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# Remarks on the boundary set of spectral equipartitions

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## 1 Introduction

Let  $\Omega$  be a bounded open set in  $\mathbb{R}^n$  (or in a Riemannian manifold), and  $k$  a positive integer  $k$ .

We denote by  $\mathcal{D}_k(\Omega)$  the set of  $k$ -tuples  $\mathcal{D} = \{\omega_1, \dots, \omega_k\}$  of open, pairwise disjoint subsets of  $\Omega$ . Denote by  $\phi$  a positive function on open subsets of  $\Omega$ , and assume that it satisfies the following two basic assumptions (i) if  $\omega_1 \subset \omega_2$ , then  $\phi(\omega_2) \leq \phi(\omega_1)$ , with strict inequality provided that  $\omega_2 \setminus \omega_1$  is not too small; (ii)  $\phi(\omega)$  tends to infinity if  $\omega$  shrinks to a point. Introduce the function  $\Phi$  defined on  $\mathcal{D}_k(\Omega)$  by  $\Phi(\mathcal{D}) = \max_{1 \leq j \leq k} \phi(\omega_j)$ , if  $\mathcal{D} = \{\omega_1, \dots, \omega_k\}$ . Minimizing the function  $\Phi$  over  $\mathcal{D}_k(\Omega)$  appears naturally when one considers  $k$  non-mixing populations (represented by the sets  $\omega_j$ ) competing for space (with the constraint represented by the function  $\phi$ ). As possible examples of constraint functions, we mention the functions  $\omega \rightarrow \text{Vol}(\omega)^{-1}$  and  $\omega \rightarrow \lambda_1(\omega)$ , where  $\lambda_1(\omega)$  is the ground state energy of the Dirichlet realization of the Laplacian in  $\omega$ . It is easy to see that if there exists a minimal  $k$ -tuple  $\mathcal{D} = \{\omega_1, \dots, \omega_k\}$ , *i.e.* one which realizes the infimum of  $\Phi$  over  $\mathcal{D}_k(\Omega)$ , then  $\mathcal{D}$  must actually be a partition of  $\Omega$  and, more precisely, an equipartition, which means that  $\mathcal{D}$  also satisfies  $\phi(\omega_1) = \dots = \phi(\omega_k)$ . For more details on this subject, we refer to the papers [BBO, CL, CTV1, CTV2, CTV3, Ha, Hel].

In this paper, we consider the constraint function  $D \rightarrow \lambda_1(D)$ . In this framework, it can be shown that minimal partitions do exist, and that they have regular representatives (see Properties 2.7). Although the analysis of minimal  $k$ -partitions is rather standard when  $k = 2$  (we find the nodal domains of a second eigenfunction), the analysis for higher values of  $k$  becomes non trivial and quite interesting. If  $u$  is an eigenfunction of the Dirichlet realization of

the Laplacian in  $\Omega$ , the nodal domains of  $u$  form an equipartition of  $\Omega$  with boundaries the nodal set of  $u$ .

The purpose of this paper is to revisit various properties of nodal sets, and to explore if they are also true for minimal spectral partitions, or more generally for spectral equipartitions (here the word *spectral* refers to our choice of constraint function).

In Section 2, we review some basic notions related to our choice of constraint function. In Section 3, we prove a lower bound for the length of the boundary set of a partition in the 2-dimensional situation. In Section 4, we consider estimates involving the cardinality of the partition.

## 2 Definitions and notations

### 2.1 Spectral theory

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$ , or a compact Riemannian surface, possibly with boundary  $\partial\Omega$ , which we assume to be piecewise  $C^1$ . Let  $H(\Omega)$  be the realization of the Laplacian, or of the Laplace-Beltrami operator,  $-\Delta$  in  $\Omega$ , with Dirichlet boundary condition ( $u|_{\partial\Omega} = 0$ ). Let  $\{\lambda_j(\Omega)\}_{j \geq 1}$  be the increasing sequence of the eigenvalues of  $H(\Omega)$ , counted with multiplicity. The eigenspace associated with  $\lambda_k$  is denoted by  $E(\lambda_k)$ .

A groundstate  $u \in E(\lambda_1)$  does not vanish in  $\Omega$  and can be chosen to be positive. On the contrary, any non-zero eigenfunction  $u \in E(\lambda_k)$ ,  $k \geq 2$ , changes sign in  $\Omega$ , and hence has a nonempty zero set or *nodal set*,

$$N(u) = \overline{\{x \in \Omega \mid u(x) = 0\}}. \quad (2.1)$$

The connected components of  $\Omega \setminus N(u)$  are called the *nodal domains* of  $u$ . The number of nodal domains of  $u$  is denoted by  $\mu(u)$ .

