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Hybrid finite volume scheme for
two-phase flow in porous media

Konstantin Brenner *

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Abstract We apply a finite volume method on general meshes for the discretization of an incompressible and immiscible two-phase flow in porous media. The problem is considered in the global pressure formulation. Mathematically, it amounts to solve an elliptic equation for the global pressure, with an anisotropic and heterogeneous permeability tensor coupled to a parabolic degenerate convection-diffusion equation for a saturation, again with the same permeability tensor. Extending ideas which we had previously developed for the numerical solution of a degenerate parabolic convection-reaction-diffusion equation we discretize the diffusion terms by means of a hybrid finite volume scheme, while we use a Godunov scheme for the non monotone convection flux. We prove the convergence of the numerical scheme in arbitrary space dimension and we present results of a number of numerical tests in space dimension two.

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1 Introduction

The two-phase flow in porous media is an important problem arising in many engineering and scientific context as e.g the secondary oil recovery, the basin modeling and the contaminated sites remediation [9], [7], [13]. In this paper we consider the simplified dead-oil model, that is to say, we assume that there are only two incompressible and immiscible fluids. The problem is described by the mass conservation of each phase together with Darcy-Muskat law

\[
\omega \partial_t s_o - \nabla \cdot \left( K \frac{k r_o(s_o)}{\mu_o} (\nabla p_o - \rho_o g) \right) = k_o
\]

\[
\omega \partial_t s_w - \nabla \cdot \left( K \frac{k r_w(s_w)}{\mu_w} (\nabla p_w - \rho_w g) \right) = k_w
\]

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where \( \omega \) is the porosity, \( K \) absolute permeability, \( s_w \) stands for the wetting phase (water) saturation and \( s_o \) for the non-wetting phase (oil) saturation. The relative permeabilities \( k r_o(s_o) \) and \( k r_w(s_w) \) model the effects of coexistence of both phases in porous medium; \( \rho_\alpha \) and \( \mu_\alpha \) denote the densities and the viscosity of phase \( \alpha \in \{o,w\} \), \( g \) is the gravity vector. It is also assumed that the porous medium is saturated and that two phases are immiscible

\[ s_o + s_w = 1. \]
The phase pressures are related by a capillary pressure law

\[ p_o + p_w = \pi(s). \]  

(1.4)

In view of (1.3) we can take as unknown only one saturation, for instance \( s = s_o \). We denote by \( \lambda_o(s) = \frac{kr_o(s)}{\mu_o} \) and \( \lambda_w(s) = \frac{kr_w(1-s)}{\mu_w} \) the relative mobilities, which are such that \( \lambda_o(0) = \lambda_w(1) = 0 \). In this paper we transform the system (1.1)-(1.4) into the so-called global pressure or fractional flow formulation. The global pressure \( p \) which is defined by

\[ p = p_w + \int_0^s \frac{\lambda_o(a)}{\lambda_o(a) + \lambda_w(a)} \pi(a) da \]

was first introduced in [5] and [13]. We also define the fractional flow \( f \), total mobility \( \lambda \) and capillary diffusion \( \varphi \)

\[ \lambda(s) = \lambda_o(s) + \lambda_w(s), \quad f(s) = \frac{\lambda_o(s)}{\lambda(s)}, \quad \varphi(s) = \int_0^s \lambda_w(\tau)f(\tau)\pi'(\tau)d\tau. \]

The system (1.1)-(1.4) is equivalent to

\[ -\nabla \cdot K(\lambda(s)\nabla p - \xi(s)g) = k_w + k_o \text{ in } \Omega \times (0,T), \]  

(1.5)

\[ q = -K(\lambda(s)\nabla p - \xi(s)g) \text{ in } \Omega \times (0,T), \]  

(1.6)

\[ \frac{\partial s}{\partial t} + \nabla \cdot (qf(s) + \gamma(s)Kg) - \nabla \cdot (K\nabla \varphi(s)) = k_o \text{ in } \Omega \times (0,T), \]  

(1.7)

where

\[ \gamma(s) = (\rho_o - \rho_w)\frac{\lambda_o(s)\lambda_w(s)}{\lambda_o(s) + \lambda_w(s)} \quad \text{and} \quad \xi(s) = (\lambda_o(s)\rho_o + \lambda_w(s)\rho_w). \]

The usual assumption \( \lambda_o + \lambda_w \geq \Lambda > 0 \) implies that the first equation is uniformly elliptic in \( p \), whereas the second one is parabolic degenerate with respect to the saturation \( s \). The system (1.5)-(1.7) has a remarkable mathematical structure, which permits to obtain a number of energy estimates. Many numerical methods were proposed for solving two-phase problem, such as finite element methods (see e.g. [21], [20], [13], [6], [14], [15], [16], [17]), discontinuous Galerkin (see e.g. [8], [19]) and finite volume methods (see e.g. [1], [23], [29], [25], [2], [30], [4]). The convergence results for the different finite elements schemes were obtained in [6], [14] and [17]. For finite volume schemes convergence was shown in [1], [29] and [25], where the initial formulation (1.1)-(1.4) was considered. We also would like to mention some interesting convergence results which were obtained in the case of miscible displacement [31], [33], [12].

The heterogeneity and anisotropy of porous media is a numerical challenge even when studying the elliptic problem derived from Darcy’s law for one-phase problem. Many schemes was proposed and analyzed in last decades for its discretization. See [27] and [24] for more references and for the detailed description and comparison of those numerical methods. In this paper we propose an implicit fully coupled Hybrid finite volume scheme for the system (1.5)-(1.7). The discretization of the diffusion terms is based upon a hybrid finite volume method [22], which allows the tensor \( K \) to be anisotropic and highly variable in space. Remark that heterogeneity also affects the first order terms \( K(x)g\xi(\cdot) \) and \( qf(\cdot) + K(x)g\gamma(\cdot) \), which may be discontinuous with respect to the space variable \( x \); in that case they require a suitable treatment see e.g. [35], [34], [26]. We apply the Godunov scheme proposed in [28], which seems to be a natural choice, since the hybrid
Assumptions on the data: \((H_1)\) \(\varphi \in C(\mathbb{R}), \varphi(0) = 0\), is a strictly increasing piecewise continuously differentiable Lipschitz continuous function with a Lipschitz constant \(L_\varphi\). We assume that the function \(\varphi^{-1}\) is Hölder continuous, namely that there exists \(H_\varphi > 0\) and \(\alpha \in (0,1]\) such that \(|s_1 - s_2| \leq H_\varphi|\varphi(s_1) - \varphi(s_2)|^\alpha\). It is also such that \(\varphi(s) = s\) for all \(s < 0\) and \(\varphi(s) - \varphi(1) = s - 1\) for all \(s > 1\);

\((H_2)\) The functions \(\lambda, \xi, \gamma, f \in C([0,1])\) are Lipschitz continuous; for \(f = \lambda, \xi, \gamma, f\) we denote by \(L_f\) the corresponding Lipschitz constant.

\((H_{\lambda})\) \(\lambda\) is such that \(0 < \lambda \leq \lambda(s)\);

\((H_{\xi})\) \(\xi\) and \(\gamma\) are convex functions, moreover \(\gamma\) is such that \(\gamma(0) = \gamma(1) = 0\); \(g\) is a constant vector from \(\mathbb{R}^d\);

\((H_{f})\) \(f\) is a nondecreasing function and it satisfies \(f(0) = 0, f(1) = 1\);

\((H_{\varphi})\) The functions \(\lambda, \xi, \gamma, f\) are constant outside of \((0,1)\), i.e. for \(f = \lambda, \xi, \gamma, f\) we assume that \(f(s) = f(0)\) for all \(s < 0\) and \(f(s) = f(1)\) for all \(s > 1\).

\((H_{s})\) \(s_0 \in L^\infty(\Omega); \omega \in L^\infty(\Omega)\) and such that \(0 < \omega \leq \omega(x) \leq \overline{\omega}\) for a.e. in \(x \in \Omega\);

\((H_{k})\) \(k_o, k_w \in L^\infty(0, T; L^2(\Omega))\) and such that \(k_o + k_w \geq 0\) a.e. in \(Q_T\).

Remark 1.1 The physical range of values for the saturation is \([0,1]\), however we extend the definition of all nonlinear functions in \((1.5)-(1.7)\) outside of \([0,1]\), since the numerical scheme which we study does not preserve neither the maximum principle, nor the positivity of \(s\) (nor the bound \(s \leq 1\)). It is worth noting that one can replace \((H_3)\) by the assumption that the functions \(\lambda, \xi, \gamma, f\) are in \(L^\infty(\mathbb{R})\). In turn the assumption \((H_1)\) can also weakened, namely the capillary diffusion \(\varphi\) can be extended by an arbitrary strictly increasing function, such that \(\varphi^{-1}\) is Lipschitz continuous on \(\mathbb{R} \setminus (0,1)\).

Assumptions on the geometry:

\((H_{\Omega})\) \(\Omega\) is a polyhedral open bounded connected subset of \(\mathbb{R}^d\), with \(d \in \mathbb{N}^*\), and \(\partial \Omega = \overline{\Omega} \setminus \Omega\) its boundary.

\((H_{K})\) \(K\) is a piecewise constant function from \(\Omega\) to \(M_d(\mathbb{R})\), where \(M_d(\mathbb{R})\) denotes the set of real \(d \times d\) matrices. More precisely we assume that there exist a finite family \(\{(\overline{\Omega}_j)_{i \in \{1, \ldots, I\}}\}\) of open connected polyhedral in \(\mathbb{R}^d\), such that \(\overline{\Omega} = \bigcup_{i \in \{1, \ldots, I\}} \overline{\Omega}_i, \Omega \cap \Omega_j = \emptyset\) if \(i \neq j\) and such that \(K(x)|_{\Omega_i} = K_i \in M_d(\mathbb{R})\). By \(\Gamma_{i,j}\) we denote the interface between the sub-domains \(i, j\), \(\Gamma_{i,j} = \overline{\Omega_i} \cap \overline{\Omega_j}\). We suppose that there exist two positive constants \(K\) and \(\overline{K}\) such that for all \(i \in \{1, \ldots, I\}\) the eigenvalues of the symmetric positive definite 
\(K_i\) are included in \([K, \overline{K}]\).

Remark 1.2 For the sake of simplicity we have assumed that the heterogeneity of the medium is only expressed through the \(x\)-dependence of the absolute permeability tensor \(K = K(x)\). However it is very simple to extend the analysis to the case where \(\lambda, \xi, \gamma\) and \(f\) depend on the rock type. On the other hand if we suppose that \(\varphi\) is discontinuous in space, this may lead to significant difficulties. The analysis of the case where the capillary pressure field (and also \(\varphi\)) is discontinuous was carried out in [11] and [10]. It is also worth noting that the partitioning of \(\Omega\) introduced in \(H_{K}\) is only used in order to provide a control on the gravity terms (see Remark 3.2 below). In the case that the gravity effects are neglected, one can consider a fully heterogeneous permeability field \(K\).

Let \(T > 0\), we consider the system \((1.5)-(1.7)\) in the domain \(Q_T = \Omega \times (0, T)\) together with the initial condition
\(s(\cdot, 0) = s_0\) in \(\Omega\) \hspace{1cm} (1.8)
Let \( p = 0 \) and \( s = 0 \) on \( \partial \Omega \times (0, T) \). (1.9)

We now present the definition of a weak solution of the problem (1.5)-(1.9).

