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► **To cite this version:**

| Stavros Vakeroudis, Ron A. Doney. Windings of planar stable processes. 2012. hal-00679710v1

**HAL Id: hal-00679710**

**<https://hal.science/hal-00679710v1>**

Preprint submitted on 16 Mar 2012 (v1), last revised 22 Dec 2012 (v3)

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# Windings of planar stable processes

R.A. Doney<sup>\*</sup> and S. Vakeroudis<sup>\*†</sup>

March 16, 2012

## Abstract

Using a generalization of the skew-product representation of planar Brownian motion and the analogue of Spitzer's celebrated asymptotic Theorem for stable processes due to Bertoin and Werner, for which we provide a new easy proof, we obtain some limit Theorems for the exit time from a cone of stable processes of index  $\alpha \in (0, 2)$ . We also study the case  $t \rightarrow 0$  and we prove some Laws of the Iterated Logarithm (LIL) for the (well-defined) winding process associated to our planar stable process.

**AMS 2010 subject classification:** Primary: 60G52, 60G51, 60F05, 60J65; secondary: 60E07, 60B12, 60G18.

**Key words:** Stable processes, Lévy processes, windings, exit time from a cone, Spitzer's Theorem, skew-product representation, Lamperti's relation, Law of the Iterated Logarithm (LIL) for small times.

## 1 Introduction

In this paper, we study the windings of planar isotropic stable processes. More precisely, having as a starting point a work of Bertoin and Werner [BeW96] concerning this subject (following their previous work on windings of planar Brownian motion<sup>§</sup> [BeW94a]) and motivated by some works of Shi [Shi98], we attempt to generalize some results obtained recently for the case of planar Brownian motion (see e.g. [Vak11, Vakth11, VaY12] and the references therein). In particular, we investigate the behaviour of stable processes for small time, an aspect which has already been investigated e.g. by Doney [Don04]

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<sup>§</sup>When we simply write: Brownian motion, we always mean real-valued Brownian motion, starting from 0. For 2-dimensional Brownian motion, we indicate planar or complex BM.

in terms of Spitzer's condition for stable processes (see e.g. [BeD97] and the references therein).

In Section 2, we recall some facts about standard isotropic stable processes of index  $\alpha \in (0, 2)$  taking values in the complex plane. Then, we follow Bertoin and Werner [BeW96] to define the process of its winding number, we generalize the skew-product representation of planar BM (see e.g. [Kiu80, ReY99, Chy06]) and we present two Lemmas for the winding process of isotropic stable Lévy processes obtained in [BeW96]. Finally, we mention some properties of the positive and the negative moments of the exit times from a cone of this process.

In Section 3, we use some continuity arguments of the composition function due to Whitt [Whi80] and we obtain a new simple proof of the analogue of Spitzer's Theorem for isotropic stable Lévy processes, initially proven by Bertoin and Werner [BeW96]. We reformulate and we extend this result in terms of the exit times from a cone. More precisely, Spitzer's asymptotic Theorem says that, if  $(\theta_t, t \geq 0)$  denotes the continuous determination of the argument of a planar BM  $\beta$  starting away from the origin, then:

$$\frac{2}{\log t} \theta_t \xrightarrow[t \rightarrow \infty]{(law)} C_1, \quad (1)$$

where  $C_1$  is a standard Cauchy variable. For other proofs of (1), see e.g. [Wil74, Dur82, MeY82, PiY86, BeW94a, Yor97, Vak11, VaY12]. Bertoin and Werner state that because an isotropic stable Lévy process is transient, we expect that it winds more slowly than planar Brownian motion and prove that, with  $\theta$  now denoting the process of its winding number (appropriately defined, see e.g. Section 2),  $\theta_t/\sqrt{\log t}$  converges in distribution to some centered Gaussian law as  $t \rightarrow \infty$  (Theorem 1 in Bertoin and Werner [BeW96], stated here as Theorem 3.2).

In Section 4, and more precisely in Theorems 4.1 and 4.4, we study the asymptotics of a symmetric Lévy process and of the winding process of isotropic stable Lévy processes for  $t \rightarrow 0$ , respectively, which are the main results of this article. In particular, we show that  $t^{-1/\alpha}\theta_t$  converges in distribution to an  $\alpha$ -stable law as  $t \rightarrow 0$ . Using this result, in Proposition 4.5 we obtain the (weak) limit in distribution of the process of the exit times from a cone with narrow amplitude and we further obtain several generalizations. We also study the windings of planar stable processes in  $(t, 1]$ , for  $t \rightarrow 0$  and we note that Spitzer's law is still valid for  $\theta_{1/t}$  (see Remark 4.7). Section 5 deals with the Law of the Iterated Logarithm LIL for Lévy processes for small times, in the spirit of some well-known (LIL) for Brownian motion for  $t \rightarrow \infty$  from Bertoin and Werner [BeW94a, BeW94b] and from Shi [Shi94, Shi98], and for stable subordinators with index  $\alpha \in (0, 1)$  for  $t \rightarrow 0$  from Fristedt [Fri64, Fri67] and Khintchine [Khi39] (see also [Ber96]). Moreover, we prove a LIL for the winding number process of stable processes, for  $t \rightarrow 0$ .

Finally, in Section 6 we discuss the planar Brownian motion case and in Theorem 6.1 we obtain the asymptotic behaviour of the winding process as  $t \rightarrow 0$ . More precisely, with obvious notation, the process  $(c^{-1/2}\theta_{ct}, t \geq 0)$  converges in law to a 1-dimensional Brownian motion as  $c \rightarrow 0$ .

**Notation:** In the following text, with the symbol " $\implies$ " we shall denote the weak convergence in distribution on the appropriate space, endowed with the Skorohod topology.

## 2 Preliminaries

Following Lamperti [Lam72], a Markov process  $X$  with values in  $\mathbb{R}^d$ ,  $d \geq 2$  is called isotropic or  $O(d)$ -invariant ( $O(d)$  stands for the group of orthogonal transformations on  $\mathbb{R}^d$ ) if its transition satisfies:

$$P_t(\phi(x), \phi(B)) = P_t(x, B), \quad (2)$$

for any  $\phi \in O(d)$ ,  $x \in \mathbb{R}^d$  and Borel subset  $B \subset \mathbb{R}^d$ .

Moreover,  $X$  is said to be  $\alpha$ -self-similar if, for  $\alpha > 0$ ,

$$P_{\lambda t}(x, B) = P_t(\lambda^{-\alpha}x, \lambda^{-\alpha}B), \quad (3)$$

for any  $\lambda > 0$ ,  $x \in \mathbb{R}^d$  and  $B \subset \mathbb{R}^d$ .

We focus now our study on the 2-dimensional case ( $d = 2$ ), where (3) holds, and we denote by  $(Z_t, t \geq 0)$  a standard isotropic stable process of index  $\alpha \in (0, 2)$  taking values in the complex plane and starting from  $z_0 + i0$ ,  $z_0 > 0$ . A scaling argument shows that we may assume  $z_0 = 1$ , without loss of generality, since, with obvious notation:

$$\left( Z_t^{(z_0)}, t \geq 0 \right) \stackrel{(law)}{=} \left( z_0 Z_{(t/z_0^\alpha)}^{(1)}, t \geq 0 \right). \quad (4)$$

Thus, from now on, we shall take  $z_0 = 1$ . More precisely,  $Z$  has stationary independent increments, its sample path is right continuous and has left limits (cadlag) and, with  $\langle \cdot, \cdot \rangle$  standing for the Euclidean inner product,  $E[\exp(i\langle \lambda, Z_t \rangle)] = \exp(-t|\lambda|^\alpha)$ , for all  $t \geq 0$  and  $\lambda \in \mathbb{C}$ .  $Z$  is transient,  $\lim_{t \rightarrow \infty} |Z_t| = \infty$  a.s. and it a.s. never visits single points. We remark that for  $\alpha = 2$ , we are in the Brownian motion case.

We are now going to recall some properties of stable processes and Lévy processes (for more details see e.g. [Ber96] or [Kyp06]).

