text{sign}(\sin c \frac{\pi}{4}) = (-1)^{n+\lambda}$. We also obtain

$$\begin{aligned} \sin(\frac{c}{b}(n + k + 1)\pi) &= \sin\left(\left(3 - \frac{4\lambda}{b}\right)(n + k + 1)\pi\right) \\ &= (-1)^{n+k} \sin\left(\frac{\lambda}{b}(b + 4k + 1)\pi\right) = (-1)^{n+k+\lambda} \sin(4k + 1)\theta. \end{aligned}$$

Hence the sign of A_k is $s(A_k) = -\text{sign}(\sin(4k + 1)\theta)$.

- For B_k we have $z(t) - z(s) = 2 \sin(\frac{c}{b}(2n + 1 - k)\pi) \sin c \frac{\pi}{2}$.

We have $\sin(c \frac{\pi}{2}) = 1 > 0$, and

$$\begin{aligned} \sin(\frac{c}{b}(2n + 1 - k)\pi) &= \sin\left(\left(3 - \frac{4\lambda}{b}\right)(2n + 1 - k)\pi\right) \\ &= (-1)^{k+1} \sin\left(\frac{\lambda}{b}(4k - 8n - 4)\pi\right) = (-1)^{k+1} \sin(4k + 2)\theta. \end{aligned}$$

Then, the sign of B_k is $s(B_k) = -\text{sign}(\sin(4k + 2)\theta)$.

- For C_k we have $z(t) - z(s) = 2 \sin(\frac{c}{b}(n - k)\pi) \sin c \frac{\pi}{4}$.

We obtain

$$\begin{aligned} \sin(\frac{c}{b}(n - k)\pi) &= \sin\left(\left(3 - \frac{4\lambda}{b}\right)(n - k)\pi\right) \\ &= (-1)^{n+k} \sin\left(\frac{4k - 4n}{b} \lambda \pi\right) = (-1)^{n+k+\lambda} \sin(4k + 3)\theta. \end{aligned}$$

The sign of C_k is then $s(C_k) = -\text{sign}(\sin(4k + 3)\theta)$.

- For D_k we have $z(t) - z(s) = 2 \sin(-\frac{c}{b}(k+1)\pi) \sin c\frac{\pi}{2}$. We have $\sin c\frac{\pi}{2} > 0$. We also have

$$\sin(-\frac{c}{b}(k+1)\pi) = \sin\left(\left(\frac{4\lambda}{b} - 3\right) = (k+1)\pi\right) (-1)^{k+1} \sin(4k+4)\theta.$$

Consequently, the sign of D_k is $s(D_k) = -\text{sign}(\sin(4k+4)\theta)$.

This concludes the computation of the Conway normal form of $H(4, b, c)$. In both cases it is $C(e_1, 2e_2, \dots, e_{b-2}, 2e_{b-1})$ where $e_j = \text{sign}(\sin(j-b)\theta)$.

If $b < c < 3b$, then we get $\lambda < \frac{b}{2}$, and $\theta < \frac{\pi}{2}$. Consequently, there is no two consecutive sign changes in our sequence. Furthermore, the total number of sign changes is $\lambda - 1$. We conclude by Proposition 2.2 that the crossing number is $N = \frac{3(b-1)}{2} - (\lambda - 1) = \frac{3b + c - 2}{4}$. The fact that $\beta^2 \equiv \pm 2 \pmod{\alpha}$ is a consequence of Proposition 2.6. \square

4 Some families with $a \geq 5$

We will consider Chebyshev curves as trajectories in a rectangular billiard (see [KP3]).