**Definition 1.1 (Weak solution)** A function pair \((s, p)\) is a weak solution of the problem (1.5)-(1.6) if:

(i) \( s \in L^\infty(0, T; L^2(\Omega)) \);

(ii) \( \varphi(s) \in L^2(0, T; H^1_0(\Omega)) \);

(iii) \( p \in L^\infty(0, T; H^1_0(\Omega)) \);

(iv) for all \( \psi, \chi \in L^2(0, T; H^1_0(\Omega)) \) with \( \varphi_t \in L^\infty(Q_T) \), \( \varphi(\cdot, T) = 0 \), \( s \) and \( p \) satisfy the integral equalities

\[
- \int_0^T \int_\Omega s \psi_t \, dx \, dt - \int_\Omega s_0 \psi(\cdot, 0) \, dx - \int_0^T \int_\Omega K(f(s)q + \gamma(s)g) \cdot \nabla \psi \, dx \, dt + \int_0^T \int_\Omega k_o \psi \, dx \, dt
\]

and

\[
- \int_0^T \int_\Omega q \cdot \nabla \chi \, dx \, dt = \int_0^T \int_\Omega (k_w + k_o) \chi \, dx \, dt,
\]

where \( q \) is given by (1.6).

In Section 2 we present the finite volume scheme and some technical lemmas. In Section 3 we provide the a priori estimates and we prove an existence of a discrete solution. We prove the estimates on space and time translates of a discrete solution in Section 4, those estimates allow to establish a strong convergence property for a subsequence of discrete saturation. The convergence result is shown in Section 5. Finally in Section 6 we present a number of numerical results obtained on different two-dimensional meshes.

## 2 The finite volume scheme

### 2.1 The main definitions

In order to describe the numerical scheme we introduce below some notations related to the space and time discretization, which follows [22].

**Definition 2.1 (Discretization of \( \Omega \))** Let \( \Omega \) be a polyhedral open bounded connected subset of \( \mathbb{R}^d \), with \( d \in \mathbb{N}^* \), \( \partial \Omega = \overline{\Omega} \setminus \Omega \) its boundary, and \((\Omega_i)_{i \in \{1, \ldots, I\}}\) its partition in the sense of \((H_4)\). A discretization of \( \Omega \), denoted by \( \mathcal{D} \), is defined as the triplet \( \mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P}) \), where:

1. \( \mathcal{M} \) is a finite family of non empty connex open disjoint subsets of \( \Omega \) (the "control volumes") such that \( \overline{\Omega} = \bigcup_{K \in \mathcal{M}} \overline{K} \). For any \( K \in \mathcal{M} \), let \( \partial K = \overline{K} \setminus K \) be the boundary of \( K \); we define \( m(K) > 0 \) as the measure of \( K \) and \( h_K \) as the diameter of \( K \). We also assume that the mesh resolve the structure of the medium, i.e. for all \( K \in \mathcal{M} \) there exist \( i \in \{1, \ldots, I\} \) such that \( K \subset \Omega_i \).

2. \( \mathcal{E} \) is a finite family of disjoint subsets of \( \overline{\Omega} \) (the "edges" of the mesh), such that, for all \( \sigma \in \mathcal{E} \), \( \sigma \) is a non empty open subset of a hyperplane of \( \mathbb{R}^d \), whose \((d-1)\)-dimensional measure \( m(\sigma) \) is strictly positive. We also assume that, for all \( K \in \mathcal{M} \), there exists a subset \( \mathcal{E}_K \) of \( \mathcal{E} \) such that \( \partial K = \bigcup_{\sigma \in \mathcal{E}_K} \overline{\sigma} \). For each \( \sigma \in \mathcal{E} \), we set \( \mathcal{M}_\sigma = \{ K \in \mathcal{M} | \sigma \in \mathcal{E}_K \} \). We then assume that, for all \( \sigma \in \mathcal{E} \), either \( \mathcal{M}_\sigma \) has exactly one element and then \( \sigma \in \partial \Omega \).
Moreover we define the following semi-norm on $X$ and spaces $T$ taking into account the time discretization leads us to define of the following discrete time steps of length $\delta t$.

**Definition 2.4 (Time discretization)**

Next we introduce the time discretization. Imposing a uniform bound on $\theta_D$ forces the mesh to be sufficiently regular. Next, we define several discrete spaces, which are going to be used in the sequel.

**Definition 2.2 (The hybrid space $X_D(\Omega)$)** Let $D = (M, E, P)$ be a discretization of $\Omega$. We define

$$X_D = \{(v_K)_{K \in M}, (v_\sigma)_{\sigma \in E}; v_K \in \mathbb{R}, v_\sigma \in \mathbb{R}\},$$

$$X_{D,0} = \{v \in X_D \text{ such that } (v_\sigma)_{\sigma \in E_{ext}} = 0\}. \quad (2.3)$$

The space $X_D$ is equipped with the semi-norm $\| \cdot \|_{X_D}$ defined by

$$|v|^2_{X_D} = \sum_{K \in M} |v|_{X_{D,K}}^2, \text{ where } |v|_{X_{D,K}}^2 = \sum_{\sigma \in E_K} m(\sigma) d_{K,\sigma}^2 (v_\sigma - v_K)^2 \text{ for all } v \in X_D. \quad (2.4)$$

Note that $\| \cdot \|_{X_D}$ is a norm on the space $X_{D,0}$.

Moreover, for each function $\psi = \psi(x)$ regular enough we define its projection $P_D \psi \in X_D$ on the space $X_D$ in following way

$$(P_D \psi)_K = \psi(x_K) \quad \text{for all } K \in M, \quad (P_D \psi)_\sigma = \psi(x_\sigma) \quad \text{for all } \sigma \in E.$$

**Definition 2.3 (The discrete flux space $Q_D(\Omega)$)** Let $D = (M, E, P)$ be a discretization of $\Omega$. We define

$$Q_D = \{(q_K, \sigma)_{K \in M, \sigma \in E_K}; q_K, \sigma \in \mathbb{R}\}. \quad (2.5)$$

Next we introduce the time discretization.

**Definition 2.4 (Time discretization)** We divide the time interval $(0, T)$ into $N$ equal time steps of length $\delta t = T/N$, where $\delta t$ is the uniform time step defined by $\delta t = t_n - t_{n-1}$.

Taking into account the time discretization leads us to define of the following discrete spaces

$$X_{D,\delta t} = X_{D,\delta t}^N = \{(v^n)_{n \in \{1, \ldots, N\}}; v^n \in X_D\}$$

and

$$X_{D,\delta t,0} = X_{D,\delta t,0}^N = \{(v^n)_{n \in \{1, \ldots, N\}}; v^n \in X_{D,0}\};$$

moreover we define the following semi-norm on $X_{D,\delta t}$

$$|v|^2_{X_{D,\delta t}} = \sum_{n=1}^N \delta t |v|^2_{X_D}. \quad (2.6)$$
2.2 The numerical scheme

2.2.1 The discrete problem

In this section we present the fully implicit finite volume scheme for the problem (1.5)-(1.9). Let us introduce the discrete saturation \( (s^n_K)_{K \in M}, (s^n_{\sigma})_{\sigma \in \mathcal{E}} \) \( n \in \{1, \ldots, N\} \) \( X_D, \delta t \) and the discrete global pressure \( (p^n_K)_{K \in M}, (p^n_{\sigma})_{\sigma \in \mathcal{E}} \) \( n \in \{1, \ldots, N\} \) \( X_D, \delta t \), which are the main discrete unknowns. Moreover let \( f \) denote \( \lambda, \xi, \gamma \) or \( f \) we introduce the following notation \( f^n_K = f(s^n_K) \) and \( f^n_{\sigma} = f(s^n_{\sigma}) \) for all \( K \in M, \sigma \in \mathcal{E} \) and \( n \in \{1, \ldots, N\} \). Let \( k^n_{i,K} \) denote the mean value of the source term \( k_i(x,t) \) over a cell \( K \times (t_{n-1}, t_n) \), i.e.

\[
k^n_{i,K} = \frac{1}{m(K)\delta t} \int_{t_{n-1}}^{t_n} \int_K k_i(x,t) \, dxdt \quad \text{with} \quad i \in \{w,n\}. \tag{2.7}
\]

We denote the porous volume of the element \( K \) by \( \omega(K) \),

\[
\omega(K) = \int_K \omega(x) \, dx.
\]

Next, let \( Q^n_{K,\sigma} \) be an approximation of the total flux through the interface \( \sigma \)

\[
Q^n_{K,\sigma} \approx \frac{1}{\delta t} \int_{t_{n-1}}^{t_n} \int_{\sigma} \mathbf{K}(\lambda(s)\nabla p - \xi(s)\mathbf{g}) \cdot \mathbf{n}_{K,\sigma} \, d\nu dt \tag{2.8}
\]

and let \( F^n_{K,\sigma} \) be an approximation of the non-wetting phase flux

\[
F^n_{K,\sigma} \approx \frac{1}{\delta t} \int_{t_{n-1}}^{t_n} \int_{\sigma} (qf(s) + \gamma(s)K \mathbf{g} - K \nabla \varphi(s)) \cdot \mathbf{n}_{K,\sigma} \, d\nu dt. \tag{2.9}
\]

The numerical fluxes \( Q^n_{K,\sigma} \) and \( F^n_{K,\sigma} \) have to be constructed as functions of the discrete unknowns. Using the notations (2.7) and (2.8) we discretize the equation (1.5) by

\[
\sum_{\sigma \in \mathcal{E}_K} Q^n_{K,\sigma} = m(K)(k^n_{w,K} + k^n_{o,K}) \quad \text{for all} \quad K \in M. \tag{2.10}
\]

We also prescribe the continuity of the fluxes

\[
Q^n_{K,\sigma} + Q^n_{L,\sigma} = 0 \quad \text{for all} \quad \sigma \in \mathcal{E}_{int} \quad \text{with} \quad \{K,L\} = M_{\sigma}. \tag{2.11}
\]

Similarly, the equation (1.7) is discretized by

\[
\omega(K) \frac{s^n_K - s^{n-1}_K}{\delta t} + \sum_{\sigma \in \mathcal{E}_K} F^n_{K,\sigma} = m(K)k^n_{o,K} \quad \text{for all} \quad K \in M, \tag{2.12}
\]

and

\[
F^n_{K,\sigma} + F^n_{L,\sigma} = 0 \quad \text{for all} \quad \sigma \in \mathcal{E}_{int} \quad \text{with} \quad \{K,L\} = M_{\sigma}. \tag{2.13}
\]

The discrete equations (2.16)-(2.13) have to be prescribed at each time step \( n \in \{1, \ldots, N\} \). We prescribe the initial and the boundary conditions for the numerical scheme by setting

\[
s^n_K = \frac{1}{m(K)} \int_K s_0(x) \, dx \quad \text{for all} \quad K \in M \tag{2.14}
\]

and

\[
s^n_{\sigma} = p^n_{\sigma} = 0 \quad \text{for all} \quad \sigma \in \mathcal{E}_{ext}. \tag{2.15}
\]

Remark that opposite to the classical two-point flux approximation, the discrete fluxes \( Q^n_{K,\sigma} \) and \( F^n_{K,\sigma} \) (which still remain to be constructed) are not \textit{a priori} continuous across the element’s interfaces, so that the continuity is prescribed in the scheme by (2.11) and (2.13).
2.2.2 The discrete weak formulation

Following the ideas of [22] we write the scheme in the variational form. For each \( n \in \{1, \ldots, N\} \) find \( s^n \in X_{D,0} \) and \( p^n \in X_{D,0} \) such that for all \( v^n, w^n \in X_{D,0} \):

\[
\sum_{K \in M} \sum_{\sigma \in \mathcal{E}_K} (v^n_K - v^n_\sigma) Q^n_{K,\sigma} = \sum_{K \in M} m(K) v^n_K (k^n_w + k^n_{0,K}), 
\]

\[
\sum_{K \in M} \omega(K) w^n_K \frac{s^n_K - s^{n-1}_K}{\delta t} + \sum_{K \in M} \sum_{\sigma \in \mathcal{E}_K} (w^n_K - w^n_\sigma) F^n_{K,\sigma} = \sum_{K \in M} m(K) w^n_K k^n_{o,K},
\]

\[
s^n_K = \frac{1}{m(K)} \int_K s_0(x) \, dx. \tag{2.18}
\]

