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# Optimal observation of the one-dimensional wave equation

Yannick Privat\*      Emmanuel Trélat†      Enrique Zuazua‡§

## Abstract

In this paper, we consider the homogeneous one-dimensional wave equation on  $[0, \pi]$  with Dirichlet boundary conditions, and observe its solutions on a subset  $\omega$  of  $[0, \pi]$ . Let  $L \in (0, 1)$ . We investigate the problem of maximizing the observability constant, or its asymptotic average in time, over all possible subsets  $\omega$  of  $[0, \pi]$  of Lebesgue measure  $L\pi$ . We solve this problem by means of Fourier series considerations, give the precise optimal value and prove that there does not exist any optimal set except for  $L = 1/2$ . When  $L \neq 1/2$  we prove the existence of solutions of a convexified minimization problem, proving a no gap result. We then provide and solve a modal approximation of this problem, show the oscillatory character of the optimal sets, the so called spillover phenomenon, which explains the lack of existence of classical solutions for the original problem.

**Keywords:** wave equation, observability, optimal design, harmonic analysis.

**AMS classification:** 93B07, 35L05, 49K20, 42B37.

## 1 Introduction and main results

### 1.1 Observability of the one-dimensional wave equation

Consider the one-dimensional wave equation with Dirichlet boundary conditions

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} &= 0 & (t, x) \in (0, T) \times (0, \pi), \\ y(t, 0) &= y(t, \pi) = 0 & t \in [0, T], \\ y(0, x) &= y^0(x), \quad \frac{\partial y}{\partial t}(0, x) = y^1(x) & x \in [0, \pi], \end{aligned} \tag{1}$$

where  $T$  be an arbitrary positive real number. For all  $y^0(\cdot) \in L^2(0, \pi)$  and  $y^1(\cdot) \in H^{-1}(0, \pi)$ , there exists a unique solution  $y$  of (1), satisfying  $y \in C^0([0, T], L^2(0, \pi)) \times C^1([0, T], H^{-1}(0, \pi))$ . For a given measurable subset  $\omega$  of  $[0, \pi]$  of positive Lebesgue measure, consider the observable variable

$$z(t, x) = \chi_\omega(x)y(t, x), \tag{2}$$

where  $\chi_\omega$  denotes the characteristic function of  $\omega$ . It is well known that the system (1)-(2) is observable whenever  $T$  is large enough (see [18, 19, 25]), that is, the following observability inequality holds: there exists  $C > 0$  such that

$$C\|(y^0, y^1)\|_{L^2(0, \pi) \times H^{-1}(0, \pi)}^2 \leq \int_0^T \int_\omega y(t, x)^2 dx dt, \tag{3}$$

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for every solution of (1) and all  $y^0(\cdot) \in L^2(0, \pi)$  and  $y^1(\cdot) \in H^{-1}(0, \pi)$ . We denote by  $C_T(\chi_\omega)$  the largest observability constant in the above inequality, that is

$$C_T(\chi_\omega) = \inf \left\{ \frac{G_T(\chi_\omega)}{\|(y^0, y^1)\|_{L^2(0, \pi) \times H^{-1}(0, \pi)}^2} \mid (y^0, y^1) \in L^2(0, \pi) \times H^{-1}(0, \pi) \setminus \{(0, 0)\} \right\} \quad (4)$$

where

$$G_T(\chi_\omega) = \int_0^T \int_0^\pi z(t, x)^2 dx dt = \int_0^T \int_\omega y(t, x)^2 dx dt. \quad (5)$$

Note that, for every subset  $\omega$  of  $[0, \pi]$  of positive measure, the observability inequality (3) is satisfied for every  $T \geq 2\pi$ . However  $2\pi$  is not the smallest possible time for a specific choice of  $\omega$ . For example, if  $\omega$  is a subinterval of  $[0, \pi]$ , a propagation argument along the characteristics shows that the smallest such time is  $2 \text{diam}(\omega)$ .

In this article we are interested in the problem of maximizing the observability features of (1)-(2), in a sense to be made precise, over all possible measurable subsets of  $[0, \pi]$  of given measure. The measure of the subsets has of course to be fixed, otherwise the best possible observation consists of taking  $\omega = [0, \pi]$ . Hence, throughout the article, we consider an arbitrary real number  $L \in (0, 1)$ , and consider the class of all measurable subsets  $\omega$  of  $[0, \pi]$  of Lebesgue measure  $|\omega| = L\pi$ . We can now be more precise. We investigate the problem of maximizing either the observability constant  $C_T(\omega)$ , for  $T > 0$  fixed, or its asymptotic average in time

$$\lim_{T \rightarrow +\infty} \frac{C_T(\chi_\omega)}{T},$$

over all measurable subsets  $\omega$  of  $[0, \pi]$  of Lebesgue measure  $|\omega| = L\pi$ , and of determining the optimal set  $\omega$  whenever it exists. Using Fourier expansions, this question is settled more precisely in Section 1.2.

The article is structured as follows. Section 1.2 is devoted to writing the above problem in terms of Fourier series and providing a more explicit criterion adapted to our analysis. The main results are gathered in Section 1.3. In particular, we define and completely solve a convexified version of this problem, underline a generic non existence result with respect to the values of the measure constraint  $L$ , and then investigate a truncated shape optimization problem. The proofs are gathered in Sections 2 and 3. Further comments are provided in Section 4.

## 1.2 Fourier expansion representation of the functional $G_T$

In this section, using series expansions in a Hilbertian basis of  $L^2(0, \pi)$ , our objective is to write the functional  $G_T$  defined by (5) in a more suitable way for our mathematical analysis. For all initial data  $(y^0, y^1) \in L^2(0, \pi) \times H^{-1}(0, \pi)$ , the solution  $y \in C^0(0, T; L^2(0, \pi)) \cap C^1(0, T; H^{-1}(0, \pi))$  of (1) can be expanded as

$$y(t, x) = \sum_{j=1}^{+\infty} (a_j \cos(jt) + b_j \sin(jt)) \sin(jx), \quad (6)$$

where the sequences  $(a_j)_{j \in \mathbb{N}^*}$  and  $(b_j)_{j \in \mathbb{N}^*}$  belong to  $\ell^2(\mathbb{R})$  and are determined in function of the initial data  $(y^0, y^1)$  by

$$a_j = \frac{2}{\pi} \int_0^\pi y^0(x) \sin(jx) dx, \quad b_j = \frac{2}{j\pi} \int_0^\pi y^1(x) \sin(jx) dx, \quad (7)$$

for every  $j \in \mathbb{N}^*$ . By the way, note that

$$\|(y^0, y^1)\|_{L^2 \times H^{-1}}^2 = \frac{\pi}{2} \sum_{j=1}^{+\infty} (a_j^2 + b_j^2). \quad (8)$$

Let  $\omega$  be an arbitrary measurable subset of  $[0, \pi]$ . Plugging (6) into (5) leads to

$$\begin{aligned} G_T(\chi_\omega) &= \int_0^T \int_\omega \left( \sum_{j=1}^{+\infty} (a_j \cos(jt) + b_j \sin(jt)) \sin(jx) \right)^2 dx dt \\ &= \sum_{i,j=1}^{+\infty} \alpha_{ij} \int_\omega \sin(ix) \sin(jx) dx \end{aligned} \quad (9)$$

where

$$\alpha_{ij} = \int_0^T (a_i \cos(it) + b_i \sin(it))(a_j \cos(jt) + b_j \sin(jt)) dt, \quad (10)$$

for all  $\alpha_{ij}$ ,  $(i, j) \in (\mathbb{N}^*)^2$ . More precisely, we have

$$\begin{aligned} \alpha_{ij} &= a_i a_j \left( \frac{\sin(i+j)T}{2(i+j)} + \frac{\sin(i-j)T}{2(i-j)} \right) + a_i b_j \left( \frac{1 - \cos(i+j)T}{2(i+j)} - \frac{1 - \cos(i-j)T}{2(i-j)} \right) \\ &\quad + a_j b_i \left( \frac{1 - \cos(i+j)T}{2(i+j)} + \frac{1 - \cos(i-j)T}{2(i-j)} \right) + b_i b_j \left( -\frac{\sin(i+j)T}{2(i+j)} + \frac{\sin(i-j)T}{2(i-j)} \right) \end{aligned} \quad (11)$$

whenever  $i \neq j$ , and

$$\alpha_{jj} = a_j^2 \left( \frac{T}{2} + \frac{\sin 2jT}{4j} \right) + a_j b_j \left( \frac{1 - \cos 2jT}{2j} \right) + b_j^2 \left( \frac{T}{2} - \frac{\sin 2jT}{4j} \right) \quad (12)$$

whenever  $i = j$ .

**Remark 1.** The case where the time  $T$  is a multiple of  $2\pi$  is particular because in that case all nondiagonal terms are zero. Indeed, if  $T = 2p\pi$  with  $p \in \mathbb{N}^*$ , then  $\alpha_{ij} = 0$  whenever  $i \neq j$ , and

$$\alpha_{jj} = p\pi(a_j^2 + b_j^2), \quad (13)$$

for all  $(i, j) \in (\mathbb{N}^*)^2$ , and therefore

$$G_{2p\pi}(\chi_\omega) = p\pi \sum_{j=1}^{+\infty} (a_j^2 + b_j^2) \int_\omega \sin^2(jx) dx. \quad (14)$$

Hence in that case the functional  $G_{2p\pi}$  does not involve any crossed terms.

For general values of  $T > 0$ , note that there holds obviously

$$G_{2\pi \lfloor \frac{T}{2\pi} \rfloor}(\chi_\omega) \leq G_T(\chi_\omega) \leq G_{2\pi(\lfloor \frac{T}{2\pi} \rfloor + 1)}(\chi_\omega),$$

where the bracket notation stands for the integer floor. Then it follows from Remark 1 that

$$\pi \left\lfloor \frac{T}{2\pi} \right\rfloor \sum_{j=1}^{+\infty} (a_j^2 + b_j^2) \int_\omega \sin^2(jx) dx \leq G_T(\chi_\omega) \leq \pi \left( \left\lfloor \frac{T}{2\pi} \right\rfloor + 1 \right) \sum_{j=1}^{+\infty} (a_j^2 + b_j^2) \int_\omega \sin^2(jx) dx.$$

Using (8), it follows immediately that

$$2 \left\lfloor \frac{T}{2\pi} \right\rfloor J(\chi_\omega) \leq C_T(\chi_\omega) \leq 2 \left( \left\lfloor \frac{T}{2\pi} \right\rfloor + 1 \right) J(\chi_\omega), \quad (15)$$

where

$$J(\chi_\omega) = \inf \left\{ \sum_{j=1}^{+\infty} (a_j^2 + b_j^2) \int_{\omega} \sin^2(jx) dx \mid (a_j)_{j \geq 1}, (b_j)_{j \geq 1} \in \ell^2(\mathbb{R}), \sum_{j=1}^{+\infty} (a_j^2 + b_j^2) = 1 \right\}.$$

Clearly, there holds

$$J(\chi_\omega) = \inf_{j \in \mathbb{N}^*} \int_{\omega} \sin^2(jx) dx. \quad (16)$$

Moreover, dividing (15) by  $T$  and making  $T$  tend to  $+\infty$ , one gets the following result.

**Lemma 1.** *For every  $\chi_\omega \in \mathcal{U}_L$ , one has*

$$\lim_{T \rightarrow +\infty} \frac{C_T(\chi_\omega)}{T} = \frac{1}{\pi} J(\chi_\omega).$$

Furthermore, in the case where  $T = 2p\pi$  with  $p \in \mathbb{N}^*$ , investigated in Remark 1, using (8) and (14) one gets that  $C_{2p\pi}(\chi_\omega) = 2pJ(\chi_\omega)$ .

