

On the definability of radicals in supersimple groups Cédric Milliet

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ABSTRACT. G is a group with supersimple theory having finite SU-rank and Fit(G) is the subgroup of G generated by all of its normal nilpotent subgroups. We show that Fit(G) is definable and nilpotent. This answers a question asked by Elwes, Jaligot, Macpherson and Ryten in [2]. We also show that the subgroup of G generated by all of its normal solvable subgroups is definable and solvable.

1. INTRODUCTION

In [2], Elwes, Jaligot, Macpherson and Ryten have begun an investigation of groups definable in supersimple theories of finite SU-rank. They often have been working under the extra assumption that T^{eq} eliminates \exists^{∞} , meaning that there is a finite upper bound for the size of the finite members of any familly of sets uniformly definable in T^{eq} . Among the main results, the following was obtained : if G is a group definable in a supersimple theory of finite SU-rank such that T^{eq} eliminates \exists^{∞} , then the soluble radical of G is a definable and solvable subgroup. It was asked if such a G had a largest nilpotent normal subgroup F and if F would be definable. We give a positive answer here and prove more generally that the Fitting subgroup of a supersimple group of finite SU-rank is definable and nilpotent. We also show that the soluble radical of a supersimple group of finite SU-rank is definable and solvable.

2. Preliminaries

The FC-centre of a group G is defined by

$$FC(G) = \{g \in G : g^G \text{ is finite}\}$$

Inductively on n, we call $FC_{n+1}(G)$ the preimage in G of $FC(G/FC_n(G))$, with the convention that $FC_0(G)$ is $\{1\}$. This defines an ascending chain of characteristic subgroups of G. The group G is called FC-nilpotent if G equals $FC_n(G)$ for some natural number n, the least such we call the FC-nilpotency class of G. Finite groups and nilpotent ones are both examples of FC-nilpotent groups. If G/N is a quotient group modulo a normal subgroup N of G, we write $FC_G(G/N)$ for the preimage of FC(G/N) in G by the canonical surjection from G onto G/N.

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The paper arose after a conversation with Professor Macpherson in the Neostability theory conference in Banff, Canada. L'auteur voudrait remercier le centre de Banff pour son hospitalité, ainsi que l'institut Camille Jordan qui lui a permis d'entreprendre ce long voyage.

Lemma 2.1. If N is a finite normal subgroup of G, then FC(G) equals $FC_G(G/N)$.

Proof. The canonical surjection from G onto G/N has a finite kernel. It follows that the conjugacy class x^G is finite if and only if $(xN)^{G/N}$ is finite.

Lemma 2.2. If H and N are two normal subgroups of G with $N \leq H$, then $FC_G(G/H)/N$ equals $FC_{G/N}(G/N/H/N)$.

Proof. There is a canonical homomorphism from G/H onto G/N/H/N. It follows that $(xH)^{G/H}$ is finite if and only if $((xN)H/N)^{G/N/H/N}$ is finite. This means precisely that $x \in FC_G(G/H)$ if and only if $xN \in FC_G/N(G/N/H/N)$.

Lemma 2.3. If for some natural number n the quotient $FC_{n+1}(G)/FC_n(G)$ is finite, then $FC_{n+2}(G)$ equals $FC_{n+1}(G)$.

Proof. We have

$$FC_{n+2}(G)/FC_n(G) = FC_G(G/FC_{n+1}(G))/FC_n(G)$$

By Lemma 2.2

$$FC_{n+2}(G)/FC_n(G) = FC_{G/FC_nG}(G/FC_n(G)/FC_{n+1}(G)/FC_n(G))$$

As $FC_{n+1}(G)/FC_n(G)$ is finite, applying Lemma 2.1 we get
 $FC_{n+2}(G)/FC_n(G) = FC(G/FC_n(G)) = FC_{n+1}(G)/FC_n(G)$