Lemma 4.1. *Let $C(t)$ be the plane curve parametrized by $x(t) = T_a(t)$, $y(t) = T_b(t)$, and let F be the function defined by $F(x) = \frac{2}{\pi}(\arccos(x) - 1)$. The mapping $(x, y) \mapsto (bF(x), aF(y))$ is an homeomorphism from the square $I = (-1, 1)^2$ onto the rectangle $(-b, b) \times (-a, a)$. The curve $B(t)$ obtained from $C(t)$ is a “billiard trajectory”. The slopes of its segments are ± 1 .*

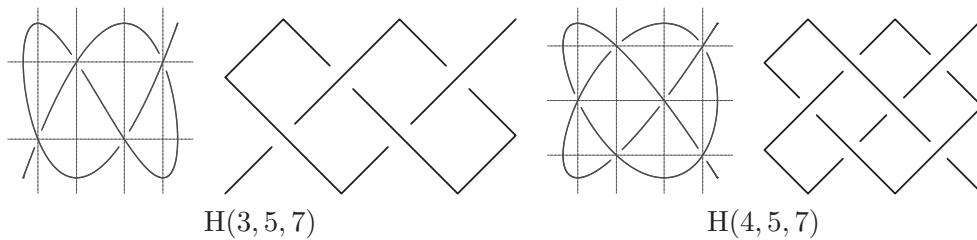


Figure 7: Billiard representations of 4_1 and 5_2

4.1 The harmonic knots $H(2n - 1, 2n, 2n + 1)$

Let us begin with some simple observations on the diagram of $K_n = H(2n - 1, 2n, 2n + 1)$.

We have $z(t) = 2t \cdot y(t) - x(t)$. Consequently, if (t, s) is a parameter pair corresponding to a crossing, we have: $z(t) - z(s) = 2(t - s)y(t)$. This simple rule allows us to draw by hand the billiard picture of the knot K_n (see Figure 8):

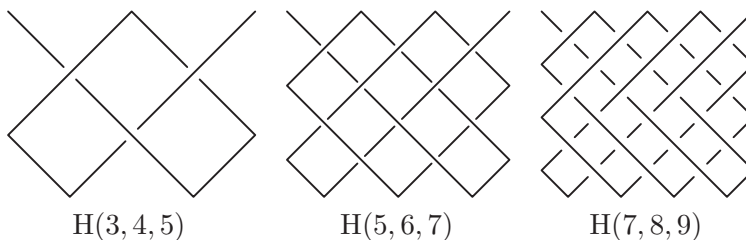


Figure 8: The knots K_n for $n = 2, 3, 4$

We can even deduce a simpler rule as follows.

Lemma 4.2. *Let $K = H(a, b, c)$ with $b = a + 1$. Then the sign of a crossing of parameters (s, t) is $\text{sign}(D) = \text{sign}((z(t) - z(s))(t - s))$.*

Proof. Let (s, t) be the parameter pair of a crossing. We have

$$t = \cos\left(\frac{k}{a} + \frac{h}{b}\right)\pi, \quad s = \cos\left(\frac{k}{a} - \frac{h}{b}\right)\pi, \quad 0 < \frac{k}{a} + \frac{h}{b} < 1.$$

An easy calculation shows that, when $b = a + 1$ then

$$x'(t)y'(t) \sim -\sin\left(\frac{k}{a}\pi\right) \sin\left(\frac{h}{b}\pi\right) \sim t - s,$$

which concludes the proof, using Lemma 3.2. □

Corollary 4.3. *The sign of a crossing of $H(2n - 1, 2n, 2n + 1)$ is $\text{sign}(D) = \text{sign}(y)$.*

Remark 4.4. This result is still true when (a, b, c) are in a more general arithmetic progression (see Figures 25 or 24).

Theorem 4.5. *The knot $H(2n - 1, 2n, 2n + 1)$ is isotopic to $H(4, 2n - 1, 2n + 1)$ if n is odd, and to $H(4, 2n + 1, 2n - 1)$ if n is even. Its crossing number is $2n - 1$.*

Proof. Our proof is by induction on n . We shall prove that K_n is isotopic to the two-bridge knot of Conway form $C(1, 2, (-1)^1, 2(-1)^1, \dots, (-1)^{n-2}, 2(-1)^{n-2})$.