In order to complete the scheme we have to define the numerical fluxes \( Q^n_{K,\sigma} \) and \( F^n_{K,\sigma} \). Let \( K_K \) denote the mean value of \( K(x) \) over a cell \( K \),

\[
K_K = \frac{1}{m(K)} \int_K K(x) \, dx \tag{2.19}
\]

and let

\[
g_{K,\sigma} = m(\sigma) K_K g \cdot n_{K,\sigma}. \tag{2.20}
\]

Note that \( g_{K,\sigma} \) satisfies

\[
\sum_{\sigma \in \mathcal{E}_K} g_{K,\sigma} = 0 \text{ for all } K \in \mathcal{M}, \tag{2.21}
\]

but not necessarily

\[
g_{K,\sigma} + g_{L,\sigma} = 0 \text{ with } \{K, L\} = \mathcal{M}_\sigma.
\]

The above equality remains true for the interfaces which are ”interior” with respect to some sub-domain \( \Omega_i \), that is to say

\[
g_{K,\sigma} + g_{L,\sigma} = 0 \text{ with } \{K, L\} = \mathcal{M}_\sigma \text{ for all } \sigma \notin \Gamma_{i,j}, \tag{2.22}
\]

for any \( i, j \). Next, we define \( Q^n_{K,\sigma} \) and \( F^n_{K,\sigma} \) by

\[
Q^n_{K,\sigma} = \lambda^n_K \mathcal{F}_{K,\sigma}(p^n) + \mathcal{G}(\xi(\cdot) g_{K,\sigma}; s^n_K, s^n_\sigma), \tag{2.23}
\]

\[
F^n_{K,\sigma} = \mathcal{G}(Q^n_{K,\sigma} f(\cdot) + \gamma(\cdot) g_{K,\sigma}; s^n_K, s^n_\sigma) + \mathcal{F}_{K,\sigma}(\varphi(s^n)), \tag{2.24}
\]

where \( \lambda^n_K = \lambda^n_K \) for all \( K \in \mathcal{M} \) and \( n \in \{1, \ldots, N\} \), in general. The terms \( \mathcal{F}_{K,\sigma}(\cdot) \) correspond to the diffusive fluxes which are discretized using the SUSHI scheme (Section 2.2.4). The terms \( \mathcal{G}(\cdot) \) stand for the discretization of the convective fluxes, using the Godunov scheme (Section 2.2.3 below, see also [28, p. 3]).

2.2.3 The Godunov scheme and the convection term

Let \( a, b \in \mathbb{R} \) and \( f \in L^\infty(\mathbb{R}) \) we define the Godunov flux by

\[
\mathcal{G}(f; a, b) = \begin{cases} 
\min_{s \in [a, b]} f(s) & \text{if } a \leq b, \\
\max_{s \in [b, a]} f(s) & \text{if } b \leq a.
\end{cases} \tag{2.25}
\]

Since it is often useful to write the discrete flux in more explicit form we define

\[
\mathcal{S}(f; a, b) = \begin{cases} 
\arg\min_{s \in [a, b]} f(s) & \text{if } a \leq b, \\
\arg\max_{s \in [b, a]} f(s) & \text{if } b \leq a,
\end{cases}
\]
and we introduce the following notations

\[
\begin{align*}
\xi^n_{K,\sigma} & = \xi(S(\xi(\cdot)g_{K,\sigma}; s^n_K, s^n_\sigma)), \\
f^n_{K,\sigma} & = f(S(Q^n_{K,\sigma} f(\cdot) + \gamma(\cdot)g_{K,\sigma}; s^n_K, s^n_\sigma)), \\
\gamma^n_{K,\sigma} & = \gamma(S(Q^n_{K,\sigma} f(\cdot) + \gamma(\cdot)g_{K,\sigma}; s^n_K, s^n_\sigma)).
\end{align*}
\]

(2.26)

Using the notations (2.26) we can write the discrete fluxes in the form

\[
Q^n_{K,\sigma} = \lambda^n_K F_{K,\sigma}(p^n) + \xi^n_{K,\sigma} g_{K,\sigma}
\]

and

\[
F^n_{K,\sigma} = Q^n_{K,\sigma} f^n_{K,\sigma} + \gamma^n_{K,\sigma} g_{K,\sigma} + F_{K,\sigma}(\phi(s^n)),
\]

(2.27) (2.28)

2.2.4 The discrete gradient and the diffusion term

In this section we recall a construction of the discrete gradient and of the numerical flux $F_{K,\sigma}(\cdot)$ proposed in [22]. Let $u \in X_D$, for all $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}_K$ we define

\[
\nabla_{K,\sigma} u = \nabla_K u + R_{K,\sigma} u \cdot n_{K,\sigma},
\]

(2.29)

where

\[
\nabla_K u = \frac{1}{m(K)} \sum_{\sigma \in \mathcal{E}_K} m(\sigma)(u_\sigma - u_K)n_{K,\sigma}
\]

(2.30)

and

\[
R_{K,\sigma} u = \sqrt{d_{K,\sigma}} (u_\sigma - u_K - \nabla_K u \cdot (x_\sigma - x_K)).
\]

(2.31)

Note that the stabilizing term $R_{K,\sigma}$ is a second order error term, which vanishes for piecewise linear functions. We define the discrete gradient $\nabla_D u$ as the piecewise constant function equal to $\nabla_{K,\sigma} u$ in the cone $D_{K,\sigma}$ with vertex $x_K$ and basis $\sigma$

\[
\nabla_D u(x)|_{x \in D_{K,\sigma}} = \nabla_{K,\sigma} u.
\]

Let $u = (u^n)_{n \in \{1, \ldots, N\}} \in X_D,\delta t$, taking into account the time discretization, we define the discrete gradient $\nabla_{D,\delta t} u(x, t)$ by

\[
\nabla_{D,\delta t} u(x, t)|_{t \in [t_{n-1}, t_n]} = \nabla_D u^n(x),
\]

(2.32)

for all $x \in \Omega$ and all $n \in \{1, \ldots, N\}$. For an arbitrary $u \in X_D$ the numerical flux $F_{K,\sigma}(u)$ can be defined through the following discrete integration by parts formula

\[
\sum_{\sigma \in \mathcal{E}_K} (v_K - v_\sigma) F_{K,\sigma}(u) = \sum_{\sigma \in \mathcal{E}_K} m(D_{K,\sigma}) K_K \nabla_{K,\sigma} u \cdot \nabla_{K,\sigma} v \text{ for all } v \in X_D,
\]

(2.33)

which in particular implies that

\[
\sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} (v_K - v_\sigma) F_{K,\sigma}(u) = \int_{\Omega} K \nabla_D u \cdot \nabla_D v \, dx \text{ for all } v \in X_D.
\]

(2.34)

The explicit form of $F_{K,\sigma}$ can be obtained by setting $v_K - v_\sigma = 1$ and $v_K - v_{\sigma'} = 0$ for all $\sigma' \neq \sigma$. We refer to [22] for more details on construction of $F_{K,\sigma}$ and its practical implementation. Next we state without proof three results from [22].
Lemma 2.1 (Strong consistency) Let $\mathcal{D}$ be a discretization of $\Omega$ in sense of Definition 2.1, moreover let $\theta \geq \theta_{\mathcal{D}}$ be given. Then for all $\psi \in C^2(\Omega)$, there exist a positive constant $C$ only depending on $d$, $\theta$ and $\varphi$ such that

$$|\nabla K P_{\mathcal{D}} \psi - \nabla \psi(x)| \leq C h \text{ for all } x \in K$$

and also

$$\|\nabla P_{\mathcal{D}} \psi - \nabla \psi\|_{(L^\infty(\Omega))^d} \leq C h.$$  \hspace{1cm} (2.36)

The following lemma, which is the slightly modified version of [22, Lemma 4.1] shows the equivalence between the semi-norm in $X_{\mathcal{D}}$ and the $L^2$-norm of the discrete gradient.

Lemma 2.2 Let $\mathcal{D}$ be a discretization of $\Omega$ and let $\theta \geq \theta_{\mathcal{D}}$ be given. Then there exists $m > 0$ and $M > 0$ only depending on $\theta$ and $d$ such that

$$m|v|_{X_{\mathcal{D},K}} \leq \|\nabla D v\|_{L^2(K)} \leq M|v|_{X_{\mathcal{D},K}} \quad \text{for all } K \in \mathcal{M} \text{ for all } v \in X_{\mathcal{D}}.$$  \hspace{1cm} (2.37)

Proof: The result follows from Lemma 5.3 of [22]. \hfill \square

The direct consequence of (2.33) and Lemma 2.2 is the lemma below.

Lemma 2.3 (Discrete Poincaré inequality) There exists a positive constant $C$, independent of the mesh size $h_{\mathcal{D}}$ such that

$$\|\Pi_{D,\delta} u\|_{L^2(\Omega)} \leq C|u|_{X_{\mathcal{D}}} \text{ for all } u \in X_{\mathcal{D}}.$$  \hspace{1cm} (2.37)

Proof: Let $K$ be an element of $\mathcal{M}$, setting $v_K = 0$ and $v_\sigma = f_{K,\sigma}$ in (2.38) we obtain

$$|\sum_{\sigma \in \mathcal{E}_K} (v_K - v_\sigma) F_{K,\sigma}(u)| \leq C_1 |u|_{X_{\mathcal{D}}} |v|_{X_{\mathcal{D}}}.$$  \hspace{1cm} (2.38)

and

$$\sum_{\sigma \in \mathcal{E}_K} (u_K - u_\sigma) F_{K,\sigma}(u) \geq C_2 |u|_{X_{\mathcal{D}}}^2.$$  \hspace{1cm} (2.39)

Lemma 2.4 implies the following useful technical result.

Lemma 2.5 Let $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ be a discretization of $\Omega$, let $q \in Q_{\mathcal{D}}$ and $u \in X_{\mathcal{D}}$. Then

$$\left| \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} q_{K,\sigma} F_{K,\sigma}(u) \right| \leq C_1 |u|_{X_{\mathcal{D}}} \left( \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_{K,\sigma}} \right)^{\frac{1}{2}}.$$  \hspace{1cm} (2.40)

Proof: Let $K$ be an element of $\mathcal{M}$, setting $v_K = 0$ and $v_\sigma = f_{K,\sigma}$ in (2.38) we obtain

$$\left| \sum_{\sigma \in \mathcal{E}_K} q_{K,\sigma} F_{K,\sigma}(u) \right| \leq |u|_{X_{\mathcal{D}}} \left( \sum_{\sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_{K,\sigma}} \right)^{\frac{1}{2}}.$$  \hspace{1cm} (2.40)

Proceeding the same way for all $K \in \mathcal{M}$ we get

$$\left| \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} q_{K,\sigma} F_{K,\sigma}(u) \right| \leq \sum_{K \in \mathcal{M}} \left| \sum_{\sigma \in \mathcal{E}_K} q_{K,\sigma} F_{K,\sigma}(u) \right| \leq C_1 \sum_{K \in \mathcal{M}} |u|_{X_{\mathcal{D}}} \left( \sum_{\sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_{K,\sigma}} \right)^{\frac{1}{2}}.$$  \hspace{1cm} (2.40)

We use the Cauchy-Schwarz inequality to complete the proof. \hfill \square
Lemma 2.6 Let $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ be a discretization of $\Omega$, let $v \in X_\mathcal{D}$ then there exists a positive constant $C$ independent of the mesh size such that for all $n \in \{1 \ldots N\}$

$$\left| \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} (v_K - v_{\sigma}) \mathcal{G}(Q_{K,\sigma}^n f(\cdot) + \gamma(\cdot) g_{K,\sigma}; s_{K,\sigma}^n, s_{\sigma}^n) \right| \leq C|v|_{X_\mathcal{D}}(|p|_{X_\mathcal{D}} + 1).$$

