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ON THE FITTING SUBGROUP AND SOLUBLE RADICAL OF A SUPERSIMPLE GROUP OF FINITE RANK

CÉDRIC MILLIET

ABSTRACT. G is a group with supersimple theory having finite SU -rank and $Fit(G)$ is the subgroup of G generated by all of its normal nilpotent subgroups. We show that $Fit(G)$ is definable and nilpotent. This answers a question asked by Elwes, Jaligot, Macpherson and Ryten in [2]. We also show that the subgroup of G generated by all of its normal solvable subgroups is definable and solvable.

1. INTRODUCTION

In [2], Elwes, Jaligot, Macpherson and Ryten have begun an investigation of groups definable in supersimple theories of finite SU -rank. They often have been working under the extra assumption that T^{eq} eliminates \exists^∞ , meaning that there is a finite upper bound for the size of the finite members of any family of sets uniformly definable in T^{eq} . Among the main results, the following was obtained : if G is a group definable in a supersimple theory of finite SU -rank such that T^{eq} eliminates \exists^∞ , then the soluble radical of G is a definable and solvable subgroup. It was asked if such a G had a largest nilpotent normal subgroup F and if F would be definable. We give a positive answer here and prove more generally that the Fitting subgroup of a supersimple group of finite SU -rank is definable and nilpotent. We also show that the soluble radical of a supersimple group of finite SU -rank is definable and solvable.

2. PRELIMINARIES

The FC -centre of a group G is defined by

$$FC(G) = \{g \in G : g^G \text{ is finite}\}$$

Inductively on n , we call $FC_{n+1}(G)$ the preimage in G of $FC(G/FC_n(G))$, with the convention that $FC_0(G)$ is $\{1\}$. This defines an ascending chain of characteristic subgroups of G . The group G is called FC -nilpotent if G equals $FC_n(G)$ for some natural number n , the least such we call the FC -nilpotency class of G . Finite groups and nilpotent ones are both examples of FC -nilpotent groups. If G/N is a quotient group modulo a normal subgroup N of G , we write $FC_G(G/N)$ for the preimage of $FC(G/N)$ in G by the canonical surjection from G onto G/N .

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The paper arose after a conversation with Professor Macpherson in the Neostability theory conference in Banff, Canada. L'auteur voudrait remercier le centre de Banff pour son hospitalité, ainsi que l'institut Camille Jordan qui lui a permis d'entreprendre ce long voyage.

Lemma 2.1. *If N is a finite normal subgroup of G , then $FC(G)$ equals $FC_G(G/N)$.*

Proof. The canonical surjection from G onto G/N has a finite kernel. It follows that the conjugacy class x^G is finite if and only if $(xN)^{G/N}$ is finite. \square

Lemma 2.2. *If H and N are two normal subgroups of G with $N \leq H$, then $FC_G(G/H)/N$ equals $FC_{G/N}(G/N/H/N)$.*

Proof. There is a canonical homomorphism from G/H onto $G/N/H/N$. It follows that $(xH)^{G/H}$ is finite if and only if $((xN)H/N)^{G/N/H/N}$ is finite. This means precisely that $x \in FC_G(G/H)$ if and only if $xN \in FC_{G/N}(G/N/H/N)$. \square

Lemma 2.3. *If for some natural number n the quotient $FC_{n+1}(G)/FC_n(G)$ is finite, then $FC_{n+2}(G)$ equals $FC_{n+1}(G)$.*

Proof. We have

$$FC_{n+2}(G)/FC_n(G) = FC_G(G/FC_{n+1}(G))/FC_n(G)$$

By Lemma 2.2

$$FC_{n+2}(G)/FC_n(G) = FC_{G/FC_n(G)}(G/FC_n(G)/FC_{n+1}(G)/FC_n(G))$$

As $FC_{n+1}(G)/FC_n(G)$ is finite, applying Lemma 2.1 we get

$$FC_{n+2}(G)/FC_n(G) = FC(G/FC_n(G)) = FC_{n+1}(G)/FC_n(G)$$

\square

3. SUPERSIMPLE GROUPS

A supersimple group G of finite SU -rank is equipped with a rank function taking values in the natural numbers, and ranking any definable subset X of G . We write $SU(X)$ for the rank of X . The SU -rank is increasing : if $X \subset Y$ are two definable subsets of G , then $SU(X)$ is smaller or equal to $SU(Y)$. If G is supersimple of finite rank, then so is any of its elementary extension, and so is G^{eq} , meaning that any quotient group G/N by a definable subgroup N has finite SU -rank. A definable set (in T^{eq}) has SU -rank zero if and only if it is finite. In particular, if H is a definable subgroup of G , then $SU(G/H)$ equals zero if and only if H has finite index in G .