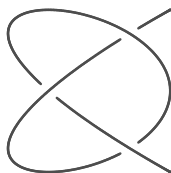
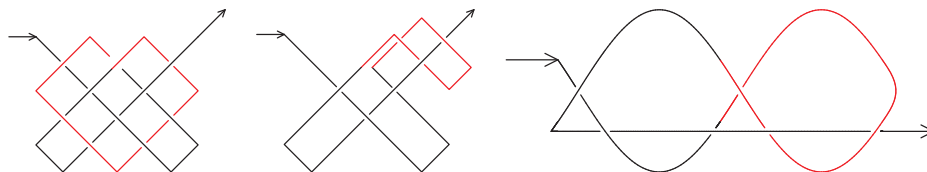


Figure 9: The knot K_2 is a trefoil

For $n = 2$, the knot $H(3, 4, 5)$ is the trefoil $K_2 = C(1, 2) = \overline{H}(4, 3, 5)$.

Figure 10: An isotopy of K_3

For $n = 3$, Figure 10 shows that $K_3 = C(1, 2, -1, -2)$. It also gives an idea of our proof.

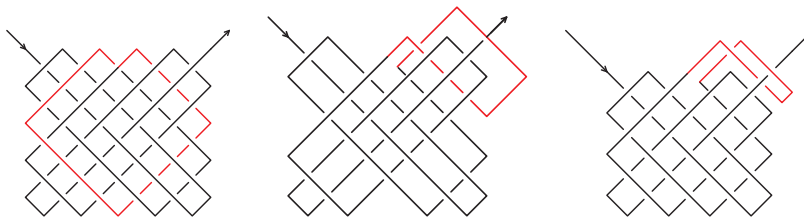
By induction, let us suppose that $K_{n-1} = C(1, 2, (-1)^1, (-1)^1 2, \dots, (-1)^{n-3}, (-1)^{n-3} 2)$. We shall consider K_n to be composed of two parts.

The first part L is a loop (the red loop of Figure 11) which is symmetrical about the y -axis, and consists of the points of parameters $t \in I = (\pi(\frac{1}{2} - \frac{1}{2n-1}), \pi(\frac{1}{2} + \frac{1}{2n-1}))$. It contains exactly $2(2n - 3)$ crossing points, which are the points of parameters

$$t = \cos \tau, \quad \tau = \frac{\pi}{2} + \frac{k\pi}{2n(2n-1)}, \quad |k| \leq 2n-2, \quad k \neq 0, \pm n.$$

The other part K' consists of the points of parameters $t \in \mathbf{R} - I$.

When n is odd, the part of the loop L where $t < \frac{\pi}{2}$ is over the bounded part of K' , and the other part of L is under the bounded part of K' . When n is even, the first part of L is under and the second part of L is over the bounded part of K' . Consequently, it is possible

Figure 11: Pulling the loop L away from K_n , we obtain K_{n-1} .

to move the loop L away from the bounded part of K' , and we see that K_n is obtained from K' by a weaving process (see [Ka, p. 50]).

Now let us look at the billiard drawing of K' . It is clear (see Figure 11) that, inside the rectangle $|X| \leq 2n - 1, |Y| \leq 2n - 2$, the diagram of K' coincides with the billiard diagram of K_{n-1} .

Consequently, our weaving are illustrated in figure 12.

If n is even, then using the induction hypothesis, we obtain the Conway form $K_n = C(1, 2, -1, -2, \dots, 1, 2)$ of length $2n - 2$. If n is odd, then we obtain the Conway form $K_n = C(1, 2, -1, -2, \dots, -1, -2)$ of length $2n - 2$. This completes our induction proof.