**Proof:** In view of (2.24), (2.27) and (2.28) we have that

$$\sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} (v_K - v_{\sigma}) \mathcal{G}(Q_{K,\sigma}^n f(\cdot) + \gamma(\cdot) g_{K,\sigma}; s_{K,\sigma}^n, s_{\sigma}^n) = T_1 + T_2 + T_3,$$

where

$$T_1 = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} (v_K - v_{\sigma}) \gamma_{K,\sigma}^n g_{K,\sigma},$$

$$T_2 = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} (v_K - v_{\sigma}) \xi_{K,\sigma}^n f_{K,\sigma}^n g_{K,\sigma},$$

$$T_3 = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} (v_K - v_{\sigma}) \lambda_{K,\sigma}^n F_{K,\sigma}^n (p^n)^{f_{K,\sigma}^n}.$$

In view of (2.20)

$$T_1 = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} m(\sigma)(v_K - v_{\sigma}) \gamma_{K,\sigma}^n K_K g \cdot n_{K,\sigma}.$$

Remark that for all $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}_K$ one has $m(\sigma)d_{K,\sigma} = m(D_{K,\sigma})d$, where $d$ is the space dimension. Using Cauchy-Schwarz inequality

$$(T_1)^2 \leq \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_{K,\sigma}} (v_K - v_{\sigma})^2 \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} m(\sigma)d_{K,\sigma}(\gamma_{K,\sigma}^n)^2 |g|^2K^2 \leq \overline{K}^2 |g|^2 m(\Omega)d\|\gamma\|^2_{L^\infty((0,1))} |v|_{X_\mathcal{D}}^2. \quad (2.41)$$

In the same way

$$(T_2)^2 \leq \overline{K}^2 |g|^2 m(\Omega)d\|\xi\|^2_{L^\infty((0,1))} \|f\|^2_{L^\infty((0,1))} |v|_{X_\mathcal{D}}^2. \quad (2.42)$$

Applying Lemma 2.5 to the term $T_3$ we obtain

$$(T_3)^2 \leq |p|_{X_\mathcal{D}}^2 \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_{K,\sigma}} (\lambda_{K,\sigma}^n(v_K - v_{\sigma})f_{K,\sigma}^n)^2 \leq \|\lambda\|^2_{L^\infty((0,1))} \|f\|^2_{L^\infty((0,1))} |p|_{X_\mathcal{D}}^2 |v|_{X_\mathcal{D}}^2. \quad (2.43)$$

Gathering (2.41)-(2.43) we complete the proof. \qed

Finally we present the technical lemma which is used in the proof of the a priori estimates.

**Lemma 2.7** Let $\varphi(s)$ satisfying the hypothesis ($\mathcal{H}_1$) and let the function $\Phi$ be defined by

$$\Phi(s) = \int_0^s \varphi(\tau) d\tau. \quad (2.44)$$

Then,

$$\frac{1}{2L cuff(\varphi(s))^2} \leq \Phi(s) \leq \frac{L \varphi s^2}{2}. \quad (2.45)$$
Proof: The function $\varphi$ is invertible, then setting $\xi = \varphi(\tau)$ in (2.44) gives
\[ \Phi(s) = \int_0^{\varphi(s)} \frac{\xi d\xi}{\varphi'(\varphi^{-1}(\xi))} \geq \frac{1}{L_\varphi} \int_0^{\varphi(s)} \xi d\xi = \frac{1}{2L_\varphi}(\varphi(s))^2. \]
On the other hand
\[ \Phi(s) \leq L_\varphi \int_0^s \tau d\tau = \frac{L_\varphi}{2} s^2. \]

\[ \square \]

3 A priori estimates and existence of discrete solution

3.1 A priori estimates

Definition 3.1 (Approximate solution) Let $D$ be a discretization of $\Omega$, $N \in \mathbb{N}^*$ and $\delta t = T/N > 0$. We say that the sequence $(s_{D,\delta t}, p_{D,\delta t}) = (s^n, p^n)_{n \in \{1, \ldots, N\}} \in (X_{D,\delta t,0})$ is an approximate solution of the problem (1.5)-(1.9) if for all $n \in \{1, \ldots, N\}$, $(s^n, p^n)$ satisfies (2.16)-(2.18). We also denote by $s_{D,\delta t}$ and $p_{D,\delta t}$ the function pair defined by
\begin{align*}
  s_{D,\delta t}(x, 0) &= s^0_K \text{ for all } x \in K, \\
  s_{D,\delta t}(x, t) &= s^n_K \text{ for all } (x, t) \in K \times (t_{n-1}, t_n], \\
  s_{D,\delta t}(x, t) &= s^n_\sigma \text{ for all } (x, t) \in \sigma \times (t_{n-1}, t_n]
\end{align*}
and
\begin{align*}
  p_{D,\delta t}(x, t) &= p^n_K \text{ for all } (x, t) \in K \times (t_{n-1}, t_n], \\
  p_{D,\delta t}(x, t) &= p^n_\sigma \text{ for all } (x, t) \in \sigma \times (t_{n-1}, t_n].
\end{align*}

Remark 3.1 In general the capillary diffusion $\varphi(s_{D,\delta t})$ is more regular than $s_{D,\delta t}$ and it is easier to work with the function $\varphi_{D,\delta t}$, thus the element $\varphi_{D,\delta t} = \varphi(s_{D,\delta t})$ of $X_{D,\delta t,0}$ can be considered as a set of primary discrete unknowns. We also define the function $\varphi_{D,\delta t}(x, t)$
\[ \varphi_{D,\delta t}(x, t) = \varphi(s_{D,\delta t}(x, t)) \text{ for all } (x, t) \in Q_T. \]

Theorem 3.1 (A priori estimate) Let $(s_{D,\delta t}, p_{D,\delta t})$ be a solution of the discrete problem (2.16)-(2.18). Then there exists a positive constant $C$ independent of $h_D$ and $\delta t$ such that
\[ \|\nabla_{D,\delta t} p_{D,\delta t}\|_{L^\infty(0,T;L^2(\Omega))} + \|\varphi_{D,\delta t}\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla_{D,\delta t} \varphi_{D,\delta t}\|_{L^2(Q_T)} \leq C. \]

Proof: Pressure equation. In order to obtain the estimate on the first term in (3.7) we use $p^n$ as a test element in the pressure equation (2.16), which implies
\[ \sum_{K \in M} \sum_{\sigma \in E_K} (p^n_K - p^n_\sigma)(\lambda^n_K F_{K,\sigma}(p^n) + \xi^n_{K,\sigma g_{K,\sigma}}) = \sum_{K \in M} m(K) p^n_K (k_{w,K} + k_{o,K}). \]
Let us first estimate the term $T_\xi = \sum_{K \in M} \sum_{\sigma \in E_K} (p^n_K - p^n_\sigma) \xi^n_{K,\sigma g_{K,\sigma}}$. In view of (2.20) we deduce from Cauchy-Schwarz inequality that
\[ (T_\xi)^2 = \left( \sum_{K \in M} \sum_{\sigma \in E_K} m(\sigma)(p^n_K - p^n_\sigma) n_{K,\sigma} \cdot \xi^n_{K,\sigma g_{K,\sigma}} d_{K,\sigma} \right)^2 \leq \sum_{K \in M} \sum_{\sigma \in E_K} \frac{m(\sigma)}{d_{K,\sigma}} (p^n_K - p^n_\sigma)^2 \sum_{K \in M} \sum_{\sigma \in E_K} m(\sigma) d_{K,\sigma} (|K g| \xi^n_{K,\sigma})^2. \]
Then, in view of the assumptions $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ and $(\mathcal{H}_{4b})$, and also thanks to (2.4) and Lemma 2.2 we deduce that

$$|T_\xi| \leq dm(\Omega)\overline{K}\|g\|_{L^\infty((0,1))}\|\nabla_{\mathcal{D},\delta t}p(\cdot, t_n)\|_{L^2(\Omega)}.$$  \hspace{1cm} (3.9)

In view of (2.7) and (3.4) the right-hand side of (3.8) can be written as

$$\sum_{K \in \mathcal{M}} m(K)p^n_K(k^n_{w,K} + k^n_{o,K}) = \sum_{K \in \mathcal{M}} \frac{1}{\delta t} \int_{t_{n-1}}^{t_n} \int_\Omega p_{\mathcal{D},\delta t}(x, t)(k_w(x, t) + k_o(x, t)) \, dx \, dt$$

Applying Cauchy-Schwarz inequality and the discrete Poincaré inequality (Lemma 2.3) we obtain

$$\sum_{K \in \mathcal{M}} m(K)p^n_K(k^n_{w,K} + k^n_{o,K}) \leq \|p_{\mathcal{D},\delta t}(\cdot, t_n)\|_{L^2(\Omega)}\|k_w + k_o\|_{L^\infty(0,T;L^2(\Omega))} \leq C\|\nabla_{\mathcal{D},\delta t}p_{\mathcal{D},\delta t}(\cdot, t_n)\|_{L^2(\Omega)}\|k_w + k_o\|_{L^\infty(0,T;L^2(\Omega))},$$

so that in view of $(\mathcal{H}_6)$,

$$\sum_{K \in \mathcal{M}} m(K)p^n_K(k^n_{w,K} + k^n_{o,K}) \leq C\|\nabla_{\mathcal{D},\delta t}p_{\mathcal{D},\delta t}(\cdot, t_n)\|_{L^2(\Omega)}.$$  \hspace{1cm} (3.10)

Gathering (2.39), (3.8), (3.9), (3.10) we obtain

$$\Delta\|\nabla_{\mathcal{D},\delta t}p_{\mathcal{D},\delta t}(\cdot, t_n)\|^2_{L^2(\Omega)} \leq \sum_{K \in \mathcal{M}, \sigma \in \mathcal{E}_K} (p^n_K - p^n_{\sigma})\lambda^n_K \mathcal{F}_{K,\sigma}(p^n) \leq C\|\nabla_{\mathcal{D},\delta t}p_{\mathcal{D},\delta t}(\cdot, t_n)\|_{L^2(\Omega)},$$

with some positive $C$; the proof of the estimate on the first term of (3.7) is complete. The estimate on the other terms of (3.7) can be obtained by setting $w^n = \varphi(s^n)$ in the saturation equation (2.17) and summing over $n \in \{1, \ldots, m\}$, which yields

$$\sum_{n=1}^m \sum_{K \in \mathcal{M}} \omega(K)\varphi^n_K(s^n_K - s^{n-1}_K) + \delta t \sum_{n=1}^m \sum_{K \in \mathcal{M}, \sigma \in \mathcal{E}_K} (\varphi^n_K - \varphi^n_{\sigma})\mathcal{F}_{K,\sigma}(\varphi^n) = \sum_{n=1}^m \sum_{K \in \mathcal{M}} \delta tm(K)\varphi^n_Kk^n_{o,K}.$$  

We define the terms

$$T_t = \sum_{n=1}^m \sum_{K \in \mathcal{M}} \omega(K)\varphi^n_K(s^n_K - s^{n-1}_K),$$

$$T_C = \sum_{n=1}^m \sum_{K \in \mathcal{M}, \sigma \in \mathcal{E}_K} \delta t(\varphi^n_K - \varphi^n_{\sigma})\mathcal{G}(Q^n_{K,\sigma}f(\cdot) + \gamma(\cdot)g_{K,\sigma}; s^n_K, s^n_{\sigma}),$$

and

$$T_D = \sum_{n=1}^m \sum_{K \in \mathcal{M}, \sigma \in \mathcal{E}_K} \delta t(\varphi^n_K - \varphi^n_{\sigma})\mathcal{F}_{K,\sigma}(\varphi(s^n)).$$

**Accumulation term.** Remark that using the notations of Lemma 2.7

$$\Phi(s^n_K) - \Phi(s^{n-1}_K) = \varphi^n_K(s^n_K - s^{n-1}_K) + \int_{s^{n-1}_K}^{s^n_K} (\varphi(\tau) - \varphi^n_K) \, d\tau.$$  \hspace{1cm} (3.11)
Since the function $\varphi$ is increasing, the second term in the right-hand side of (3.11) is negative, which implies that

$$T_t \geq \sum_{n=1}^{m} \sum_{K \in \mathcal{M}} \omega(K)(\Phi(s^n_K) - \Phi(s^{n-1}_K)) = \sum_{K \in \mathcal{M}} \omega(K)\Phi(s^n_K) - \sum_{K \in \mathcal{M}} \omega(K)\Phi(s^n_K).$$