Therefore, the problem of maximizing  $C_T(\chi_\omega)$  in the case where  $T$  is a multiple of  $2\pi$  and the problem of maximizing  $\lim_{T \rightarrow +\infty} \frac{C_T(\chi_\omega)}{T}$  over all subsets  $\omega$  of  $[0, \pi]$  of measure  $L\pi$  is equivalent to the problem of maximizing the functional  $J$  defined by (16) over this class of sets. In the sequel we will actually focus on this problem:

*Maximize*

$$J(\chi_\omega) = \inf_{j \in \mathbb{N}^*} \int_{\omega} \sin^2(jx) dx \quad (17)$$

*over all possible subsets  $\omega$  of  $[0, \pi]$  of Lebesgue measure  $L\pi$ .*

**Remark 2.** It is not difficult to prove that

$$0 < \frac{L\pi - \sin(L\pi)}{2} \leq J(\chi_\omega) \leq \frac{L\pi + \sin(L\pi)}{2},$$

for every  $L \in (0, 1)$  and every subset  $\omega$  of Lebesgue measure  $L\pi$ .

**Remark 3.** According to the above considerations, in the case where  $T = 2p\pi$  with  $p \in \mathbb{N}^*$ , the maximal value of the observability constant over all subsets of  $[0, \pi]$  of measure  $L\pi$  is

$$2p \sup_{\substack{\omega \subset [0, \pi] \\ |\omega| = L\pi}} \inf_{j \in \mathbb{N}^*} \int_{\omega} \sin^2(jx) dx.$$

**Remark 4.** We stress that the optimization problem (17), consisting of maximizing the asymptotic observability constant, does not depend on the initial data. If we fix some initial data  $(y^0, y^1) \in L^2(0, \pi) \times H^{-1}(0, \pi)$  and some observability time  $T \geq 2\pi$ , we can investigate the problem of maximizing the functional  $G_T(\chi_\omega)$  defined by (5) over all possible subsets  $\omega$  of  $[0, \pi]$  of measure

equal to  $L\pi$ . According to the expression (9), there exists at least an optimal set  $\omega$  which can be characterized in terms of a level set of the function

$$\varphi(x) = \sum_{i,j=1}^{+\infty} \alpha_{ij} \sin(ix) \sin(jx),$$

and of course  $\omega$  depends on the initial data  $(y^0, y^1)$ . It is interesting to investigate the regularity of this kind of optimal set in function of the regularity of the initial data. This study is made in details in [24]. Nevertheless this problem is of little practical interest since it depends on the initial data.

Another interesting problem that can be investigated is the one of maximizing the  $L^2$ -norm of the worst observation among a given finite set  $\mathcal{C}$  of initial data, in other words, maximize

$$\inf \left\{ \frac{G_T(\chi_\omega)}{\|(y^0, y^1)\|_{L^2 \times H^{-1}}^2} \mid (y^0, y^1) \in \mathcal{C} \right\} \quad (18)$$

over all possible subsets  $\omega$  of  $[0, \pi]$  of Lebesgue measure  $L\pi$ . Note that the problem (24) investigated further is of this kind.

### 1.3 Main results

We define  $\mathcal{U}_L$  by

$$\mathcal{U}_L = \{\chi_\omega \mid \omega \text{ is a measurable subset of } [0, \pi] \text{ of measure } |\omega| = L\pi\}. \quad (19)$$

**Theorem 1.** *For every  $L \in (0, 1)$ , there holds*

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \int_{\omega} \sin^2(jx) dx = \frac{L\pi}{2}, \quad (20)$$

and the supremum is reached if and only if  $L = 1/2$ . Moreover, if  $L = 1/2$  then the problem (17) has an infinite number of solutions, consisting of all measurable subsets  $\omega \subset [0, \pi]$  of measure  $\pi/2$  such that  $\omega$  and its symmetric set  $\omega' = \pi - \omega$  are disjoint and complementary in  $[0, \pi]$ .

We have the following corollary for the observability maximization problems.

**Corollary 1.** *There holds*

$$\sup_{\omega \in \mathcal{U}_L} \lim_{T \rightarrow +\infty} \frac{C_T(\chi_\omega)}{T} = \frac{L}{2},$$

and for every  $p \in \mathbb{N}^*$ ,

$$\sup_{\omega \in \mathcal{U}_L} C_{2p\pi}(\chi_\omega) = pL\pi = \frac{LT}{2}.$$

Moreover, an optimal set exists if and only if  $L = 1/2$ , and in that case all optimal sets are given by Theorem 1.

**Remark 5.** The case where  $T$  is not an integer multiple of  $2\pi$  is open, and is commented in Section 4.3 hereafter.

**Remark 6.** It follows from this result that if  $L \neq 1/2$  then the optimization problem (17) does not have any solution. We mention that this result generalizes [8, Lemma 3.1] where the non-existence was proved within the class of subsets of  $[0, \pi]$  of measure  $L\pi$  and that are unions of a finite number of subintervals; their result does however not cover the case of more general subsets of  $[0, \pi]$ , for instance unions of an infinite number of subintervals (like Cantor sets), and does not make precise the optimal value (20).

To overcome this non-existence result, it is usual in shape optimization problems to consider convexified formulations permitting to derive existence results (see e.g. [2]). Here, the convex closure of  $\mathcal{U}_L$  for the weak star topology of  $L^\infty$  is the set

$$\bar{\mathcal{U}}_L = \{a \in L^\infty([0, \pi], [0, 1]) \mid \int_0^\pi a(x) dx = L\pi\}. \quad (21)$$

Note that such a convexification was used in [20, 21] for getting an existence result of an optimal control domain of a string. Replacing  $\chi_\omega \in \mathcal{U}_L$  with  $a \in \bar{\mathcal{U}}_L$ , we define a convexified formulation of the second problem (17) by

$$\sup_{a \in \bar{\mathcal{U}}_L} \inf_{j \in \mathbb{N}^*} \int_0^\pi a(x) \sin^2(jx) dx. \quad (22)$$

We have the following result.

**Proposition 1.** *The convexified problem (22) has an infinite number of solutions, and*

$$\sup_{a \in \bar{\mathcal{U}}_L} \inf_{j \in \mathbb{N}^*} \int_0^\pi a(x) \sin^2(jx) dx = \frac{L\pi}{2}.$$

Moreover, all solutions of (22) are given by the functions of  $\bar{\mathcal{U}}_L$  whose Fourier expansion series is of the form

$$a(x) = L + \sum_{j=1}^{+\infty} (a_j \cos(2jx) + b_j \sin(2jx)),$$

with coefficients  $a_j \leq 0$ .

**Remark 7.** Using Theorem 1, there is no gap between the problem (17) and its convexified formulation (22). It is however worth noticing that this issue cannot be treated with standard limit arguments, since the functional defined by

$$J(a) = \inf_{j \in \mathbb{N}^*} \int_0^\pi a(x) \sin^2(jx) dx$$

is not lower semi-continuous for the weak star topology of  $L^\infty$  (it is however upper semi-continuous for that topology, as an infimum of linear functions). Indeed, consider the sequence of subsets  $\omega_N$  of  $[0, \pi]$  of measure  $L\pi$  defined by

$$\omega_N = \bigcup_{k=1}^N \left[ \frac{k\pi}{N+1} - \frac{L\pi}{2N}, \frac{k\pi}{N+1} + \frac{L\pi}{2N} \right],$$

for every  $N \in \mathbb{N}^*$ . Clearly, the sequence of functions  $\chi_{\omega_N}$  converges to the constant function  $a(\cdot) = L$  for the weak star topology of  $L^\infty$ , but nevertheless, an easy computation shows that

$$\begin{aligned} \int_{\omega_N} \sin^2(jx) dx &= \frac{L\pi}{2} - \frac{1}{2j} \sin\left(\frac{jL\pi}{N}\right) \sum_{k=1}^N \cos\left(\frac{2jk\pi}{N+1}\right) \\ &= \begin{cases} \frac{L\pi}{2} - \frac{N}{2j} \sin\left(\frac{jL\pi}{N}\right) & \text{if } j = 0 \pmod{N+1}, \\ \frac{L\pi}{2} + \frac{1}{2j} \sin\left(\frac{jL\pi}{N}\right) & \text{otherwise,} \end{cases} \end{aligned}$$

and hence, considering  $N$  large enough and  $j = N + 1$ , we get

$$\limsup_{N \rightarrow +\infty} J(\chi_{\omega_N}) = \limsup_{N \rightarrow +\infty} \inf_{j \in \mathbb{N}^*} \int_{\omega_N} \sin^2(jx) dx \leq \frac{L\pi}{2} - \frac{\sin(L\pi)}{2} < \frac{L\pi}{2} = J(L).$$

Due to this lack of semi-continuity, a gap could have been expected in the convexification procedure, in the sense that it could have been expected that

$$\sup_{\chi_\omega \in \mathcal{U}_L} J(\chi_\omega) < \sup_{a \in \bar{\mathcal{U}}_L} J(a) = \frac{L\pi}{2}.$$

The results above show that it is not the case, and that however the (nonconvexified) second problem (17) does not have any solution (except for  $L = 1/2$ ).

As a byproduct, it is interesting to note that we obtain the following corollary in harmonic analysis, which is new up to our knowledge.

**Corollary 2.** *Denote by  $\mathcal{F}$  the set of the functions  $f \in L^2(0, \pi)$  whose Fourier series expansion is of the form*

$$f(x) = L + \sum_{j=1}^{+\infty} (a_j \cos(2jx) + b_j \sin(2jx)), \quad (23)$$

with  $a_j \leq 0$  for every  $j \in \mathbb{N}^*$ . There holds

$$d(\mathcal{F}, \mathcal{U}_L) = 0,$$

where  $d(\mathcal{F}, \mathcal{U}_L)$  denotes the distance of the set  $\mathcal{U}_L$  to the set  $\mathcal{F}$  in  $L^2$  topology. If  $L \in (0, 1) \setminus \{1/2\}$  then  $\mathcal{F} \cap \mathcal{U}_L = \emptyset$ , and if  $L = 1/2$  then  $\mathcal{F} \cap \mathcal{U}_L$  consists of the characteristic functions of all measurable subsets  $\omega \subset [0, \pi]$  of measure  $\pi/2$  such that  $\omega$  and its symmetric set  $\omega' = \pi - \omega$  are disjoint and complementary in  $[0, \pi]$ .

Based on the fact that, for  $L \neq 1/2$ , the optimization problem (17) does not have any solution, and based on the observation that it is more realistic from an engineering point of view to take into consideration only a finite number of modes, it is natural to consider as in [9] a truncated version of (17) involving only the first  $N$  modes, for a given  $N \in \mathbb{N}^*$ , and to investigate the optimization problem

$$\sup_{\substack{\omega \subset [0, \pi] \\ |\omega| = L\pi}} \min_{1 \leq j \leq N} \int_0^\pi \chi_\omega(x) \sin^2(jx) dx. \quad (24)$$

We have the following result.

**Theorem 2.** *For every  $N \in \mathbb{N}^*$ , the problem (24) has a unique<sup>1</sup> solution  $\chi_{\omega^N}$ , where  $\omega^N$  is a subset of  $[0, \pi]$  of measure  $L\pi$  that is the union of at most  $N$  intervals and is symmetric with respect to  $\pi/2$ . Moreover there exists  $L_N \in (0, 1]$  such that, for every  $L \in (0, L_N]$ , the optimal domain  $\omega^N$  satisfies*

$$\int_{\omega^N} \sin^2 x dx = \int_{\omega^N} \sin^2(2x) dx = \dots = \int_{\omega^N} \sin^2(Nx) dx. \quad (25)$$

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<sup>1</sup>Here the uniqueness must be understood up to some subset of zero Lebesgue measure. In other words if  $\omega$  is optimal then  $\omega \cup \mathcal{N}$  and  $\omega \setminus \mathcal{N}$  where  $\mathcal{N}$  denotes any subset of zero measure is also a solution.



