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NEW FAULT TOLERANT CONTROL STRATEGIES FOR NONLINEAR TAKAGI-SUGENO SYSTEMS

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In this paper new methodologies for fault tolerant control (FTC) are proposed in order to compensate actuator faults in nonlinear systems. These approaches are based on the representation of the nonlinear system by a Takagi-Sugeno model. Two control laws are proposed requiring the simultaneous estimation of the system states and of the occurring actuator faults. The first approach concerns the stabilization problem in the presence of actuator faults. In the second one, the system state is forced to track a reference trajectory even in the faulty situation. The control performance depends on the estimation quality, indeed, it is important to accurately and rapidly estimate the states and the faults. This task is then performed with an Adaptive Fast State and Fault Observer (AFSFO) for the first case, and a Proportional-Integral Observer (PIO) in the second case. The stability conditions are established with Lyapunov theory and expressed in Linear Matrix Inequality (LMI) formulation to ease the design of the FTC. Furthermore, relaxed stability conditions are given with the use of the Polya's theorem. Some simulation examples are given in order to illustrate the proposed approaches.

Keywords: Takagi-Sugeno model (T-S), fault tolerant control (FTC), simultaneous fault and state estimations, Polya's theorem, Lyapunov theory, input-to-state stability (ISS).

1. Introduction

Since several years, the problem of fault tolerance has been treated from many points of view. Two classes can then be considered: passive control and active control. The first class may be viewed as a robust control. It requires the *a priori* knowledge of the possible faults which may affect the system. The principal idea of this kind of control is based on the consideration of all possible faults as uncertainties which are taken into account for the design of the tolerant control by using different techniques like $\mathcal{H}_\infty$ (Patton, 1997), (Niemann and Stoustrup, 2005). The interest of this approach lies in the fact that no on line information is needed and the structure of the control law remains unchanged. Generally, the structure of the uncertainties (faults) are not taken into account in order to lead to a convex optimization problem. Furthermore, the class of considered faults is limited, it becomes then risky to use only the passive fault tolerant control (see (Mufeed et al., 2003) for more details).

The second class concerns active fault tolerant control which is more interesting due to its possibility to take into account a large class of faults, because of its variable structure which may change in the presence of faults. The knowledge of some informations about these last are required and are obtained from a Fault Detection and Diagnosis (FDD) block. Different ideas are developed in the literature, for example, a Control Law Rescheduling (Ocampo-Martinez et al., 2010), (Leith and Leithead, 1999), (Stilwell and Rugh, 1997). This approach requires a very robust Fault Detection and Isolation (FDI) block which constitutes its major disadvantage. Indeed, a false alarm or a non detected fault can lead to degraded performance or even to instability. Other smooth fault tolerant control laws are proposed in (Ichalal et al., 2010) for Takagi-Sugeno systems and in (Patton and Klinkhieo, 2009a) for LPV systems.

Many efforts have been dedicated to the problem of designing active fault tolerant controllers for nonlinear

systems, the obtained results are linked to the system representation. Some works can be mentioned in the FTC field for nonlinear systems. For example, in (Gao and Ding, 2007), the authors took into account actuator faults for nonlinear descriptor systems with Lipschitz nonlinearities. In (Ocampo-Martinez et al., 2010), a method which requires only the fault isolation was proposed for T-S systems. It was based on a bank of observer based controllers. A switching mechanism is then designed depending on the obtained residuals. An efficient way to deal with the system nonlinearity in the FTC design is the T-S approach introduced by (Takagi and Sugeno, 1985). This approach is used in (Ichalal et al., 2010) where the FTC is based on trajectory tracking and Proportional-Integral Observer (PIO) design for T-S systems with weighting functions depending on the state of the system which is not accessible for measure.

In this paper, a new actuator fault tolerant control is proposed. Using a fast adaptive observer proposed in (Zhang et al., 2008) and extended here to nonlinear T-S systems, the state and the fault affecting the system are rapidly estimated. The use of such an observer is motivated by the fact that if a fault occurs, it is important to quickly and accurately detect it in order to take it into account and preserve the system performances. With the use of Lyapunov theory, sufficient conditions are obtained for asymptotic stability in the constant fault case and for input-to-state stability (ISS) in the case of time varying faults. The LMI formulation is used for representing the obtained stability conditions in an adequate form for existing LMI solvers. Relaxed stability conditions are obtained with the use of Polya's theorem (Sala and Ariño, 2007). Finally, another FTC strategy is proposed, based on trajectory tracking. Put in other words, the system state is kept as close as possible to a reference state, even in the faulty situation. Some simulations illustrate the obtained results.

2. Takagi-Sugeno structure for modeling

The T-S modeling allows to represent the behavior of nonlinear systems by the interpolation of a set of linear submodels. Each sub-model contributes to the global behavior of the nonlinear system through a weighting function $\mu_i(\xi(t))$. The T-S structure is given by

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} \mu_i(\xi(t))(A_i x(t) + B_i u(t)) \\ y(t) = Cx(t) \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the input vector, $y(t) \in \mathbb{R}^{n_y}$ represents the output vector. $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times n_u}$ and $C \in \mathbb{R}^{n_y\times n}$ are known matrices. The functions $\mu_i(\xi(t))$ are the weighting functions depending on the variable $\xi(t)$ which is, in the sequel of the paper, accessible for measure (as the input or the output of the system). These functions verify the following properties:

$$\begin{cases} \sum_{i=1}^{r} \mu_i(\xi(t)) = 1 \\ 0 \leq \mu_i(\xi(t)) \leq 1 \quad \forall i \in \{1, 2, ..., r\} \end{cases} \quad (2)$$

Obtaining a T-S model (1) can be performed from different methods such as linearization around some operating points and using adequate weighting functions. It can also be obtained by black-box approaches which allow to identify the parameters of the model from input-output data. Finally, a T-S model can be obtained from the well-known nonlinear sector transformations (Tanaka and Wang, 2001), (Nagy et al., 2009). These transformations allow to obtain an exact T-S representation of a nonlinear model with no information loss on a compact set of the state space.

Thanks to the convex sum property of the weighing functions (2), it is possible to generalize some tools developed in the linear domain to the nonlinear systems. The representation (1) is very interesting in the sense that it simplifies the stability studies of nonlinear systems and the design of control laws and observers. In (Tanaka and Wang, 2001), (Kruszewski et al., 2008), the stability and stabilization tools are inspired from the study of linear systems. In (Akhenak et al., 2008), (Marx et al., 2007), the authors worked on the problem of state estimation and diagnosis of T-S systems. The proposed approaches in these last papers rely on the generalization of the classical observers (Luenberger Observer (Luenberger, 1971) and Unknown Input Observer (UIO) (Darouach et al., 1994)) to the nonlinear systems. Recently in (Sala and Ariño, 2007), a new approach, derived from the Polya's theorem, leads to asymptotic necessary and sufficient stability conditions. FTC was also envisaged for T-S systems with time-delay in (Zhang et al., 2009), but no reference tracking was considered.

In the remaining of the paper, the two following lemmas and notations are used.

Lemma 1. *Consider two matrices X and Y with appropriate dimensions and G a symmetric positive definite matrix. The following property is verified:*

$$X^T Y + Y^T X \leq X^T G X + Y^T G^{-1} Y, \quad G > 0. \quad (3)$$

Lemma 2. (Congruence) *Consider two matrices P and Q, if P is positive definite and if Q is a full column rank matrix, then the matrix QPQ^T is positive definite.*

Notations. For any square matrix M, $\lambda_{max}(M)$ represents the maximum singular value of the matrix M and $\mathbb{S}(M)$ is defined by $\mathbb{S}(M) = M + M^T$. A block diagonal

matrix with the block matrices M_i on the diagonal entries is denoted $diag(M_1, M_2, \ldots, M_n)$. In a partitionned matrix, the sign $*$ denotes the terms induced by symmetry.

3. Problem statement

Under actuator faults, the system (1) can be rewritten in the following form:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} \mu_i(\xi(t)) \left(A_i x(t) + B_i \left(u(t) + f(t)\right)\right) \\ y(t) = Cx(t) \end{cases} \tag{4}$$

where $f(t)$ is an actuator fault. Faults can affect a system in many different ways. They can be represented by an additive or a multiplicative external signal. It can be pointed out that if the fault depends on the system state, it can change the model structure and cause its instability. For instance, malfunctions of an actuator can be represented by a faulty control input defined by $u_f(t) = (I_{n_u} - \gamma)u(t)$ which can easily be rewritten as an external additive signal: $(u(t) + f(t))$ where $f(t) = -\gamma u(t)$ and $\gamma = diag\left(\gamma_1, \gamma_2, \quad \cdots \gamma_{n_u}\right), 0 \leq \gamma_i \leq 1$ $(i = 1, ..., n_u)$ where:

$$\begin{cases} \gamma_i = 1 \Rightarrow \text{a total failure of the } i^{th} \text{ actuator,} \\ \gamma_i = 0 \Rightarrow \text{the } i^{th} \text{ actuator is healthy,} \\ \gamma_i \in]0 \;\; 1[\Rightarrow \text{a loss of effectiveness of the } i^{th} \text{ actuator.} \end{cases}$$

For example if $\gamma_2 = 0.4$, there is a 40% loss of effectiveness of the second actuator. Note that such multiplicative faults can cause the system instability.