It follows from (2.45) and the assumption $\mathcal{H}_0$ that

$$T_t \geq \frac{1}{2L_\varphi} \sum_{K \in \mathcal{M}} \omega(K)(\varphi_K^m)^2 - \frac{L_\varphi}{2} \sum_{K \in \mathcal{M}} \omega(K)(s^n_K)^2 \geq \frac{\omega}{2L_\varphi}\|\varphi_D,\delta \|_{L^2(\Omega)}^2 - \frac{\omega L_\varphi}{2}\|s_D,\delta\|_{L^2(\Omega)}^2.$$

**Convection term.** In this subsection we use the simplified notation

$$\mathcal{G}_{K,\sigma}^n(a, b) = \mathcal{G}(Q_{K,\sigma}^n f(\cdot) + \gamma(\cdot)g_{K,\sigma}; a, b) \text{ for all } a, b \in \mathbb{R}. \quad (3.12)$$

For all $K \in \mathcal{M}, \sigma \in \mathcal{E}_K$ and $s \in \mathbb{R}$ we define the function

$$G_{K,\sigma}^n(s) = \int_0^s \mathcal{G}_{K,\sigma}^n(\tau, \tau)\varphi'(\tau)d\tau, \quad (3.13)$$

which is such that

$$G_{K,\sigma}^n(s^n_K) - G_{K,\sigma}^n(s^n_\sigma) = (\varphi_K - \varphi_\sigma)G_{K,\sigma}^n(s^n_K, s^n_\sigma) + \int_{s^n_\sigma}^{s^n_K} (G_{K,\sigma}^n(\tau, \tau) - G_{K,\sigma}^n(s^n_K, s^n_\sigma)) \varphi'(\tau)d\tau. \quad (3.14)$$

In view of (2.25) the second term on the right-hand side of (3.14) is negative, which implies the following estimate

$$T_C \geq \sum_{n=1}^{m} \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} \delta t \left( G_{K,\sigma}^n(s^n_K) - G_{K,\sigma}^n(s^n_\sigma) \right).$$

In view of (3.12), (3.13) and (2.25) we have that

$$G_{K,\sigma}^n(s^n_K) - G_{K,\sigma}^n(s^n_\sigma) = Q_{K,\sigma}^n \int_{s^n_\sigma}^{s^n_K} f(\tau)\varphi'(\tau)d\tau + g_{K,\sigma} \int_{s^n_\sigma}^{s^n_K} \gamma(\tau)\varphi'(\tau)d\tau.$$

It follows from (2.16) and the hypotheses $\mathcal{H}_1, \mathcal{H}_2$ that

$$\sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} Q_{K,\sigma}^n \int_{s^n_\sigma}^{s^n_K} f(\tau)\varphi'(\tau)d\tau = \sum_{K \in \mathcal{M}} m(K)(k^n_{u,K} + k^n_{o,K}) \int_{0}^{s^n_K} f(\tau)\varphi'(\tau)d\tau \geq 0,$$

where we have set

$$v^n_K = \int_{0}^{s^n_K} f(\tau)\varphi'(\tau)d\tau \quad \text{and} \quad v^n_\sigma = \int_{0}^{s^n_\sigma} f(\tau)\varphi'(\tau)d\tau.$$

Therefore we have that

$$T_C \geq \sum_{n=1}^{N} \delta t \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} g_{K,\sigma} \int_{s^n_\sigma}^{s^n_K} \gamma(\tau)\varphi'(\tau)d\tau \geq \sum_{n=1}^{N} \delta t \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} g_{K,\sigma} \int_{0}^{s^n_\sigma} \gamma(\tau)\varphi'(\tau)d\tau - \sum_{n=1}^{N} \delta t \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} g_{K,\sigma} \int_{0}^{s^n_K} \gamma(\tau)\varphi'(\tau)d\tau$$
In view of (2.21), (2.22) and using the homogeneous Dirichlet boundary condition we obtain

\[ T_C \geq - \sum_{n=1}^{N} \delta t \sum_{K \in \mathcal{M}} \sum_{0 \leq i \leq j \leq J} \sum_{\sigma \in E_{K \cap \Gamma_{i,j}}} g_{K,\sigma} \int_{0}^{\sigma} \gamma(\tau) \varphi'(\tau) d\tau. \]

Therefore

\[ T_C \geq -2T \sum_{0 \leq i \leq j \leq J} m(\Gamma_{i,j}) K |g| L_{\varphi} \|\gamma\| L^{1}(0,1)). \]  

(3.15)

**Diffusion and source terms.** It follows from the equality (2.33)

\[ T_D \geq K \|\nabla_{\mathcal{D},\delta t}\varphi_{\mathcal{D},\delta t}\|_{L^{2}(Q_{T})}. \]

In view of the estimates on \( T_{t} \) and \( T_{D} \) we deduce that

\[
\frac{\omega}{2L_{\varphi}}\|\varphi_{\mathcal{D},\delta t}\|_{L^{\infty}(0,T;L^{2}(\Omega))}^{2} + K \|\nabla_{\mathcal{D},\delta t}\varphi_{\mathcal{D},\delta t}\|_{L^{2}(Q_{T})}^{2} \\
\leq \frac{\omega}{2} s_{\mathcal{D},\delta t}(\cdot,0)\|\varphi_{\mathcal{D},\delta t}\|_{L^{2}(\Omega)}^{2} + \sum_{n=1}^{N} \sum_{K \in \mathcal{M}} \delta t m(K) \varphi_{K}^{n} k_{o,K} - T_{C}.
\]

Applying Cauchy-Schwarz and Young’s inequality to the last term in (3.16) leads to

\[
\frac{\omega}{2L_{\varphi}}\|\varphi_{\mathcal{D},\delta t}\|_{L^{\infty}(0,T;L^{2}(\Omega))}^{2} + K \|\nabla_{\mathcal{D},\delta t}\varphi_{\mathcal{D},\delta t}\|_{L^{2}(Q_{T})}^{2} \\
\leq \frac{\varepsilon}{2} \|\varphi_{\mathcal{D},\delta t}\|_{L^{2}(Q_{T})}^{2} + \frac{1}{2\varepsilon} k_{o}\|\varphi_{\mathcal{D},\delta t}\|_{L^{2}(Q_{T})} + \frac{\omega L_{\varphi}}{2} s_{\mathcal{D},\delta t}(\cdot,0)\|\varphi_{\mathcal{D},\delta t}\|_{L^{2}(\Omega)}^{2} - T_{C} \\
\leq \frac{\varepsilon}{2} \|\varphi_{\mathcal{D},\delta t}\|_{L^{\infty}(0,T;L^{2}(\Omega))}^{2} + \frac{1}{2\varepsilon} k_{o}\|\varphi_{\mathcal{D},\delta t}\|_{L^{2}(Q_{T})} + \frac{\omega L_{\varphi}}{2} s_{\mathcal{D},\delta t}(\cdot,0)\|\varphi_{\mathcal{D},\delta t}\|_{L^{2}(\Omega)}^{2} - T_{C}.
\]

Setting \( \varepsilon = \omega/(2TL_{\varphi}) \) we complete the proof. \( \square \)

**Remark 3.2** Remark that the partitioning of \( \Omega \) in to the set of homogeneous media \( \{(\Omega_{i})_{i \in \{1,\ldots,N\}}\} \) serves to control the term \( T_{C} \) in (3.15). Therefore if the gravity effects are neglected the assumption \( (H_{4b}) \) can be weakened.

### 3.2 Existence of a discrete solution

In this subsection we show that there exists a sequence \( ((p^{n},s^{n}))_{n \in \{1,\ldots,N\}} \in X_{\mathcal{D},\delta t,0}^{2} \) satisfying the variational discrete problem. The existence result makes use of the topological degree argumentation [[18], Theorem 3.1].

**Theorem 3.2** The problem (2.16)-(2.18) has at least one solution.

**Proof:** In view of Remark 3.1 the discrete problem (2.16)-(2.18) can be written as a system of \( 2N(\text{card}(\mathcal{M}) + \text{card}(\mathcal{E})) \) nonlinear equations \( H_{\mathcal{D},\delta t}(\varphi,p) = 0 \). For \( \nu \in [0,1] \) we define

- \( f^{\nu}(s) = \nu f(s) \),
- \( \xi^{\nu}(s) = \nu \xi(s) \),
- \( \gamma^{\nu}(s) = \nu \gamma(s) \),
- \( \lambda^{\nu}(s) = \nu \lambda(s) + (1 - \nu) \Lambda \)

and we consider the following extended problem

\[
- \nabla \cdot (K(\lambda(s^{\nu}) \nabla p^{\nu} - \xi^{\nu}(s^{\nu}))g) = k_{w} + k_{o} \quad \text{in} \quad \Omega \times (0,T), \quad (3.17)
\]

\[
q^{\nu} = -K(\lambda^{\nu}(s^{\nu}) \nabla p^{\nu} - \xi^{\nu}(s^{\nu}))g \quad \text{in} \quad \Omega \times (0,T), \quad (3.18)
\]

\[
\omega \frac{\partial s^{\nu}}{\partial t} + \nabla \cdot (u^{\nu} f^{\nu}(s^{\nu}) + \gamma^{\nu}(s^{\nu})Kg) - \nabla \cdot (K \nabla \varphi(s^{\nu})) = k_{o} \quad \text{in} \quad \Omega \times (0,T), \quad (3.19)
\]
together with initial and boundary conditions (1.8), (1.9). We denote by \( H_{\nu,\delta t}(\phi,p) = 0 \) the corresponding discrete problem and \((\varphi_{\nu,\delta t},p_{\nu,\delta t})\) its solution, which satisfies the a priori estimates (3.7) instead of \((\varphi_{\nu,\delta t},p_{\nu,\delta t})\). Then, there exists \( R > 0 \) such that

\[
(\varphi_{\nu,\delta t},p_{\nu,\delta t}) \in B_R = \{ (u,v) \in X_{\nu,\delta t,0}^2 \mid \|\nabla_{\nu,\delta t}u\|_{L^2(Q_T)} + \|\nabla_{\nu,\delta t}v\|_{L^2(Q_T)} < R \}
\]

for all \( \nu \in [0, 1] \). Therefore for any \( \nu \in [0, 1] \), \((\varphi_{\nu,\delta t},p_{\nu,\delta t})\) does not belongs to \( \partial B_R \), which implies that the topological degree of \( H_{\nu,\delta t}(\phi,p) \) with respect to \( B_R \) and right hand side \( 0 \) is constant. Remark that for \( \nu = 0 \) the system (3.17)-(3.19), becomes uncoupled

\[
-\nabla \cdot \mathbf{K} \nabla \rho^0 = k_w + k_o \quad \text{in} \quad \Omega \times (0,T), \tag{3.20}
\]

\[
\omega \frac{\partial s^0}{\partial t} - \nabla \cdot (\mathbf{K} \nabla \varphi(s^0)) = k_o \quad \text{in} \quad \Omega \times (0,T). \tag{3.21}
\]

We first remark that since the equation (3.20) does not contain \( s \). It follows from [22] that the discrete problem corresponding to (3.20) and (1.9) admits the unique solution \( p_{\nu,\delta t}^0 \); moreover the existence and uniqueness of \( \varphi_{\nu,\delta t}^0 \) satisfying the discrete version of the problem (3.21), (1.8) and (1.9) has been shown in [3] where a family of schemes, containing in particular the SUSHI scheme was applied to a scalar parabolic degenerate convection-reaction-diffusion equation. Consequently there exists the unique solution \((p_{\nu,\delta t}^0, \varphi_{\nu,\delta t}^0)\) to the problem \( H_{\nu,\delta t}(\phi,p) \), and the corresponding topological degree is equal to \( \pm 1 \). From the homotopy invariance of the topological degree we deduce that here exists at least one solution to \( H_{\nu,\delta t}(\phi,p) = 0 \).

\[\square\]

4 Estimates on space and time translates

4.1 Estimates on space translates

We state here an estimate on space translates, which is a consequence of the a priori estimate (given by Theorem 3.1), Lemma 2.2 and [3, Lemma 3.4, Lemma 6.5] (see also the basic article [22]).