To start with, if  $\beta = (\beta_t, t \geq 0)$  denotes a planar Brownian motion starting from 1 and  $S = (S(t), t \geq 0)$  an independent stable subordinator with index  $\alpha/2$  starting from 0, i.e.:

$$E[\exp(-\mu S(t))] = \exp(-t\mu^{\alpha/2}), \quad (5)$$

for all  $t \geq 0$  and  $\mu \geq 0$ , then the subordinated planar BM  $\beta = (\beta_{2S(t)}, t \geq 0)$  is a standard isotropic stable process of index  $\alpha$ . The Lévy measure of  $S$  is:

$$\frac{\alpha}{2\Gamma(1 - \alpha/2)} s^{-1-\alpha/2} 1_{\{s>0\}} ds$$

thus, the Lévy measure  $\nu$  of  $Z$  is:

$$\begin{aligned} \nu(dx) &= \frac{\alpha}{2\Gamma(1 - \alpha/2)} \int_0^\infty s^{-1-\alpha/2} P(\beta_{2s} - 1 \in dx) ds \\ &= \frac{\alpha}{8\pi\Gamma(1 - \alpha/2)} \left( \int_0^\infty s^{-2-\alpha/2} \exp(-|x|^2/(4s)) ds \right) dx \\ &= \frac{\alpha 2^{-1+\alpha/2} \Gamma(1 + \alpha/2)}{\pi\Gamma(1 - \alpha/2)} |x|^{-2-\alpha} dx. \end{aligned} \quad (6)$$

Contrary to planar Brownian motion, as  $Z$  is discontinuous, we cannot define its winding number (recall that, as is well known [ItMK65], for planar BM, since it starts away from the origin, it does not visit a.s. the point 0 but keeps winding around it infinitely often. In particular, the winding process is well defined, for further details see also e.g. [PiY86]). However, following [BeW96], we can consider a path on a finite time interval  $[0, t]$  and "fill in" the gaps with line segments in order to obtain the curve of a continuous function  $f : [0, 1] \rightarrow \mathbb{C}$  with  $f(0) = 1$ . Now, since 0 is polar and  $Z$  has no jumps across 0 a.s., we have  $f(u) \neq 0$  for every  $u \in [0, 1]$ . Hence, we can define the process of the winding number of  $Z$  around 0, which we denote by  $\theta = (\theta_t, t \geq 0)$ . It has cadlag paths of absolute length greater than  $\pi$  and, for all  $t \geq 0$ ,

$$\exp(i\theta_t) = \frac{Z_t}{|Z_t|} . \quad (7)$$

We also introduce the clock:

$$H(t) \equiv \int_0^t \frac{ds}{|Z_s|^\alpha} , \quad (8)$$

and its inverse:

$$A(u) \equiv \inf\{t \geq 0, H(t) > u\} . \quad (9)$$

Bertoin and Werner following [GVA86] obtained these two Lemmas for  $\alpha \in (0, 2)$  (for the proofs see [BeW94a]):

**Lemma 2.1.** *The time-changed process  $(\theta_{A(u)}, u \geq 0)$  is a real-valued symmetric Lévy process. It has no Gaussian component and its Lévy measure has support in  $[-\pi, \pi]$ .*

We now denote by  $dz$  the Lebesgue measure on  $\mathbb{C}$ . Then, for every complex number  $z \neq 0$ ,  $\phi(z)$  denotes the determination of its argument valued in  $(-\pi, \pi]$ .

**Lemma 2.2.** *The Lévy measure of  $\theta_{A(\cdot)}$  is the image of the Lévy measure  $\nu$  of  $Z$  by the mapping  $z \rightarrow \phi(1+z)$ . As a consequence,  $E[(\theta_{A(u)})^2] = uk(\alpha)$ , where*

$$k(\alpha) = \frac{\alpha 2^{-1+\alpha/2} \Gamma(1+\alpha/2)}{\pi \Gamma(1-\alpha/2)} \int_{\mathbb{C}} |z|^{-2-\alpha} |\phi(1+z)|^2 dz . \quad (10)$$

Using Lemma 2.1, we can obtain the analogue of the skew product representation for planar BM which is the Lamperti correspondence for stable processes. Indeed, following [GVA86] and using Lamperti's relation (see e.g. [Lam72, Kiu80, Chy06, CPP11] or [ReY99]) and Lemma 2.1, there exist two real-valued Lévy processes  $(\xi_u, u \geq 0)$  and  $(\rho_u, u \geq 0)$ , the first one non-symmetric whereas the second one symmetric, both starting from 0, such that:

$$\log |Z_t| + i\theta_t \equiv \int_0^t \frac{dZ_s}{Z_s} = (\xi_u + i\rho_u) \Big|_{u=H_t=\int_0^t \frac{ds}{|Z_s|^\alpha}} . \quad (11)$$

We remark here that  $|Z|$  and  $Z_{A(\cdot)}/|Z_{A(\cdot)}|$  are NOT independent. Indeed, the processes  $|Z_{A(\cdot)}|$  and  $Z_{A(\cdot)}/|Z_{A(\cdot)}|$  jump at the same times hence they cannot be independent. Moreover,  $A(\cdot)$  depends only upon  $|Z|$ , hence  $|Z|$  and  $Z_{A(\cdot)}/|Z_{A(\cdot)}|$  are not independent. For

further discussion on the independence, see e.g. [LiW11], where is shown that an isotropic  $\alpha$ -self-similar Markov process has a skew-product structure if and only if its radial and its angular part do not jump at the same time.

We also remark that

$$H^{-1}(u) \equiv A(u) \equiv \inf\{t \geq 0 : H(t) > u\} = \int_0^u \exp\{\alpha \xi_s\} ds . \quad (12)$$

Hence, (11) may be equivalently written as:

$$\begin{cases} |Z_t| = \exp(\xi(H_t)) \Leftrightarrow |Z_{A(t)}| = \exp(\xi_t), & (\text{extension of Lamperti's identity}) \\ \theta_t = \rho(H_t) \Leftrightarrow \theta(A(t)) = \rho(t) . \end{cases} \quad (13)$$

We also define the random times  $T_c^{|\theta|} \equiv \inf\{t : |\theta_t| \geq c\}$  and  $T_c^{|\rho|} \equiv \inf\{t : |\rho_t| \geq c\}$ , ( $c > 0$ ). Using the "generalized" skew-product representation (11) (or (13)), we obtain:

$$T_c^{|\theta|} = H_u^{-1} \Big|_{u=T_c^{|\rho|}} = \int_0^{T_c^{|\rho|}} ds \exp(\alpha \xi_s) \equiv A_{T_c^{|\rho|}} . \quad (14)$$

Following [VaY12], for the random times  $T_{-d,c}^\theta \equiv \inf\{t : \theta_t \notin (-d, c)\}$ ,  $d, c > 0$ , and  $T_c^\theta \equiv \inf\{t : \theta_t \geq c\}$ , we have:

**Remark 2.3.** For  $0 < c < d$ , the random times  $T_{-d,c}^\theta$ ,  $T_c^{|\theta|}$  and  $T_c^\theta$  satisfy the trivial inequality:

$$T_c^{|\theta|} \leq T_{-d,c}^\theta \leq T_c^\theta . \quad (15)$$

Hence, with  $p > 0$ :

$$E \left[ (T_c^{|\theta|})^p \right] \leq E \left[ (T_{-d,c}^\theta)^p \right] \leq E \left[ (T_c^\theta)^p \right] , \quad (16)$$

and for the negative moments:

$$E \left[ (T_c^\theta)^{-p} \right] \leq E \left[ (T_{-d,c}^\theta)^{-p} \right] \leq E \left[ (T_c^{|\theta|})^{-p} \right] . \quad (17)$$

**Remark 2.4.** For further details concerning the finiteness of the positive moments of  $T_c^{|\theta|}$ , see e.g. [DeB90, BaB04]. Recall also that for the positive moments of the exit time from a cone of planar Brownian motion, Spitzer showed that (with obvious notation) [Spi58, Bur77]:

$$E \left[ \left( T_c^{|\theta^\beta|} \right)^p \right] < \infty \Leftrightarrow p < \frac{\pi}{4c} , \quad (18)$$

whereas all the negative moments  $E \left[ \left( T_c^{|\theta^\beta|} \right)^{-p} \right]$  are finite [VaY12].