**Remark 8.** This result is the same as the one in [9, Theorem 3.2]. Note however that the proof in [9] is not completely correct. In order to prove (25), the authors of [9] use a first-order expansion of  $\int_{\omega} \sin^2 x dx$ , with  $\omega = \bigcup_{k=1}^N [\frac{k}{N+1} - \frac{l}{2N}, \frac{k}{N+1} + \frac{l}{2N}]$ , and find  $l + \frac{1}{\pi} \sin(\frac{\pi l}{2N})$  instead of  $l + \frac{1}{\pi} \sin(\frac{\pi l}{N})$ . The serious consequence of this misprint is that the asymptotic expansion of the quantity denoted  $\alpha$  in their paper is wrong and their method unfortunately does not permit to establish (25). In this article we provide a proof of that result, which is however unexpectedly difficult and technical.

**Remark 9.** Note that the necessity of  $L_N$  being small enough for (25) to hold has been illustrated on numerical simulations in [9]. The problem of fully understanding the minimization problem (24) and its limit behavior as  $N$  tends to infinity for all possible values of  $L$  is widely open.

**Remark 10.** As observed in [9], the equalities (25) are the main ingredient to characterize the optimal set  $\omega^N$ . It follows from (25) (but it also follows from our proof, in particular from Lemma 4) that the optimal domain  $\omega^N$  concentrates around the nodes  $\frac{k\pi}{N+1}$ ,  $k = 1, \dots, N$ . This causes the well-known spillover phenomenon, according to which the optimal domain  $\omega^N$  solution of (24) with the  $N$  first modes is the worst possible domain for the problem with the  $N + 1$  first modes. Note that this phenomenon is rather a bad news for practical purposes, but from the mathematical point of view this is in accordance with the fact that the problem (17) has no solution whenever  $L \neq 1/2$ . Besides, the optimal solution  $\chi_{\omega^N}$  of (24) converges for the weak star topology of  $L^\infty$  to some function  $a(\cdot) \in L^\infty([0, \pi], [0, 1])$  such that  $\int_0^\pi a(x) dx = L\pi$ . This function  $a(\cdot)$  is actually a solution of the convexified formulation (22) of our problem introduced further.

**Remark 11.** Since the set of optimal solutions (23) of the convexified problem (22) is not enumerable whereas the set of closure points of the sequence  $(\chi_{\omega^N})_{N \in \mathbb{N}^*}$  is enumerable, it follows that not every optimal solution of the convexified problem is a weak limit of a subsequence of  $(\chi_{\omega^N})_{N \in \mathbb{N}^*}$ . We recover the fact mentioned previously that we are not in the conditions of usual  $\Gamma$ -convergence arguments.

## 1.4 State of the art

This kind of shape optimization problem has been widely considered from the application point of view, in particular in engineering problems where the problem of optimal measurement locations for state estimation in linear partial differential equations was much investigated (see e.g. [7, 16, 17] and the many references therein). In these applications the goal is to optimize the number, the place and the type of sensors in order to improve the estimation or more generally some performance index. Practical approaches consist either of solving the above kind of problem with a finite number of possible initial data, or of recasting the optimal sensor location problem for distributed systems as an optimal control problem with an infinite dimensional Riccati equation, having a statistical model interpretation, and then of computing approximations with optimization techniques. However, on the one part, their techniques rely on an exhaustive search over a predefined set of possible candidates and are faced with combinatorial difficulties due to the selection problem and thus with the usual flaws of combinatorial optimization methods. On the other part, in all these references approximations are used to determine the optimal sensor location. The optimal performance and the corresponding sensor or actuator location of the approximating sequence are then expected to converge to the exact optimal performance and location. Among the possible approximation processes, the closest one to our present study consists of considering Fourier expansion representations and of using modal approximation schemes.

However, in these references there is no systematic mathematical study of the optimal design problem. The search of optimal domains relies on finite-dimensional approximations and no convergence analysis is led. By the way we will show in the present article that there may exist serious

problems in the sense that optimal domains may exist for any modal approximation but do not exist for the infinite dimensional model. In other words,  $\Gamma$ -convergence properties may not hold.

There exist only few mathematical results. An important difficulty arising when focusing on an optimal shape problem is the generic non-existence of classical solutions, as explained and surveyed in [1], thus leading to consider convexification procedures. In [6] the authors discuss several possible criteria for optimizing the damping of abstract wave equations in Hilbert spaces, and derive optimality conditions for a certain criterion related to a Lyapunov equation. In [8, 9], the authors consider the problem of determining the best possible shape and position of the damping subdomain of given measure for a one-dimensional wave equation. More precisely, in these two references the authors consider the damped wave equation

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} + 2k\chi_\omega \frac{\partial y}{\partial t} &= 0 & (t, x) \in (0, T) \times (0, \pi), \\ y(t, 0) = y(t, \pi) &= 0 & t \in [0, T], \\ y(0, x) = y^0(x), \quad \frac{\partial y}{\partial t}(0, x) &= y^1(x) & x \in [0, \pi], \end{aligned}$$

where  $k > 0$ , and investigate the problem of determining the best possible subset  $\omega$  of  $[0, \pi]$ , of Lebesgue measure  $L\pi$ , maximizing the decay rate of the total energy of the system. The overdamping phenomenon is underlined in [8] (see also [4]), meaning that if  $k$  is too large then the decay rate tends to zero. According to a result of [4], if the set  $\omega$  has a finite number of connected components and if  $k$  is small enough, then at the first order the decay rate is equivalent to  $k \inf_{j \in \mathbb{N}^*} \int_\omega \sin^2(jx) dx$ . We then recover exactly the problem (17) but with the additional restriction that the subsets  $\omega$  be a finite union of intervals.

Other quite similar questions have been investigated for control issues. In [20, 21] the authors investigate numerically the optimal location of the support of the control determined by the Hilbert Uniqueness Method for the 1D wave equation with Dirichlet conditions, using gradient techniques or level sets methods combined with shape and topological derivatives. In [23] this optimal location problem is solved both theoretically and numerically using an approach based on Fourier expansion series like in the present article.

## 2 Proof of Theorem 1

The proof of the result consists of proving that there is no gap between the problem (17) and its convexified version (22).

### 2.1 Proof of Proposition 1

First of all, noting that  $\sin^2(jx) = \frac{1}{2} - \frac{1}{2} \cos(2jx)$ , it follows from Riemann-Lebesgue's lemma that, for every  $a \in \overline{\mathcal{U}}_L$ , the integral  $\int_0^\pi a(x) \sin^2(jx) dx$  tends to  $\frac{L\pi}{2}$  as  $j$  tends to  $+\infty$ . Therefore,

$$\sup_{a \in \overline{\mathcal{U}}_L} \inf_{j \in \mathbb{N}^*} \int_0^\pi a(x) \sin^2(jx) dx \leq \frac{L\pi}{2},$$

and the equality holds for instance with the constant function  $a(x) = L$ . More precisely, the equality holds in the above inequality if and only if  $\int_0^\pi a(x) \cos(2jx) dx \leq 0$  for every  $j \in \mathbb{N}^*$ . The result follows.

### 2.2 The supremum is reached only for $L = 1/2$

Let us prove that the supremum of  $J$  on  $\mathcal{U}_L$  is reached (and then equal to  $\frac{L\pi}{2}$ ) if and only if  $L = 1/2$ . First, if  $L = 1/2$  then it is easy to see that the supremum is reached, and is achieved for

all measurable subsets  $\omega \subset [0, \pi]$  of measure  $\pi/2$  such that  $\omega$  and its symmetric  $\omega' = \pi - \omega$  are disjoint and complementary in  $[0, \pi]$ .

Conversely, assume that the supremum be reached for  $L \neq 1/2$ . Then, as in the proof of Proposition 1, this implies in particular that the Fourier series expansion of  $\chi_\omega$  on  $[0, \pi]$  is of the form

$$\chi_\omega(x) = L + \sum_{j=1}^{+\infty} (a_j \cos(2jx) + b_j \sin(2jx)),$$

with coefficients  $a_j \leq 0$ . The argument is then standard. Let  $\omega' = \pi - \omega$  be the symmetric set of  $\omega$  with respect to  $\pi/2$ . Then, the Fourier series expansion of  $\chi_{\omega'}$  is

$$\chi_{\omega'}(x) = L + \sum_{j=1}^{+\infty} (a_j \cos(2jx) - b_j \sin(2jx)).$$

For every  $x \in [0, \pi]$ , define  $g(x) = L - \frac{1}{2}(\chi_\omega(x) + \chi_{\omega'}(x))$ . The Fourier series expansion of  $g$  is

$$g(x) = - \sum_{j=1}^{+\infty} a_j \cos(2jx),$$

with  $a_j \leq 0$  for every  $j \in \mathbb{N}^*$ . Then the sets  $\omega$  and  $\omega'$  are not disjoint and complementary, and hence  $g$  is discontinuous. It then follows that  $\sum_{j=1}^{+\infty} a_j = -\infty$ . Besides, the sum  $\sum_{j=1}^{+\infty} a_j$  is also the limit of  $\sum_{k=1}^{+\infty} a_k \widehat{\Delta}_n(k)$  as  $n \rightarrow +\infty$ , where  $\widehat{\Delta}_n$  is the Fourier transform of the positive function  $\Delta_n$  whose graph is the triangle joining the points  $(-\frac{1}{n}, 0)$ ,  $(0, 2n)$  and  $(\frac{1}{n}, 0)$  (note that  $\Delta_n$  is an approximation of the Dirac measure, with area equal to 1). But this raises a contradiction with the following identity obtained by applying Plancherel's Theorem:

$$\int_0^\pi g(t) \Delta_n(t) dt = \sum_{k=1}^{+\infty} a_k \widehat{\Delta}_n(k).$$

### 2.3 Proof of the no-gap statement

We now assume that  $L \neq 1/2$  and prove the no-gap statement, that is,  $\sup_{\chi_\omega \in \mathcal{U}_L} J(\chi_\omega) = \frac{L\pi}{2}$ . In what follows, for every subset  $\omega$  of  $[0, \pi]$ , set

$$I_j(\omega) = \int_\omega \sin^2(jx) dx \quad \text{and} \quad \bar{I}_j(\omega) = \int_\omega \cos(2jx) dx$$

for every  $j \in \mathbb{N}^*$ , so that there holds

$$I_j(\omega) = \frac{L\pi}{2} - \frac{1}{2} \bar{I}_j(\omega). \tag{26}$$

Our proof below is inspired from the proof of [8, Theorem 3.2] in which the idea of domain perturbation by making some holes in the subsets under consideration was introduced. It consists of getting a refined estimation of the evolution of  $I_j(\omega)$  when perturbing  $\omega$ , precise enough to consider the infimum of these quantities. In what follows, for every open subset  $\omega$  we denote by  $\#\omega$  the number of its connected components. For  $p \in \mathbb{N}^*$ , define

$$J_p = \sup \{ J(\omega) \mid \omega \text{ open subset of } [0, \pi], |\omega| = \pi L, \#\omega \leq p \}.$$

where  $J(\omega) = \inf_{j \in \mathbb{N}^*} \int_\omega \sin^2(jx) dx$ . Since  $J$  is upper semi-continuous and since the set of open subsets of  $[0, \pi]$  of measure  $L\pi$  whose number of connected components is lower than or equal to

$p$  can be written as a compact set, it is obvious that  $J_p$  is attained at some open subset  $\omega_p$ . Using the arguments of Section 2.2, it is clear that  $J_p < \frac{L\pi}{2}$  for every  $p \in \mathbb{N}^*$ . Denote by  $\overline{\omega}_p$  the closure of  $\omega_p$ , and by  $\omega_p^c$  the complement of  $\omega_p$  in  $[0, \pi]$ .