**Lemma 4.1** Let \( D \) be a discretization of \( \Omega \) and let \( \delta t = T/N > 0 \) with some \( N \in \mathbb{N}^* \); let \((s_{\nu,\delta t},p_{\nu,\delta t}) \in X_{\nu,\delta t,0}^2 \) be a solution of the discrete problem (2.16)-(2.18). We recall that the function \( \varphi_{\nu,\delta t} \) is defined by \( \varphi_{\nu,\delta t}(x,t) = \varphi(s_{\nu,\delta t}(x,t)) \). Let also \( \theta \geq \theta_D \) be given, then there exist \( C > 0 \) and \( \rho > 0 \), which depend on \( \theta \), but does not depend on mesh and time step size, such that

\[
\|\varphi_{\nu,\delta t}(\cdot + y, \cdot) - \varphi_{\nu,\delta t}\|_{L^2(\mathbb{R}^d \times (0,T))} \leq C|y|^{\rho},
\]

taking \( \varphi_{\nu,\delta t} = 0 \) outside of \( Q_T \).

4.2 Estimates on time translates

To begin with we state without proof two technical lemmas which will be useful for proving the estimate on time translates.

**Lemma 4.2** Let \( T > 0, \tau \in (0,T), N \in \mathbb{N}^* \), \( \delta t = T/N \) be given and \((a^n)_{n \in \mathbb{N}^*} \) be a family of non negative real values. Let \([s]\) denotes the smallest integer larger or equal to \( s \). Then

\[
\int_0^{T-\tau} \sum_{[t/\delta t] + 1 \leq n \leq [((t+\tau)/\delta t)]} a^n dt \leq \tau \sum_{n=1}^{N} a^n.
\]
Lemma 4.3 Let $T > 0$, $\tau \in (0, T)$, $N \in \mathbb{N}^*$, $\delta t = T/N$, $\zeta \in [0, \tau]$ be given and $(a^n)_{n \in \mathbb{N}^*}$ be a family of nonnegative real values. Let $[s]$ denotes the smallest integer larger or equal to $s$. Then

$$\int_0^{T-\tau} \sum_{\lceil t/\delta t \rceil+1 \leq n \leq \lceil (t+\zeta)/\delta t \rceil} a^{\lceil (t+\zeta)/\delta t \rceil} dt \leq \tau \sum_{n=1}^{N} a^n.$$ 

Next we prove the estimate on differences of time translates.

Lemma 4.4 Let $\mathcal{D}$ be a discretization of $\Omega$ and let $\delta t = T/N > 0$ with some $N \in \mathbb{N}^*$; let $(s_{\mathcal{D},\delta t}, p_{\mathcal{D},\delta t}) \in X^2_{\mathcal{D},\delta t}$ be a solution of the discrete problem (2.16)-(2.18). Let also $\theta \geq \theta_\mathcal{D}$ be given, then there exists a positive constant $C$ only depending on $\theta$ such that

$$\int_{\Omega \times (0, T-\tau)} (\varphi_{\mathcal{D},\delta t}(x, t+\tau) - \varphi_{\mathcal{D},\delta t}(x, t))^2 \, dx dt \leq C \tau$$

for all $\tau \in (0, T)$.

Proof: The discrete saturation equation (2.17) yields

$$\sum_{K \in \mathcal{M}} \omega(K) w_K(s^n_K - s_{K}^{n-1}) =$$

$$- \delta t \sum_{K \in \mathcal{M}, \sigma \in \mathcal{E}_K} (w_K - w_\sigma) \mathcal{G}(Q^n_{K,\sigma} f(\cdot) + \gamma(\cdot) g_{K,\sigma}; s^n_K, s^0_\sigma)$$

$$- \delta t \sum_{K \in \mathcal{M}, \sigma \in \mathcal{E}_K} (w_K - w_\sigma) \mathcal{F}_{K,\sigma}(\varphi(s^n)) + \delta t \sum_{K \in \mathcal{M}} m(K) w_K k_{o,K}.$$ 

for all $w \in X_{\mathcal{D},0}$. In view of Lemma 2.6 and Lemma 2.4 one has

$$| \sum_{K \in \mathcal{M}} \omega(K) w_K(s^n_K - s_{K}^{n-1}) |$$

$$\leq C \delta t |w|_{\mathcal{D}} (|p^n|_{\mathcal{D}} + |\varphi^n|_{\mathcal{D}} + 1) + \delta t \sum_{K \in \mathcal{M}} m(K) w_K k_{o,K}.$$ 

Next, applying the Cauchy-Schwarz and discrete Poincaré inequality (cf. inequality (2.37)) to the last term of (4.3)

$$| \sum_{K \in \mathcal{M}} \omega(K) w_K(s^n_K - s_{K}^{n-1}) |$$

$$\leq C \delta t |w|_{\mathcal{D}} (|p^n|_{\mathcal{D}} + |\varphi^n|_{\mathcal{D}} + \frac{1}{\delta t} \| k_o \|_{L^2(\Omega) \times (t_{n-1}, t_n)} + 1).$$

In view of hypothesis ($\mathcal{H}_1$) we obtain

$$\frac{1}{L_{\varphi}} \int_0^{T-\tau} \int_{\Omega} (\varphi_{\mathcal{D},\delta t}(x, t+\tau) - \varphi_{\mathcal{D},\delta t}(x, t))^2 \, dx dt$$

$$= \frac{1}{L_{\varphi}} \int_0^{T-\tau} \sum_{K \in \mathcal{M}} \omega(K) \left( \varphi_{K}^{\lceil (t+\tau)/\delta t \rceil} - \varphi_{K}^{\lceil t/\delta t \rceil} \right)^2 \, dt$$

$$\leq \int_0^{T-\tau} \sum_{K \in \mathcal{M}} \omega(K) \left( \varphi_{K}^{\lceil (t+\tau)/\delta t \rceil} - \varphi_{K}^{\lceil t/\delta t \rceil} \right) \left( s_{K}^{\lceil (t+\tau)/\delta t \rceil} - s_{K}^{\lceil t/\delta t \rceil} \right) \, dt$$

$$= \int_0^{T-\tau} \sum_{\lceil t/\delta t \rceil+1 \leq n \leq \lceil (t+\tau)/\delta t \rceil} \sum_{K \in \mathcal{M}} \omega(K) \left( \varphi_{K}^{\lceil (t+\tau)/\delta t \rceil} - \varphi_{K}^{\lceil t/\delta t \rceil} \right) (s^n_K - s_{K}^{n-1}) \, dt.$$
For a given $\delta t$ and for all real $t$ and $\tau$ we define the following set
\[ n(t, \tau) = \{ n \in \mathbb{N}, \; [t/\delta t] + 1 \leq n \leq [(t+\tau)/\delta t] \}, \]
which can be empty. Then, in view of (4.4), Lemma 4.2 and Lemma 4.3 we obtain
\[
\frac{1}{L} \int_0^{T-\tau} \int_0^T (\varphi_{D,\delta t}(x, t + \tau) - \varphi_{D,\delta t}(x, t))^2 \, dx \, dt \\
\leq C \int_0^{T-\tau} \sum_{n \in n(t, \tau)} \delta t \left( |\varphi^{[t+\tau]/\delta t}]_{X_D} + |\varphi^{[t]/\delta t}]_{X_D} \right) \\
+ |\varphi^{[t]/\delta t}]_{X_D}^2 + |p^n|_{X_D}^2 + |\varphi^n|_{X_D}^2 + \frac{1}{\delta t} \|k_o\|_{L^2(\Omega \times (t_n-1, t_n))}^2 + 1) \\
\leq C \tau \sum_{n=1}^N \delta t \left( |\varphi^n|_{X_D}^2 + |p^n|_{X_D}^2 + \frac{1}{\delta t} \|k_o\|_{L^2(\Omega \times (t_n-1, t_n))}^2 + 1 \right).
\]

Finally we use Theorem 3.1 and the hypothesis $(\mathcal{H}_\theta)$ to complete the proof. □

5 Convergence result

This section is devoted to the proof of the following theorem.

Theorem 5.1 Let $\mathcal{D}$ be a sequence of discretizations of $\Omega$ and such that there exists a positive constant $\theta$ satisfying $\theta_D \leq \theta$ for all $D \in \mathcal{D}$. Let $\delta t$ be a sequence of real positive numbers, such that $T/\delta t \in \mathbb{N}$ for all $\delta t \in \delta t$ and such that $\delta t$ tends to zero along $\delta t$. Let $(s_{\mathcal{D},\delta t}, p_{\mathcal{D},\delta t}) = (s_{\mathcal{D},\delta t}, p_{\mathcal{D},\delta t})_{D \in \mathcal{D}, \delta t \in \delta t}$ be a sequence of approximate solutions corresponding to $\mathcal{D}$ and $\delta t$, and let $\varphi_{\mathcal{D},\delta t} = \varphi(s_{\mathcal{D},\delta t})$. Then there exists a subsequence of $(s_{\mathcal{D},\delta t}, p_{\mathcal{D},\delta t})$, which we denote again by $(s_{\mathcal{D},\delta t}, p_{\mathcal{D},\delta t})$, such that $s_{\mathcal{D},\delta t} \to s$ strongly in $L^2(Q_T)$ and $p_{\mathcal{D},\delta t} \to p$ weakly in $L^2(Q_T)$ as $h_D, \delta t \to 0$, where $(s, p)$ is a weak solution of the problem (1.5) – (1.7).

5.1 Relative compactness of $\varphi_{\mathcal{D},\delta t}$

The relative compactness of $\varphi_{\mathcal{D},\delta t}$ follows from Theorem 3.1 and the estimates on time and space translates. Indeed, let us prolonge $\varphi_{\mathcal{D},\delta t}$ by zero outside of $Q_T$ and remark that we may prolonge as well $\nabla D,\delta t \varphi_{\mathcal{D},\delta t}$ by zero outside of $Q_T$. The Fréchet-Kolmogorov Compactness Theorem implies that the family $\varphi_{\mathcal{D},\delta t}$ is relatively compact in $L^2(Q_T)$, which yields in turn that $\varphi_{\mathcal{D},\delta t}$ contains a subsequence strongly converging in $L^2(Q_T)$ (and also in $L^2(\mathbb{R}^d \times (0, T))$) to some $\phi \in L^\infty(0, T; L^2(\mathbb{R}^d))$. In view of (3.7), we deduce that $\nabla D,\delta t \varphi_{\mathcal{D},\delta t}$ is weakly relatively compact in $L^2(\mathbb{R}^d \times (0, T))$ and it can be shown (see [3, Theorem 7.1] and [22, Lemma 4.2]) that $\nabla D,\delta t \varphi_{\mathcal{D},\delta t} \rightharpoonup \nabla \phi$ weakly in $L^2(\mathbb{R}^d \times (0, T))$. Thus $\phi \in L^2(0, T; H^1(\mathbb{R}^d))$, since $\phi = 0$ for all $x \in \mathbb{R}^d \setminus \Omega$, it follows that $\phi \in L^2(0, T; H^1_0(\Omega))$.