Assumption 1 *In this paper, it is assumed that:*

- ***A1.*** *the faults have norm bounded first time derivative*

$$\left\| \dot{f}(t) \right\| \leq f_{1max}, \;\; 0 \leq f_{1max} < \infty. \tag{5}$$

- ***A2.*** $rank(CB_i) = n_u, i = 1, \ldots, r.$

- ***A3.*** *only partial actuator failures are considered, i.e.* $\gamma_i \in [0 \;\; 1[, i = 1, \ldots, r.$

The objective of the FTC design is to find the control law $u(t)$ in (4) such that the system remains stable even in the faulty case. For this purpose, the state and fault estimations will be used in order to minimize the fault influence on the system stability.

4. Stabilizing fault tolerant control for nonlinear systems

In order to estimate the state and the faults of the system (4), the following adaptive observer is proposed

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{i=1}^{r} \mu_i(\xi(t))(A_i\hat{x}(t) + B_i(u(t) + \hat{f}(t)) + L_i e_y(t)) \\ \hat{y}(t) = C\hat{x}(t) \\ \dot{\hat{f}}(t) = \Gamma \sum_{i=1}^{r} \mu_i(\xi(t)) F_i(\dot{e}_y(t) + \sigma e_y(t)) \\ e_y(t) = y(t) - \hat{y}(t) \end{cases} \tag{6}$$

and the active fault tolerant control is chosen as

$$u(t) = -\sum_{i=1}^{r} \mu_i(\xi(t)) K_i \hat{x}(t) - \hat{f}(t). \tag{7}$$

This control law can be seen as a generalization to the nonlinear case of the approach proposed in (Patton and Klinkhieo, 2009b). The solution of the FTC problem is obtained by setting $\Gamma \in \mathbb{R}^{n_u \times n_u}$ and the scalars σ and $\beta \in \mathbb{R}$ and determining the gains $L_i \in \mathbb{R}^{n \times n_y}$, $F_i \in \mathbb{R}^{n_u \times n_y}$ and $K_i \in \mathbb{R}^{n_u \times n}$ by LMI optimization such that the state of the system asymptotically converges to zero if the fault $f(t)$ is constant or to a small set around the origin when $f(t)$ is time varying with norm bounded first time derivative. The expression describing the dynamic of the estimated fault $\hat{f}(t)$ given in (6) depends on both the output error and the derivative of the output error.

Let us consider the state and fault estimation errors $e_x(t)$ and $e_f(t)$ defined by

$$e_x(t) = x(t) - \hat{x}(t) \tag{8}$$

$$e_f(t) = f(t) - \hat{f}(t) \tag{9}$$

The dynamics of the state estimation error and the closed-loop system with the control (7) obey the differential equations:

$$\dot{e}_x(t) = \sum_{i=1}^{r} \mu_i(\xi(t)) \left(\Phi_i e_x(t) + B_i e_f(t)\right) \tag{10}$$

$$\begin{aligned} \dot{x}(t) = &\sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i(\xi(t)) \mu_j(\xi(t)) \left(\Xi_{ij} x(t)\right. \\ &\left. + B_i e_f(t) + B_i K_j e_x(t)\right) \end{aligned} \tag{11}$$

where $\Phi_i = A_i - L_i C$ and $\Xi_{ij} = A_i - B_i K_j$.

Theorem 1. *Under the assumptions 1, given positive scalars σ and β, if there exists symmetric and positive definite matrices $\mathcal{X} \in \mathbb{R}^{n \times n}$, $P_2 \in \mathbb{R}^{n \times n}$, a positive definite matrix $G \in \mathbb{R}^{n_u \times n_u}$, matrices $M_i \in \mathbb{R}^{n_u \times n}$ and $N_i \in \mathbb{R}^{n \times n_y}$ and a positive scalar η solution to the optimization problem:*

$$\min \quad \eta \quad s.t. \tag{12}$$

$$\begin{pmatrix} \eta I & B_i^T P_2 - F_i C \\ * & \eta I \end{pmatrix} > 0 \tag{13}$$

$$\mathcal{Q}_{ij} = \begin{pmatrix} S_{ij} & B_i M_j & B_i & 0 & 0 \\ * & -2\beta\mathcal{X} & 0 & \beta I & 0 \\ * & * & -2\beta I & 0 & \beta I \\ * & * & * & \Omega_i & \mathcal{R}_{ij} \\ * & * & * & * & \Psi_{ij} \end{pmatrix} < 0 \tag{14}$$

$$S_{ij} = \mathbb{S}(A_i\mathcal{X} - B_i M_j) \tag{15}$$
$$\Omega_i = \mathbb{S}(P_2 A_i - N_i C) \tag{16}$$
$$\mathcal{R}_{ij} = -\frac{1}{\sigma}(A_j^T P_2 - C^T N_j^T) B_i \tag{17}$$
$$\Psi_{ij} = -\frac{1}{\sigma}\left(B_i^T P_2 B_j + B_j^T P_2 B_i\right) + \frac{1}{\sigma} G \tag{18}$$

then the state of the system $x(t)$, the state estimation error $e_x(t)$ and the fault estimation error $e_f(t)$ are bounded. Furthermore, if the bound of the first time derivative of $f(t)$ is zero i.e. $f_{1max} = 0$, these variables converge asymptotically to zero. The gains of the observer and the fault tolerant control are given by F_i, $L_i = P_2^{-1} N_i$ and $K_i = M_i \mathcal{X}^{-1}$.

Proof. In order to prove both the stability of the closed-loop system and the convergence of the state and fault estimation errors, consider a Lyapunov function depending on $x(t)$, $e_x(t)$ and $e_f(t)$ defined by:

$$V(t) = x^T(t) P_1 x(t) + e_x^T(t) P_2 e_x(t) + \frac{1}{\sigma} e_f(t) \Gamma^{-1} e_f(t) \tag{19}$$

where P_1, P_2 and Γ are symmetric and positive definite matrices with appropriate dimensions.

According to the equations (10) and (11), the time derivative of $V(t)$ is given by:

$$\begin{aligned} \dot{V}(t) = & \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi(t))\mu_j(\xi(t))(x^T(t)\Pi_{ij} x(t) \\ & + e_x^T(t)\Omega_i e_x(t) + 2x^T(t) P_1 B_i K_j e_x(t) \\ & + 2x^T(t) P_1 B_i e_f(t) + 2e_x^T(t) P_2 B_i e_f(t) \\ & + \frac{2}{\sigma} e_f^T(t)\Gamma^{-1}\dot{e}_f(t)) \end{aligned} \tag{20}$$

where

$$\Pi_{ij} = \Xi_{ij}^T P_1 + P_1 \Xi_{ij} \tag{21}$$

and

$$\Omega_i = \Phi_i^T P_2 + P_2 \Phi_i \tag{22}$$

Knowing that $\dot{e}_f(t) = \dot{f}(t) - \dot{\hat{f}}(t)$ and using the expression of $\hat{f}(t)$ in (6), one obtains:

$$\begin{aligned} \dot{V}(t) = & \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi(t))\mu_j(\xi(t))(x^T(t)\Pi_{ij} x(t) \\ & + e_x^T(t)\Omega_i e_x(t) + 2x^T(t) P_1 B_i K_j e_x(t) \\ & + 2x^T(t) P_1 B_i e_f(t) + 2e_x^T(t) P_2 B_i e_f(t) \\ & - \frac{2}{\sigma} e_f^T(t) F_i(\dot{e}_y(t) + \sigma e_y(t)) + \frac{2}{\sigma} e_f^T(t)\Gamma^{-1}\dot{f}(t)) \end{aligned} \tag{23}$$

where $e_y(t) = C e_x(t)$.

Using the differential equation (10) generating $e_x(t)$, the following is obtained:

$$\begin{aligned} \dot{V}(t) = & \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi(t))\mu_j(\xi(t))(x^T(t)\Pi_{ij} x(t) \\ & + e_x^T(t)\Omega_i e_x(t) + 2x^T(t) P_1 B_i K_j e_x(t) \\ & + 2x^T(t) P_1 B_i e_f(t) + 2e_x^T(t) P_2 B_i e_f(t) \\ & - \frac{2}{\sigma} e_f^T(t) F_i C \Phi_j e_x(t) - \frac{2}{\sigma} e_f^T(t) F_i C B_j e_f(t) \\ & - 2e_f^T(t) F_i C e_x(t) + \frac{2}{\sigma} e_f^T(t)\Gamma^{-1}\dot{f}(t)). \end{aligned} \tag{24}$$

Using Lemma 1 and assumption ***A1***, we deduce that:

$$\begin{aligned} & 2\frac{1}{\sigma} e_f^T(t)\Gamma^{-1}\dot{f}(t) \\ \leq & \frac{1}{\sigma} e_f^T(t) G e_f + \frac{1}{\sigma}\dot{f}^T(t)\Gamma^{-1} G^{-1}\Gamma^{-1}\dot{f}(t) \\ \leq & \frac{1}{\sigma} e_f^T(t) G e_f + \frac{1}{\sigma} f_{1\max}^2 \lambda_{max}\left(\Gamma^{-1} G^{-1}\Gamma^{-1}\right) \end{aligned} \tag{25}$$

and using assumption ***A2***, it is possible to obtain F_i and P_2 such that $B_i^T P_2 = F_i C$ holds. The time derivative of the Lyapunov function (24) is bounded as follows

$$\dot{V}(t) \leq \tilde{x}^T(t) \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi(t))\mu_j(\xi(t))\Delta_{ij}\tilde{x}(t) + \delta \tag{26}$$

where

$$\tilde{x}^T(t) = (x^T(t) \;\; e_x^T(t) \;\; e_f^T(t))^T \tag{27}$$
$$\delta = \frac{1}{\sigma} f_{1\max}^2 \lambda_{\max}\left(\Gamma^{-1} G^{-1}\Gamma^{-1}\right) \tag{28}$$

$$\Delta_{ij} = \begin{pmatrix} \Pi_{ij} & P_1 B_i K_j & P_1 B_i \\ * & \Omega_i & -\frac{1}{\sigma}\Phi_j^T P_2 B_i \\ * & * & \Psi_{ij} \end{pmatrix}. \tag{29}$$