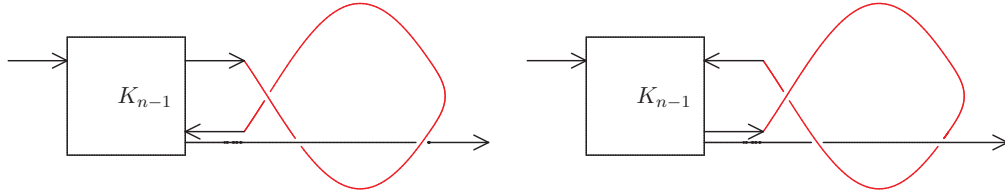


Figure 12: The weaving process: n odd (left), n even (right)

By the proof of corollary 3.9, we deduce that K_n is isotopic to $H(4, 2n - 1, 2n + 1)$ if n is odd, and to $H(4, 2n + 1, 2n - 1)$ if n is even. \square

The result of this inductive weaving process is illustrated in Figure 13 for the knot K_5 .

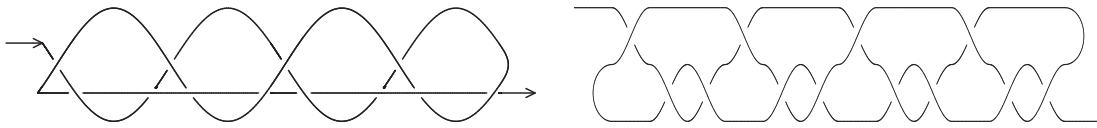


Figure 13: The knot K_5 is a two-bridge knot

4.2 The harmonic knots $H(5, k, k + 1)$.

The bridge number of such a knot is at most three, and there is no obvious reason for it to be smaller. This is why the following result surprised us.

Theorem 4.6.

*The knot $H(5, 5n + 1, 5n + 2)$ is the two-bridge knot of Conway form $C(2n + 1, 2n)$.
 The knot $H(5, 5n + 3, 5n + 4)$ is the two-bridge knot of Conway form $C(2n + 1, 2n + 2)$.
 Besides $H(5, 6, 7) = H(4, 5, 7)$ and $H(5, 3, 4)$, these knots are not of the form $H(a, b, c)$ with $a \leq 4$.*

Proof. We shall often need the toric move shown in Figure 14. The equivalence of the two

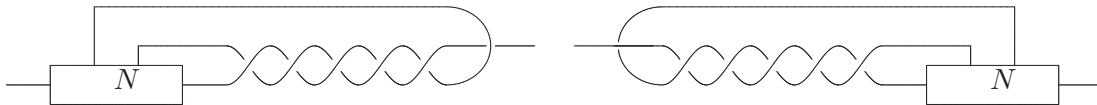


Figure 14: The toric move

diagrams is clear if we consider the compactification in S^3 of this knot.

First, we shall study the knots $H_n = H(5, 5n + 1, 5n + 2)$. We shall provide drawings of an isotopy showing that $H_3 = C(7, 6)$, and show how the different steps generalize when $n \neq 3$.

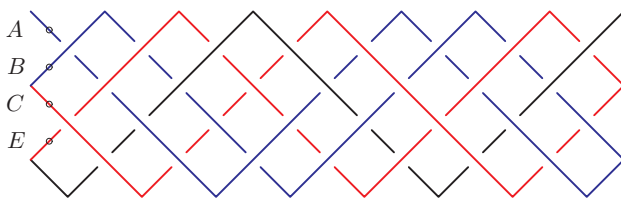


Figure 15: The knot $H_3 = H(5, 16, 17)$

Let us introduce the points A, B, C, E as shown in Figure 15. We shall consider the knot to be divided in three parts: the loop α from A to B , the loop β from C to E , the "stick" γ is the rest of the knot.

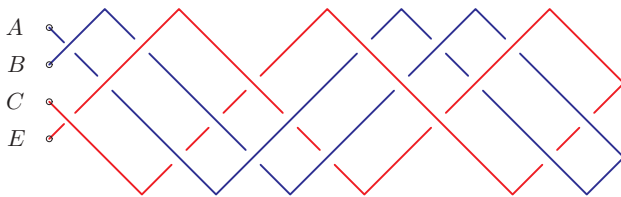


Figure 16: The loops $\tilde{\alpha}$ and $\tilde{\beta}$ are unlinked

First, we notice that the loops $\tilde{\alpha} = \alpha \cup [A, B]$ and $\tilde{\beta} = \beta \cup [C, E]$ are unlinked.