5.2 Weak relative compactness of $p_{\mathcal{D},\delta t}$

Since the pressure equation does not involves any time derivatives on $p$, we can not obtain any estimate time translates. Nevertheless, Theorem 3.1 and discrete Poincaré
inequality implies that $p_{\mathcal{D},\delta t}$ is bounded in $L^\infty(0,T;L^2(\Omega))$. Then up to a sequence $p_{\mathcal{D},\delta t}$ converges weakly in $L^2(Q_T)$ (and once again in $L^2(\mathbb{R}^d \times (0,T))$ taking $p_{\mathcal{D},\delta t} = 0$ outside of $Q_T$) to some $p \in L^\infty(0,T;L^2(\mathbb{R}^d))$; moreover using same arguments as for $\varphi_{\mathcal{D},\delta t}$, we deduce that $\nabla_{\mathcal{D},\delta t}p_{\mathcal{D},\delta t} \rightharpoonup \nabla p$ weakly in $L^2(\mathbb{R}^d \times (0,T))$ and $p \in L^2(0,T;H^1_0(\Omega))$.

### 5.3 Saturation equation

The assumption $(\mathcal{H}_1)$ implies that

$$
\|s_{\mathcal{D},\delta t}\|_{L^\infty(0,T;L^2(\Omega))} \leq m(\Omega) + \|\varphi_{\mathcal{D},\delta t}\|_{L^\infty(0,T;L^2(\Omega))}.
$$

Therefore in view of Theorem 3.1 we have that $\{s_{\mathcal{D},\delta t}\}$ contains a subsequence (denoted again by $\{s_{\mathcal{D},\delta t}\}$) such that $s_{\mathcal{D},\delta t}$ converges to some $s \in L^\infty(0,T;L^2(\Omega))$ strongly in $L^2(Q_T)$ as $h_{\mathcal{D}},\delta t \to 0$. It remains to show that $(s,p)$ satisfies the integral equalities (iv) of Definition 1.1, for this purpose we introduce the function space

$$
\Psi = \{\psi \in C^{2,1}(\overline{\Omega} \times [0,T]), \ \psi = 0 \text{ on } \partial\Omega \times [0,T], \ \psi(\cdot, T) = 0\}.
$$

Taking an arbitrary $\psi \in \Psi$, we define the sequence of elements of $X_{\mathcal{D},0}$

$$
\psi^n = P_{\mathcal{D}}\psi(\cdot, t_n) \text{ for all } n \in \{1, \ldots, N\},
$$

which implies $\psi^n_K = \psi(x_K, t_n)$ and $\psi^n_\sigma = \psi(x_\sigma, t_n)$. Setting $w^n = \psi^{n-1}$ in the discrete saturation equation we multiply it by the time step $\delta t$ and we sum over $n \in \{1, \ldots, N\}$ to obtain

$$
S_T + S_C + S_D = S_S,
$$

where

$$
T_T = \sum_{n=1}^{N} \sum_{K \in \mathcal{M}} \omega(K)\psi^n_K(s^n_K - s^{n-1}_K),
$$

$$
T_C = \sum_{n=1}^{N} \delta t \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} \left(\psi^n_K - \psi^n_\sigma\right)\mathcal{G}(Q^n_K, f(\cdot) + \gamma(\cdot)g_K, s^n_K, s^n_\sigma)
$$

and

$$
T_D = \sum_{n=1}^{N} \delta t \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} (\psi^n_{K-1} - \psi^n_\sigma)F_{K,\sigma}(\varphi(s^n))
$$

$$
T_S = \sum_{n=1}^{N} \delta t \sum_{K \in \mathcal{M}} m(K)\psi^n_K k^n_{o,K}.
$$

### 5.3.1 Time evolution term

Using discrete integration by parts and the fact that $\varphi(x,T) = 0$ we obtain

$$
T_T = -\sum_{n=1}^{N} \sum_{K \in \mathcal{M}} \omega(K)(\psi^n_K - \psi^{n-1}_K)s^n_K - \sum_{K \in \mathcal{M}} \omega(K)\psi^n_K s^n_K.
$$

Clearly

$$
\sum_{K \in \mathcal{M}} \omega(K)\psi^n_K s^n_K \to \int_{\Omega} \omega(x)\psi(x,0)s_0(x) \, dx;
$$

$$
\sum_{K \in \mathcal{M}} \omega(K)\psi^n_K s^n_K \to \int_{\Omega} \omega(x)\psi(x,0)s_0(x) \, dx;
$$

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next we define
\[
T_1^T = \sum_{n=1}^N \sum_{K \in M} \omega(K)(\psi^n_K - \psi^{n-1}_K) s^n_K - \int_0^T \int_{\Omega} s(x, t) \psi_t(x, t) \, dx dt,
\]
and we add and subtract \(\int_{t_{n-1}}^{t_n} \int_K s^n_K \psi_t(x, t) \, dx dt\) in each term to obtain
\[
T_1^T = \sum_{n=1}^N \sum_{K \in M} m(K)s^n_K \int_{t_{n-1}}^{t_n} (\psi_t(x_K, t) - \psi_t(x, t)) \, dt
+ \int_0^T \int_{\Omega} (s_{D, \delta t}(x, t) - s(x, t)) \psi_t(x, t) \, dx dt.
\]

In view of the regularity of \(\psi\) we have that for all \(x \in K\) and all \(K \in M\) it holds
\[
|\psi_t(x_K, t) - \psi_t(x, t)| \leq Ch_D.
\]
Since \(s_{D, \delta t}\) is bounded in \(L^\infty(0, T; L^2(\Omega))\) the first term in (5.1) tends to zero as \(h_D, \delta t \to 0\).
Further, since \(|\psi_t(x, t)| \leq C_\psi\), the second term in (5.1) also tends to zero in view of the strong convergence of \(s_{D, \delta t}\).

5.3.2 Convection term

Using the notation of (2.28) we split \(T_C\) into \(T_C = T_C^1 + T_C^2\) with
\[
T_C^1 = \sum_{n=1}^N \delta t \sum_{K \in M} \sum_{\sigma \in E_K} (\psi_{n-1}^{K} - \psi_{n-1}^{\sigma}) Q^n_{K, \sigma} f^n_{K, \sigma},
T_C^2 = \sum_{n=1}^N \delta t \sum_{K \in M} \sum_{\sigma \in E_K} (\psi_{n-1}^{K} - \psi_{n-1}^{\sigma}) g_{K, \sigma} \gamma^n_{K, \sigma}.
\]

First we study the limit of the term \(T_C^2\), which can be written as \(T_C^2 = T_C^{21} + T_C^{22}\),
\[
T_C^{21} = \sum_{n=1}^N \delta t \sum_{K \in M} \sum_{\sigma \in E_K} (\psi_{n-1}^{K} - \psi_{n-1}^{\sigma}) g_{K, \sigma} \gamma^n_{K, \sigma},
T_C^{22} = \sum_{n=1}^N \delta t \sum_{K \in M} \sum_{\sigma \in E_K} (\psi_{n-1}^{K} - \psi_{n-1}^{\sigma}) g_{K, \sigma} (\gamma^n_{K, \sigma} - \gamma^n_{K}).
\]

Using (2.20) and (2.30) one has
\[
T_C^{21} = \sum_{n=1}^N \delta t \sum_{K \in M} \gamma^n_{K} Kg \cdot \left( \sum_{\sigma \in E_K} m(\sigma)(\psi_{n-1}^{K} - \psi_{n-1}^{\sigma}) n_{K, \sigma} \right)
= - \sum_{n=1}^N \sum_{K \in M} \int_{t_{n-1}}^{t_n} \int_K \gamma(s_{D, \delta t}) Kg \cdot \nabla K \psi_{n-1}.
\]

We deduce from the regularity of \(\psi\) and Lemma 2.1 that
\[
|\nabla K \psi^n - \nabla \psi(x, t)| \leq C(h_D + \delta t)
\]
for all \((x, t) \in K \times (t_{n-1}, t_n)\), which implies in view of the strong convergence of \(s_{D,\delta t}\) to \(s\) and the Lipschitz continuity of \(\gamma\) that
\[
T_C^{21} \to - \int_0^T \int_\Omega \gamma(s) K \cdot \nabla \psi \, dx dt
\]
as \(\delta t\) and \(h_D\) tend to zero. Next we show that \(\lim_{h_D, \delta t \to 0} T_C^{22} = 0\). Thanks to the regularity of \(\psi\) we have
\[
|\psi_K^n - \psi_{\sigma}^n| \leq C_{\psi} |x_K - x_{\sigma}| \leq C_{\psi} \theta_D d_{K,\sigma},
\]
therefore in view of \((H_{2b})\) and \((H_{4b})\) there exists a positive constant \(C\) such that
\[
|T_C^{22}| = \left| \sum_{n=1}^N \delta t \sum_{K \in M} \sum_{\sigma \in E_K} m(\sigma) (\psi_K^n - \psi_{\sigma}^n) K \psi (\gamma_{K,\sigma} - \gamma_K^n) \right| \\
\leq C \sum_{n=1}^N \delta t \sum_{K \in M} \sum_{\sigma \in E_K} m(\sigma) d_{K,\sigma} |\gamma_K^n - \gamma_{\sigma}^n|.
\]
Remark that \(S(\phi^n_{K,\sigma} f(\cdot) + \gamma(\cdot) g_{K,\sigma}; s_K^n, s_\sigma^n) \in [\min(s_K^n, s_\sigma^n), \max(s_K^n, s_\sigma^n)]\) therefore in view of \((2.26)\) and the Lipschitz continuity of \(\gamma\) we obtain
\[
|T_C^{22}| \leq C \sum_{n=1}^N \delta t \sum_{K \in M} \sum_{\sigma \in E_K} m(\sigma) d_{K,\sigma} |s_K^n - s_\sigma^n|.
\]
The Hölder continuity of function \(\varphi^{-1}\) (the assumption \((H_1)\)) implies that
\[
|T_C^{22}| \leq C \sum_{n=1}^N \sum_{K \in M} \sum_{\sigma \in E_K} \delta t m(\sigma) d_{K,\sigma} |\varphi_K^n - \varphi_\sigma^n| \alpha \\
\leq C \left( \sum_{n=1}^N \sum_{K \in M} \sum_{\sigma \in E_K} \delta t m(\sigma) d_{K,\sigma} \left| \frac{\varphi_K^n - \varphi_\sigma^n}{d_{K,\sigma}} \right| \alpha \right) h_D^{-\beta},
\]
Applying Hölder inequality we obtain
\[
|T_C^{22}| \leq C \left( \sum_{n=1}^N \sum_{K \in M} \sum_{\sigma \in E_K} \delta t m(\sigma) d_{K,\sigma} \left| \frac{\varphi_K^n - \varphi_\sigma^n}{d_{K,\sigma}} \right| \alpha \right)^{\frac{2}{\beta}} \\
\cdot \left( \sum_{n=1}^N \sum_{K \in M} \sum_{\sigma \in E_K} \left( \delta t m(\sigma) d_{K,\sigma} \right)^{\frac{1-\alpha}{\beta}} \right)^{\frac{\beta}{2}},
\]
where \(\frac{2}{\beta} + \beta = 1\). Using the a priori estimates we finally obtain that
\[
|T_C^{22}| \leq C (dm(\Omega)T)^{1-\frac{\alpha}{2}} \|\varphi\|_{X_{D,\delta t},h_D}^\alpha,\]
where \(C = C_{\psi} \theta_D L \gamma H_{\varphi} K |\psi|\). Thus, \(\lim_{h_D \to 0} T_C^{22} = 0\). Next we show that
\[
T_C^{1} \to \int_0^T \int_\Omega f(s) K (\lambda(s) \nabla p - \xi(s) g) \cdot \nabla \psi \, dx dt
\]
as \(h_D\), \(\delta t\) → 0. In view of \((2.27)\) we can write \(T_C^{1}\) as
\[
T_C^{11} = \sum_{n=1}^N \delta t \sum_{K \in M} \sum_{\sigma \in E_K} (\psi_K^{n-1} - \psi_{\sigma}^{n-1}) \xi_K^n f_K^n g_{K,\sigma},
\]
\[
T_C^{12} = \sum_{n=1}^N \delta t \sum_{K \in M} \sum_{\sigma \in E_K} (\psi_K^{n-1} - \psi_{\sigma}^{n-1}) \lambda_K^n f_{K,\sigma} (p^n) f_K^n,
\]
\[
T_C^{13} = \sum_{n=1}^N \delta t \sum_{K \in M} \sum_{\sigma \in E_K} (\psi_K^{n-1} - \psi_{\sigma}^{n-1}) \lambda_K^n f_{K,\sigma} (p^n) (f_{K,\sigma}^n - f_K^n).
\]
Using the same arguments as for the term $T_{22}^2$ one can show that

$$T_{C}^{11} \to - \int_0^T \int_\Omega f(s)\xi(s)Kg \cdot \nabla \psi \, d\mathbf{x} dt.$$ 