If the following inequality holds

$$\sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi(t))\mu_j(\xi(t))\Delta_{ij} < 0 \tag{30}$$

it is established that:

$$\dot{V}(t) < -\varepsilon \left\|\tilde{x}(t)\right\|^2 + \delta \tag{31}$$

where $\varepsilon > 0$ is defined by:

$$\varepsilon = \min_{t>0} \lambda_{min} \left(-\sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi(t))\mu_j(\xi(t))\Delta_{ij} \right) \tag{32}$$

or can also be bounded by

$$\varepsilon \leq \min_{i,j} \lambda_{min}(-\Delta_{ij}) \tag{33}$$

It follows that $\dot{V}(t) < 0$ if $\varepsilon \left\|\tilde{x}(t)\right\|^2 > \delta$, and according to Lyapunov stability theory the state $x(t)$, the state estimation error $e_x(t)$ and the fault estimation error $e_f(t)$ converge to a small set around the origin and then lies into it. This set is smaller as the constant δ converges to zero.

In order to achieve the proof, it remains to establish some LMI conditions to ensure that (30) and $B_i^T P_2 = F_i C$ hold. The latter is first considered.

As pointed out in (Zhang et al., 2008), it is difficult to solve simultaneously the inequality $\sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi(t))\mu_j(\xi(t))\Delta_{ij} < 0$ with the equality constraint $B_i^T P_2 = F_i C$. A technique for reducing this difficulty is to formulate the equality constraint as an optimization problem (Corless and Tu, 1998):

$$\min\ \eta\ \ s.t.\ \ \begin{pmatrix} \eta I & B_i^T P_2 - F_i C \\ * & \eta I \end{pmatrix} > 0. \tag{34}$$

For the sake of simplicity, the following notations will be used:

$$Y_\xi = \sum_{i=1}^{r} \mu_i(\xi(t)) Y_i, \tag{35}$$

$$Y_{\xi\xi} = \sum_{i=1}^{r}\sum_{i=1}^{r} \mu_i(\xi(t))\mu_j(\xi(t)) Y_{ij}, \tag{36}$$

where Y_i and Y_{ij} are given matrices. Using this representation, the inequality (30) becomes:

$$\Delta_{\xi\xi} = \begin{pmatrix} \Pi_{\xi\xi} & \Theta_{\xi\xi} \\ \Theta_{\xi\xi}^T & \Lambda_{\xi\xi} \end{pmatrix} < 0 \tag{37}$$

where:

$$\Theta_{ij} = \begin{pmatrix} P_1 B_i K_j & P_1 B_i \end{pmatrix} \tag{38}$$

$$\Lambda_{ij} = \begin{pmatrix} \Omega_i & -\frac{1}{\sigma}\Phi_j^T P_2 B_i \\ * & \Psi_{ij} \end{pmatrix}. \tag{39}$$

Consider a symmetric matrix X defined as follows:

$$X = \begin{pmatrix} P_1^{-1} & 0 \\ 0 & X_1 \end{pmatrix},\ X_1 = \begin{pmatrix} P_1^{-1} & 0 \\ 0 & I \end{pmatrix}. \tag{40}$$

Using Lemma 2, post and pre-multiplying the inequality (37) by X, it follows that (37) is equivalent to the following inequality:

$$\begin{pmatrix} P_1^{-1}\Pi_{\xi\xi}P_1^{-1} & P_1^{-1}\Theta_{\xi\xi}X_1 \\ * & X_1\Lambda_{\xi\xi}X_1 \end{pmatrix} < 0. \tag{41}$$

Since the following inequality holds for any scalar β

$$\begin{aligned} & \left(X_1 + \beta\Lambda_{\xi\xi}^{-1}\right)^T \Lambda_{\xi\xi} \left(X_1 + \beta\Lambda_{\xi\xi}^{-1}\right) \leq 0 \\ \Leftrightarrow \quad & X_1\Lambda_{\xi\xi}X_1 \leq -2\beta X_1 - \beta^2\Lambda_{\xi\xi}^{-1} \end{aligned} \tag{42}$$

and with a Schur complement, it follows that the inequality (41) holds if (43) is satisfied

$$\begin{pmatrix} P_1^{-1}\Pi_{\xi\xi}P_1^{-1} & P_1^{-1}\Theta_{\xi\xi}X_1 & 0 \\ * & -2\beta X_1 & \beta I \\ * & * & \Lambda_{\xi\xi} \end{pmatrix} < 0. \tag{43}$$

Using the notations (35)-(36) and the definitions of the matrices $\Pi_{\xi\xi}$, $\Theta_{\xi\xi}$ and $\Lambda_{\xi\xi}$ given by (21), (38) and (39), and with the changes of variables $\mathcal{X} = P_1^{-1}$, $M_i = K_i\mathcal{X}$, $N_i = P_2 L_i$ it is easy to obtain the inequalities given in the theorem 1. Finally, the inequality (31) is satisfied, if the optimization problem given by (12) under LMI constraints (13)-(14) has a solution, which ends the proof. ■

Remark 1. After solving the optimization problem given in the theorem 1, the input-to-state stability condition given in (31) is satisfied. Thus, in the case of time varying faults with bounded first time derivative, the state $x(t)$, the state estimation error $e_x(t)$ and the fault estimation error $e_f(t)$ converge to an origin centered ball defined by the terms δ and ε. The radius of the ball in which $\tilde{x}$ converges can be minimized by a choice of the parameter Γ that minimizes δ without changing ε (that does not depend on Γ). It thus improves the accuracy of the estimation.

Remark 2. The objective of fault tolerant control is to compensate the faults, so it is important to estimate them as soon as possible with a good accuracy. The adaptive observer studied in this paper can be considered as an improvement of the classical PI observer, in the sense that convergence of the state and fault estimations is proved (in an origine centered ball) even in non constant fault case, whereas the assumption of constant fault is needed to prove the estimation error convergence when using a PI observer (Koenig and Mammar, 2002), (Ichalal et al., 2009a). Note that if the fault $f(t)$ is constant, then $f_{1max} = 0$ and with (28) $\delta = 0$, consequently the asymptotic stability is achieved, since $\dot{V}(t) < 0$ for every $\tilde{x}(t)$.

5. Simulation example

To illustrate the performances of the proposed approach, let us consider the system (4) defined by the matrices:

$$A_1 = \begin{pmatrix} 0 & 1 \\ 17.2941 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 1 \\ 3.5361 & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 \\ -0.1765 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ -0.1763 \end{pmatrix}, \quad C = I_2.$$

The weighting functions are given by $\mu_1(x(t)) = 1 - \frac{2}{\pi}|x_1(t)|$ and $\mu_2(x(t)) = 1 - \mu_1(x(t))$. Let us consider the fault $f(t)$ defined as follows:

$$f(t) = \begin{cases} 0 & t \leq 20 \\ 7.5\sin(2t + 2.1) + 15 & 20 \leq t \leq 70 \\ -0.89u(t) & 70 \leq t \leq 100 \end{cases} \tag{44}$$

For $t \geq 70$ s, the fault $f(t)$ describes a loss of effectiveness of the actuator, satisfying assumption ***A3***. The first simulation is obtained by synthesizing a classical controller without taking the faults into account the faults: $u(t) = -\sum_{i=1}^{r} \mu_i(x(t))K_i x(t)$, using an approach proposed in (Tanaka and Wang, 2001). The gains K_i are obtained by $K_i = M_i P^{-1}$ where P and M_i are solution of the LMIs

$$\mathbb{S}(PA_i - B_i M_j) < 0, \quad i, j = 1, 2. \tag{45}$$

With this control law, as shown in the figure 1, the states of the system converge to zero in fault free case (i.e. for $t < 20$), but in the faulty case the system performances are degraded from $t = 20$ s to $t = 70$ s and the system becomes unstable for $t \geq 70$ s.