This is clear when $n = 3$ (see Figure 16), and can be proved by induction for the general case. It is also possible to deduce this from the fact that $\tilde{\alpha} \cup \tilde{\beta}$ is a 2-bridge link of Schubert fraction 0, because its Conway normal form is $C(0, 1, 0, -1, \dots, 0, -1)$. Using this fact, we can shrink α towards the left as shown in Figure 17.

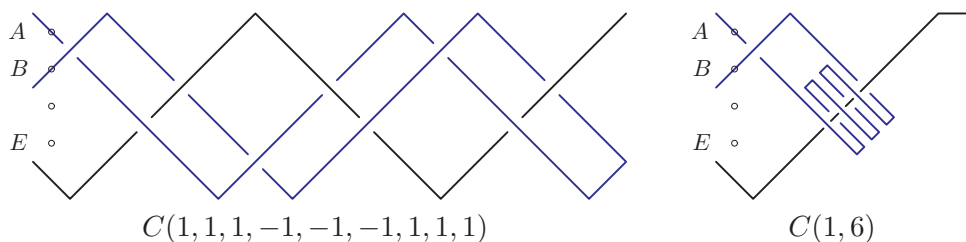


Figure 17: α is the rational tangle of Conway form $C(1, 1, 1, -1, -1, -1, 1, 1, 1)$

Then we simplify it using the fact that the tangle α is the rational tangle of Conway form $C(1, 1, 1, -1, -1, -1, 1, 1, 1)$, and consequently is isotopic to the tangle $C(1, 6)$. In the general case, this tangle is isotopic to $C(1, 2n)$. Now, our knot resembles Figure 18 (left). The right part of the knot is a tangle of Conway form $C(2, 1, -1, -1, -1, 1, 1)$ and then it is isotopic to the tangle $C(6)$. Using this isotopy, we obtain the diagram of figure 18 (right). In the general case this right part is of Conway form $C(1, 2, x)$ where

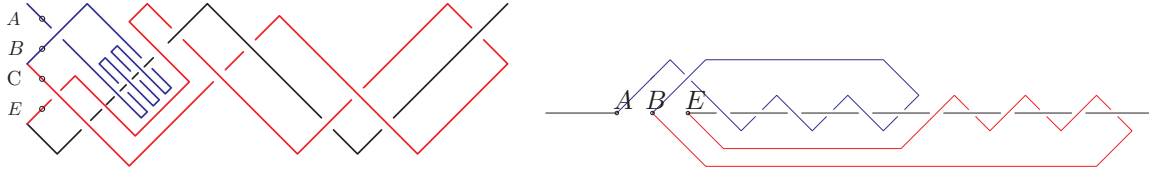


Figure 18: A simplified version of the knot H_3

$x = (-1, -1, -1, 1, 1, 1, \dots, (-1)^{n-2}, (-1)^{n-2}, (-1)^{n-2}, (-1)^{n-1}, (-1)^{n-1})$ and is isotopic to $C(2n)$.

Finally we slide the loop β from right to left by a toric move. The resulting diagram is shown in Figure 19; it is of Conway form $C(7, 6)$ ($C(2n + 1, 2n)$ in the general case).

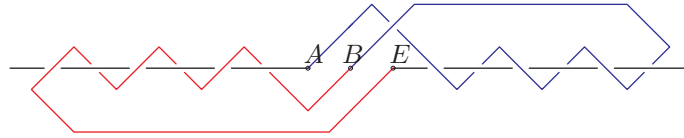


Figure 19: The knot $H_3 = H(5, 16, 17)$ is isotopic to the knot of Conway form $C(7, 6)$

The study of the knots $H(5, 5n + 3, 5n + 4)$ is similar; we only give a few figures showing the isotopy $H(5, 18, 19) = C(7, 8)$. First, we consider the points A, B, C as shown in Figure