The following comes from [3, Remark 3.5] as a particular case of [11, Theorem 5.5.4]

Fact 3.1 (Indecomposability Theorem). *G is a supersimple group of finite rank, $(X_i)_{i \in I}$ a family of definable subsets of G . There exists a definable subgroup H of G such that*

- (1) $H \leq \langle X_i : i \in I \rangle$
- (2) *Finitely many translates of H cover X_i for every i .*

If the sets X_i are normal in G , the group H may be chosen normal in G .

Corollary 3.2. *If G is a supersimple group with finite SU -rank, then the derived subgroup G' is definable.*

Proof. We follow exactly the proof of [5, Corollary 7.5]. Let C be the set of commutators of G . By Fact 3.1, there is a definable subgroup H of G' with H normal in G such that finitely many translates of H cover C . It follows that the set of commutators in G/H is finite, so the derived group $(G/H)'$ is finite by [6, p. 110]. \square

Fact 3.3 (Lascar equality). *G is a supersimple group with finite SU -rank, and H a definable normal subgroup of G . Then*

$$SU(G/H) + SU(H) = SU(G)$$

Fact 3.4 ([8]). *If G is a saturated group with simple theory, its FC -centre is definable.*

4. FITTING SUBGROUP

Let G be any group. We call the *Fitting subgroup* of G the subgroup generated by all nilpotent normal subgroups. We write it $Fit(G)$. It is worth mentioning that the Fitting subgroup is definable if and only if it is nilpotent. Namely x belongs to $Fit(G)$ if and only if the subgroup generated by its conjugacy class x^G is nilpotent. It follows that $Fit(G)$ equals $\bigcup_{n \geq 1} \{x \in G : [x^G \cup (x^{-1})^G, \dots, {}_n x^G \cup (x^{-1})^G] = 1\}$, where for any subset X of G the set $[X, \dots, {}_n X]$ is defined inductively by $[X, {}_1 X] = \{[x_1, x_2] : (x_1, x_2) \in X^2\}$ and $[X, \dots, {}_{n+1} X] = \{[y, x] : (y, x) \in [X, \dots, {}_n X] \times X\}$. The observation that $Fit(G)$ is definable if it is nilpotent was first made by Ould Houcine, and the simple proof above was independently provided by the referee of [1].

Proposition 4.1. *G is a group and F is a normal subgroup of G . Assume that $F \leq FC_n(G)$ for some natural number n (in particular, F is FC -nilpotent). If G/F is FC -nilpotent, then so is G .*

Proof. Assume that G/F is FC -nilpotent of class m . There is a surjection from G/F onto $G/FC_n(G)$. As recalled in [8], the image of an FC -nilpotent group by a group homomorphism is FC -nilpotent. It follows that $G/FC_n(G)$ is FC -nilpotent of class at most m , so that $FC_{m+n}(G)$ equals G . \square

We recall Hall's criterion for nilpotence.

Fact 4.2 (Hall [4]). *G is any group and N is a normal subgroup of G . If G/N' and N are nilpotent, then G is nilpotent.*

Two other proofs of Fact 4.2 can be found in [10] and [7], with a bound on the nilpotency class of G depending on the classes of N and G/N' in [7]. Note that since $G/N''/N'/N''$ and G/N' are isomorphic, a straightforward induction on the nilpotency class of N reduces the proof to the case where N is 2-nilpotent.

Fact 4.3 ([8]). *G is a group with simple theory. If G is FC -nilpotent of class n , then G has a definable normal subgroup of finite index which is nilpotent of class at most $2n$.*

Fact 4.4 ([8]). *G is a group with simple theory. If N is a normal nilpotent subgroup of class n , then N is contained in a normal definable nilpotent subgroup of class at most $3n$.*

Theorem 4.5. *G is a supersimple group with finite SU -rank. The Fitting subgroup of G is definable and nilpotent.*

Proof. We may assume that the ambient group is saturated, so that $FC_n(G)$ is definable for all natural number n by Fact 3.4. By Lemma 2.3, for big enough n , the quotient $FC_{n+1}(G)/FC_n(G)$ is either trivial or infinite. By Lascar's equality, there exists a natural number such that $FC_n(G) = FC_{n+1}(G)$. We call \overline{G} the quotient group $G/FC_n(G)$ so that \overline{G} has a trivial FC -centre. Let \overline{F} be its Fitting subgroup.