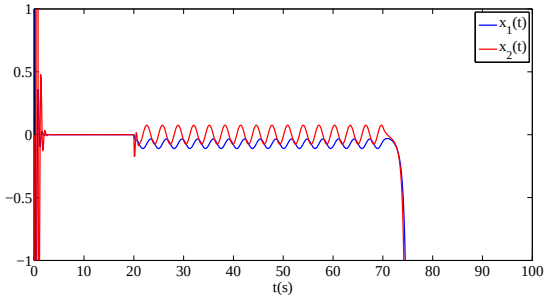

Fig. 1. System states with classical control

The proposed fault tolerant control is designed by solving the optimization problem of theorem 1. For that, the parameter values $\sigma = 0.8$, $\Gamma = 100$ and $\beta = 10$ are chosen. The obtained gains of the observer and the controller are:

$$L_1 = \begin{pmatrix} 0.52 & 1.22 \\ 17.24 & 0.27 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0.52 & 1.21 \\ 3.48 & 0.26 \end{pmatrix}$$

$$F_1 = \begin{pmatrix} 3.63 & -43.25 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 3.62 & -43.20 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} -161.89 & -65.95 \end{pmatrix},$$

$$K_2 = \begin{pmatrix} -156.25 & -65.24 \end{pmatrix}$$

The figure 2 illustrates the results of the proposed control law obtained after solving the optimization problem of theorem 1. One can note that, with the fault $f(t)$ defined in (44), the performances are better than those of the classical control and the system remains stable for $t \geq 70$ (figure 2 (top)). The observer rapidly and accurately estimates the fault as shown in the figure 2 (bottom). In this example, the classical control cannot

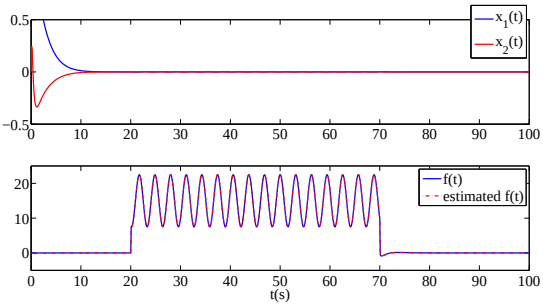

Fig. 2. Fault tolerant control : states of the system (top) - fault and its estimation (bottom)

preserve the stability of the system when $\gamma \geq 0.89$ however (based on simulations not displayed here due to space limitation it can be claimed that) the proposed FTC strategy can tolerate faults until $\gamma = 0.992$ which means that if the loss of effectiveness of the actuator is less than 99.2%, the proposed controller makes the system stable. In figure 3, after $t > 70$ the loss of effectiveness of the actuator is considered with $\gamma = 0.99$ and we see that the controller compensates it despite of its severity.

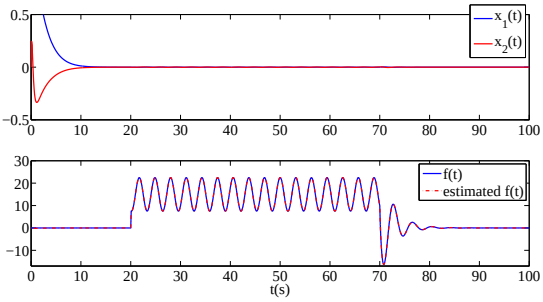

Fig. 3. Fault tolerant control with $\gamma = 0.99$: states of the system (top) - fault and its estimation (bottom)

In addition, this approach provides a rapid and accurate estimation of occurred actuator faults with the adaptive observer (figure 2 (bottom)) which constitutes a FDI block for diagnosis. If $f(t) = 7.5\sin(2t + 2.1) + 15$, its derivative over the time is bounded by 15, then in this simulation example, the term $\delta = \frac{1}{\sigma} f_{1max}^2 \lambda_{max}(\Gamma^{-1} G^{-1} \Gamma^{-1}) = 0.0186$, and the term ε can be minimized by an appropriate choice of Γ to reduce the radius of the ball in which converge the estimation errors and then obtain a more accurate fault estimation.

6. Conservatism reduction with Polya's theorem

In the previous section, the proposed result may be conservative in the sense that common Lyapunov matrices were sought to satisfy r^2 LMIs. Recently, a new interesting method to reduce the conservativeness of the matrix summations inequality has been proposed with the use of Polya's theorem (Sala and Ariño, 2007).

Let us consider the inequality (46):

$$\Delta_{\xi\xi} = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi(t))\mu_j(\xi(t))\Delta_{ij} < 0 \qquad (46)$$

where Δ_{ij} is defined in equation (29).

Noticing that $(\sum_{i=1}^{r} \mu_i(\xi(t)))^p = 1$ for any positive integer p, it can be deduced that inequalities like $\Delta_{\xi\xi} < 0$ can be rewritten as $(\sum_{i=1}^{r} \mu_i(\xi(t)))^p \Delta_{\xi\xi} < 0$. By gathering all terms in the left hand side of the inequality with the same coefficient, less conservative LMI conditions than $\Delta_{ij} < 0$ for $i, j = 1, \ldots, r$ can be obtained. As proved in (Sala and Ariño, 2007), if a solution exists for a given p, this solution satisfies the inequalities obtained for $p+1$. As a consequence, increasing p will provide less restrictive conditions and if $p \to +\infty$ asymptotic necessary and sufficient conditions for the negativity of (46) are obtained. The authors proposed also an algorithm to compute finite values of p which gives necessary and sufficient conditions with a given accuracy. The reader can refer to the paper (Sala and Ariño, 2007) for more details on Polya's theorem based relaxation approach.

Theorem 2. *Under the assumptions 1, given positive scalars σ and β, if there exists symmetric and positive definite matrices $\mathcal{X} \in \mathbb{R}^{n\times n}$, $P_2 \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n_u\times n_u}$ and matrices $M_i \in \mathbb{R}^{n_u\times n}$ and $N_i \in \mathbb{R}^{n\times n_y}$ and a positive scalar η solution to the optimization problem*

$$\min\ \eta\ s.t. \qquad (47)$$

$$\begin{pmatrix} \eta I & B_i^T P_2 - F_i C \\ * & \eta I \end{pmatrix} > 0 \qquad (48)$$

$$\mathcal{Q}_{ii} < 0$$
$$i = 1, ..., r$$

$$3\mathcal{Q}_{ii} + \mathcal{Q}_{ij} + \mathcal{Q}_{ji} < 0$$
$$i, j = 1, ..., r,\ \ i \neq j$$

$$3\mathcal{Q}_{ii} + \mathcal{Q}_{jj} + 3\mathcal{Q}_{ij} + 3\mathcal{Q}_{ji} < 0$$
$$i, j = 1, ..., r, i \neq j$$

$$6\mathcal{Q}_{ii} + 3\mathcal{Q}_{ij} + 3\mathcal{Q}_{ik} + 3\mathcal{Q}_{ji}$$
$$+3\mathcal{Q}_{ki} + \mathcal{Q}_{jk} + \mathcal{Q}_{kj} < 0$$
$$i, j, k = 1, ..., r, i < j < k$$

$$3\mathcal{Q}_{ii} + 3\mathcal{Q}_{jj} + 6\mathcal{Q}_{ij} + 6\mathcal{Q}_{ji} + 3\mathcal{Q}_{ik}$$
$$+3\mathcal{Q}_{ki} + 3\mathcal{Q}_{jk} + 3\mathcal{Q}_{kj} < 0$$
$$i, j, k = 1, ..., r, i < j < k$$

$$6\mathcal{Q}_{ii} + 6\mathcal{Q}_{ij} + 6\mathcal{Q}_{ji} + 6\mathcal{Q}_{ik}$$
$$+6\mathcal{Q}_{ki} + 6\mathcal{Q}_{il} + 6\mathcal{Q}_{li} + 3\mathcal{Q}_{jk}$$
$$+3\mathcal{Q}_{kj} + 3\mathcal{Q}_{jl} + 3\mathcal{Q}_{lj} + 3\mathcal{Q}_{kl} + 3\mathcal{Q}_{lk} < 0$$
$$i, j, k = 1, ..., r, i < j < k < l$$

$$6(\mathcal{Q}_{ij} + \mathcal{Q}_{ji} + \mathcal{Q}_{ik} + \mathcal{Q}_{ki} + \mathcal{Q}_{il} + \mathcal{Q}_{li} + \mathcal{Q}_{im}$$
$$+\mathcal{Q}_{mi} + \mathcal{Q}_{jk} + \mathcal{Q}_{kj} + \mathcal{Q}_{jl} + \mathcal{Q}_{lj} + \mathcal{Q}_{jm}$$
$$+\mathcal{Q}_{mj} + \mathcal{Q}_{kl} + \mathcal{Q}_{lk} + \mathcal{Q}_{km} + \mathcal{Q}_{mk}) < 0$$
$$i, j, k, l, m = 1, ..., r,\ \ i < j < k < l < m$$

where $\mathcal{Q}_{ij}$ is defined in (14)*. Then the state of the system $x(t)$, the state estimation error $e_x(t)$ and the fault estimation error $e_f(t)$ are bounded. The gains of the observer and the fault tolerant control are given by F_i, $L_i = P_2^{-1}N_i$ and $K_i = M_i\mathcal{X}^{-1}$.*

Proof. According to theorem 1, the solution of the FTC problem is obtained by minimizing η under the constraints (13) and $\sum_{i=1}^{r}\sum_{j=i}^{r} \mu_i(\xi(t))\mu_j(\xi(t))Q_{ij} < 0$, which due to the convex property of the weighting functions is equivalent to

$$\left(\sum_{k=1}^{r} \mu_k(\xi(t))\right)^p \sum_{i=1}^{r}\sum_{j=i}^{r} \mu_i(\xi(t))\mu_j(\xi(t))\mathcal{Q}_{ij} < 0 \qquad (49)$$