3. Supersimple groups

A supersimple group G of finite SU-rank is equiped with a rank function taking values in the natural numbers, and ranking any definable subset X of G. We write SU(X) for the rank of X. The SU-rank is increasing : if $X \subset Y$ are two definable subsets of G, then SU(X) is smaller or equal to SU(Y). If G is supersimple of finite rank, then so is any of its elementary extension, and so is G^{eq} , meaning that any quotient group G/N by a definable subgroup N has finite SU-rank. A definable set (in T^{eq}) has SU-rank zero if and only if it is finite. In particular, if H is a definable subgroup of G, then SU(G/H) equals zero if and only if H has finite index in G. The following comes from [3, Remark 3.5] as a particular case of [11, Theorem 5.5.4]

Fact 3.1 (Indecomposability Theorem). *G* is a supersimple group of finite rank, $(X_i)_{i \in I}$ a family of definable subsets of *G*. There exists a definable subgroup *H* of *G* such that

- (1) $H \leq \langle X_i : i \in I \rangle$
- (2) Finitely many translates of H cover X_i for every i.

If the sets X_i are normal in G, the group H may be choosen normal in G.

Corollary 3.2. If G is a supersimple group with finite SU-rank, then the derived subgroup G' is definable.

Proof. We follow exactly the proof of [5, Corollary 7.5]. Let C be the set of commutators of G. By Fact 3.1, there is a definable subgroup H of G' with H normal in G such that finitely many translates of H cover C. It follows that the set of commutators in G/H is finite, so the derived group (G/H)' is finite by [6, p. 110]. \Box

Fact 3.3 (Lascar equality). G is a supersimple group with finite SU-rank, and H a definable normal subgroup of G. Then

$$SU(G/H) + SU(H) = SU(G)$$

Fact 3.4 ([8]). If G is a saturated group with simple theory, its FC-centre is definable.

4. Fitting subgroup

Let G be any group. We call the *Fitting subgroup of* G the subgroup generated by all nilpotent normal subgroups. We write it Fit(G). It is worth mentioning that the Fitting subgroup is definable if and only if it is nilpotent. Namely x belongs to Fit(G) if and only if the subgroup generated by its conjugacy class x^G is nilpotent. It follows that Fit(G) equals $\bigcup_{n\geq 1} \{x \in G : [x^G \cup (x^{-1})^G, \ldots, x^G \cup (x^{-1})^G] = 1\}$, where for any subset X of G the set $[X, \ldots, x]$ is defined inductively by $[X_{,1}X] = \{[x_1, x_2] : (x_1, x_2) \in X^2\}$ and $[X, \ldots, x^{n+1}X] = \{[y, x] : (y, x) \in [X, \ldots, x^{n+1}X] \times X\}$. The observation that Fit(G) is definable if it is nilpotent was first made by Ould Houcine, and the simple proof above was independently provided by the referee of [1].

Proposition 4.1. G is a group and F is a normal subgroup of G. Assume that $F \leq FC_n(G)$ for some natural number n (in particular, F is FC-nilpotent). If G/F is FC-nilpotent, then so is G.

Proof. Assume that G/F is FC-nilpotent of class m. There is a surjection from G/F onto $G/FC_n(G)$. As recalled in [8], the image of an FC-nilpotent group by a group homomorphism is FC-nilpotent. It follows that $G/FC_n(G)$ is FC-nilpotent of class at most m, so that $FC_{m+n}(G)$ equals G.

We recall Hall's criterion for nilpotence.

Fact 4.2 (Hall [4]). G is any group and N is a normal subgroup of G. If G/N' and N are nilpotent, then G is nilpotent.

Two other proofs of Fact 4.2 can be found in [10] and [7], with a bound on the nilpotency class of G depending on the classes of N and G/N' in [7]. Note that since G/N''/N'/N'' and G/N' are isomorphic, a straightforward induction on the nilpotency class of N reduces the proof to the case where N is 2-nilpotent.

Fact 4.3 ([8]). *G* is a group with simple theory. If *G* is *FC*-nilpotent of class *n*, then *G* has a definable normal subgroup of finite index which is nilpotent of class at most 2n.

Fact 4.4 ([8]). *G* is a group with simple theory. If *N* is a normal nilpotent subgroup of class *n*, then *N* is contained in a normal definable nilpotent subgroup of class at most 3n.

Theorem 4.5. G is a supersimple group with finite SU-rank. The Fitting subgroup of G is definable and nilpotent.