Consider subdivisions of  $\overline{\omega}_p$  and  $\omega_p^c$ , to be chosen later:

$$\overline{\omega}_p = \bigcup_{i=1}^K [a_i, b_i] \quad \text{and} \quad \omega_p^c = \bigcup_{i=1}^M [c_i, d_i]. \quad (27)$$

Using the Taylor Lagrange inequality, one gets

$$\left| \int_{\omega_p} \cos(2jx) dx - \sum_{i=1}^K (b_i - a_i) \cos\left(2j \frac{a_i + b_i}{2}\right) \right| \leq 4j^2 \sum_{i=1}^K \frac{(b_i - a_i)^3}{24},$$

and

$$\left| \int_{\omega_p^c} \cos(2jx) dx - \sum_{i=1}^M (d_i - c_i) \cos\left(2j \frac{c_i + d_i}{2}\right) \right| \leq 4j^2 \sum_{i=1}^M \frac{(d_i - c_i)^3}{24},$$

for every  $j \in \mathbb{N}^*$ . Note that, since  $\omega_p^c$  is the complement of  $\omega_p$ , there holds

$$\int_{\omega_p^c} \cos(2jx) dx = - \int_{\omega_p} \cos(2jx) dx, \quad (28)$$

for every  $j \in \mathbb{N}^*$ . Set  $h_i = (1-L)(b_i - a_i)\pi$ ,  $\ell_i = L\pi(d_i - c_i)$ ,  $x_i = \frac{a_i + b_i}{2}$ , and  $y_i = \frac{c_i + d_i}{2}$ . Set also  $h = \max(h_1, \dots, h_K, \ell_1, \dots, \ell_M)$ . Then, using the fact that  $\sum_{i=1}^K (b_i - a_i) = L\pi$ , one gets

$$\begin{aligned} (1-L)\pi \bar{I}_j(\omega_p) &= \sum_{i=1}^K h_i \cos(2jx_i) + O(j^2 h^2), \\ L\pi \bar{I}_j(\omega_p) &= - \sum_{i=1}^M \ell_i \cos(2jy_i) + O(j^2 h^2), \end{aligned} \quad (29)$$

for every  $j \in \mathbb{N}^*$ . Now, for  $\varepsilon \in (0, 1)$ , define the perturbation  $\omega^\varepsilon$  of  $\omega_p$  by

$$\omega^\varepsilon = \left( \omega_p \setminus \bigcup_{i=1}^K \left( x_i - \frac{\varepsilon}{2} h_i, x_i + \frac{\varepsilon}{2} h_i \right) \right) \cup \bigcup_{i=1}^M \left[ y_i - \frac{\varepsilon}{2} \ell_i, y_i + \frac{\varepsilon}{2} \ell_i \right].$$

Note that, by construction,  $\#\omega^\varepsilon = p + K + M$  and that  $|\omega^\varepsilon| = L\pi - \varepsilon \sum_{i=1}^K h_i + \varepsilon \sum_{i=1}^M \ell_i = L\pi$ . Moreover, one has

$$\bar{I}_j(\omega^\varepsilon) = \int_{\omega^\varepsilon} \cos(2jx) dx = \bar{I}_j(\omega_p) - \sum_{i=1}^K \int_{x_i - \frac{\varepsilon}{2} h_i}^{x_i + \frac{\varepsilon}{2} h_i} \cos(2jx) dx + \sum_{i=1}^M \int_{y_i - \frac{\varepsilon}{2} \ell_i}^{y_i + \frac{\varepsilon}{2} \ell_i} \cos(2jx) dx.$$

Therefore,

$$\begin{aligned} \frac{d}{d\varepsilon} \bar{I}_j(\omega^\varepsilon) &= -\frac{1}{2} \sum_{i=1}^K h_i \left( \cos\left(2j\left(x_i + \frac{\varepsilon}{2} h_i\right)\right) + \cos\left(2j\left(x_i - \frac{\varepsilon}{2} h_i\right)\right) \right) \\ &\quad + \frac{1}{2} \sum_{i=1}^M \ell_i \left( \cos\left(2j\left(y_i + \frac{\varepsilon}{2} \ell_i\right)\right) + \cos\left(2j\left(y_i - \frac{\varepsilon}{2} \ell_i\right)\right) \right), \end{aligned}$$

and in particular, using (28) and (29),

$$\begin{aligned}
\frac{d}{d\varepsilon} \bar{I}_j(\omega^\varepsilon)|_{\varepsilon=0} &= -\sum_{i=1}^K h_i \cos(2jx_i) + \sum_{i=1}^M \ell_i \cos(2jy_i) \\
&= -(1-L)\pi \bar{I}_j(\omega_p) - L\pi \bar{I}_j(\omega_p) + O(j^2 h^2) \\
&= -\bar{I}_j(\omega_p) + O(j^2 h^2)
\end{aligned} \tag{30}$$

and furthermore,

$$\begin{aligned}
\frac{d^2}{d\varepsilon^2} \bar{I}_j(\omega^\varepsilon) &= \frac{1}{2} \sum_{i=1}^K j h_i^2 \left( \sin\left(2j\left(x_i + \frac{\varepsilon}{2} h_i\right)\right) - \sin\left(2j\left(x_i - \frac{\varepsilon}{2} h_i\right)\right) \right) \\
&\quad - \frac{1}{2} \sum_{i=1}^M j \ell_i^2 \left( \sin\left(2j\left(y_i + \frac{\varepsilon}{2} \ell_i\right)\right) + \sin\left(2j\left(y_i - \frac{\varepsilon}{2} \ell_i\right)\right) \right),
\end{aligned}$$

hence,

$$\left| \frac{d^2}{d\varepsilon^2} \bar{I}_j(\omega^\varepsilon) \right| \leq 2L(1-L)\pi^2 j h. \tag{31}$$

From (26), (30) and (31), we infer that

$$\begin{aligned}
I_j(\omega^\varepsilon) &= I_j(\omega_p) + \varepsilon \frac{d}{d\varepsilon} I_j(\omega^\varepsilon)|_{\varepsilon=0} + \int_0^\varepsilon (\varepsilon - s) \frac{d^2}{ds^2} I_j(\omega^s) ds \\
&= I_j(\omega_p) + \varepsilon \left( \frac{L\pi}{2} - I_j(\omega_p) \right) + O(\varepsilon j^2 h^2) + O(\varepsilon^2 j h),
\end{aligned}$$

for every  $j \in \mathbb{N}^*$  and every  $\varepsilon \in (0, 1)$ . Since, by definition,  $J_p = J(\omega_p) = \inf_{j \in \mathbb{N}^*} I_j(\omega_p)$ , we get the inequality

$$I_j(\omega^\varepsilon) \geq J_p + \varepsilon \left( \frac{L\pi}{2} - J_p \right) + O(\varepsilon j^2 h^2) + O(\varepsilon^2 j h), \tag{32}$$

for every  $j \in \mathbb{N}^*$  and every  $\varepsilon \in (0, 1)$ .

Besides, it follows from Riemann-Lebesgue's Lemma that  $I_j(\omega_p)$  tends to  $\frac{L\pi}{2}$  as  $j$  tends to  $+\infty$ . Therefore, there exists an integer  $j_0$  such that

$$I_j(\omega_p) \geq \frac{L\pi}{2} - \frac{1}{4} \left( \frac{L\pi}{2} - J_p \right),$$

for every  $j \geq j_0$ . Since there holds

$$\begin{aligned}
|I_j(\omega^\varepsilon) - I_j(\omega_p)| &= \left| \int_0^\pi (\chi_{\omega^\varepsilon}(x) - \chi_{\omega_p}(x)) \cos(2jx) dx \right| \\
&\leq \varepsilon \left( \sum_{i=1}^K h_i + \sum_{i=1}^M \ell_i \right) = 2\varepsilon L(1-L)\pi^2,
\end{aligned}$$

for every  $\varepsilon \in (0, 1)$ , we infer that

$$\forall \varepsilon \in \left( 0, \min \left( 1, \frac{1}{2L(1-L)\pi^2} \left( \frac{L\pi}{2} - J_p \right) \right) \right) \quad \forall j \geq j_0 \quad I_j(\omega^\varepsilon) \geq \frac{L\pi}{2} - \frac{1}{2} \left( \frac{L\pi}{2} - J_p \right). \tag{33}$$

Note that this inequality does not depend on the choice of the subdivisions (27), and in particular does not depend on the number of connected components of  $\omega^\varepsilon$ .

We now choose the subdivisions (27) fine enough so that  $hj_0 \leq \frac{1}{2} \left( \frac{L\pi}{2} - J_p \right)$ . Then one has

$$\varepsilon^2 hj \leq \frac{\varepsilon}{2} \left( \frac{L\pi}{2} - J_p \right),$$

for every  $\varepsilon \in (0, 1)$  and every  $j \in \{1, \dots, j_0\}$ , and it follows from (32) that

$$I_j(\omega^\varepsilon) \geq J_p + \frac{\varepsilon}{2} \left( \frac{L\pi}{2} - J_p \right), \quad (34)$$

for every  $j \in \{1, \dots, j_0\}$ .

Set  $C_L = \frac{1}{16L(1-L)\pi^2}$ , and choose

$$\varepsilon = \min \left( 1, C_L \left( \frac{L\pi}{2} - J_p \right) \right).$$

For this specific choice of  $\varepsilon$ , we obtain, using (33) and (34),

$$I_j(\omega^\varepsilon) \geq J_p + \frac{1}{2} \min \left( 1, C_L \left( \frac{L\pi}{2} - J_p \right) \right) \left( \frac{L\pi}{2} - J_p \right),$$

for every  $j \in \mathbb{N}^*$ , and therefore, passing to the infimum over  $j$ ,

$$J(\omega^\varepsilon) \geq J_p + \frac{1}{2} \min \left( 1, C_L \left( \frac{L\pi}{2} - J_p \right) \right) \left( \frac{L\pi}{2} - J_p \right).$$

We have thus constructed a new open set  $\omega^\varepsilon$  having  $q = p + K + M$  connected components. Set  $\varphi(p) = q = p + K + M$  and set  $\omega_q = \omega^\varepsilon$ . Define the sequence  $u_k = J(\varphi^k(p))$ ; thus, one has

$$u_{k+1} \geq u_k + \frac{1}{2} \min \left( 1, C_L \left( \frac{L\pi}{2} - J_p \right) \right) \left( \frac{L\pi}{2} - J_p \right),$$

for every  $k$ . It follows easily from that inequality that this sequence is increasing, bounded above by  $\frac{L\pi}{2}$ , and converges to  $\frac{L\pi}{2}$ . This finishes the proof of Theorem 1

## 2.4 Proof of Corollary 2

For every subset  $\omega$  of  $[0, \pi]$  of measure  $L\pi$ , consider the Fourier series expansion of its characteristic function

$$\chi_\omega(x) = L + \sum_{j=1}^{+\infty} (a_j \cos(2jx) + b_j \sin(2jx)).$$

Since  $\mathcal{F}$  is a closed convex of  $L^2(0, \pi)$ , the projection of  $\chi_\omega$  on  $\mathcal{F}$  is

$$P_{\mathcal{F}}\chi_\omega(x) = L + \sum_{j=1}^{+\infty} (\min(a_j, 0) \cos(2jx) + b_j \sin(2jx)),$$

and

$$d(\chi_\omega, \mathcal{F})^2 = \sum_{j=1}^{+\infty} \max(a_j, 0)^2.$$

From Theorem 1,  $\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \int_0^\pi \chi_\omega(x) \sin^2(jx) dx = \frac{L\pi}{2}$ . Then, let  $(\omega_n)_{n \in \mathbb{N}}$  be a sequence of  $\mathcal{U}_L$  such that  $\inf_{j \in \mathbb{N}^*} \int_{\omega_n} \sin^2(jx) dx$  tends to  $\frac{L\pi}{2}$  as  $n$  tends to  $+\infty$ . Since  $\sin^2(jx) = \frac{1}{2}(1 - \cos(2jx))$ , this implies that  $\sup_{j \in \mathbb{N}^*} \int_{\omega_n} \cos(2jx) dx$  tends to 0 as  $n$  tends to  $+\infty$ . Denoting by  $a_j^n$  and  $b_j^n$  the Fourier coefficients of  $\chi_{\omega_n}$ , it follows in particular that  $\sup_{j \in \mathbb{N}^*} a_j^n$  tends to 0 as  $n$  tends to  $+\infty$ . Combined with the fact that the sequence  $(a_j^n)_{j \in \mathbb{N}^*}$  is of summable squares, this implies that  $d(\chi_{\omega_n}, \mathcal{F})$  tends to 0 as  $n$  tends to  $+\infty$ . The rest of the statement is obvious to prove.