Setting $p = 3$ and gathering the terms sharing the same

combinations of weighting functions, we obtain

$$\begin{aligned}
&\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\mathcal{Q}_{ij} = \left(\sum_{i=1}^{r}\mu_i\right)^3\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\mathcal{Q}_{ij}\\
=&\sum_{i=1}^{r}\mu_i^5\mathcal{Q}_{ii} + \sum_{\substack{i,j=1\\ i\neq j}}^{r}\mu_i^4\mu_j\left(3\mathcal{Q}_{ii}+\mathcal{Q}_{ij}+\mathcal{Q}_{ji}\right)\\
+&\sum_{\substack{i,j=1\\ i\neq j}}^{r}\mu_i^3\mu_j^2\tilde{Q}_{ij} + \sum_{i=1}^{r}\sum_{\substack{j=1\\ i<j}}^{r}\sum_{\substack{k=1\\ j<k}}^{r}\mu_i^3\mu_j\mu_k\tilde{Q}_{ijk}\\
+&\sum_{i=1}^{r}\sum_{\substack{j=1\\ i<j}}^{r}\sum_{\substack{k=1\\ j<k}}^{r}\mu_i^2\mu_j^2\mu_k\tilde{Q}^*_{ijk}\\
+&\sum_{i=1}^{r}\sum_{\substack{j=1\\ i<j}}^{r}\sum_{\substack{k=1\\ j<k}}^{r}\sum_{\substack{l=1\\ k<l}}^{r}\mu_i^2\mu_j\mu_k\mu_l\tilde{Q}_{ijkl}\\
+&\sum_{i=1}^{r}\sum_{\substack{j=1\\ i<j}}^{r}\sum_{\substack{k=1\\ j<k}}^{r}\sum_{\substack{l=1\\ k<l}}^{r}\sum_{\substack{m=1\\ l<m}}^{r}\mu_i\mu_j\mu_k\mu_l\mu_m\tilde{Q}_{ijklm} \qquad (50)
\end{aligned}$$

with

$$\begin{aligned}
\tilde{Q}_{ij} =&3\mathcal{Q}_{ii}+\mathcal{Q}_{jj}+3\mathcal{Q}_{ij}+3\mathcal{Q}_{ji}\\
\tilde{Q}_{ijk} =&6\mathcal{Q}_{ii}+3(\mathcal{Q}_{ij}+\mathcal{Q}_{ji}+\mathcal{Q}_{ik}+\mathcal{Q}_{ki})\\
&+\mathcal{Q}_{jk}+\mathcal{Q}_{kj}
\end{aligned}$$

$$\begin{aligned}
\tilde{Q}^*_{ijk} =&3\mathcal{Q}_{ii}+3\mathcal{Q}_{jj}+6\mathcal{Q}_{ij}+6\mathcal{Q}_{ji}+3\mathcal{Q}_{ik}\\
&+3\mathcal{Q}_{ki}+3\mathcal{Q}_{jk}+3\mathcal{Q}_{kj}\\
\tilde{Q}_{ijkl} =&6(\mathcal{Q}_{ii}+\mathcal{Q}_{ij}+\mathcal{Q}_{ji}+\mathcal{Q}_{ik}+\mathcal{Q}_{ki}+\mathcal{Q}_{il}+\mathcal{Q}_{li})
\end{aligned}$$

$$\begin{aligned}
\tilde{Q}_{ijklm} =&6(\mathcal{Q}_{ij}+\mathcal{Q}_{ji}+\mathcal{Q}_{ik}+\mathcal{Q}_{ki}+\mathcal{Q}_{il}+\mathcal{Q}_{li}+\mathcal{Q}_{im}\\
&+\mathcal{Q}_{mi}+\mathcal{Q}_{jk}+\mathcal{Q}_{kj}+\mathcal{Q}_{jl}+\mathcal{Q}_{lj}+\mathcal{Q}_{jm}\\
&+\mathcal{Q}_{mj}+\mathcal{Q}_{kl}+\mathcal{Q}_{lk}+\mathcal{Q}_{km}+\mathcal{Q}_{mk})
\end{aligned}$$

Finally, with the same reasoning as in the theorem 1, the optimization problem with relaxed LMI constraints of theorem 2 is obtained. ■

7. Fault tolerant control by trajectory tracking

In this section the control objective is not only closed-loop stabilization but also trajectory tracking. First, the FTC strategy is detailed, before addressing the FTC design.

7.1. FTC strategy. The state trajectory $x(t)$ to be followed by the controlled system is given by a reference model which corresponds to the model of the fault-free system with nominal input $u(t)$ defined by

$$\begin{cases}
\dot{x}(t) = \sum_{i=1}^{r}\mu_i(\xi(t))\left(A_ix(t)+B_iu(t)\right)\\
y(t) = \sum_{i=1}^{r}\mu_i(\xi(t))C_ix(t)
\end{cases} \qquad (51)$$

Because of the fault $f(t)$, the state of the faulty controlled system, denoted $x_f(t)$, may differ from the reference trajectory. As a consequence, the control input of the system has to be modified, the resulting applied input is denoted $u_f(t)$. The faulty system is given by

$$\begin{cases}
\dot{x}_f(t) = \sum_{i=1}^{r}\mu_i(\xi_f(t))\left(A_ix_f(t)+B_i(u_f(t)+f(t))\right)\\
y_f(t) = \sum_{i=1}^{r}\mu_i(\xi_f(t))C_ix_f(t)
\end{cases} \qquad (52)$$

The considered faults are the same as in previous sections. Note that, the weighting functions depend on a faulty premise variable $\xi_f(t)$. These last may be the input of the system, which in closed-loop depend on the state $x_f(t)$, or the output $y_f(t)$, consequently the fault necessarily affects these variables. In order to minimize the state deviation due to the fault, the controlled input $u_f(t)$ encompasses two additive terms respectively depending on the fault estimate and on the estimated state deviation $x(t)-\hat{x}_f(t)$. One should note that the reference state is simulated from (51), and thus is accessible for the control law, whereas the faulty system state has to be estimated. The FTC law $u_f(t)$ is given by

$$u_f(t) = -\hat{f}(t) + K(x(t)-\hat{x}_f(t)) + u(t) \qquad (53)$$

A PI observer is designed to provide the controller with both the estimates of the actuator fault and of the faulty system state. The proposed PI observer is given by

$$\begin{cases}
\dot{\hat{x}}_f(t) = \sum_{i=1}^{r}\mu_i(\xi_f(t))(A_i\hat{x}_f(t)+B_i(u_f(t)+\hat{f}(t))\\
\qquad +H_{1i}(y_f(t)-\hat{y}_f(t)))\\
\dot{\hat{f}}(t) = \sum_{i=1}^{r}\mu_i(\xi_f(t))\left(H_{2i}(y_f(t)-\hat{y}_f(t))\right)\\
\hat{y}_f(t) = \sum_{i=1}^{r}\mu_i(\xi_f(t))C_i\hat{x}_f(t)
\end{cases} \qquad (54)$$

The premise variable $\xi_f(t)$ is assumed to be known and the observer weighting functions depend on the same premise variable as the system (52).

The overall scheme of the proposed FTC strategy is depicted by the figure 4. The FTC design consists in determining the control law (53) and the PIO (54), such the controlled system state $x_f(t)$ is as closed as possible from the reference state $x(t)$.

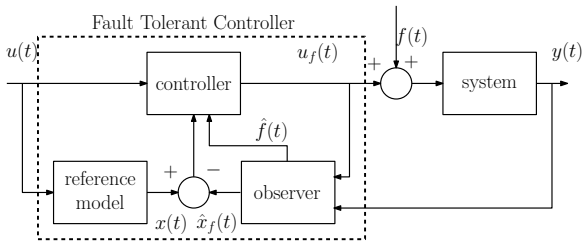

Fig. 4. Fault tolerant control scheme

7.2. FTC design. The FTC design consists in determining the gains K in (53) and H_{1i} and H_{2i} in (54) that minimize the trajectory tracking error defined by $e(t) = x(t) - x_f(t)$ and the state and fault estimation errors, respectively defined by $e_x(t) = x_f(t) - \hat{x}_f(t)$ and $e_f(t) = f(t) - \hat{f}(t)$. From (51), (52), (53) and (54) and the definition of the errors, it follows

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i(\xi_f(t))(A_i x(t) + B_i u(t)) + \delta(t) \quad (55)$$

$$\dot{x}_f(t) = \sum_{i=1}^{r} \mu_i(\xi_f(t)) \left(A_i x_f(t) + B_i(u(t) + e_f(t) + K(e(t) + e_x(t)))\right) \quad (56)$$

$$\dot{\hat{x}}_f(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi_f(t))\mu_j(\xi_f(t)) \left(A_i \hat{x}_f(t) + B_i(u(t) + K(e(t) + e_x(t))) + H_{1i} C_j e_x(t)\right) \quad (57)$$

where $\delta(t)$ is defined by

$$\delta(t) = \sum_{i=1}^{r} (\mu_i(\xi(t)) - \mu_i(\xi_f(t)))(A_i x(t) + B_i u(t)) \quad (58)$$

Assuming $\dot{f}(t) = 0$ the dynamics of the errors is given by

$$\dot{\tilde{e}}(t) = \begin{pmatrix} A_{\xi_f} - B_{\xi_f} K & -\tilde{L}_{\xi_f} \\ 0 & \tilde{A}_{\xi_f} - H_{\xi_f} \tilde{C}_{\xi_f} \end{pmatrix} \tilde{e}(t) + \tilde{\Gamma}\delta(t) \quad (59)$$

with the following notations

$$\tilde{e}(t) = \begin{pmatrix} e(t) \\ e_x(t) \\ e_f(t) \end{pmatrix} \quad \tilde{\Gamma} = \begin{pmatrix} I_n \\ 0 \\ 0 \end{pmatrix} \quad H_i = \begin{pmatrix} H_{1i} \\ H_{2i} \end{pmatrix}$$

$$\tilde{L}_i = \begin{pmatrix} B_i K & B_i \end{pmatrix} \quad \tilde{A}_i = \begin{pmatrix} A_i & B_i \\ 0 & 0 \end{pmatrix} \quad \tilde{C}_i = \begin{pmatrix} C_i & 0 \end{pmatrix}$$

In order to design the PIO, it is necessary that the pairs $(\tilde{C}, \tilde{A}_i)$ are observable or at least detectable.