We denote now by  $\Psi(u)$  the exponent of the symmetric Lévy process  $\rho$ , hence (Lévy-Khintchine formula)  $E[e^{iu\rho t}] = e^{-t\Psi(u)}$ , with:

$$\Psi(u) = \int_{(-\infty, \infty)} (1 - e^{iux} + iux 1_{\{|x| \leq 1\}}) \mu(dx), \quad u \in \mathbb{R}, \quad (19)$$

where  $\mu$  is a Radon measure on  $\mathbb{R} \setminus \{0\}$  such that:

$$\int_{(-\infty, \infty)} (x^2 \wedge 1) \mu(dx) < \infty .$$

$\mu$  is the Lévy measure of  $\rho$  and is symmetric.

### 3 Large time asymptotics

Concerning the clock  $H$ , we have the almost sure convergence (see Corollary 1 in Bertoin and Werner [BeW96]):

$$\frac{H(e^u)}{u} \xrightarrow[u \rightarrow \infty]{a.s.} 2^{-\alpha} \frac{\Gamma(1 - \alpha/2)}{\Gamma(1 + \alpha/2)} \equiv K(\alpha) = E[|Z_1|^{-\alpha}] . \quad (20)$$

Moreover, we have the following:

**Proposition 3.1.** *The family of processes*

$$H_x^{(u)} \equiv \left( \frac{H(e^{ux})}{u}, x \geq 0 \right)$$

is tight, as  $u \rightarrow \infty$ .

**Proof of Proposition 3.1:**

To prove this, we could repeat some arguments of Pitman and Yor [PiY89] (see the estimates in their proof of Theorem 6.4), however, we give here a straightforward proof, using the definition of tightness:

for every  $\varepsilon, \eta > 0$ , there exist  $\delta > 0$  and  $C_\delta > 0$  such that, for every  $0 < x < y$ :

$$P \left( \sup_{|x-y| \leq \delta} |H(e^{uy}) - H(e^{ux})| \geq u\varepsilon \right) \leq \eta , \quad \text{for } u \geq C_\delta , \quad (21)$$

or equivalently:

$$P \left( \frac{1}{u} |H(e^{u(x+\delta)}) - H(e^{ux})| \geq \varepsilon \right) \leq \eta , \quad \text{for } u \geq C_\delta . \quad (22)$$

First, following Bertoin and Werner [BeW96], we introduce the "Ornstein-Uhlenbeck type" process:

$$\tilde{Z}_u = \exp(-u/\alpha) Z_{\exp(u)}, \quad u \geq 0 , \quad (23)$$

which is a stationary Markov process under  $P_0$  (see e.g. [Bre68]). We denote by  $p_t(\cdot)$  the semigroup of  $\tilde{Z}$ :

$$p_t(\bar{z}) = P_0(Z_t \in d\bar{z})/d\bar{z}, \quad \bar{z} \in \mathbb{C}.$$

We denote by  $Z^{(0)}$  another stable process starting at 0. Then, using the scaling property, given that  $\tilde{Z}_0 \equiv Z_1 \equiv 1 + Z_1^{(0)} = \bar{x}$ , the semigroup  $q_u(\cdot)$  of  $\tilde{Z}$  is given by:

$$\begin{aligned} q_u(\bar{x}, \bar{y}) &= p_{\exp(u)-1}(e^{u/\alpha}\bar{y} - \bar{x}) e^{2u/\alpha} \\ &= (e^u - 1)^{-2/\alpha} e^{2u/\alpha} p_1((e^u - 1)^{-1/\alpha}(e^{u/\alpha}\bar{y} - \bar{x})) \\ &= (l(v))^2 p_1(l(v)(\bar{y} - e^{-u/\alpha}\bar{x})), \end{aligned} \quad (24)$$

where  $l(v) = e^{v/\alpha}(e^v - 1)^{-1/\alpha}$ . For every  $\delta > 0$  and changing variables  $s = \exp(v)$ , with obvious notation, we have:

$$E[|H(e^{u(x+\delta)}) - H(e^{ux})|] = \int_{e^{ux}}^{e^{u(x+\delta)}} E[|Z_s|^{-\alpha}] ds = \int_{ux}^{u(x+\delta)} E_{\tilde{Z}_0}[|\tilde{Z}_v|^{-\alpha}] dv . \quad (25)$$

We also define  $\varepsilon(v) \equiv l(v)e^{-v/\alpha} = (e^v - 1)^{-1/\alpha}$ . From (23), using the stability of  $Z$ , we have:

$$\begin{aligned}\tilde{Z}_v &= e^{-v/\alpha} Z_{\exp(v)} = e^{-v/\alpha} (Z_{\exp(v)-1} + Z_1) \stackrel{(law)}{=} e^{-v/\alpha} \left( (e^v - 1)^{1/\alpha} Z_1^{(0)} + Z_1 \right) \\ &= (l(v))^{-1} \left( Z_1^{(0)} + \varepsilon(v) Z_1 \right).\end{aligned}\quad (26)$$

Hence (for simplicity, we use  $E \equiv E_0$ ):

$$E_{\tilde{Z}_0} \left[ |\tilde{Z}_v|^{-\alpha} \right] = (l(v))^\alpha E \left[ |Z_1^{(0)} + \varepsilon(v) \tilde{Z}_0|^{-\alpha} \right] \equiv (l(v))^\alpha (E_1 + E_2), \quad (27)$$

where, with  $\delta' > 0$ ,

$$\begin{aligned}E_1 &= E \left[ |Z_1^{(0)} + \varepsilon(v) \tilde{Z}_0|^{-\alpha} : |Z_1^{(0)} + \varepsilon(v) \tilde{Z}_0| \geq \delta' \right], \\ E_2 &= E \left[ |Z_1^{(0)} + \varepsilon(v) \tilde{Z}_0|^{-\alpha} : |Z_1^{(0)} + \varepsilon(v) \tilde{Z}_0| \leq \delta' \right].\end{aligned}$$

We have:  $l(v) \xrightarrow{v \rightarrow \infty} 1$  and  $\varepsilon(v) \xrightarrow{v \rightarrow \infty} 0$ , thus, by Dominated Convergence Theorem:

$$E_1 \xrightarrow{v \rightarrow \infty} E \left[ |Z_1^{(0)}|^{-\alpha} : |Z_1^{(0)}| \geq \delta' \right] \xrightarrow{\delta' \rightarrow 0} E \left[ |Z_1^{(0)}|^{-\alpha} \right]. \quad (28)$$

Moreover, changing the variables:  $\bar{w} = \bar{z} + \varepsilon(v)\bar{x}$ , we have:

$$\begin{aligned}E_2 &= \int_{\bar{x}, \bar{z}: |\bar{z} + \varepsilon(v)\bar{x}| \leq \delta'} P(\tilde{Z}_0 \in d\bar{x}) P(Z_1^{(0)} \in d\bar{z}) |\bar{z} + \varepsilon(v)\bar{x}|^{-\alpha} \\ &= \int_{\bar{x}, \bar{w}: |\bar{w}| \leq \delta'} P(\tilde{Z}_0 \in d\bar{x}) P(Z_1^{(0)} \in d\bar{w}) |\bar{w}|^{-\alpha}.\end{aligned}$$

Remarking now that for stable processes:  $P(Z_1^{(0)} \in d\bar{y}) \leq C' d\bar{y}$ , where  $C'$  stands for a positive constant and using  $w = (w_1, w_2)$ , we have:

$$E_2 \leq C' \int_{\bar{x}, \bar{w}: |\bar{w}| \leq \delta'} P(\tilde{Z}_0 \in d\bar{x}) \frac{dw_1 dw_2}{|\bar{w}|^\alpha} = C' \int_{\bar{z}: |\bar{w}| \leq \delta'} \frac{dw_1 dw_2}{|\bar{w}|^\alpha} \xrightarrow{\delta' \rightarrow 0} 0. \quad (29)$$