Claim 1. *We may assume that $FC(G) = \{1\}$.*

To prove Claim 1, we need just assume that \overline{F} is definable and nilpotent and show that $Fit(G)$ is definable and nilpotent. Note that \overline{F} is FC -nilpotent. Let F be its pull-back in G so that we have $F/FC_n(G) = \overline{F}$. The group F is FC -nilpotent by Proposition 4.1. By Fact 4.3, F has a definable subgroup of finite index which is nilpotent, so it must have a normal one N . It follows that F contains a maximal normal (in G) nilpotent subgroup H of finite index so that H equals $Fit(G)$. Being a finite extension of N , $Fit(G)$ is definable.

Claim 2. *We may assume that G has a definable normal 2-nilpotent subgroup.*

On the one hand, if every nilpotent normal subgroup of G is abelian, then $Fit(G)$ is abelian. In this case, by the remark made at the beginning of this section (or Fact 4.4), $Fit(G)$ must be definable. On the other hand, if there is a non abelian nilpotent normal subgroup, then there is a definable one by Fact 4.4. Call it N . The group $Z_2(N)$ has the required properties.

Conclusion.

We proceed by induction on $SU(G)$. If $SU(G)$ is zero, G is finite and so is $Fit(G)$. If $SU(G)$ equals $n + 1$, by Claim 2, there is a normal nilpotent definable subgroup N of G of nilpotency class 2. By Fact 3.2, the derived subgroup N' is definable. As N' is normal in G , it is infinite by Claim 1, so $SU(N') \geq 1$. By Lascar's equality, $SU(G/N') \leq n$, and we may apply the induction hypothesis to G/N' . It follows that $Fit(G/N')$ is definable and nilpotent. Let F be its preimage in G so that we have $F/N' = Fit(G/N')$. By Fact 4.2, F is nilpotent. Thus F equals $Fit(G)$. \square

5. SOLUBLE RADICAL

We call the *soluble radical* of G the subgroup generated by all soluble normal subgroups. We write it $R(G)$. Similarly to $Fit(G)$, the soluble radical $R(G)$ is definable if and only if it is solvable. Namely x belongs to $R(G)$ if and only if the subgroup generated by its conjugacy class x^G is soluble. It follows that $R(G)$ equals $\bigcup_{n \geq 1} \{x \in G : [x^G \cup (x^{-1})^G, \dots, {}^{2^n} x^G \cup (x^{-1})^G] = 1\}$, where for any subset X of G the set $[X, \dots, {}^{2^n} X]$ is defined inductively by $[X, {}^2 X] = \{[x_1, x_2] : (x_1, x_2) \in X^2\}$ and $[X, \dots, {}^{2^{n+1}} X] = [[X, \dots, {}^{2^n} X], [X, \dots, {}^{2^n} X]]$. Again, the fact that if $R(G)$ is solvable then it must be definable was first observed by Ould Houcine.

Fact 5.1 (Neumann [9]). *An FC -group with bounded conjugacy classes has a class 2 nilpotent subgroup of finite index.*

Fact 5.2 ([8]). *G is a group with simple theory, S a normal soluble subgroup of derived length n . Then S is contained in a definable soluble group of derived length at most $3n$.*

Theorem 5.3. *A supersimple group of finite SU -rank has a solvable and definable soluble radical.*

Proof. We may assume that the ambient group G is saturated. Let n be the maximal rank of a definable normal soluble subgroup S . We proceed by induction on n . If n is zero, then every normal soluble subgroup is finite by Fact 5.2. The group $FC(G)$ is exactly the union of all normal finite subgroups of G . By Fact 3.4 and 5.1, the group $FC(G)$ has a normal nilpotent subgroup of finite index, so it must be finite. It follows that the soluble radical is finite, hence definable.

if $n > 0$, then by Lascar's equality, we may apply induction hypothesis to G/S so G/S has a definable soluble radical R . The preimage of R in G is solvable and is the soluble radical of G . □

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