Remark 3. One can note that in the previous section, the weighting functions depend on the premise variable $\xi_f(t)$. It can be an external known variable which is not affected by faults. Indeed, in (Witczak et al., 2008), the authors proposed a method for this case with an application to the three tank system in open-loop control. In this case, $\xi(t) = \xi_f(t)$ and the equation (59) becomes an autonomous system

$$\dot{\tilde{e}}(t) = \begin{pmatrix} A_{\xi_f} - B_{\xi_f} K & -\tilde{L}_{\xi_f} \\ 0 & \tilde{A}_{\xi_f} - H_{\xi_f} \tilde{C}_{\xi_f} \end{pmatrix} \tilde{e}(t) \quad (60)$$

In Takagi-Sugeno modeling, it is often considered that the premise variable $\xi(t)$ is the input, the output or the state of the system, which are necessarily affected by faults, consequently $\xi(t) \neq \xi_f(t)$ and the fault and state estimation errors and the state tracking error are expressed by (59). When $\xi(t) = u(t)$ and $\xi_f(t) = u_f(t)$, the term $\delta(t)$ does not converge to zero if $x_f(t)$ converges to the reference state $x(t)$ but if $\xi(t) = y(t)$ and $\xi_f(t) = y_f(t)$, the tolerant control allows the convergence of $x_f(t)$ to $x(t)$ and $y_f(t)$ to $y(t)$, then the term $\delta(t)$ converges also to zero which gives better results compared to the case where $\xi(t) = u(t)$. The same problem can appear if the output is also affected by faults. In these cases, the fault tolerant control design aims to minimize the difference between $x_f(t)$ and $x(t)$ and to minimize the $\mathcal{L}_2$ gain of the transfer from $\delta(t)$ to the state tracking error.

The gains K, H_{1i} and H_{2i} are determined by solving the optimization problem under LMI constraints given in the next theorem.

Theorem 3. *Consider λ a positive scalar. The system* (59) *that generates the state tracking error $e(t)$ and the state and fault estimation errors $e_x(t)$ and $e_f(t)$ is stable and the $\mathcal{L}_2-$gain of the transfer from $\delta(t)$ to $e(t)$ is bounded if there exists symmetric and positive definite matrices X_1, X_2 and P_2, matrices $\bar{H}_i$ and $\bar{K}$ and a positive scalar $\bar{\gamma}$ solution to the following optimization problem*

$$\min_{X_1, X_2, P_2, \bar{K}_i, \bar{H}_i} \quad \bar{\gamma} \quad (61)$$

s.t.

$$\begin{array}{l} \mathrm{Y_{ii}} < 0, i = 1, ..., r \\ \frac{1}{r-1} Y_{ii} + Y_{ij} + Y_{ji} < 0, i < j \end{array} \quad (62)$$

where

$$Y_{ij} = \begin{pmatrix} \Psi_i & -B_i M & 0 & I_n & X_1 \\ * & -2\lambda X & \lambda I_{n+n_u} & 0 & 0 \\ * & * & \Delta_{ij} & 0 & 0 \\ * & * & * & -\bar{\gamma} I_n & 0 \\ * & * & * & * & -I_n \end{pmatrix} < 0 \quad (63)$$

$$\Psi_i = \mathbb{S}(A_i X_1 - B_i \bar{K}) \quad (64)$$

$$\Delta_{ij} = \mathbb{S}\left(P_2 \tilde{A}_i - \bar{H}_i \tilde{C}_{ij}\right) \quad (65)$$

$$M = \begin{pmatrix} \bar{K} & X_2 \end{pmatrix}, \quad X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \quad (66)$$

The controller and observer gains are computed from

$$H_i = \begin{pmatrix} H_{1i} \\ H_{2i} \end{pmatrix} = P_2^{-1}\bar{H}_i, \;\; K = \bar{K}X_1^{-1} \tag{67}$$

and the $\mathcal{L}_2$-gain from $\delta(t)$ (58) *to the tracking error $e(t)$ is obtained by*

$$\gamma = \sqrt{\bar{\gamma}} \tag{68}$$

Proof. With the variable changes $X_1 = P_1^{-1}$, $\bar{K} = KX_1$, $\bar{H}_i = P_2H_i$, $M = \begin{bmatrix}\bar{K} & X_2\end{bmatrix}$ which implies

$$B_iM = \tilde{L}_i \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} = \tilde{L}_iX$$

and a Schur complement, the inequalities (63) are equivalent to

$$\begin{pmatrix} \Xi_i & -\tilde{L}_iX & I_n \\ * & -2\lambda X - \lambda^2\Delta_{ij}^{-1} & 0 \\ * & * & -\gamma^2 I_n \end{pmatrix} < 0 \tag{69}$$

where Δ_i is defined in (65) and Ξ_i given by

$$\Xi_i = \mathbb{S}\left(A_iP_1^{-1} - B_iKP_1^{-1}\right) + P_1^{-1}P_1^{-1} \tag{70}$$

If (63) hold, then the $(3,3)$ blocks of Y_i, i.e. Δ_i, are negative definite and the two following equivalent inequalities yield

$$\begin{aligned} & \left(X + \lambda\Delta_{ij}^{-1}\right)^T \Delta_{ij} \left(X + \lambda\Delta_{ij}^{-1}\right) \leq 0 \\ \Leftrightarrow \quad & X\Delta_{ij}X \leq -\lambda\left(X + X^T\right) - \lambda^2\Delta_{ij}^{-1} \end{aligned} \tag{71}$$

Consequently, (69) implies

$$\begin{pmatrix} \Xi_i & -\tilde{L}_iX & I_n \\ * & X\Delta_{ij}X & 0 \\ * & * & -\gamma^2 I_n \end{pmatrix} < 0 \tag{72}$$

Pre- and post-multiplying (72) by $diag(P_1, X^{-1}, I_n)$, it follows that (63) implies

$$\mathcal{N}_{ij} = \begin{pmatrix} \mathbb{S}(\Lambda_i) + I_n & -P_1\tilde{L}_i & P_1 \\ * & \Delta_{ij} & 0 \\ * & * & -\gamma^2 I_n \end{pmatrix} < 0 \tag{73}$$

with Λ_i defined by $\Lambda_i = P_1A_i - P_1B_iK$. Choosing a quadratic Lyapunov function defined by

$$V(\tilde{e}(t)) = \tilde{e}^T(t) \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \tilde{e}(t)$$

with $P_1 \in \mathbb{R}^{n\times n}$ and $P_2 \in \mathbb{R}^{(n+n_u)\times(n+n_u)}$ real symmetric positive definite matrices, it is easily derived from (59) that

$$\dot{V}(\tilde{e}(t)) + e^T(t)e(t) - \gamma^2\delta^T(t)\delta(t) = \begin{pmatrix} e(t) \\ \delta(t) \end{pmatrix}^T \mathcal{N}_{ij} \begin{pmatrix} e(t) \\ \delta(t) \end{pmatrix} \tag{74}$$

As a consequence, (63) implies

$$\dot{V}(\tilde{e}(t)) + e^T(t)e(t) - \gamma^2\delta^T(t)\delta(t) < 0$$

which is well known to be a sufficient condition for the $\mathcal{L}_2$-gain from $\delta(t)$ to $e(t)$ to be bounded by γ and for $\tilde{e}(t)$ to asymptotically converges toward zero when $\delta(t)$ is identically null. Finlay, the application of Tuan's lemma, given in (Tuan et al., 2001), and minimization of γ leads to the the optimization problem with LMI constraints given in theorem 3. ■

Remark 4. The conservatism of the results given in the previous theorem could be reduced by applying, in similar way, the Polya's theorem presented in the first strategy.

Remark 5. The assumption that the fault signal is constant over the time is restrictive, but in many practical situations where the faults are slowly time-varying signals, the estimation of the faults is correct, and the proposed FTC scheme can be applied. In the case where the faults are not slowly time-varying or constant, the PIO can be replaced by a Proportional Multiple Integral Observer (PMIO). Such an observer was introduced in order to filter high-frequency disturbance in (Ibrir, 2004). It is able to estimate a large class of time-varying signals which satisfies the assumption $f^{(q+1)}(t) = 0$. The principle of this observer is based on the estimation of the q first derivatives of the signal $f(t)$. This observer can also be extended to the case where $f^{(q+1)}(t)$ is bounded (see (Ichalal et al., 2009b)). It is also possible to use the adaptive observer given in the first part of the paper, since, as shown in the example, this observer can estimate faults with fast variations.

8. Simulation examples

In this section, two examples are proposed to illustrate the proposed FTC. The first one discusses some aspects of the proposed approach and the second one is dedicated to actuator fault tolerant control of lateral dynamics of a vehicle.

8.1. Example 1. To illustrate the proposed actuator fault tolerant control strategy for T-S systems with measurable premise variables and affected by actuator faults, two academic examples are presented.