Thus, from (27), (28) and (29), invoking the Dominated Convergence Theorem, we deduce:

$$\lim_{u \rightarrow \infty} E_{\tilde{Z}_0} \left[ |\tilde{Z}_v|^{-\alpha} \right] = E \left[ |Z_1^{(0)}|^{-\alpha} \right], \quad (30)$$

which is a constant. Hence, for every  $\varepsilon, \eta > 0$ , there exist  $\delta > 0$  and  $C_\delta > 0$  such that (22) is satisfied for  $u \geq C_\delta$ .  $\blacksquare$

Bertoin and Werner in [BeW96] obtained the analogue of Spitzer's asymptotic Theorem [Spi58] for isotropic stable Lévy processes of index  $\alpha \in (0, 2)$ :

**Theorem 3.2.** *The family of processes*

$$\Theta_t^{(c)} \equiv (c^{-1/2} \theta_{\exp(ct)}, t \geq 0)$$



converges in distribution on  $D([0, \infty), \mathbb{R})$  endowed with the Skorohod topology, as  $c \rightarrow \infty$ , to  $(\sqrt{r(\alpha)}B_t, t \geq 0)$ , where  $(B_s, s \geq 0)$  is a real valued Brownian motion and

$$r(\alpha) = \frac{\alpha 2^{-1-\alpha/2}}{\pi} \int_{\mathbb{C}} |z|^{-2-\alpha} |\phi(1+z)|^2 dz . \quad (31)$$

Using some results due to Whitt [Whi80], we can obtain a simple proof of this Theorem.

**Proof of Theorem 3.2: (new proof)**

Essentially, an argument of continuity of the composition function (Theorem 3.1 in [Whi80]) may replace the martingale argument in the lines of the proof for  $t \rightarrow \infty$  from Bertoin and Werner. We split the proof in three parts:

i) Concerning the clock  $H$ , we have the almost sure convergence (20):

$$\frac{H(e^u)}{u} \xrightarrow[u \rightarrow \infty]{a.s.} K(\alpha) = E [|Z_1|^{-\alpha}] .$$

From this result follows the convergence of the finite dimensional distributions of  $v^{-1}H(\exp(vt))$ , for  $v \rightarrow \infty$  and every  $t > 0$ .

Moreover, from Proposition 3.1, the family of processes

$$H_x^{(u)} \equiv \left( \frac{H(e^{ux})}{u}, x \geq 0 \right)$$

is tight as  $u \rightarrow \infty$ . Hence, from (20) and (21), finally,  $H^{(u)}(t) \equiv (u^{-1}H(\exp(ut)), t \geq 0)$  converges weakly to  $(tK(\alpha), t \geq 0)$  as  $u \rightarrow \infty$ , in other words:

$$\left( \frac{H(e^{ut})}{u}, t \geq 0 \right) \xrightarrow[u \rightarrow \infty]{(d)} (tK(\alpha), t \geq 0), \quad (32)$$

where the convergence in distribution is viewed on  $D([0, \infty), \mathbb{R})$  endowed with the Skorohod topology.

ii) Using the skew product representation (11) and Lemma 2.2, we have:

$$\left( \frac{\rho_{tu}}{\sqrt{u}}, t \geq 0 \right) \xrightarrow[u \rightarrow \infty]{(d)} \left( \sqrt{k(\alpha)} B_t, t \geq 0 \right) , \quad (33)$$

where the convergence in distribution is viewed again on  $D([0, \infty), \mathbb{R})$  endowed with the Skorohod topology and  $k(\alpha)$  is given by (10). This follows from the convergence of the finite dimensional distributions:

$$\frac{\rho_u}{\sqrt{u}} = \frac{\theta_{A(u)}}{\sqrt{u}} \xrightarrow[u \rightarrow \infty]{(d)} \sqrt{k(\alpha)} B_1 , \quad (34)$$

a condition which is sufficient for the weak convergence (33), since Lévy processes are semimartingales with stationary independent increments; for further details see e.g. [Sko91] or [JaS03] (Corollary 3.6, Chapter VII, p. 415).

iii) Theorem 3.1 in [Whi80] states that the composition function on  $D([0, \infty), \mathbb{R}) \times D([0, \infty), [0, \infty))$  is continuous at each  $(\rho, H) \in (C([0, \infty), \mathbb{R}) \times D_0([0, \infty), [0, \infty)))$ , with  $C$  denoting the set of continuous functions and  $D_0$  the subset of increasing cadlag functions in  $D$  (hence the subset of non-decreasing cadlag functions in  $D$ ). Hence, from (32) and (33), we have: for every  $t > 0$ ,

$$\frac{\theta_{\exp(ct)}}{\sqrt{c}} = \frac{\rho_{H(\exp(ct))}}{\sqrt{c}} = \frac{\rho_{c(H(e^{ct})/c)}}{\sqrt{c}}. \quad (35)$$

The result now follows from the continuity of the composition function together with (35) and the weak convergence of  $H^{(c)}(\cdot)$  and  $c^{-1/2}\rho_c$ , as  $c \rightarrow \infty$ . ■

From Theorem 3.2, we can obtain the asymptotic behaviour of the exit times from a cone for isotropic stable processes which generalizes a recent result in [VaY12]:

**Proposition 3.3.** *For  $c \rightarrow \infty$ , for every  $x > 0$ , we have the weak convergence:*

$$\left( \frac{1}{c} \log \left( T_{x\sqrt{c}}^\theta \right), x \geq 0 \right) \xrightarrow[c \rightarrow \infty]{(d)} \left( \tau_{\sqrt{1/r(\alpha)}}^{(1/2)}, x \geq 0 \right), \quad (36)$$

where for every  $y > 0$ ,  $\tau_y^{(1/2)}$  stands for the  $\frac{1}{2}$ -stable process defined by:  $\tau_y^{(1/2)} \equiv \inf\{t : B_t = y\}$ .

**Proof of Proposition 3.3:**

We rely now upon Theorem 3.2, the analogue of Spitzer's Theorem for stable processes by Bertoin and Werner:

$$\Theta_t^{(c)} \equiv (c^{-1/2}\theta_{\exp(ct)}, t \geq 0) \xrightarrow[c \rightarrow \infty]{(d)} (B_{r(\alpha)t}, t \geq 0). \quad (37)$$

Hence, for every  $x > 0$ ,

$$\begin{aligned} \frac{1}{c} \log \left( T_{x\sqrt{c}}^\theta \right) &= \frac{1}{c} \log \left( \inf \{ t : \theta_t > x\sqrt{c} \} \right) \\ &\stackrel{t=\exp(cs)}{=} \frac{1}{c} \log \left( \inf \left\{ e^{cs} : \frac{1}{\sqrt{c}} \theta_{\exp(cs)} > x \right\} \right) \\ &= \inf \left\{ s : \frac{1}{\sqrt{c}} \theta_{\exp(cs)} > x \right\} \\ &\stackrel{c \rightarrow \infty}{\longrightarrow} \inf \{ s : B_{r(\alpha)s} > x \} \\ &= \inf \left\{ s : \sqrt{r(\alpha)} B_s > x \right\} \equiv \tau_{x/\sqrt{r(\alpha)}}^{(1/2)}. \end{aligned} \quad (38)$$

Moreover, from Theorem 7.1 in [Whi80], we know that the first passage time function mapping is continuous, thus, we deduce (36). ■

If we replace  $c$  by  $ac$ , we can obtain several variants of Proposition 3.3 for the random times  $T_{-bc, ac}^\theta$ ,  $0 < a, b \leq \infty$ , for  $c \rightarrow \infty$ , and  $a, b > 0$  fixed:

**Corollary 3.4.** *The following asymptotic results hold:*

$$\frac{1}{c} \log \left( T_{\sqrt{ac}}^\theta \right) \xrightarrow{c \rightarrow \infty} \tau_{\sqrt{a/r(\alpha)}}^{(1/2)}, \quad (39)$$

$$\frac{1}{c} \log \left( T_{\sqrt{ac}}^{|\theta|} \right) \xrightarrow{c \rightarrow \infty} \tau_{\sqrt{a/r(\alpha)}}^{|B|}, \quad (40)$$

$$\frac{1}{c} \log \left( T_{-\sqrt{bc}, \sqrt{ac}}^\theta \right) \xrightarrow{c \rightarrow \infty} \tau_{-\sqrt{b/r(\alpha)}, \sqrt{a/r(\alpha)}}^B, \quad (41)$$

where for every  $x, y > 0$ ,  $\tau_x^{|B|} \equiv \inf\{t : |B_t| = x\}$  and  $\tau_{-y,x}^B \equiv \inf\{t : B_t \notin (-y, x)\}$ .