### 3 Proof of Theorem 2

First of all, writing  $\sin^2(jx) = \frac{1}{2} - \frac{1}{2} \cos(2jx)$ , one has

$$\sup_{\chi_\omega \in \mathcal{U}_L} \min_{1 \leq j \leq N} \int_\omega \sin^2(jx) dx = \frac{L\pi}{2} - \frac{1}{2} \inf_{\chi_\omega \in \mathcal{U}_L} \max_{1 \leq j \leq N} \int_\omega \cos(2jx) dx,$$

and hence in what follows we are concerned with the problem

$$P_N(L) = \inf_{\chi_\omega \in \mathcal{U}_L} \max_{1 \leq j \leq N} \int_\omega \cos(2jx) dx. \quad (35)$$

Our objective is to prove that this problem has a unique solution  $\omega^N(L)$ , satisfying the properties stated in the theorem, and that

$$\int_{\omega^N(L)} \cos(2x) dx = \int_{\omega^N(L)} \cos(4x) dx = \cdots = \int_{\omega^N(L)} \cos(2Nx) dx, \quad (36)$$

provided that  $L$  is small enough. We proceed in three steps. The two first steps are straightforward and can already be found as well in [9] (although the method is slightly different). The third step, consisting of proving (36), is much more technical. As already mentioned in Remark 8, the equality (36) is claimed in [9] but the proof is erroneous and cannot be corrected. We propose here another approach for the proof, which is unexpectedly difficult. Note that the derivation of the equality (36) is an important issue for the optimal set  $\omega^N(L)$  because it permits to put in evidence the spillover phenomenon discussed in Remark 10.

#### 3.1 Convexification procedure

As in the proof of Theorem 1, the convexification consists here of replacing  $\mathcal{U}_L$  with  $\bar{\mathcal{U}}_L$ . The convexified formulation of the problem (35) is

$$\inf_{a \in \bar{\mathcal{U}}_L} \max_{1 \leq j \leq N} \int_0^\pi a(x) \cos(2jx) dx. \quad (37)$$

Note that, by compactness, it is obvious that there exists an optimum, and thus the infimum is attained for some  $a \in \bar{\mathcal{U}}_L$ .

#### 3.2 Interpretation in terms of an optimal control problem

We change of point of view and consider the functions  $a(\cdot)$  of  $\bar{\mathcal{U}}_L$  as controls. Consider the control system

$$\begin{aligned} y'(x) &= a(x), \\ y'_j(x) &= a(x) \cos(2jx), \quad j \in \{1, \dots, N\}, \\ z'(x) &= 0, \end{aligned} \quad (38)$$

for almost every  $x \in [0, \pi]$ , with initial conditions

$$y(0) = 0, \quad y_j(0) = 0, \quad j \in \{1, \dots, N\}. \quad (39)$$

The convexified problem (37) is then equivalent to the optimal control problem of determining a control  $a \in \bar{\mathcal{U}}_L$  steering the control system (38) from the initial conditions (39) to the final condition

$$y(\pi) = L\pi, \quad (40)$$

and minimizing the quantity  $z(\pi)$ , with the additional final conditions

$$z(\pi) \geq y_j(\pi), \quad j \in \{1, \dots, N\}. \quad (41)$$

Expressed in such a way, this problem is a usual optimal control problem in finite dimension. From Section 3.1,  $a$  is an optimal control solution of that problem. Note anyway that the existence of an optimal control follows immediately from standard results in optimal control theory. According to the Pontryagin Maximum Principle (see [22]), if  $a$  is optimal then there exist real numbers<sup>2</sup>  $(p_y, p_1, \dots, p_N, p_z, p^0) \neq (0, \dots, 0)$ , with  $p^0 \leq 0$ , such that

$$a(x) = \begin{cases} 1 & \text{if } \varphi^N(x) > 0, \\ 0 & \text{if } \varphi^N(x) < 0, \end{cases} \quad (42)$$

for almost every  $x \in [0, \pi]$ , where the so-called switching function  $\varphi^N$  is defined by

$$\varphi^N(x) = p_y + \sum_{j=1}^N p_j \cos(2jx). \quad (43)$$

Note that, in the application of the Pontryagin Maximum Principle, it could happen that the control  $a$  be undetermined whenever the switching function  $\varphi^N$  were to vanish on some subset of positive measure (singular case; see [27]). This does not happen here since  $\varphi^N$  is a finite trigonometric sum. In particular, this implies that the optimal control  $a$  is the characteristic function of a measurable subset  $\omega^N(L)$  of  $[0, \pi]$  of measure  $L\pi$ . Note that the minimum of  $\varphi^N$  on  $[0, \pi]$  is reached at 0 and  $\pi$ , hence from (42) the optimal set  $\omega^N$  does not contain 0 and  $\pi$ .

To prove uniqueness, assume the existence of two distinct minimizers  $\chi_{\omega_1}$  and  $\chi_{\omega_2}$  (it indeed follows from the discussion above that every minimizer is an extremal point of  $\mathcal{U}_L$ , or in other words, is the characteristic function of some subset of  $[0, \pi]$ ). As a maximum of linear functionals, the functional  $a \mapsto \max_{1 \leq j \leq N} \int_0^\pi a(x) \sin^2(jx) dx$  is convex on  $\bar{\mathcal{U}}_L$ , and it follows that for every  $t \in (0, 1)$  the function  $t\chi_{\omega_1} + (1-t)\chi_{\omega_2}$  is also a solution of the problem (37), which is in contradiction with the fact that any solution of this problem is extremal.

Finally, the fact that  $\omega^N(L)$  has at most  $N$  connected components follows from the facts that the elements of  $\partial\omega^N(L)$  are the solutions of  $\varphi^N(x) = 0$  and that  $\varphi^N$  can be written as

$$\varphi^N(x) = p_y + \sum_{j=1}^N p_j T_{2j}(\cos x),$$

where  $T_{2j}$  denotes the  $2j$ -th Chebychev polynomial of the first kind. The degree of the polynomial  $\varphi^N(\arccos X)$  (in the argument  $X$ ) is at most  $2N$ , whence the result.

<sup>2</sup>Note that, since the dynamics of (38) do not depend on the state, it follows that the adjoint states of the Pontryagin Maximum Principle are constant.



### 3.3 Equality of the criteria for $L$ small enough

This is the most technical and difficult part of the proof of the theorem. Let us first show how the minimum and the maximum can be inverted in (35). In order to apply a minimax theorem, it is required to convexify the criteria and the constraints under consideration. In accordance with the convexified formulation (37), we define the convex set

$$K_N^L = \left\{ \left( \int_0^\pi a(x) \cos(2x) dx, \dots, \int_0^\pi a(x) \cos(2Nx) dx \right) \mid a \in \overline{\mathcal{U}}_L \right\}.$$

From the previous step, the optimization problem (35) coincides with its convexified formulation (37), and thus can be written as

$$P_N(L) = \min_{(x_1, \dots, x_N) \in K_N^L} \max_{1 \leq j \leq N} x_j.$$

Denote by  $\mathcal{A}_N$  the simplex of  $\mathbb{R}^N$ , defined by

$$\mathcal{A}_N = \left\{ \alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}_+^N \mid \sum_{j=1}^N \alpha_j = 1 \right\}$$

Then, for every  $(x_1, \dots, x_N) \in K_N^L$  there holds obviously

$$\max_{1 \leq j \leq N} x_j = \max_{\alpha \in \mathcal{A}_N} \sum_{j=1}^N \alpha_j x_j,$$

and therefore,

$$P_N(L) = \min_{(x_1, \dots, x_N) \in K_N^L} \max_{\alpha \in \mathcal{A}_N} \sum_{j=1}^N \alpha_j x_j.$$

The set  $\mathcal{A}_N$  is compact and convex, the set  $K_N^L$  is convex, the function  $\alpha \mapsto \sum_{j=1}^N \alpha_j x_j$  is convex on  $\mathcal{A}_N$  and lower-semicontinuous for every  $(x_1, \dots, x_N) \in K_N^L$ , and the function  $x \mapsto \sum_{j=1}^N \alpha_j x_j$  is concave on  $K_N^L$  for every  $\alpha \in \mathcal{A}_N$ . Then, according to Sion's Minimax Theorem (see e.g. [14, 26]) we can invert the minimum and the maximum, and we get

$$P_N(L) = \max_{\alpha \in \mathcal{A}_N} \min_{(x_1, \dots, x_N) \in K_N^L} \sum_{j=1}^N \alpha_j x_j = \max_{\alpha \in \mathcal{A}_N} \min_{a \in \overline{\mathcal{U}}_L} \int_0^\pi a(x) \sum_{j=1}^N \alpha_j \cos(2jx) dx.$$

For every  $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathcal{A}_N$  and every  $x \in [0, \pi]$ , define

$$F_N(\alpha, x) = \sum_{j=1}^N \alpha_j \cos(2jx),$$

**Lemma 2.** *For every  $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathcal{A}_N$ , the problem*

$$\min_{a \in \overline{\mathcal{U}}_L} \int_0^\pi a(x) F_N(\alpha, x) dx \tag{44}$$

*has a unique solution  $a \in \overline{\mathcal{U}}_L$ . Moreover,  $a$  belongs actually to  $\mathcal{U}_L$  and thus is the characteristic function of a subset of  $[0, \pi]$  of measure  $L\pi$ .*

*Proof.* The proof is done as previously by interpreting the optimization problem (44) as the optimal control problem of determining an optimal control  $a \in \overline{\mathcal{U}}_L$  steering the two-dimensional control system

$$\begin{aligned} y'(x) &= a(x), \\ z'(x) &= a(x)F_N(\alpha, x), \end{aligned} \tag{45}$$

from initial conditions  $y(0) = z(0) = 0$  to the final condition  $y(\pi) = L\pi$ , and minimizing  $z(\pi)$ . The application of the Pontryagin Maximum Principle, on which we do not give details, implies immediately that the optimal control is unique and is a characteristic function. The conclusion follows. Note that the optimal set can be characterized in terms of the level sets of the function  $F_N(\alpha, \cdot)$ .  $\square$

It follows that

$$P_N(L) = \max_{\alpha \in \mathcal{A}_N} \min_{\chi_\omega \in \mathcal{U}_L} \int_0^\pi \chi_\omega(x) F_N(\alpha, x) dx, \tag{46}$$

for every  $L \in (0, 1)$ . We denote by  $\alpha^N(L)$  a maximizer in  $\mathcal{A}_N$  of this problem (it is not unique a priori). The minimizer is  $\omega^N(L)$  and is unique.

We define the mapping

$$\begin{aligned} \psi : \mathcal{A}_N \times (0, 1) &\longrightarrow \mathbb{R} \\ (\alpha, L) &\longmapsto \min_{\chi_\omega \in \mathcal{U}_L} \frac{1}{L} \int_0^\pi \chi_\omega(x) F_N(\alpha, x) dx. \end{aligned}$$

It is clear that, for every  $\alpha \in \mathcal{A}_N$ , there exists an optimal set minimizing  $\int_\omega F_N(\alpha, x) dx$  over  $\mathcal{U}_L$ , and moreover this set is unique since the function  $F_N(\alpha, \cdot)$  cannot be piecewise constant. Moreover, the optimal set is obviously characterized as a level set of  $F_N(\alpha, \cdot)$ , and concentrates around the minima of  $F_N(\alpha, \cdot)$  whenever  $L$  tends to 0. The following lemma is then obvious.