8.2. First case : $\xi(t) = u(t)$. Consider the T-S system described by

$$\begin{cases} \dot{x}_f(t) = \sum\limits_{i=1}^{r} \mu_i(u(t)) \left(A_ix_f(t) + B_i(u_f(t) + f(t))\right) \\ y_f(t) = Cx_f(t) + \omega(t) \end{cases} \tag{75}$$

where

$$A_1 = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 1 & -8 \end{bmatrix}, \; A_2 = \begin{bmatrix} -3 & 2 & -2 \\ 0 & -3 & 0 \\ 5 & 2 & -4 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

The weighting functions depend on the input $u(t)$ which is the nominal control of the system in the fault-free case; they are defined by $\mu_1(u(t)) = (1 - u(t))/2$ and $\mu_2(u(t)) = 1 - \mu_1(u(t))$. To apply the proposed FTC strategy, the following reference model is considered

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} \mu_i(u(t)) \left(A_i x(t) + B_i u(t)\right) \\ y(t) = Cx(t) \end{cases} \tag{76}$$

The fault $f(t)$ is time varying and defined as follows

$$f(t) = \begin{cases} 0, & t < 10 \\ -0.5u(t), & 10 \leq t \leq 20 \\ 1, & 20 < t \end{cases} \tag{77}$$

and $\omega(t)$ is a random signal with maximum value corresponds to 10% of the maximum amplitude of the measured signal $y(t)$. Notice that even if the assumption $\dot{f}(t) = 0$ is not satisfied, the PIO is able to reconstruct time varying signals with slow variation.

Solving the optimization problem under LMI constraints in theorem 3 with $\lambda = 20$, results in the following matrices

$$H_{11} = \begin{bmatrix} -24.84 & 59.47 \\ 30.05 & -29.75 \\ 31.54 & -43.02 \end{bmatrix}, \; H_{12} = \begin{bmatrix} -11.03 & 45.34 \\ 31.58 & -33.25 \\ 17.80 & -26.25 \end{bmatrix}$$

$$H_{21} = \begin{bmatrix} 337.82 & -356.67 \end{bmatrix}, \; H_{21} = \begin{bmatrix} 338.57 & -353.93 \end{bmatrix}$$

$$K = \begin{bmatrix} 6.5179 & 4.9204 & 1.2659 \end{bmatrix}, \; \gamma = 0.4721$$

The figure 5 (top) shows the time evolution of the fault $f(t)$ and its estimate $\hat{f}(t)$, while the bottom part depicts the nominal control $u(t)$ and the FTC $u_f(t)$. The state estimation errors, $x_f(t) - \hat{x}_f(t)$ are shown in the top of figure 6, while the bottom part shows the state tracking errors $x(t) - x_f(t)$. Finally, figure 7 allows the comparison of the reference model states with the state obtained when the system is faulty without any modification of the control law and those of the system when using FTC.

Even if a fault occurs, the system trajectory follows the trajectory of the reference model which represents the trajectory of the system in the fault-free situation. Thus, the FTC control law compensates the fault and allows a normal functioning of the system in the presence of faults.

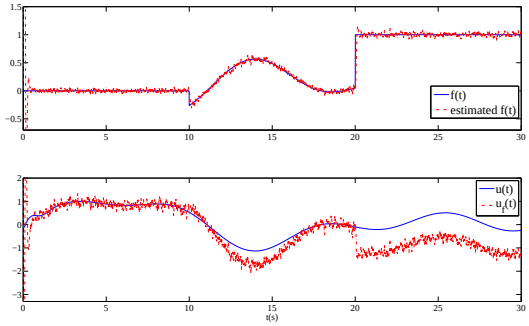

Fig. 5. Fault and its estimates (top) Nominal control and FTC (bottom)

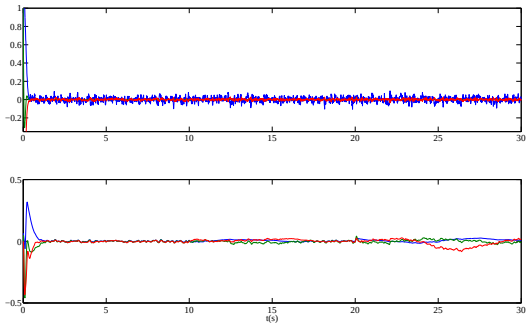

Fig. 6. State estimation errors (top) State tracking errors (bottom)

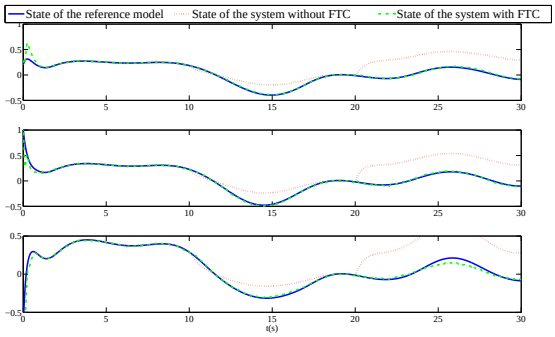

Fig. 7. Comparison between states of the system without fault, states with fault and nominal control and states with fault and FTC

8.3. Second case : $\xi(t) = y(t)$. In this subsection, the previous system is considered, but with weighting functions depending on the first component of the system output vector. The figure 8 illustrates the state estimation errors (top) and the state tracking errors (bottom). It is clear that the use of weighting functions depending on the output of the system provides better results than the case where they are depending on the control input. This is

due to the fact that the system is only affected by actuator faults and the perturbation like term $\delta(t)$ converges to zero when $y_f(t)$ converges to the reference $y(t)$. But in the previous simulation, the term $\delta(t)$ did not converge to zero, in the presence of fault, because $u(t) \neq u_f(t)$ which leads to $\mu_i(u(t)) \neq \mu_i(u_f(t))$. As a conclusion, consid-

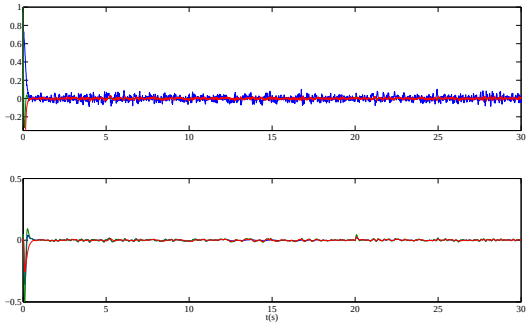

Fig. 8. State estimation errors (top) State tracking errors (bottom)

ering the problem of fault tolerant control of T-S systems with actuator faults, it is more interesting to use the output of the system as a premise variable. However, when actuator and sensor faults simultaneously occur, better results are obtained by using the state of the system as a premise variable. This is a more difficult and general case but the obtained state tracking error is less than the ones obtained above, first results on this point are published in (Ichalal et al., 2010).

8.4. Example 2. In this second example, a interesting model is used. Security and assistance in vehicles are important especially in dangerous situations like cornering and land keeping with high speed. For that purpose, this example deals with the problem of fault tolerant control of lateral dynamics of a vehicle in cornering situation (for example). Let us consider the nonlinear model of the vehicle lateral dynamics given by the following differential equations

$$\begin{cases} m(\dot{v} + ur) = 2\left(F_f + F_r\right) \\ J\dot{r} = 2\left(a_f F_f - a_r F_r\right) \end{cases} \tag{78}$$

where u and v are the longitudinal and lateral velocities ($v = \beta u$), r is the yaw rate, β denotes the side slip angle, m and J are the mass and the yaw moment of inertia respectively, a_f and a_r are the distances of the front and rear axle from the center of gravity. F_f and F_r are front and rear lateral forces. A T-S simplified model as given in (Oudghiri et al., 2008) is represented by

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{2} \mu_i\left(|\alpha_f|\right)\left(A_i x(t) + B_i \delta_f\right) \\ y(t) = \sum_{i=1}^{2} \mu_i\left(|\alpha_f|\right)\left(C_i x(t) + D_i \delta_f\right) \end{cases} \tag{79}$$

where $x^T = [v \;\; r]$, δ_f is the steering angle of the front wheel. The output y contains the lateral acceleration and yaw rate given by the inertial unit and expressed as functions of state variables. The matrices defining the model are

$$A_1 = \begin{pmatrix} -6.9426 & -0.8775 \\ 26.5175 & -7.8343 \end{pmatrix}, \; B_1 = \begin{pmatrix} 3.4892 \\ 39.2914 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -0.4735 & -0.9971 \\ 0.6356 & -0.4921 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0.2767 \\ 3.1155 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} -6.9426 & 2.4491 \\ 0 & 1 \end{pmatrix}, \; D_1 = \begin{pmatrix} 69.7847 \\ 0 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} -0.4735 & 0.0587 \\ 0 & 1 \end{pmatrix}, \; D_2 = \begin{pmatrix} 5.5333 \\ 0 \end{pmatrix}$$

The weighting functions depend on the slip angle α_f which is assumed to be known. An additive actuator fault signal affecting the vehicle is given in the form

$$f(t) = \begin{cases} 0 & t < 10 \\ -0.1\sin(0.314t) & 10 \leq t < 14.95 \\ 0.1 & t \geq 14.95 \end{cases} \tag{80}$$

Furthermore, a random noise with maximal magnitude 0.05 is added to output measurements. Figure 9 illustrates a comparison between the states of the reference model (without fault), the states of the faulty vehicle without FTC and finally the states of the vehicle with FTC. Clearly, the proposed strategy is robust with respect to actuator additive fault $f(t)$. In figure 10, we can observe the estimation of the fault. Moreover, the FTC scheme provides good results in the presence of measurement noises as shown in the figures. Figure 11 illustrates the state estimation of vehicle states.