**Proposition 3.5.** *The following asymptotic result for  $\alpha \in (0, 2)$ , holds: for every  $b > 0$ ,*

$$P \left( T_{b\sqrt{\log t}}^\theta > t \right) \xrightarrow{t \rightarrow \infty} \operatorname{erf} \left( \frac{b}{\sqrt{2r(\alpha)}} \right), \quad (42)$$

where  $\operatorname{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$  is the error function.

**Proof of Proposition 3.5:**

Using the notation of Theorem 3.2, for every  $b > 0$ , we have:

$$\begin{aligned} P \left( T_{b\sqrt{\log t}}^\theta > t \right) &= P \left( \sup_{u \leq t} \theta_u < b\sqrt{\log t} \right) \stackrel{u=tv}{=} P \left( \sup_{v \leq 1} (\log t)^{-1/2} \theta_{tv} < b \right) \\ &\stackrel{t=e^c}{=} P \left( \sup_{v \leq 1} c^{-1/2} \theta_{\exp(cv)} < b \right) \end{aligned}$$

Hence, using Theorem 3.2 for  $t \rightarrow \infty$ , we deduce:

$$\begin{aligned} P \left( T_{b\sqrt{\log t}}^\theta > t \right) &\xrightarrow{t \rightarrow \infty} P \left( \sup_{v \leq 1} \sqrt{r(\alpha)} B_v < b \right) = P \left( |B_1| < \frac{b}{\sqrt{r(\alpha)}} \right) \\ &= 2 \int_0^{b/\sqrt{r(\alpha)}} \frac{dw}{\sqrt{2\pi}} e^{-w^2/2}, \end{aligned}$$

and changing the variables  $w = y\sqrt{2}$ , we obtain (42). ■

As mentioned in [BeW96], because an isotropic stable Lévy process  $Z$  is transient, the difference between  $\theta$  and the winding number around an arbitrary fixed  $z \neq 1$  is bounded and converges as  $t \rightarrow \infty$ . Hence, with  $(\theta_t^i, t > 0)$ ,  $1 \leq i \leq n$  denoting the continuous total angle wound of  $Z$  of index  $\alpha \in (0, 2)$  around  $z^i$  ( $z^1, \dots, z^n$  are  $n$  distinct points in the complex plane  $\mathbb{C}$ ) up to time  $t$ , we obtain the following concerning the finite dimensional distributions (windings around several points):

**Proposition 3.6.** *For isotropic stable Lévy processes of index  $\alpha \in (0, 2)$ , we have:*

$$\left( \frac{\theta_t^i}{\sqrt{\log t}}, 1 \leq i \leq n \right) \xrightarrow[t \rightarrow \infty]{(d)} \left( \sqrt{r(\alpha)} B_1^i, 1 \leq i \leq n \right), \quad (43)$$

where  $(B_s^i, 1 \leq i \leq n, s \geq 0)$  is an  $n$ -dimensional Brownian motion and  $r(\alpha)$  is given by (31).

## 4 Small time asymptotics

We turn now our study to the behaviour of  $\theta_t$  for  $t \rightarrow 0$ .

**Theorem 4.1.** *For  $\alpha \in (0, 2)$ , the following convergence in law holds:*

$$(t^{-1/\alpha} \rho_{ts}, s \geq 0) \xrightarrow[t \rightarrow 0]{(d)} (\zeta_s, s \geq 0) , \quad (44)$$

where  $(\zeta_s, s \geq 0)$  is a symmetric 1-dimensional  $\alpha$ -stable process and the convergence in distribution is considered on  $D([0, \infty), \mathbb{R})$  endowed with the Skorohod topology.

**Proof of Theorem 4.1:**

From Lemma 2.2, we use the Lévy measure, say  $\tilde{\pi}$ , of  $\theta_{A(\cdot)}$  (thus the Lévy measure of  $\rho$ ) and we prove that for  $t \rightarrow 0$  it converges to the Lévy measure of a 1-dimensional  $\alpha$ -stable process. Indeed, with

$$L \equiv \frac{\alpha 2^{-1+\alpha/2} \Gamma(1 + \alpha/2)}{\pi \Gamma(1 - \alpha/2)},$$

and  $z$  denoting a number in  $\mathbb{C}$ , using polar coordinates, we have:

$$\phi(1+z) = \int_{\mathbb{C}} \frac{dz}{1+z} = 2L \int_0^\pi \int_0^\infty \frac{r dr d\varphi}{(1+r^2-2r \cos \varphi)^{1+\alpha/2}} .$$

We remark that:

$$1+r^2-2r \cos \varphi = (r - \cos \varphi)^2 + \sin^2 \varphi ,$$

hence, changing the variables  $(r - \cos \varphi)^2 = t^{-1} \sin^2 \varphi$  and denoting by:

$$B(y; a, b) = \int_0^y u^{a-1} (1-u)^{b-1} du ,$$

the incomplete Beta function, for  $\varphi > 0$  (we can repeat the same arguments for  $\varphi < 0$ ) we have:

$$\begin{aligned} \tilde{\pi}(d\varphi) &= d\varphi 2L \int_0^\infty \frac{r dr}{((r - \cos \varphi)^2 + \sin^2 \varphi)^{1+\alpha/2}} \\ &= d\varphi \frac{2L}{2} \left( \frac{2}{\alpha} + \cos \varphi (1 - \cos^2 \varphi)^{-\frac{1}{2}-\frac{\alpha}{2}} \int_0^{1-\frac{1}{\cos^2 \varphi}} t^{-\frac{1}{2}+\frac{\alpha}{2}} (1+t)^{-\frac{\alpha}{2}-1} dt \right) \\ &\stackrel{u=-t}{=} d\varphi L \left( \frac{2}{\alpha} + \cos \varphi (-1 + \cos^2 \varphi)^{-\frac{1}{2}-\frac{\alpha}{2}} \int_0^{1-\frac{1}{\cos^2 \varphi}} u^{-\frac{1}{2}+\frac{\alpha}{2}} (1-u)^{-\frac{\alpha}{2}-1} du \right) \\ &= d\varphi L \left( \frac{2}{\alpha} + \cos \varphi (-1 + \cos^2 \varphi)^{-\frac{1}{2}-\frac{\alpha}{2}} B \left( 1 - \frac{1}{\cos^2 \varphi}; \frac{1}{2} + \frac{\alpha}{2}, -\frac{\alpha}{2} \right) \right) \\ &\stackrel{\varphi \sim 0}{\sim} \tilde{L} \varphi^{-1-\alpha} d\varphi , \end{aligned} \quad (45)$$

which is the Lévy measure of an  $\alpha$ -stable process. The result now follows by standard arguments. ■

Concerning the clock  $H$  and its increments, we have:

**Theorem 4.2.** *The following a.s. convergence holds:*

$$\left( \frac{H(ux)}{u}, x \geq 0 \right) \xrightarrow[u \rightarrow 0]{a.s.} (x, x \geq 0) . \quad (46)$$