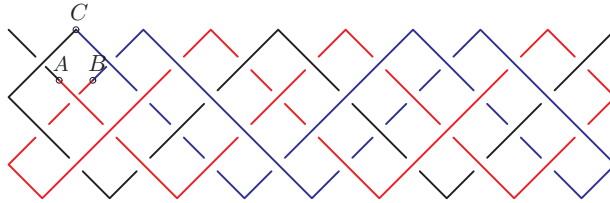
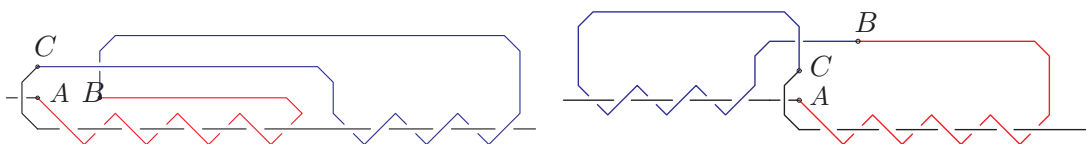
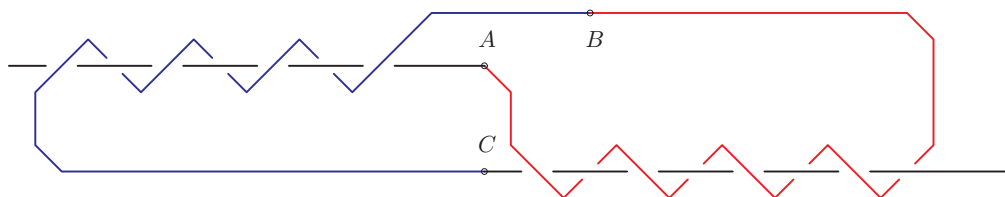


Figure 20: The knot $H(5, 18, 19)$

20. The knot is composed of three parts: the loop α from A to B , the loop β from B to C , and the rest of the knot.

As before, the loops α (from A to B) and β (from B to C) are unlinked, and therefore α can be shrunk towards the left and simplified. The loop β is simplified too, and we obtain the diagram shown in Figure 21 (left). Applying a toric move to the loop β , we obtain the diagram of Figure 21 (right). The last move is easy to see, we simply pull the half-loop containing C downwards. The resulting diagram shown in Figure 22 is of Conway form $C(7, 8)$.

Except for $H(5, 6, 7)$ and $H(5, 3, 4)$, these knots do not have any Schubert fraction $\frac{\alpha}{\beta}$ such that $\beta^2 \equiv \pm 1$ or $\beta^2 \equiv \pm 2 \pmod{\alpha}$; therefore they are not harmonic for $a \leq 4$. \square

Figure 21: Applying a toric move to the loop β Figure 22: The knot $H(5, 18, 19)$ is isotopic to the knot of Conway form $C(7, 8)$

4.3 Some new findings on harmonic knots

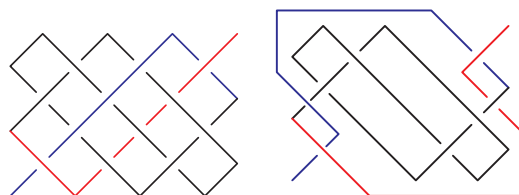
Thanks to the simplicity of our billiard diagrams, we can easily compute the Alexander polynomials of our knots (see [Li]). On the other hand, there is a list of the Alexander polynomials of the first prime knots with 15 or fewer crossings in [KA].

Using some evident simplifications, we can estimate the crossing number N and identify the knot.

We first give some specific examples, then an exhaustive list of harmonic knots $H(a, b, c)$ with $(a - 1)(b - 1) \leq 30$. Their diagrams have 15 or fewer crossing points.

Harmonic knots are not necessarily prime.

G. and J. Freudentburg conjectured that every harmonic knot is prime. This conjecture is not true. The knot $H(5, 7, 11)$ is not prime; it is the connected sum of two figure-eight knots.