**Lemma 3.** *For every  $\alpha \in \mathcal{A}_N$ , there holds*

$$\lim_{L \rightarrow 0} \psi(\alpha, L) = \lim_{L \rightarrow 0} \min_{\chi_\omega \in \mathcal{U}_L} \frac{1}{L} \int_0^\pi \chi_\omega(x) F_N(\alpha, x) dx = \pi \min_{0 \leq x \leq \pi} F_N(\alpha, x).$$

The function  $\psi$  is in such a way extended to a continuous function on  $\mathcal{A}_N \times [0, 1)$ . Now, we claim that

$$\lim_{L \rightarrow 0} \max_{\alpha \in \mathcal{A}_N} \psi(\alpha, L) = \max_{\alpha \in \mathcal{A}_N} \lim_{L \rightarrow 0} \psi(\alpha, L). \tag{47}$$

Indeed, note first that, for every  $L \in (0, 1)$ , one has  $\psi(\alpha^N(L), L) = \max_{\alpha \in \mathcal{A}_N} \psi(\alpha, L)$ . For  $L = 0$ , let  $\bar{\alpha} \in \mathcal{A}_N$  be such that  $\psi(\bar{\alpha}, 0) = \max_{\alpha \in \mathcal{A}_N} \psi(\alpha, 0)$ . Note that  $\alpha^N(L)$  does not necessarily converge to  $\bar{\alpha}$ , however we will prove that  $\psi(\alpha^N(L), L)$  tends to  $\psi(\bar{\alpha}, 0)$  as  $L$  tends to 0. Let  $\alpha^* \in \mathcal{A}_N$  be a closure point of the family  $(\alpha^N(L))_{L \in (0, 1)}$  as  $L$  tends to 0. Then, by definition of the maximum, one has  $\psi(\alpha^*, 0) \leq \psi(\bar{\alpha}, 0)$ . On the other hand, since  $\psi$  is continuous,  $\psi(\bar{\alpha}, L)$  tends to  $\psi(\bar{\alpha}, 0)$  as  $L$  tends to 0. By definition of the maximum,  $\psi(\bar{\alpha}, L) \leq \psi(\alpha^N(L), L)$  for every  $L \in (0, 1)$ . Therefore, passing to the limit, one gets  $\psi(\bar{\alpha}, 0) \leq \psi(\alpha^*, 0)$ . It follows that  $\psi(\bar{\alpha}, 0) = \psi(\alpha^*, 0)$ . We have thus proved that the (bounded) family  $(\psi(\alpha^N(L), L))_{L \in (0, 1)}$  of real numbers has a unique closure point at  $L = 0$ , which is  $\psi(\bar{\alpha}, 0)$ . The formula (47) follows.

Now, combining Lemma 3 and (47), we infer that

$$\lim_{L \rightarrow 0} \max_{\alpha \in \mathcal{A}_N} \min_{\chi_\omega \in \mathcal{U}_L} \frac{1}{L} \int_0^\pi \chi_\omega(x) F_N(\alpha, x) dx = \pi \max_{\alpha \in \mathcal{A}_N} \min_{0 \leq x \leq \pi} F_N(\alpha, x).$$

Hence, we have put in evidence a new auxiliary optimization problem,

$$P_N = \max_{\alpha \in \mathcal{A}_N} \min_{0 \leq x \leq \pi} F_N(\alpha, x), \quad (48)$$

which is the limit problem (at  $L = 0$ ) of the problems  $\frac{P_N(L)}{L}$ , i.e.,

$$\lim_{L \rightarrow 0} \frac{P_N(L)}{L} = P_N.$$

The next lemma provides the solution of the limit problem  $P_N$ .

**Lemma 4.** *The problem (48) has a unique solution  $\bar{\alpha}^N$  given by*

$$\bar{\alpha}_j^N = \frac{2(N+1-j)}{N(N+1)}, \quad j = 1, \dots, N. \quad (49)$$

Moreover,  $P_N = -\frac{1}{N}$  and  $F_N(\bar{\alpha}^N, \cdot)$  attains its minimum  $N$  times on  $[0, \pi]$ , at the points

$$\bar{x}_k^N = \frac{k\pi}{N+1}, \quad k = 1, \dots, N. \quad (50)$$

The proof of this lemma, which is long and technical, is postponed to Section 3.4. It can be noticed that Sion's Minimax Theorem cannot be applied to (48), and indeed it is wrong that the minimum and the maximum can be inverted.

Let us end the proof of the theorem, with the use of this lemma. First, note that, using the same arguments as previously and Sion's Minimax Theorem, there holds

$$\begin{aligned} \frac{P_N(L)}{L} &= \max_{\alpha \in \mathcal{A}_N} \min_{\chi_\omega \in \mathcal{U}_L} \frac{1}{L} \int_0^\pi \chi_\omega(x) F_N(\alpha, x) dx \\ &= \min_{\chi_\omega \in \mathcal{U}_L} \max_{\alpha \in \mathcal{A}_N} \frac{1}{L} \int_0^\pi \chi_\omega(x) F_N(\alpha, x) dx \\ &= \max_{\alpha \in \mathcal{A}_N} \frac{1}{L} \int_{\omega^N(L)} \sum_{j=1}^N \alpha_j \cos(2jx) dx \end{aligned} \quad (51)$$

The reasoning made to prove (47) shows that every closure point  $\alpha^* \in \mathcal{A}_N$  of the family  $(\alpha^N(L))_{L \in (0,1)}$  as  $L$  tends to 0 satisfies  $\psi(\alpha^*, 0) = \max_{\alpha \in \mathcal{A}_N} \psi(\alpha, 0)$ . Therefore, by continuity of  $\psi$  at  $L = 0$ , there exists a sequence  $(L_k)_{k \in \mathbb{N}}$  converging to 0 such that  $\alpha^N(L_k)$  tends to  $\alpha^*$  and  $\psi(\alpha^N(L_k), L_k)$  tends to  $\psi(\alpha^*, 0)$  as  $k$  tends to  $+\infty$ .

By Lemma 4, the solution  $\bar{\alpha}^N$  of the limit problem is unique, therefore every closure point of the family  $(\alpha^N(L))_{L \in (0,1)}$  as  $L$  tends to 0 is equal to  $\bar{\alpha}^N$ , and hence  $\alpha^N(L)$  converges to  $\bar{\alpha}^N$  as  $L$  tends to 0. Since  $\bar{\alpha}^N$  clearly belongs to the interior of  $\mathcal{A}_N$ , it follows that there exists  $L_N \in (0, 1]$  such that, for every  $L \in (0, L_N]$ ,  $\alpha^N(L)$  belongs to the interior of  $\mathcal{A}_N$  as well. Using (51), for every  $L$  fixed  $\alpha^N(L)$  is a solution of the maximization problem

$$\max_{\alpha \in \mathcal{A}_N} \sum_{j=1}^N \alpha_j \int_{\omega^N(L)} \cos(2jx) dx,$$

and since the optimal  $\alpha^N(L)$  belongs to the interior of the simplex, it follows from the Lagrange multipliers rule that all integrals  $\int_{\omega^N(L)} \cos(2jx) dx$  are equal. This ends the proof of Theorem 2.

### 3.4 Proof of Lemma 4

For every  $\alpha \in \mathcal{A}_N$ , we define

$$F^N(\alpha) = \min_{0 \leq x \leq \pi} F_N(\alpha, x) = \min_{0 \leq x \leq \pi} \sum_{j=1}^N \alpha_j \cos(2jx),$$

so that  $P_N = \max_{\alpha \in \mathcal{A}_N} F^N(\alpha)$ . The function  $F^N$  is continuous and concave on the convex set  $\mathcal{A}_N$  as a minimum of linear functions. Let  $\underline{\alpha} \in \mathcal{A}_N$  be a maximizer of  $F^N$ . Note that the functions  $x \mapsto \cos(2jx)$  are symmetric with respect to the axis  $x = \frac{\pi}{2}$ , and hence the minima of  $F_N(\underline{\alpha}, \cdot)$  on  $[0, \pi]$  share this property as well. We denote by  $\underline{x}_1 < \dots < \underline{x}_k$  the points of  $[0, \pi/2]$  at which  $F_N(\underline{\alpha}, \cdot)$  attains its minimum.

Note that the number  $k$  of such minima can be determined using Chebychev polynomials. For every  $j \in \{1, \dots, N\}$  we denote respectively by  $T_j$  and  $U_j$  the  $j^{\text{th}}$  Chebychev polynomial of the first and second kind, i.e. the polynomials satisfying

$$T_j(\cos \theta) = \cos(j\theta) \quad \text{and} \quad \sin \theta U_j(\cos \theta) = \sin(j\theta),$$

for every  $\theta \in \mathbb{R}$ . Setting  $y = \cos x$  and  $q(y) = F_N(\underline{\alpha}, x) = \sum_{j=1}^N \alpha_j T_{2j}(y)$ . The degree of  $q$  is less than or equal to  $2N$ , and there holds  $q(1) = 1$  and  $|q(y)| \leq 1$  on  $[-1, 1]$ . Distinguishing between the cases  $N$  odd or even, it is easy to see that there are at most  $p = \lceil \frac{N+1}{2} \rceil$  local minimizers of  $q$  on  $[-1, 1]$ , and therefore  $k \leq \lceil \frac{N+1}{2} \rceil$ . At the end of the proof, we will see that actually  $k = \lceil \frac{N+1}{2} \rceil$ .

Let us provide a first-order characterization of the optimal solution  $\underline{\alpha}$ . According to Danskin's Theorem (see [5]),  $F^N$  is differentiable in all directions, and

$$dF^N(\alpha) \cdot \beta = \min_{x \in S(\alpha)} \frac{\partial F_N}{\partial \alpha}(\alpha, x) \cdot \beta,$$

where  $S(\alpha)$  denotes the set of all points  $x \in [0, \pi]$  at which the minimum of  $x \mapsto F_N(\alpha, x)$  is reached. Since  $\underline{\alpha}$  realizes the maximum of  $J$  on  $\mathcal{A}_N$ , one has

$$dF^N(\underline{\alpha}) \cdot \beta = \min_{x \in S(\underline{\alpha})} \frac{\partial F_N}{\partial \alpha}(\underline{\alpha}, x) \cdot \beta \leq 0,$$

for every  $\beta \in \mathcal{O}_{\text{ad}}^{\underline{\alpha}}$ , where  $\mathcal{O}_{\text{ad}}^{\underline{\alpha}}$  is the set of admissible perturbations  $\beta = (\beta_1, \dots, \beta_N) \in \mathbb{R}^N$  that satisfy

$$\sum_{j=1}^N \beta_j = 0, \quad \text{and} \quad \beta_j \geq 0 \quad \text{whenever} \quad \underline{\alpha}_j = 0.$$

Since  $F^N$  is concave (but not strictly concave) on the convex set  $\mathcal{A}_N$ , these necessary first order optimality conditions are sufficient as well. Therefore, a necessary and sufficient condition for  $\underline{\alpha}$  to be a maximizer of  $F^N$  on  $\mathcal{A}_N$  is that

$$\min_{1 \leq \ell \leq k} \sum_{j=1}^N \beta_j \cos(2j\underline{x}_\ell) \leq 0, \tag{52}$$

for every  $\beta \in \mathcal{O}_{\text{ad}}^{\underline{\alpha}}$ .