**Proof of Theorem 4.2:**

From the definition of the clock  $H$  we have:

$$\frac{H(ux)}{u} = \frac{1}{u} \int_0^{ux} \frac{ds}{|Z_s|^\alpha} .$$

Hence, for every  $x_0 > 0$ , we have:

$$\begin{aligned} \sup_{x \leq x_0} \left| \frac{H(ux) - ux}{u} \right| &= \sup_{x \leq x_0} \frac{1}{u} \left| \int_0^{ux} \left( \frac{1}{|Z_s|^\alpha} - 1 \right) ds \right| \leq \frac{1}{u} \int_0^{ux_0} \left| \frac{1}{|Z_s|^\alpha} - 1 \right| ds \\ &\stackrel{s=uw}{=} \int_0^{x_0} \left| \frac{1}{|Z_{uw}|^\alpha} - 1 \right| dw \xrightarrow[u \rightarrow 0]{a.s.} 0 . \end{aligned} \quad (47)$$

because:

$$|Z_u|^\alpha \xrightarrow[u \rightarrow 0]{a.s.} 1 . \quad (48)$$

Hence, as (47) is true for every  $x_0 > 0$ , we obtain (46). ■

**Remark 4.3.** *We remark that this behaviour of the clock is different for the case  $t \rightarrow \infty$ , where (32) can be equivalently stated as:*

$$\left( \frac{H(ux)}{\log u}, x \geq 0 \right) \xrightarrow[u \rightarrow \infty]{(d)} \left( 2^{-\alpha} \frac{\Gamma(1 - \alpha/2)}{\Gamma(1 + \alpha/2)} x, x \geq 0 \right) . \quad (49)$$

Using Theorems 4.1 and 4.2, we obtain:

**Theorem 4.4.** *With  $\alpha \in (0, 2)$ , the family of processes*

$$(c^{-1/\alpha} \theta_{ct}, t \geq 0)$$

*converges in distribution on  $D([0, \infty), \mathbb{R})$  endowed with the Skorohod topology, as  $c \rightarrow 0$ , to a symmetric 1-dimensional  $\alpha$ -stable process  $(\zeta_t, t \geq 0)$ .*

**Proof of Theorem 4.4:**

We will use Theorems 4.1 and 4.2. More precisely, we shall rely again upon the continuity of the composition function as studied in Theorem 3.1 in [Whi80].

i) First, concerning the clock  $H$ , for every  $t > 0$ , we have the almost sure convergence (46), which yields the weak convergence (on  $D([0, \infty), \mathbb{R})$  endowed with the Skorohod topology) of the family of processes  $\tilde{H}^{(u)}(t) \equiv (u^{-1}H(ut), t \geq 0)$  to  $(t, t \geq 0)$  as  $u \rightarrow 0$ .

ii) We use another result of Whitt [Whi80] which states that the composition function on  $D([0, \infty), \mathbb{R}) \times D([0, \infty), [0, \infty))$  is continuous at each  $(\rho, H) \in (D([0, \infty), \mathbb{R}) \times C_0([0, \infty), [0, \infty)))$ , with  $D$  denoting the set of cadlag functions and  $C_0$  the subset of

strictly-increasing functions in  $C$ . Hence, from Theorem 4.1 and (46), using the weak convergence of  $\tilde{H}^{(c)}(\cdot)$  and of  $c^{-1/\alpha}\rho_c$ , as  $c \rightarrow 0$ , we deduce: for every  $t > 0$ ,

$$\frac{\theta_{ct}}{c^{1/\alpha}} = \frac{\rho_{H(ct)}}{c^{1/\alpha}} = \frac{\rho_{c(H(ct)/c)}}{c^{1/\alpha}} \xrightarrow[c \rightarrow 0]{(d)} \zeta_t, \quad (50)$$

where the convergence in distribution is viewed on  $D([0, \infty), \mathbb{R})$  endowed with the Skorohod topology.  $\blacksquare$

From the previous results, we deduce the asymptotic behaviour, for  $c \rightarrow 0$ , of the first exit times from a cone for isotropic stable processes of index  $\alpha \in (0, 2)$  taking values in the complex plane:

**Proposition 4.5.** *For  $c \rightarrow 0$ , we have the weak convergence:*

$$\left( \frac{1}{c} T_{c^{1/\alpha}x}^\theta, x \geq 0 \right) \xrightarrow[c \rightarrow 0]{(d)} (T_x^\zeta, x \geq 0), \quad (51)$$

where for every  $x$ ,  $T_x^\zeta$  is the first hitting time defined by:  $T_x^\zeta \equiv \inf\{t : \zeta_t = x\}$ .

**Proof of Proposition 4.5:**

Using Theorem 4.4, we have:

$$\begin{aligned} \frac{1}{c} T_{c^{1/\alpha}x}^\theta &= \frac{1}{c} \inf\{t : \theta_t > c^{1/\alpha}x\} \stackrel{t=cs}{=} \frac{1}{c} \inf\{cs : c^{-1/\alpha}\theta_{cs} > x\} = \inf\{s : c^{-1/\alpha}\theta_{cs} > x\} \\ &\xrightarrow{c \rightarrow 0} \inf\{s : \zeta_s > x\} \stackrel{t=cs}{=} \frac{1}{c^\alpha} \inf\left\{cs : \frac{1}{c} \theta_{cs} > x\right\}, \end{aligned}$$

which, using again the continuity of the first passage time function mapping (see Theorem 7.1 in [Whi80]), yields (51).  $\blacksquare$

Finally, we can obtain several variants of Proposition 4.5 for the random times  $T_{-bc,ac}^\theta$ ,  $0 < a, b \leq \infty$  fixed, for  $c \rightarrow 0$ :

**Corollary 4.6.** *The following asymptotic results hold:*

$$\frac{1}{c} T_{ac^{1/\alpha}}^\theta \xrightarrow[c \rightarrow 0]{(law)} T_a^\zeta, \quad (52)$$

$$\frac{1}{c} T_{ac^{1/\alpha}}^{|\theta|} \xrightarrow[c \rightarrow 0]{(law)} T_a^{|\zeta|}, \quad (53)$$

$$\frac{1}{c} T_{-bc^{1/\alpha}, ac^{1/\alpha}}^\theta \xrightarrow[c \rightarrow 0]{(law)} T_{-b,a}^\zeta, \quad (54)$$

where, for every  $x, y > 0$ ,  $T_x^{|\zeta|} \equiv \inf\{t : |\zeta_t| = x\}$  and  $T_{-y,x}^\zeta \equiv \inf\{t : \zeta_t \notin (-y, x)\}$ .

**Remark 4.7. (Windings of planar stable processes in  $(t, 1]$  for  $t \rightarrow 0$ )**

*i) We consider now our stable process  $Z$  starting from 0 and we want to investigate its windings in  $(t, 1]$  for  $t \rightarrow 0$ . We know that it doesn't visit again the origin but it winds a.s. infinitely often around it. We change variables  $u = tv$  and we apply  $Z_{tv} \stackrel{(law)}{=} t^{1/\alpha}Z_v$ . Hence, we have:*

$$\theta'_t \equiv \text{Im} \left( \int_t^1 \frac{dZ_u}{Z_u} \right) \stackrel{(law)}{=} \text{Im} \left( \int_1^{1/t} \frac{dZ_v}{Z_v} \right) \equiv \theta_{1/t}. \quad (55)$$

The only difference with respect to the "normal" stable case is that the integral starts from 1, but this doesn't provoke any problem.