Figure 23: The knot $H(5, 7, 11)$ is composite

Harmonic knots may be nonreversible.

We have identified the following knots of form $H(2n - 1, 2n + 1, 2n + 3)$, $n \geq 4$, by computing their Alexander polynomials and estimating their crossing numbers. We obtain

$$\begin{aligned} H(3, 5, 7) &= 4_1, & H(5, 7, 9) &= H(3, 7, 11) = 6_3, & H(7, 9, 11) &= H(5, 9, 13) = 8_{17}, \\ H(9, 11, 13) &= H(7, 11, 15) = 10_{115}, & H(11, 13, 15) &= H(9, 13, 17) = 12a_{1167}. \end{aligned}$$

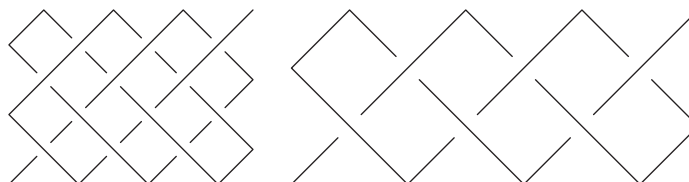


Figure 24: The knots $H(5, 7, 9)$ and $H(3, 7, 11)$ are isotopic to 6_3

Figure 25 shows that $H(7, 9, 11) = 8_{17}$ is symmetrical about the origin and therefore is strongly $(-)$ amphicheiral. It is also the first nonreversible knot (see also [Cr, p. 30]).

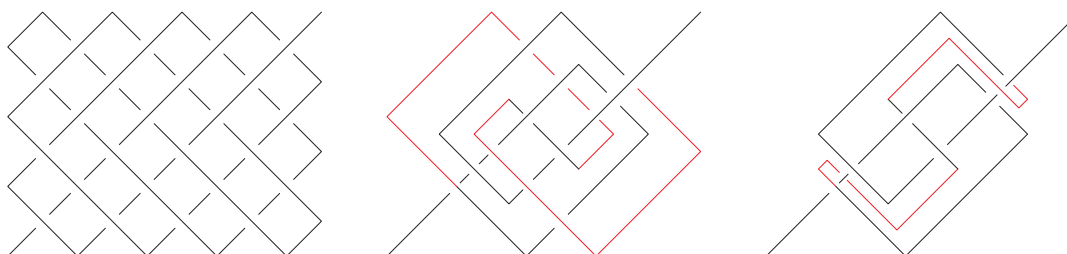


Figure 25: The knot $H(7, 9, 11)$, an unusual model of 8_{17}

A table of harmonic knots with $(a - 1)(b - 1) \leq 30$.

Here, we provide a table giving the Alexander polynomial of the harmonic knots with diagrams having 15 or fewer crossings, that is $H(a, b, c)$ with $3 \leq a < b < c$, $(a-1)(b-1) \leq 30$ and $(c, ab) = 1$. Using Lemma 3.5, we choose c such that $c \neq \lambda a + \mu b$, $\lambda, \mu > 0$ (see also [FF]). We have to consider 51 different harmonic knots.

In cases where $a = 3$ or $a = 4$, we know that $H(a, b, c)$ is a two-bridge knot. The crossing number of such a knot is $\frac{1}{3}(b + c)$, when $a = 3$ and $\frac{1}{4}(3b + c - 2)$ when $a = 4$. Furthermore, its Schubert fraction is computed using Theorem 3.7 or [KP4, Th. 6.5].