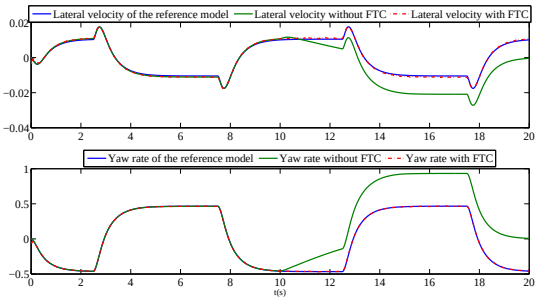

Fig. 9. States of the reference model, vehicle model without FTC and with FTC

9. Conclusions and future works

This paper is dedicated to the study of a new actuator fault tolerant control for nonlinear systems described by a Takagi-Sugeno model. The stabilizing active fault tolerant

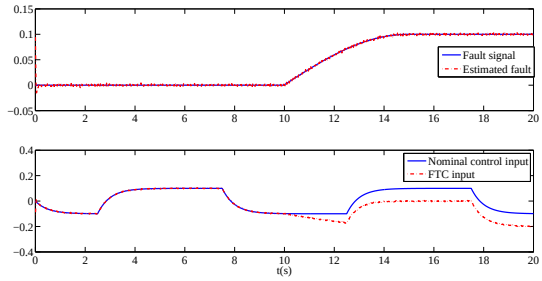

Fig. 10. Fault estimation and control inputs

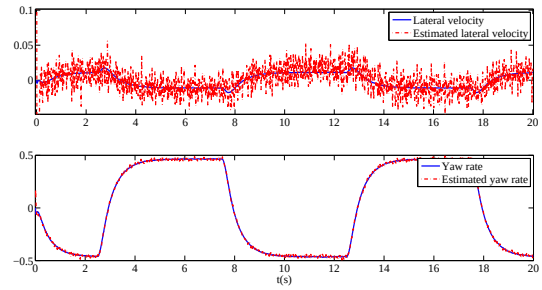

Fig. 11. State estimation

control is studied. It requires the simultaneous estimations of the state and fault, obtained by the proposed adaptive observer. This observer is able to simultaneously estimate time varying faults and state variables with a good accuracy. Furthermore, it rapidly gives the estimations, which is important to preserve the performances of the system. The stability analysis, studied with Lyapunov theory and input-to-state stability, is proved in the case of time varying faults, and furthermore asymptotic stability is guaranteed in the case of constant faults. Sufficient stability conditions are given in terms of LMI. In order to reduce the conservatism of the given conditions, Polya's theorem is used to derive relaxed conditions for FTC design for nonlinear systems. Secondly, an extension of the first work is proposed in order to provide a new control law which forces the faulty-state of the system to track a reference trajectory given by the fault free model of the system. For that purpose the control law is modified in order to take into account the trajectory tracking error and the estimated actuator faults. Finally, an application to a nonlinear model of lateral dynamics of a vehicle with additive actuator fault is given in order to illustrate the second approach. Future works will concern the FTC of systems affected by both sensor and actuator fault and/or uncertainties and/or perturbations. Secondly, the consideration of the case when the premise variables are unmeasured (like states of the system). It will also be interesting to study the case when a set of actuators is completely out of order, in this situation the dimensions of the matrices B_i and of the control vector $u(t)$ are decreased.

References

Akhenak, A., Chadli, M., Ragot, J. and Maquin, D. (2008). Fault detection and isolation using sliding mode observer for uncertain Takagi-Sugeno fuzzy model, *16th Mediterranean Conference on Control and Automation*, Ajaccio, France.

Corless, M. and Tu, J. (1998). State and input estimation for a class of uncertain systems, *Automatica* **34**(6): 757–764.

Darouach, M., Zasadzinski, M. and Xu, S. (1994). Full-order observers for linear systems with unknown inputs, *IEEE Transactions on Automatic Control* **39**(3): 606–609.

Gao, Z. and Ding, S. X. (2007). Actuator fault robust estimation and fault-tolerant control for a class of nonlinear descriptor systems, *Automatica* **43**(5): 912–920.

Ibrir, S. (2004). Robust state estimation with q-integral observers, *American Control Conference*, Boston, MA, USA.

Ichalal, D., Marx, B., Ragot, J. and Maquin, D. (2009a). Simultaneous state and unknown inputs estimation with PI and PMI observers for Takagi-Sugeno model with unmeasurable premise variables, *17th Mediterranean Conference on Control and Automation, MED'09*, Thessaloniki, Greece.

Ichalal, D., Marx, B., Ragot, J. and Maquin, D. (2009b). Simultaneous state and unknown inputs estimation with PI and PMI observers for Takagi-Sugeno model with unmeasurable premise variables., *17th Mediterranean Conference on Control and Automation, MED'09*, Thessaloniki, Greece.

Ichalal, D., Marx, B., Ragot, J. and Maquin, D. (2010). Fault tolerant control for Takagi-Sugeno systems with unmeasurable premise variables by trajectory tracking, *IEEE International Symposium on Industrial Electronics*, Bari, Italy.

Koenig, D. and Mammar, S. (2002). Design of a proportional integral observer for unknown input descriptor systems, *IEEE Transactions on Automatic Control* **47**: 2057–2063.

Kruszewski, A., Wang, R. and Guerra, T. M. (2008). Nonquadratic stabilization conditions for a class of uncertain nonlinear discrete time TS fuzzy models: A new approach, *IEEE Transactions on Automatic Control* **53**(2): 606–611.

Leith, D. J. and Leithead, W. E. (1999). Survey of gain-scheduling analysis design, *International Journal of Control* **73**(11): 1001–1025.

Luenberger, D. (1971). An introduction to observers, *IEEE Transactions on Automatic Control* **16**(6): 596–602.

Marx, B., Koenig, D. and Ragot, J. (2007). Design of observers for Takagi Sugeno descriptor systems with unknown inputs and application to fault diagnosis, *IET Control Theory and Application* **1**(5): 1487–1495.

Mufeed, M. M., Jiang, J. and Zhang, Z. (2003). *Active Fault Tolerant Control Systems : Stochastic Analysis and Synthesis*, Springer.

Nagy, A., Mourot, G., Marx, B., Schutz, G. and Ragot, J. (2009). Model structure simplification of a biological reactor, *15th IFAC Symposium on System Identification, SYSID'09*, Saint Malo, France.

Niemann, H. and Stoustrup, J. (2005). Passive fault tolerant control of a double inverted pendulum–a case study, *Control Engineering Practice* **13**(8): 1047–1059.

Ocampo-Martinez, C., De Dona, J. and Seron, M. (2010). Actuator fault-tolerant control based on set separation, *International Journal of Adaptive Control and Signal Processing* **24(12)**: 1070–1090.

Oudghiri, M., Chadli, M. and El Hajjaji, A. (2008). Robust observer-based fault tolerant control for vehicle lateral dynamics, *International Journal of Vehicle Design* **48**(3-4): 173–189.

Patton, R. (1997). Fault-tolerant control systems : The 1997 situation, *3rd IFAC Symposium on Fault Detection, Supervision and Safety of Technical Processes*, Hull, England.

Patton, R. J. and Klinkhieo, S. (2009a). An LPV approach to active FTC of a two-link manipulator, *7th Workshop on Advanced Control and Diagnosis*, Zielona Góra, Poland.

Patton, R. and Klinkhieo, S. (2009b). Actuator fault estimation and compensation based on an augmented state observer approach, *48th IEEE Conference Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009*, Shanghai, China.

Sala, A. and Ariño, C. (2007). Asymptotically necessary and sufficient conditions for stability and performance in fuzzy control: Applications of Polya's theorem, *Fuzzy Sets and Systems* **158**(24): 2671–2686.

Stilwell, D. J. and Rugh, W. J. (1997). Interpolation of observer state feedback controllers for gain scheduling, *IEEE Transactions on Automatic Control* **44**(6): 1225–1229.

Takagi, T. and Sugeno, M. (1985). Fuzzy identification of systems and its applications to modeling and control, *IEEE Transactions on Systems, Man, and Cybernetics* **15**(1): 116–132.

Tanaka, K. and Wang, H. (2001). *Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach*, John Wiley and Sons.

Tuan, H., Apkarian, P., Narikiyo, T. and Yamamoto, Y. (2001). Parameterized linear matrix inequality techniques in fuzzy control system design, *IEEE Transactions on Fuzzy Systems* **9**(2): 324–332.

Witczak, M., Dziekan, L., Puig, V. and Korbicz, J. (2008). Design of a fault-tolerant control scheme for Takagi-Sugeno fuzzy systems, *16th Mediterranean Conference on Control and Automation*, Ajaccio, France.

Zhang, K., Jiang, B. and Cocquempot, V. (2008). Adaptive observer-based fast fault estimation, *International Journal of Control, Automation, and Systems* **6**(3): 320–326.

Zhang, K., Jiang, B. and Shi, P. (2009). A new approach to observer-based fault-tolerant controller design for Takagi-Sugeno fuzzy systems with state delay, *Circuits, Systems, and Signal Processing* **28**(5): 679–697.