Hence, Bertoin and Werner's Theorem 3.2 is still valid for  $Z$  in  $(t, 1]$ ,  $t \rightarrow 0$ : for  $\alpha \in (0, 2)$ :

$$\frac{1}{\sqrt{\log(1/t)}} \theta_{1/t} \xrightarrow[t \rightarrow 0]{(d)} \sqrt{r(\alpha)} N, \quad (56)$$

with  $r(\alpha)$  defined in (31) and  $N \sim \mathcal{N}(0, 1)$ .

ii) We note that this study is also valid for a planar Brownian motion starting from 0 in  $(t, 1]$  for  $t \rightarrow 0$  and for planar stable processes or planar Brownian motion starting both from a point different from 0 (in order that its winding number  $\tilde{\theta}$  is well defined) in  $[0, 1]$ . In particular, for planar Brownian motion, we obtain that Spitzer's law is still valid for  $t \rightarrow 0$  (see e.g. [ReY99, LeG92]):

$$\frac{2}{\log(1/t)} \tilde{\theta}_{1/t} \xrightarrow[t \rightarrow 0]{(law)} C_1, \quad (57)$$

with  $C_1$  denoting a standard Cauchy variable. We also remark that this result could also be obtained from a time inversion argument, that is: with  $Z$  and  $Z'$  denoting now two planar Brownian motions, by time inversion, we have:  $Z_u = uZ'_{1/u}$ . Changing variables  $u = 1/v$ , we obtain:

$$\tilde{\theta}_t \equiv \text{Im} \left( \int_t^1 \frac{dZ_u}{Z_u} \right) = \text{Im} \left( \int_t^1 \frac{d(uZ'_{1/u})}{uZ'_{1/u}} \right) = \text{Im} \left( \int_t^1 \frac{dZ'_{1/u}}{Z'_{1/u}} \right) = \text{Im} \left( \int_1^{1/t} \frac{dZ'_v}{Z'_v} \right) \equiv \tilde{\theta}'_{1/t},$$

where  $\tilde{\theta}'$  is the continuous winding process of  $Z'$ , and we continue as before. This time inversion argument is NOT valid for planar stable processes.

## 5 The Law of the Iterated Logarithm (LIL)

In this Section, we shall use some notation introduced in [DoM02a, DoM02b]. Recall (19); then, for all  $x > 0$ , because  $\rho$  is symmetric, we define:

$$L(x) = 2\hat{\mu}(x) = 2\mu(x, +\infty), \quad U(x) = 2 \int_0^x yL(y)dy .$$

We remark that  $U$  plays essentially the role of the truncated variance in the random walk case (see e.g. [DoM00]).

Hence, from (44), for  $t \rightarrow 0$ , we have ( $\tilde{K}(\alpha)$  is a constant depending on  $\alpha$ ):

$$U(x) \stackrel{x \rightarrow 0}{\sim} \tilde{K}(\alpha)x^{2-\alpha}. \quad (58)$$

Then, we obtain the following Law of the Iterated Logarithm (LIL) for Lévy processes for small times:

**Theorem 5.1. (LIL for Lévy processes for small times)**

For any non-decreasing function  $f > 0$ ,

$$\limsup_{t \rightarrow 0} \frac{\rho_t}{f(t)} = \begin{cases} 0 ; \\ \infty \end{cases} \quad a.s. \Leftrightarrow \int_1^\infty (f(t))^{-\alpha} dt \begin{cases} < \infty ; \\ = \infty . \end{cases} \quad (59)$$

We can reformulate Theorem 5.1 by using the skew-product representation (13)  $\rho_t = \theta_{A(t)}$  in order to deduce a LIL for the winding process  $\theta_{A(\cdot)}$  for small times:

**Corollary 5.2.** For any non-decreasing function  $f > 0$ ,

$$\limsup_{t \rightarrow 0} \frac{\theta_{A(t)}}{f(t)} = \begin{cases} 0 ; \\ \infty \end{cases} \quad a.s. \Leftrightarrow \int_1^\infty (f(t))^{-\alpha} dt \begin{cases} < \infty ; \\ = \infty . \end{cases} \quad (60)$$

**Proof of Theorem 5.1:**

First, we define:

$$h(y) = y^{-2}U(y), \quad y > 0. \quad (61)$$

Then, we consider  $t_n = 2^{-n}$  and we note that (Cauchy's test):

$$I(f) \equiv \int_1^\infty dt h(f(t)) < \infty \iff \sum_{n=1}^\infty t_n h(f(t_n)) < \infty .$$

Using now Lemma 2 from Doney and Maller [DoM02a], because  $\rho$  is symmetric, there exists a positive constant  $c_1$  such that for every  $x > 0, t > 0$ ,

$$P(\rho_t \geq x) \leq P\left(\sup_{0 \leq u \leq t} \rho_u \geq x\right) \leq c_1 t h(x) = c_1 t \frac{U(x)}{x^2}. \quad (62)$$

Thus:

$$\sum_{n=1}^\infty P(\rho_{t_{n-1}} \geq f(t_n)) \leq c_1 \sum_{n=1}^\infty t_n \frac{U(f(t_n))}{(f(t_n))^2}.$$

From (58), we have that,

$$\frac{U(f(t_n))}{(f(t_n))^2} \underset{t_n \rightarrow 0}{\sim} (f(t_n))^{-\alpha}.$$

Hence, when  $I(f) < \infty$ , from Borel-Cantelli Lemma we have that with probability 1,  $\rho_{t_{n-1}} \leq f(t_n)$  for all  $n$ 's, except for a finite number of them. Now, from a monotonicity argument for  $f$ , if  $t \in [t_n, t_{n-1}]$ , we have that:  $\rho_{t_{n-1}} \leq f(t_n) \leq f(t)$  for every  $t$  sufficiently small. It follows now that  $\lim_{t \rightarrow 0} (\rho_t/f(t)) \leq 1$  a.s. Finally, we remark that as  $I(f) < \infty$ , we also have that  $I(\varepsilon f) < \infty$ , for arbitrarily small  $\varepsilon > 0$  and follows that  $\rho_t/f(t) \rightarrow 0$  a.s.

The proof of the second statement follows from the same kind of arguments. Indeed, using Lemma 2 from Doney and Maller [DoM02a] and the fact that  $\rho$  is symmetric, there exists a positive constant  $c_2$  such that for every  $x > 0, t > 0$ ,

$$P\left(\sup_{0 \leq u \leq t} \rho_u \leq x\right) \leq \frac{c_2}{t h(x)}. \quad (63)$$



Hence:

$$\begin{aligned} \sum_{n=1}^{\infty} P(\rho_{t_{n-1}} \leq f(t_{n-1})) &\leq \sum_{n=1}^{\infty} P\left(\sup_{0 \leq u \leq t_{n-1}} \rho_u \leq f(t_{n-1})\right) \\ &\leq \sum_{n=1}^{\infty} \frac{c_2}{t_{n-1} h(f(t_{n-1}))}. \end{aligned}$$

Thus, for  $I(f) = \infty$  (or equivalently  $\sum h(f(t_{n-1})) = \infty$ ), Borel-Cantelli Lemma yields that for every  $n$ , a.s.  $\rho_{t_{n-1}} > f(t_{n-1})$  infinitely often, which finishes the proof.  $\blacksquare$

**Remark 5.3.** For other kinds of LIL for Lévy processes for small times e.g. of the Chung type, see [ADS10] and the references therein.