In order to prove the lemma, we next prove that  $\bar{\alpha}^N$  defined by (49) satisfies the necessary and sufficient condition (52). Let us first prove that the minimizers of  $F_{\bar{\alpha}^N}$  are given by (50). Using the identities

$$\sum_{j=1}^N \cos(2jx) = \frac{\cos((N+1)x) \sin(Nx)}{\sin x},$$

$$\sum_{j=1}^N j \cos(2jx) = \frac{N+1}{2} \cos((N+1)x) U_N(\cos x) - \frac{\sin x}{2} \sin((N+1)x) U'_N(\cos x),$$

and

$$\sin^2 x U'_N(\cos x) = \cos x U_N(\cos x) - NT_N(\cos x),$$

for every  $x \in [0, \pi]$ , one computes

$$\begin{aligned} F_{\bar{\alpha}^N}(x) &= \frac{1}{N} \cos((N+1)x) U_N(\cos x) + \frac{1}{N(N+1)} \cos x U_{N+1}(\cos x) U_N(\cos x) \\ &\quad - \frac{1}{N+1} U_{N+1}(\cos x) - \frac{1}{N+1} U_{N+1}(\cos x) T_N(\cos x) \\ &= -\frac{1}{N} + \frac{1}{N(N+1)} (U_{N+1}(\cos x))^2. \end{aligned}$$

Therefore, the minimizers of  $F_{\bar{\alpha}^N}$  on  $[0, \pi]$  are the solutions of  $\sin((N+1)x) = 0$ , hence (50) follows. Moreover, if we denote as previously by  $k$  the number of minimizers of  $F_{\bar{\alpha}^N}$  on  $[0, \pi/2]$ , one has  $k = \lceil \frac{N+1}{2} \rceil$ . Now, showing that  $\bar{\alpha}^N$  satisfies (52) amounts to checking that

$$\min_{1 \leq \ell \leq k} \langle \xi_\ell, \beta \rangle \leq 0$$

for every  $\beta = (\beta_1, \dots, \beta_N) \in \mathbb{R}^N$  such that  $\beta_1 + \dots + \beta_N = 0$ , where  $\xi_\ell$  is the vector of coordinates

$$\xi_{\ell,j} = \cos\left(\frac{2j\ell\pi}{N+1}\right), \quad j = 1, \dots, N.$$

From Farkas' lemma, this condition is equivalent to the existence of  $(\lambda_0, \lambda_1, \dots, \lambda_k) \in \mathbb{R} \times \mathbb{R}_+^k$  such that

$$\lambda_0 u_N + \sum_{\ell=1}^k \lambda_\ell \xi_\ell = 0, \quad (53)$$

where  $u_N = (1, \dots, 1) \in \mathbb{R}^N$ . Using the fact that

$$\sum_{j=1}^k \cos\left(\frac{2j\ell\pi}{N+1}\right) = -\frac{1}{2}$$

for every  $\ell \in \{1, \dots, k\}$ , it follows that the condition (53) is satisfied. This ends the proof.

## 4 Further comments

### 4.1 Numerical simulations

As an illustration of Theorem 2 and in particular of the spillover phenomenon (see [9]), we provide on Figure 1 some numerical simulations representing the optimal set  $\omega^N$  of the truncated problem (24) in function of  $L$ .

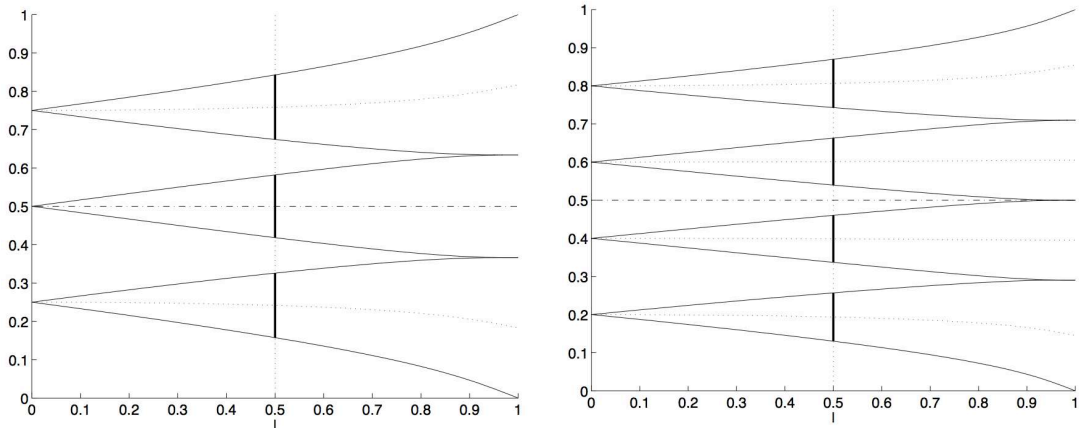


Figure 1: Optimal set  $\omega^N$  in function of  $L$ , for  $N = 3$  (left) and  $N = 4$  (right) (by courtesy of P. Hébrard and A. Henrot)

## 4.2 Further comments on the non existence result of Theorem 1

Having in mind the spillover phenomenon mentioned in Remark 10 and the fact that if  $L \neq 1/2$  then the problem (17) has no solution, we stress that the optimal solution  $\chi_{\omega^N}$  of (24) converges for the weak star topology of  $L^\infty$  to a solution  $a(\cdot) \in \bar{\mathcal{U}}_L$  of the convexified problem (22) (note that all solutions of this problem are determined by Proposition 1), however  $a(\cdot)$  is not a characteristic function. In other words, the sequence of optimization problems (24)  $\Gamma$ -converges to the convexified formulation of the second problem. Although the optimal value of (24) converges to  $L\pi/2$ , there is the spillover phenomenon (see Theorem 2 and Remark 10).

Because of the non existence of solution for this problem, there is a compromise to make between the existence of an optimal set and the optimal value. More precisely, It is claimed in [8] that bounding the number of connected components of the admissible sets permits to get an existence result. It corresponds to adding a bounded variation constraint in the maximization problem. Of course, restricting the set of maximization in such a way makes the shape optimization problem well-posed, but decreases the optimal value of the observability constant.

Another natural idea in view of trying to recover a nice existence result consists of penalizing the functional  $J$  defined by (16), and for instance of maximizing the functional

$$J_\varepsilon(\chi_\omega) = J(\chi_\omega) - \frac{1}{\varepsilon^2} \|a(1-a)\|_{L^2}^2.$$

over  $\mathcal{U}_L$ . The issue is however similar. Indeed, consider a maximizing sequence  $(\chi_{\omega_n})_{n \in \mathbb{N}^*}$  in  $\mathcal{U}_L$ , that converges  $L^\infty$  weak star to the constant function  $a(\cdot) = L$ . Thus, the penalization term vanishes and one see that  $J_\varepsilon(\chi_{\omega_n})$  converges to  $L\pi/2$  whereas  $(\chi_{\omega_n})_{n \in \mathbb{N}^*}$  does not converge to a characteristic function.

## 4.3 Comments on the case where $T$ is not an integer multiple of $2\pi$

We do not know how to solve the problem of maximizing the observability constant whenever  $T$  is not an integer multiple of  $2\pi$ . In this section we provide however two comments showing the difficulty of this problem.

**First comment.** If  $T$  is not an integer multiple of  $2\pi$  then as already mentioned the functional  $G_T(\chi_\omega)$  involves crossed terms (see (9)) that cannot be handled easily. The same kind of difficulty due to crossed terms is encountered in the problem of finding what are the best possible constants in Ingham's inequality (see [10]), according to which, for every real number  $\gamma$  and every  $T > \frac{2\pi}{\gamma}$ , there exist two positive constants  $C_1(T, \gamma)$  and  $C_2(T, \gamma)$  such that for every sequence of real numbers  $(\lambda_n)_{n \in \mathbb{N}^*}$  satisfying

$$\forall n \in \mathbb{N}^* \quad |\lambda_{n+1} - \lambda_n| \geq \gamma,$$

there holds

$$C_1(T, \gamma) \sum_{n \in \mathbb{N}^*} |a_n|^2 \leq \int_0^T \left| \sum_{n \in \mathbb{N}^*} a_n e^{i\lambda_n t} \right|^2 dt \leq C_2(T, \gamma) \sum_{n \in \mathbb{N}^*} |a_n|^2,$$

for every  $(a_n)_{n \in \mathbb{N}^*} \in \ell^2(\mathbb{C})$ . Establishing sharp constants within the framework of Ingham's method has been discussed in a number of works (see e.g. [11, 12, 13, 15]) but the problem of finding the best possible constants is still a well-known open problem.

**Second comment: on the optimality of the constant function  $\bar{a}(\cdot) = L$ .** In Proposition 1, it is stated that the constant function  $\bar{a}(\cdot) = L$  is one of the solutions of the convexified problem (22). Intuitively, this result is not surprising. It can be indeed expected that the best possible domain should be equitably spread over the interval  $[0, \pi]$ , and therefore the convexified solution  $\bar{a}(\cdot) = L$  appears as an intuitive solution.

Recall that (22) is the convexified version of the problem (17), itself being equivalent to the problem of maximizing the observability constant  $C_T(\chi_\omega)$  (defined by (54)) over  $\mathcal{U}_L$ , in the case where  $T$  is an integer multiple of  $2\pi$ . The convexified version was however not defined for general values of  $T$  and we define it now. For every  $a \in \bar{\mathcal{U}}_L$ , we define

$$C_T(a) = \inf \left\{ \frac{G_T(a)}{\|(y^0, y^1)\|_{L^2(0, \pi) \times H^{-1}(0, \pi)}^2} \mid (y^0, y^1) \in L^2(0, \pi) \times H^{-1}(0, \pi) \setminus \{(0, 0)\} \right\} \quad (54)$$

where the functional  $G_T$  initially defined on  $\mathcal{U}_L$  by (5) is naturally extended to  $\bar{\mathcal{U}}_L$  by

$$G_T(a) = \int_0^T \int_0^\pi a(x) y(t, x)^2 dx dt, \quad (55)$$

where  $y$  is the solution of (1). Note that the mapping  $a \in \bar{\mathcal{U}}_L \mapsto C_T(a)$  is upper semi-continuous as an infimum of linear functions that are continuous for the  $L^\infty$  weak star topology. The set  $\bar{\mathcal{U}}_L$  being compact for this topology, the existence of a maximizer follows immediately.

What is proved in Proposition 1 is that the constant function  $\bar{a}(\cdot) = L$  is one of the solutions of the problem of maximizing the functional  $a \mapsto C_T(a)$  over  $\bar{\mathcal{U}}_L$ , whenever  $T$  is an integer multiple of  $2\pi$ , and as said above this is quite intuitive and could be expected. However more surprisingly the constant function  $\bar{a}(\cdot) = L$  is not a solution whenever  $T$  is not an integer multiple of  $\pi$ .