It follows from (2.34) and (2.32) that the term $T_{C}^{12}$ can be written as

$$T_{C}^{12} = \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \int_K \lambda(s_{D,\delta t})f(s_{D,\delta t})K\nabla_D\delta t p_{D,\delta t} \cdot \nabla_D \psi^{n-1} \, d\mathbf{x} dt.$$

The regularity of $\psi$ combined with Lemma 2.1 implies that

$$\| \nabla_D \psi^n(x) - \nabla \psi(x, t)\|_{L^\infty(\Omega)^d} \leq C(h_D + \delta t)$$

for all $(x, t) \in K \times (t_{n-1}, t_n)$. In view of the Lipschitz continuity of the functions $\lambda$ and $f$, also in view of the weak convergence of $\nabla_D \delta t p_{D,\delta t}$ to $\nabla p$ we conclude that

$$T_{C}^{12} \to \int_0^T \int_\Omega \lambda(s)f(s)K\nabla p \cdot \nabla \psi \, d\mathbf{x} dt.$$

It remains to show that $T_{3}^2$ tends to zero as $h_D, \delta t \to 0$. Thanks to Lemma 2.5 the term $T_{C}^{13}$ can be estimated by

$$(T_{C}^{13})^2 \leq C^2 \| \nabla_D \delta t p\|_{L^2(Q_T)}^2 \sum_{n=1}^{N} \delta t \sum_{K \in M} \sum_{\sigma \in E_K} \frac{m(\sigma)}{d_{K,\sigma}^2} \lambda^2(s_{K}^n)(\psi_{K}^{n-1} - \psi_{\sigma}^{n-1})^2(f_{K,\sigma} - f_{K})^2$$

which implies

$$(T_{C}^{13})^2 \leq C_1^2 C_\psi^2 \| \lambda \|_{L^\infty((0,1))^d}\| \nabla_D \delta t p\|_{L^2(Q_T)}^2 \sum_{n=1}^{N} \delta t \sum_{K \in M} \sum_{\sigma \in E_K} m(\sigma)d_{K,\sigma}(f_{K,\sigma} - f_{K})^2.$$ 

Using the same arguments that for the term $T_{22}^2$ one shows that $|T_{C}^{13}| \leq Ch^\alpha \to 0$ as $h_D \to 0$.

### 5.3.3 Diffusion term

In view of (2.36) and (2.33) one has

$$T_D = \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \int_\Omega K\nabla_D\varphi(s^n) \cdot \nabla_D \psi^{n-1} \, d\mathbf{x} dt.$$ 

In view of (5.3) and since $\nabla_{D,\delta t} \varphi_{D,\delta t} \to \nabla \varphi(s)$ weakly in $L^2(Q_T)$ we deduce that the term $T_D$ tends to $\int_0^T \int_\Omega K\nabla \varphi(s) \cdot \nabla \psi \, d\mathbf{x} dt$ as $h_D, \delta t$ tend to zero.

### 5.3.4 Source term

In view of regularity of $\psi$ and thanks to the definition (2.7) of $k_{o,K}^n$ one has

$$\sum_{n=1}^{N} \sum_{K \in M} \delta t m(K)k_{K}^{n-1}k_{o,K}^n \to \int_0^T \int_\Omega k_{o} \, d\mathbf{x} dt$$

as $h_D, \delta t \to 0$. 

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5.4 Pressure equation

Setting $v^n = \psi^n$ in the discrete pressure equation (2.16) we multiply it by the time step $\delta t$ and we sum over $n \in \{1 \ldots N\}$, we obtain

$$\sum_{n=1}^{N} \sum_{K \in \mathcal{M}} \sum_{\sigma \in E_K} \delta t (\psi^n_K - \psi^n_\sigma) Q^n_{K,\sigma} = \sum_{n=1}^{N} \sum_{K \in \mathcal{M}} \delta t m(K) \psi^n_K (k^n_{w,K} + k^n_{o,K}).$$

Replacing $f^n_{K,\sigma}$ by 1 in the definition of $T_1^C$ allows to conclude that

$$\sum_{n=1}^{N} \sum_{K \in \mathcal{M}} \sum_{\sigma \in E_K} \delta t (\psi^n_K - \psi^n_\sigma) Q^n_{K,\sigma} \rightarrow \int_0^T \int_\Omega (\lambda(s)K \nabla p - \xi(s)Kg) \cdot \nabla \psi \, dx \, dt.$$

It is also clear that

$$\sum_{n=1}^{N} \sum_{K \in \mathcal{M}} \delta t m(K) \psi^n_K k^n_{o,K} \rightarrow \int_0^T \int_\Omega \psi k_o \, dx \, dt$$

as $h_D, \delta t \to 0$. We deduce from the density of the set $\Psi$ in the set $\{\psi \in L^2(0,T; H_0^1(\Omega)), \psi_t \in L^\infty(Q_T), \psi(\cdot,T) = 0\}$ that $(s,p)$ is a weak solution of the continuous problem (1.5)-(1.7) in the sense of Definition 1.1.

6 Numerical experiments

In this section we present some numerical results obtained using the scheme.

6.1 Five spot problem I

We consider the quarter five spot problem, which is the model problem in oil recovery; more precisely we consider a two dimensional horizontal (gravity is neglected) domain $\Omega = (0,1)^2$, which is initially saturated in oil ($s_0 = 1$ in $\Omega$). The oil-phase is then displaced by water, which is injected at the lower left corner; the production well is placed at the upper right corner. Assume that the oil-phase and water-phase mobilities and capillary pressure are respectively defined by

$$\lambda_o(s) = \frac{s^3}{2}, \quad \lambda_w(s) = \frac{(1-s)^3}{2} \quad \text{and} \quad \pi(s) = \frac{1}{2} \sqrt{\frac{s}{1-s}}.$$
Recall that the nonlinear functions in (1.5)-(1.7) are defined by

\[ \lambda(s) = \lambda_0(s) + \lambda_w(s), \quad f(s) = \frac{\lambda_0(s)}{\lambda(s)}, \quad \varphi(s) = \int_0^s \lambda_w(\tau)f(\tau)\pi'(\tau)d\tau. \]

The effects of wells are modeled by terms on the right-hand side of (1.5) and (1.7)

\[ k_o(s) = f(\bar{s})q^+ + f(s)q^- \]

and

\[ k_w(s) = (1 - f(\bar{s}))q^+ + (1 - f(s))q^-, \]

where \( q^+ = q^+(x,t) \) and \( q^- = q^-(x,t) \) denote the production and the injection rates. The value \( \bar{s} \) is the oil saturation of an injected fluid, which is set to \( \bar{s} = 0 \). We assume that \( q^+ \) and \( q^- \) are Dirac functions

\[ q^+ = \delta(x) \text{ and } q^- = \delta(x - (1,1)). \]

We recall that due to the fact that \( \varphi'(0) = 0 \) the water front propagates with a finite speed. We consider three types of meshes, namely the square mesh, the randomly perturbed quadrangular mesh and the Kershaw mesh (see Figure 1). For each type of mesh we also consider three different resolutions \( 20 \times 20, 40 \times 40 \) and \( 80 \times 80 \) elements; the meshes are constructed in such a way that the regularity parameter \( \theta_D \) remains almost constant for each class namely \( \theta_D = 2\sqrt{2} \) for orthogonal meshes, \( \theta_D \approx 30 \) for random and Kershaw meshes.

Figure 2 represents the water-phase saturation profile along the diagonal \(((0,0),(0,1))\) at time \( t = 0.6 \) for the family of orthogonal grids. Note that in this case the hybrid discretization of the diffusion operator is equivalent to the classical two-point flux approximation scheme. Since the exact solution is not available for this problem we compare the results obtained using "bad" (randomly perturbed and Kershaw) meshes with the results corresponding to the orthogonal grids (see Figures 2-3). We first remark that the solution obtained on the square and randomly perturbed meshes are in very good agreement. However the Kershaw mesh seems to be more challenging. The corresponding numerical solution has a non negligible deviation from the reference one, which we believe is due to the mesh orientations effects breaking the symmetry of the solution (see Figure 4). A positive point is that for a sufficiently small mesh size the speed of the water front propagation is correct.
card(M) = 20 × 20

card(M) = 40 × 40

card(M) = 80 × 80

Figure 2: Water profile at time $t = 0.3$ and $t = 0.6$ for a family of orthogonal meshes (left) and the same profile for three type of meshes with 20 × 20 elements (right)

Figure 3: Water profile at time $t = 0.3$ and $t = 0.6$ for three type of meshes with 40 × 40 elements (left), 80 × 80 elements (right)

Figure 4: Oil saturation at time $t = 0.6$ using 40 × 40 randomly perturbed (left) and Kershaw (right) meshes
6.2 Five spot problem II (Heterogeneous and anisotropic medium)

Next we consider two test cases proposed in [30], which allow to investigate the qualitative behavior of the scheme in the case of anisotropic and heterogeneous absolute permeability tensor $K$.

First we assume that the porous medium is homogeneous. The absolute permeability tensor $K$ is defined by $K = R_{\theta}DR_{\theta}^T$, where

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 10^{-3} \end{pmatrix} \quad \text{with} \quad R_{\theta} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$

for all real $\theta$. First we set $\theta = \pi/4$ and we present on Figure 5 the approximate water-phase saturation and the global pressure fields at time $t = 0.02$. Note that due to the anisotropy the solution form a relatively fine strip propagating along the direction $(\cos \theta, \sin \theta)$, which is an eigenvector of $K$ corresponding to the value 1.

![Figure 5: Oil saturation (left) and global pressure field (right) at time $t = 0.02$ (homogeneous case)](image)

Next let us consider a more complex geometry. We assume that the computational domain $\Omega = (0,1)^2$ is composed of four layers

$$\begin{align*}
\Omega_1 &= \Omega \cap \{|x_1| + |x_2| < 0.5\}, \\
\Omega_2 &= \Omega \cap \{0.5 < |x_1| + |x_2| < 1\}, \\
\Omega_3 &= \Omega \cap \{1 < |x_1| + |x_2| < 1.5\}, \\
\Omega_4 &= \Omega \cap \{|x_1| + |x_2| > 1.5\},
\end{align*}$$

which we represent on Figure 6 by different colors. The piecewise constant absolute permeability field is defined by $K(x)|_{x \in \Omega_i} = K_i$, with $i \in \{1, \ldots, 4\}$ and

$$K_1 = K_4 = R_{\pi/4}DR_{\pi/4}^T, \quad K_2 = D, \quad K_3 = R_{\pi/2}DR_{\pi/2}^T.$$

On Figures 7 we present the approximate water-phase saturation and global pressure field at time $t = 0.07$. As in the previous case water propagates in the most permeable direction, which changes at the interface between two different layers.

**Remark 6.1** In conclusion to this section we can state that the scheme seems to be perform well on the distorted meshes and it gives a fairly correct in the case of a highly anisotropic and heterogeneous absolute permeability fields. In future it would be interesting to compare our scheme with others methods both in the cases with and without gravity.
Figure 6: Absolute permeability field

Figure 7: Oil saturation (left) and global pressure field (right) at time $t = 0.07$ (heterogeneous case)

References