**Theorem 5.4. (LIL for the angular part of planar Lévy processes for small times)**

For any non-decreasing function  $f > 0$ ,

$$\limsup_{t \rightarrow 0} \frac{\theta_t}{f(t)} = \begin{cases} 0; \\ \infty \end{cases} \text{ a.s.} \Leftrightarrow \int_1^{\infty} (f(t))^{-\alpha} dt \begin{cases} < \infty; \\ = \infty. \end{cases} \quad (64)$$

**Proof of Theorem 5.4:**

We use the skew-product representation (13) together with (46), which essentially writes:  $t^{-1}H(t) \xrightarrow[t \rightarrow 0]{a.s.} 1$ . Thus, for every  $\varepsilon, \delta > 0$ , there exists  $t_0 > 0$  such that:

$$P\left(\frac{H(t)}{t} \leq 1 + \varepsilon\right) \geq 1 - \delta, \text{ for } t \leq t_0. \quad (65)$$

We define now the setting:

$$\mathcal{K} \equiv \mathcal{K}(\omega) \equiv \left\{ \omega : \frac{H(t)}{t} \leq 1 + \varepsilon \right\}, \text{ thus: } \bar{\mathcal{K}} \equiv \bar{\mathcal{K}}(\omega) \equiv \left\{ \omega : \frac{H(t)}{t} \geq 1 + \varepsilon \right\},$$

hence, there exists  $t_0 > 0$  such that: for every  $t \leq t_0$ ,

$$P(\mathcal{K}) \geq 1 - \delta \text{ and } P(\bar{\mathcal{K}}) \leq \delta.$$

Hence, choosing  $\delta > 0$  small enough, it suffices to restrict our study in the set  $\mathcal{K}$  and it follows that:

$$\begin{aligned} P\left(\sup_{0 \leq u \leq t} \theta_u > x\right) &= P\left(\sup_{0 \leq u \leq t} \rho_{H(u)} > x\right) = P\left(\left\{\sup_{0 \leq u \leq t} \rho_{H(u)} > x\right\} \cap \mathcal{K}\right) \\ &\leq P\left(\sup_{0 \leq u \leq t} \rho_{u(1+\varepsilon)} > x\right) \end{aligned}$$

Changing now the variables  $\tilde{u} = u(1 + \varepsilon)$ , and invoking (62), there exists another positive constant  $c_3$  such that, for every  $x > 0$  and  $t > 0$ :

$$P\left(\sup_{0 \leq u \leq t} \theta_u > x\right) \leq P\left(\sup_{0 \leq \tilde{u} \leq t(1+\varepsilon)} \rho_{\tilde{u}} > x\right) \leq c_3 t(1 + \varepsilon) \frac{U(x)}{x^2}. \quad (66)$$

Mimicking now the proof of Theorem 5.1, we obtain the first statement.

For the second statement, we use the settings:

$$\mathcal{K}' \equiv \mathcal{K}'(\omega) \equiv \left\{ \omega : \frac{H(t)}{t} \geq 1 - \varepsilon \right\}, \quad \text{thus : } \overline{\mathcal{K}}' \equiv \overline{\mathcal{K}}'(\omega) \equiv \left\{ \omega : \frac{H(t)}{t} \leq 1 - \varepsilon \right\}.$$

Hence, for every  $\varepsilon, \delta > 0$ , there exists  $t_0 > 0$  such that: for every  $t \leq t_0$ ,

$$P(\mathcal{K}') \geq 1 - \delta \quad \text{and} \quad P(\overline{\mathcal{K}}') \leq \delta.$$

As before, we choose  $\delta > 0$  small enough and we restrict our study in the set  $\mathcal{K}'$ . The proof finishes by repeating the arguments of the proof of Theorem 5.1.  $\blacksquare$

## 6 The planar Brownian motion case

Before starting, we remark that the notations used in this Section are independent from the ones used in the text up to now.

In this Section, we state and give a new proof of the analogue of Theorem 4.4 for the planar Brownian motion case, which is equivalent to a result obtained in [VaY12]. For this purpose, and in order to avoid complexity, we will use the same notation as in the "stable" case. Hence, for a planar BM  $Z$  starting from a point different  $z_0$  from 0 (without loss of generality, let  $z_0 = 1$ ) and with  $\theta = (\theta_t, t \geq 0)$  denoting now the (well defined - see eg. [ItMK65]) continuous winding process, we have the skew product representation (see e.g. [ReY99]):

$$\log |Z_t| + i\theta_t \equiv \int_0^t \frac{dZ_s}{Z_s} = (\beta_u + i\gamma_u) \Big|_{u=H_t} = \int_0^t \frac{ds}{|Z_s|^2}, \quad (67)$$

where  $(\beta_u + i\gamma_u, u \geq 0)$  is another planar Brownian motion starting from  $\log 1 + i0 = 0$ . The Bessel clock  $H$  plays a key role in many aspects of the study of the winding number process  $(\theta_t, t \geq 0)$  (see e.g. [Yor80]). We shall also make use of the inverse of  $H$ , which is given by:

$$H_u^{-1} = \inf\{t \geq 0 : H(t) > u\} = \int_0^u ds \exp(2\beta_s) = A_u. \quad (68)$$

Rewriting (67) as:

$$\log |Z_t| = \beta_{H_t}; \quad \theta_t = \gamma_{H_t}, \quad (69)$$

we easily obtain that the two  $\sigma$ -fields  $\sigma\{|Z_t|, t \geq 0\}$  and  $\sigma\{\beta_u, u \geq 0\}$  are identical, whereas  $(\gamma_u, u \geq 0)$  is independent from  $(|Z_t|, t \geq 0)$ , a fact that is in contrast to what happens in the "stable" case.

**Theorem 6.1.** *The family of processes*

$$(c^{-1/2}\theta_{ct}, t \geq 0)$$

*converges in distribution, as  $c \rightarrow 0$ , to a 1-dimensional Brownian motion  $(\gamma_t, t \geq 0)$ .*

**Proof of Theorem 6.1:**

We split the proof in two parts:

i) First, repeating the arguments in the proof of Theorem 4.2 with  $\alpha = 2$ , we obtain:

$$\left( \frac{H(ux)}{u}, x \geq 0 \right) \xrightarrow[u \rightarrow 0]{a.s.} (x, x \geq 0). \quad (70)$$

which also implies the weak convergence:

$$\left( \frac{H(ux)}{u}, x \geq 0 \right) \xrightarrow[u \rightarrow 0]{(d)} (x, x \geq 0). \quad (71)$$

ii) Using the skew product representation (69) and the scaling property of BM together with (70), we have that for every  $s > 0$ :

$$t^{-1/2} \theta_{st} = t^{-1/2} \gamma_{H(st)} \stackrel{(law)}{=} \sqrt{\frac{H(st)}{t}} \gamma_1 \xrightarrow[t \rightarrow 0]{a.s.} \sqrt{s} \gamma_1 \stackrel{(law)}{=} \gamma_s, \quad (72)$$

which finishes the proof.

We remark that for part (ii) of the proof, we could also invoke Whitt's Theorem 3.1 concerning the composition function [Whi80], however, the independence in the planar Brownian motion case simplifies the proof. ■

From Theorem 6.1 now, with  $T_c^{|\theta|} \equiv \inf\{t : |\theta_t| = c\}$  and  $T_c^{|\gamma|} \equiv \inf\{t : |\gamma_t| = c\}$ , ( $c > 0$ ), we deduce for the exit time from a cone of planar BM (this result has already been obtained in [VaY12], where one can also find several variants):

**Corollary 6.2.** *The following convergence in law holds:*

$$\left( \frac{1}{c^2} T_{cx}^{|\theta|}, x \geq 0 \right) \xrightarrow[c \rightarrow 0]{(law)} (T_x^{|\gamma|}, x \geq 0). \quad (73)$$

**Remark 6.3.** *We highlight the different behaviour of the clock  $H$  for  $t \rightarrow 0$  and for  $t \rightarrow \infty$  (for the second see e.g. [PiY84], followed by [PiY86, LGY86, LGY87], a result which is equivalent to Spitzer's Theorem [Spi58] stated in (1) ), that is:*

$$\frac{H(t)}{t} \xrightarrow[t \rightarrow 0]{a.s.} 1, \quad (74)$$

$$\frac{4H(t)}{(\log t)^2} \xrightarrow[t \rightarrow \infty]{(law)} T_1 \equiv \inf\{t : \beta_t = 1\}, \quad (75)$$

where the latter follows essentially from the classical Laplace argument:  $\|\cdot\|_p \xrightarrow{p \rightarrow \infty} \|\cdot\|_\infty$ . We also remark that, from Remark 4.3, the behaviour of the clock for  $t \rightarrow 0$  is a.s. the same for Brownian motion and for stable processes, whereas it is different for  $t \rightarrow \infty$ . In particular, for  $t \rightarrow \infty$ , compare (49) to (75).

**Acknowledgements**

The author S. Vakeroudis is very grateful to Professor M. Yor for the financial support during his stay at the University of Manchester as a Post Doc fellow invited by Professor R.A. Doney.

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