When $a \geq 5$, we compute the Alexander polynomial of the knot and compare it with the tables. Sometimes (when starred) it is also necessary to use Knotscape ([KS]) to determine the name of the knot.

| Table of the the first harmonic knots | | | | | |
|---------------------------------------|----------|----------|--------------|----------|-----------|
| | Fraction | Name | | Fraction | Name |
| $H(3,4,5)$ | 3 | 3_1 | $H(3,5,7)$ | $5/2$ | 4_1 |
| $H(3,7,8)$ | 5 | 5_1 | $H(3,7,11)$ | $13/5$ | 6_3 |
| $H(3,8,13)$ | $21/8$ | 7_7 | $H(3,10,11)$ | 7 | 7_1 |
| $H(3,10,17)$ | $55/21$ | 9_{31} | $H(3,11,13)$ | $17/4$ | 8_3 |
| $H(3,11,16)$ | $39/14$ | 9_{17} | $H(3,11,19)$ | $89/34$ | 10_{45} |

| | | | | | |
|------------|---------|-------------------|------------|---------|--------------------------------|
| H(3,13,14) | 9 | 9 ₁ | H(3,13,17) | 53/23 | 10 ₂₈ |
| H(3,13,20) | 105/41 | 11a175 | H(3,13,23) | 233/89 | 12a499 |
| H(3,14,19) | 77/34 | 11a119* | H(3,14,25) | 377/144 | 13a1739 |
| H(3,16,17) | 11 | 11a367 | H(3,16,23) | 187/67 | 13a2124* |
| H(3,16,29) | 987/377 | 15a39533* | H(4,5,7) | 7/2 | 5 ₂ |
| H(4,5,11) | 11/3 | 6 ₂ | H(4,7,9) | 17/5 | 7 ₅ |
| H(4,7,13) | 23/5 | 8 ₇ | H(4,7,17) | 41/11 | 9 ₂₀ |
| H(4,9,11) | 41/12 | 9 ₁₈ | H(4,9,19) | 89/25 | 11a180 |
| H(4,9,23) | 153/41 | 12a541 | H(4,11,13) | 99/29 | 11a236 |
| H(4,11,17) | 113/31 | 12a758 | H(4,11,21) | 187/41 | 13a2679* |
| H(4,11,25) | 329/87 | 14a7552* | H(4,11,29) | 571/153 | 15a42637* |
| H(5,6,7) | 7/4 | 5 ₂ | H(5,6,13) | | 10 ₁₅₉ |
| H(5,6,19) | | 10 ₁₁₆ | H(5,7,8) | 5/2 | 4 ₁ |
| H(5,7,9) | 13/8 | 6 ₃ | H(5,7,11) | | 4 ₁ #4 ₁ |
| H(5,7,13) | | 12n356 | H(5,7,16) | | 12n798 |
| H(5,7,18) | | 12n321 | H(5,7,23) | | 12a960 |
| H(5,8,9) | 13/4 | 7 ₃ | H(5,8,11) | 21/13 | 7 ₇ |
| H(5,8,17) | | 14n22712* | H(5,8,19) | | 14n26120* |
| H(5,8,27) | | 14a19221* | H(6,7,11) | | 10 ₁₃₄ |
| H(6,7,17) | | 15n42918* | H(6,7,23) | | 15n165258* |
| H(6,7,29) | | 15a81117 | | | |

Some isotopic harmonic knots

Theorem 4.5 asserts that the knots $H(2n-1, 2n, 2n+1)$ and $H(4, 2n-1, 2n+1)$ are isotopic and therefore can be identified.

Theorem 4.6 asserts that the knots $H(5, 5k+1, 5k+2)$ and $H(5, 5k+3, 5k+4)$ are also two-bridge knots. The knots $H(5, 5k+2, 5k+3)$, $2 \leq k \leq 8$ are not two-bridge knots because their modulo 2 Conway polynomials are not Fibonacci polynomials (see [KP5]). Note that $H(5, 7, 8)$ is the figure-eight knot 4_1 .

We have observed that for some values (a, b, c) in arithmetic progression, $H(b-k, b, b+k) = H(b-\lambda k, b, b+\lambda k)$, for some $\lambda > 1$. It happens for example with $H(5, 11, 17) = H(9, 11, 13)$, $H(3, 11, 19) = H(7, 11, 15)$, $H(5, 9, 13) = H(7, 9, 11)$, and many others. It would be interesting to explain this phenomenon.

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