Proof. We may assume that the ambient group is saturated, so that $FC_n(G)$ is definable for all natural number n by Fact 3.4. By Lemma 2.3, for big enough n, the quotient $FC_{n+1}(G)/FC_n(G)$ is either trivial or infinite. By Lascar's equality, there exists a natural number such that $FC_n(G) = FC_{n+1}(G)$. We call \overline{G} the quotient group $G/FC_n(G)$ so that \overline{G} has a trivial FC-centre. Let \overline{F} be its Fitting subgroup.

Claim 1. We may assume that $FC(G) = \{1\}$.

To prove Claim 1, we need just assume that \overline{F} is definable and nilpotent and show that Fit(G) is definable and nilpotent. Note that \overline{F} is FC-nilpotent. Let F be its pull-back in G so that we have $F/FC_n(G) = \overline{F}$. The group F is FC-nilpotent by Proposition 4.1. By Fact 4.3, F has a definable subgroup of finite index which is nilpotent, so it must have a normal one N. It follows that F contains a maximal normal (in G) nilpotent subgroup H of finite index so that H equals Fit(G). Being a finite extension of N, Fit(G) is definable.

Claim 2. We may assume that G has a definable normal 2-nilpotent subgroup. On the one hand, if every nilpotent normal subgroup of G is abelian, then Fit(G) is abelian. In this case, by the remark made at the beginning of this section (or Fact 4.4), Fit(G) must be definable. On the other hand, if there is a non abelian nilpotent normal subgroup, then there is a definable one by Fact 4.4. Call it N. The group $Z_2(N)$ has the required properties.

Conclusion.

We proceed by induction on SU(G). If SU(G) is zero, G is finite and so is Fit(G). If SU(G) equals n + 1, by Claim 2, there is a normal nilpotent definable subgroup N of G of nilpotency class 2. By Fact 3.2, the derived subgroup N' is definable. As N' is normal un G, it is infinite by Claim 1, so $SU(N') \ge 1$. By Lascar's equality, $SU(G/N') \le n$, and we may apply the induction hypothesis to G/N'. It follows that Fit(G/N') is definable and nilpotent. Let F be its preimage in G so that we have F/N' = Fit(G/N'). By Fact 4.2, F is nilpotent. Thus F equals Fit(G). \Box

5. Soluble radical

We call the soluble radical of G the subgroup generated by all soluble normal subgroups. We write it R(G). Similarly to Fit(G), the soluble radical R(G) is definable if and only if it is solvable. Namely x belongs to R(G) if and only if the subgroup generated by its conjugacy class x^G is soluble. It follows that R(G) equals $\bigcup_{n\geq 1} \{x \in G : [x^G \cup (x^{-1})^G, \dots, 2^n x^G \cup (x^{-1})^G] = 1\}$, where for any subset X of G the set $[X, \dots, 2^n X]$ is defined inductively by $[X, 2X] = \{[x_1, x_2] : (x_1, x_2) \in X^2\}$ and $[X, \dots, 2^{n+1} X] = [[X, \dots, 2^n X], [X, \dots, 2^n X]]$. Again, the fact that if R(G) is solvable then it must be definable was first observed by Ould Houcine.

Fact 5.1 (Neumann [9]). An FC-group with bounded conjugacy classes has a class 2 nilpotent subgroup of finite index.

Fact 5.2 ([8]). *G* is a group with simple theory, *S* a normal soluble subgroup of derived length *n*. Then *S* is contained in a definable soluble group of derived length at most 3n.

Theorem 5.3. A supersimple group of finite SU-rank has a solvable and definable soluble radical.

Proof. We may assume that the ambient group G is saturated. Let n be the maximal rank of a definable normal soluble subgroup S. We proceed by induction on n. If n is zero, then every normal soluble subgroup is finite by Fact 5.2. The group FC(G) is exactly the union of all normal finite subgroups of G. By Fact 3.4 and 5.1, the group FC(G) has a normal nilpotent subgroup of finite index, so it must be finite. It follows that the soluble radical is finite, hence definable.

if n > 0, then by Lascar's equality, we may apply induction hypothesis to G/S so G/S has a definable soluble radical R. The preimage of R in G is solvable and is the soluble radical of G.

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