**Proposition 2.** *For every  $L \in (0, 1)$  and every  $T > 0$  such that  $T$  is not an integer multiple of  $\pi$ , the constant function  $\bar{a}(\cdot) = L$  is not a maximizer of the functional  $a \mapsto C_T(a)$  over  $\bar{\mathcal{U}}_L$ .*

*Proof.* First of all, using (6) and (8), one has

$$C_T(a) = \frac{2}{\pi} \inf_{\substack{(a_j)_{j \in \mathbb{N}^*}, (b_j)_{j \in \mathbb{N}^*} \in \ell^2(\mathbb{R}) \\ \sum_{j=1}^{+\infty} (a_j^2 + b_j^2) = 1}} \int_0^T \int_0^\pi a(x) \left( \sum_{j=1}^{+\infty} (a_j \cos(jt) + b_j \sin(jt)) \sin(jx) \right)^2 dx dt,$$

for every  $a \in \bar{\mathcal{U}}_L$ . Setting  $a_j = \rho_j \cos \theta_j$  and  $b_j = \rho_j \sin \theta_j$  for every  $j \in \mathbb{N}^*$ , we get

$$C_T(a) = \frac{2}{\pi} \inf_{\substack{(\rho_j)_{j \in \mathbb{N}^*} \in \ell^2(\mathbb{R}) \\ \sum_{j=1}^{+\infty} \rho_j^2 = 1}} \inf_{(\theta_j)_{j \in \mathbb{N}^*} \in \mathbb{R}^{\mathbb{N}^*}} \int_0^T \int_0^\pi a(x) \left( \sum_{j=1}^{+\infty} \rho_j \cos(jt - \theta_j) \sin(jx) \right)^2 dx dt, \quad (56)$$

for every  $a \in \bar{\mathcal{U}}_L$ . Now if  $\bar{a}$  is the constant function equal to  $L$  on  $[0, \pi]$ , then

$$C_T(\bar{a}) = \frac{2L}{\pi} \inf_{\substack{(\rho_j)_{j \in \mathbb{N}^*} \in \ell^2(\mathbb{R}) \\ \sum_{j=1}^{+\infty} \rho_j^2 = 1}} \inf_{(\theta_j)_{j \in \mathbb{N}^*} \in \mathbb{R}^{\mathbb{N}^*}} \int_0^T \int_0^\pi \left( \sum_{j=1}^{+\infty} \rho_j \cos(jt - \theta_j) \sin(jx) \right)^2 dx dt,$$

and since

$$\begin{aligned} \int_0^T \int_0^\pi \left( \sum_{j=1}^{+\infty} \rho_j \cos(jt - \theta_j) \sin(jx) \right)^2 dx dt &= \frac{\pi}{2} \sum_{j=1}^{+\infty} \rho_j^2 \int_0^T \cos^2(jt - \theta_j) dt \\ &= \frac{\pi}{4} \sum_{j=1}^{+\infty} \rho_j^2 \left( T - \cos(jT - 2\theta_j) \frac{\sin(jT)}{j} \right), \end{aligned}$$

it follows that

$$C_T(\bar{a}) = \frac{L}{2} \inf_{\substack{(\rho_j)_{j \in \mathbb{N}^*} \in \ell^2(\mathbb{R}) \\ \sum_{j=1}^{+\infty} \rho_j^2 = 1}} \inf_{(\theta_j)_{j \in \mathbb{N}^*} \in \mathbb{R}^{\mathbb{N}^*}} \sum_{j=1}^{+\infty} \rho_j^2 \left( T - \cos(jT - 2\theta_j) \frac{\sin(jT)}{j} \right).$$

Since the infimum over the  $\theta_j \in \mathbb{R}$  is reached at  $\theta_j = \frac{1}{2}(jT + \varepsilon\pi)$  with  $\varepsilon = 1$  or  $0$  according to the sign of  $\sin(jT)$ , we get

$$\begin{aligned} C_T(\bar{a}) &= \frac{L}{2} \inf_{\substack{(\rho_j)_{j \in \mathbb{N}^*} \in \ell^2(\mathbb{R}) \\ \sum_{j=1}^{+\infty} \rho_j^2 = 1}} \sum_{j=1}^{+\infty} \rho_j^2 \left( T - \frac{|\sin(jT)|}{j} \right) \\ &= \frac{L}{2} \inf_{j \in \mathbb{N}^*} \left( T - \frac{|\sin(jT)|}{j} \right). \end{aligned}$$

Since  $|\sin(jT)| \leq j |\sin T|$  for every  $j \in \mathbb{N}^*$  and any value of  $T$ , we obtain finally

$$C_T(\bar{a}) = \frac{L}{2} (T - |\sin T|),$$

and moreover the infimum in the definition (56) of  $C_T(a)$  is reached at  $\bar{\rho}_1 = 1$ ,  $\bar{\rho}_j = 0$  for  $j > 1$ ,  $\bar{\theta}_1 = \frac{1}{2}(T + \varepsilon\pi)$  with  $\varepsilon = 1$  or  $0$  according to the sign of  $\sin(T)$ , and  $\bar{\theta}_j$  arbitrary for  $j > 1$ . Note that these are exactly all points at which the infimum is reached, whenever  $T$  is not a multiple integer of  $\pi$ . This fact is important to apply a usual version of Danskin's theorem (see [5]). This classical result can indeed be applied for a minimum over a finite dimensional space, and in our case it is possible to consider an observability constant  $C_{T,N}(a)$  truncated to the  $N$  first modes. In this case, when computing  $C_T(\bar{a})$  the infimum is a minimum and is reached at the same points as above. Thus, the derivative  $dC_{T,N}(\bar{a}).h$  has exactly the same expression than  $dC_T(\bar{a}).h$  below for every  $N \in \mathbb{N}^*$ , and to conclude, it suffices to let  $N$  tend to  $+\infty$  and the result follows. Let



us now provide the details of the computation of the differential of  $C_T$  at  $\bar{a}$  along any admissible direction  $h \in L^\infty(0, \pi)$ . Here  $h$  admissible means that  $\int_0^\pi h(x) dx = 0$ . Using Danskin's argument as discussed above, one has

$$dC_T(\bar{a}).h = \int_0^T \cos^2(t - \bar{\theta}_1) dt \int_0^\pi h(x) \sin^2 x dx = \frac{1}{2}(T - |\sin T|) \int_0^\pi h(x) \sin^2 x dx, \quad (57)$$

for every  $h \in L^\infty(0, \pi)$  such that  $\int_0^\pi h(x) dx = 0$ , and for every  $T$  that is not a multiple of  $\pi$ . Now, since  $\bar{a}$  belongs to the interior of  $\bar{U}_L$ , if  $\bar{a}$  were a maximizer of the functional  $a \mapsto C_T(a)$  over  $\bar{U}_L$  (for such values of  $T$ ) then it would follow the existence of a real number  $\lambda$  such that

$$\int_0^\pi h(x) \sin^2 x dx = \lambda \int_0^\pi h(x) dx,$$

for every  $h \in L^\infty(0, \pi)$ , which is absurd. The result is proved.  $\square$

**Remark 12.** From the point of view of characteristics, this result could actually be expected. Indeed, every point  $x_0 \in (0, \pi)$  generates two characteristics, one going to the left and the other one to the right. Both characteristics intersect at  $x_0$  at times  $2p\pi$ , for every  $p \in \mathbf{N}^*$ . Now, at time  $T = 2\pi + \varepsilon$ , for every  $x_0 \in (\varepsilon, \pi - \varepsilon)$ , rays emanating from  $x_0$  are at a distance  $\varepsilon$  from  $x_0$ , and symmetrically spread on the left and on the right of  $x_0$ . But the situation goes differently whenever  $x_0$  is close to the boundary of  $(0, \pi)$ , because then rays at time  $T$  are not symmetrically spread with respect to  $x_0$ , because of reflections at the boundary. This symmetry breaking intuitively explains the loss of homogeneity of the optimal solution.

## 5 Conclusion and open problems

We have studied the problem of maximizing the observability constant, or its asymptotic average in time, over all possible subsets  $\omega$  of  $[0, \pi]$  of Lebesgue measure  $L\pi$ , for the homogeneous one-dimensional wave equation on  $[0, \pi]$  with Dirichlet boundary conditions. We have obtained a precise optimal value for this asymptotic observability constant, and also for the observability constant whenever the time  $T$  is an integer multiple of  $2\pi$ . We provided and solved a convexified version of the problem and showed that there is no gap between both optimal values. The problem of computing the best observability constant for general values of  $T$  is open and as mentioned in Section 4.3 the problem is difficult and similar to the one of computing the best constants in Ingham's inequality. We defined and solved a truncated version of the initial problem and showed that the optimal sets share a spillover property. We note however that the set of closure points of these finite dimensional approximations is strictly contained in the set of optimal solutions of the convexified problem.

An interesting open problem is to investigate the situation for second-order equations with varying coefficients. Our approach here used the explicit trigonometric form of Fourier series expansions, but the extension to the more general framework of Sturm-Liouville kind equations is not clear.

The generalization to the multi-dimensional case is not easy and requires spectral considerations related to the asymptotic behavior of the energy concentration of eigenfunctions. It will be the subject of a future work (see [24]).

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## References

- [1] G. Allaire, A. Henrot, *On some recent advances in shape optimization*, C. R. Acad. Sci. Paris, t. 329, Série II b (2001), 383–396.
- [2] D. Bucur, G. Buttazzo, *Variational methods in shape optimization problems*, Progress in Non-linear Differential Equations **65**, Birkhäuser Verlag, Basel (2005).
- [3] L. Carleson, *Convergence and growth of partial sums of Fourier series*, Acta Math. **116** (1966), 135–157.
- [4] S. Cox, E. Zuazua, *The rate at which energy decays in a damped string*, Comm. Partial Differential Equations **19** (1994), 213–243.
- [5] J.M. Danskin, *The theory of max min*, Springer, Berlin, (1967).
- [6] F. Fahroo, K. Ito, *Optimum damping design for an abstract wave equation*, New directions in control and automation, I (Limassol, 1995), Kybernetika (Prague) **32** (1996), no. 6, 557–574.
- [7] M.I. Frecker, *Recent advances in optimization of smart structures and actuators*, Journal of Intelligent Material Systems and Structures **14** (2003), 207–216.
- [8] P. Hébrard, A. Henrot, *Optimal shape and position of the actuators for the stabilization of a string*, Syst. Cont. Letters **48** (2003), 199–209.
- [9] P. Hébrard, A. Henrot, *A spillover phenomenon in the optimal location of actuators*, SIAM J. Control Optim. **44** 2005, 349–366.
- [10] A.E. Ingham, *Some trigonometrical inequalities with applications to the theory of series*, Math. Zeitschrift **41** (1936), 367–379.
- [11] S. Jaffard, S. Micu, *Estimates of the constants in generalized Ingham’s inequality and applications to the control of the wave equation*, Asymptot. Anal. **28** (2001), no. 3-4, 181–214.
- [12] S. Jaffard, M. Tucsnak, E. Zuazua, *On a theorem of Ingham*, J. Fourier Anal. Appl. **3** (1997), 577–582.
- [13] J.P. Kahane, *Pseudo-périodicité et séries de Fourier lacunaires*, Ann. Sci. Ecole Norm. Sup. **37** (1962), 93–95.
- [14] H. Komiya, *Elementary proof for Sion’s minimax theorem*, Kodai Math. J. **11** (1988), no. 1, 5–7.
- [15] V. Komornik, P. Loreti *Fourier Series in Control Theory*, Springer-Verlag, New York, 2005.
- [16] C.S. Kubrusly, H. Malebranche, *Sensors and controllers location in distributed systems - a survey*, Automatica **21** (1985), 117–128.

- [17] S. Kumar, J.H. Seinfeld, *Optimal location of measurements for distributed parameter estimation*, IEEE Trans. Autom. Contr. **23**(1978), 690–698.
- [18] J.-L. Lions, *Exact controllability, stabilizability and perturbations for distributed systems*, SIAM Rev. **30** (1988), 1–68.
- [19] J.-L. Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués*, Tomes 1 & 2, Rech. Math. Appl. [Research in Applied Mathematics], Masson (1988).
- [20] A. Münch, *Optimal location of the support of the control for the 1-D wave equation: numerical investigations*, Comput. Optim. Appl. **42** (2009), 443–470.
- [21] F. Periago, *Optimal shape and position of the support for the internal exact control of a string*, Syst. Cont. Letters **58** (2009), no. 2, 136–140.
- [22] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, E.F. Mishchenko, *The mathematical theory of optimal processes*, Interscience Publishers John Wiley & Sons, Inc. New York-London (1962).
- [23] Y. Privat, E. Trélat, E. Zuazua, *Optimal location of controllers for the one-dimensional wave equation*, Preprint (2012).
- [24] Y. Privat, E. Trélat, E. Zuazua, *Optimal observability of second order equations*, work in progress.
- [25] D.L. Russell, *Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions*, SIAM Rev. **20** (1978), no. 4, 639–739.
- [26] M. Sion, *On general minimax theorems*, Pacific J. Math. **8** (1958), 171–176.
- [27] E. Trélat, *Contrôle optimal. (French) [Optimal control] Théorie & applications. [Theory and applications]*, Vuibert, Paris (2005).