Courant's nodal domain theorem says:

**Theorem 2.1 (Courant)** *Let  $k \geq 1$ , and let  $E(\lambda_k)$  be the eigenspace of  $H(\Omega)$  associated with the eigenvalue  $\lambda_k$ . Then,  $\forall u \in E(\lambda_k) \setminus \{0\}$ ,  $\mu(u) \leq k$ .*

Except in dimension 1, the inequality is strict in general. More precisely, we have:

**Theorem 2.2 (Pleijel)** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$ . There exists a constant  $k_0$  depending on  $\Omega$ , such that if  $k \geq k_0$ , then*

$$\mu(u) < k, \forall u \in E(\lambda_k) \setminus \{0\}.$$

Both theorems are proved in [Pl]. The main points in the proof of Pleijel's Theorem are the Faber-Krahn inequality and the Weyl asymptotic law. Faber-Krahn's inequality states that, for any bounded domain  $\omega$  in  $\mathbb{R}^2$ ,

$$\lambda_1(\omega) \geq \frac{\pi \mathbf{j}^2}{A(\omega)}, \quad (2.2)$$

where  $A(\omega)$  is the area of  $\omega$ , and  $\mathbf{j}$  is the least positive zero of the Bessel function of order 0 ( $\mathbf{j} \sim 2.4048$ ). Weyl's asymptotic law for the eigenvalues of  $H(\omega)$  states that

$$\lim_{k \rightarrow \infty} \frac{\lambda_k(\omega)}{k} = \frac{4\pi}{A(\omega)}. \quad (2.3)$$

Let  $\bar{\mu}(k)$  be the maximum value of  $\mu(u)$  when  $u \in E(\lambda_k) \setminus \{0\}$ . Combining the results of Faber-Krahn and Weyl, we obtain,

$$\limsup_{k \rightarrow +\infty} \frac{\bar{\mu}(k)}{k} \leq 4/\mathbf{j}^2 < 1. \quad (2.4)$$

**Remark 2.3** *Pleijel's Theorem extends to bounded domains in  $\mathbb{R}^n$ , and more generally to compact  $n$ -manifolds with boundary, with a constant  $\gamma(n) < 1$  replacing  $4/\mathbf{j}^2$  in the right-hand side of (2.4) (Peetre [Pe], Bérard-Meyer [BeMe]). It is also interesting to note that this constant is independent of the geometry.*

**Remark 2.4** *It follows from Pleijel's Theorem that the equality  $\bar{\mu}(k) = k$  can only occur for finitely many values of  $k$ . The analysis of the equality case is very interesting. We refer to [HHOT1] for more details.*

**Remark 2.5** *In dimension 1, counting the nodal domains of an eigenfunction of a Dirichlet Sturm-Liouville problem in some interval  $[a, b]$  is the same as counting the number of zeroes of the eigenfunction. An analog in dimension 2 is to consider the length of the nodal set of eigenfunctions instead of the number of their nodal domains. We shall come back to this question in Section 3.*

## 2.2 Partitions

For this section, we refer to [HHOT1]. Let  $k$  be a positive integer. A (weak)  $k$ -partition of the open bounded set  $\Omega$  is<sup>1</sup> a family  $\mathcal{D} = \{D_j\}_{j=1}^k$  of pairwise disjoint sets such that  $\cup_{j=1}^k D_j \subset \Omega$ . We denote by  $\mathfrak{D}_k = \mathfrak{D}_k(\Omega)$  the set of  $k$ -partitions such that the domains  $D_j$  are open and connected.

Given  $\mathcal{D} \in \mathfrak{D}_k$ , we define the *energy*  $\Lambda(\mathcal{D})$  of the partition as,

$$\Lambda(\mathcal{D}) = \max_j \lambda(D_j), \quad (2.5)$$

where  $\lambda(D_j)$  is the groundstate energy of  $H(D_j)$ . We now define the number  $\mathfrak{L}_k(\Omega)$  as,

$$\mathfrak{L}_k(\Omega) = \inf_{\mathcal{D} \in \mathfrak{D}_k} \Lambda(\mathcal{D}). \quad (2.6)$$

A partition  $\mathcal{D}$  is called *minimal* if  $\Lambda(\mathcal{D}) = \mathfrak{L}_k(\Omega)$ .

**Example.** The nodal domains of an eigenfunction  $u \in E(\lambda) \setminus \{0\}$  of  $H(\Omega)$  form a  $\mu(u)$ -partition of  $\Omega$  denoted by  $\mathcal{D}(u)$ . Such a partition is called a *nodal partition*.

<sup>1</sup>Note that we start from a very weak notion of partition. We refer to [HHOT1] for a more precise definition of classes of  $k$ -partitions, and for the notion of regular representatives.

It turns out that  $\mathfrak{L}_2(\Omega) = \lambda_2(\Omega)$ , and that minimal 2-partitions are nodal partitions. The situation when  $k \geq 3$  is more complicated, and more interesting, [HHOT1].

A partition  $\mathcal{D} = \{D_j\}_{j=1}^k \in \mathfrak{D}_k(\Omega)$  is called *strong* if,

$$\text{Int}(\overline{\cup_j D_j}) = \Omega. \quad (2.7)$$

The *boundary set*  $N(\mathcal{D})$  of a strong partition  $\mathcal{D}_k = \{D_j\}_{j=1}^k \in \mathfrak{D}(\Omega)$  is the closed set,

$$N(\mathcal{D}) = \overline{\cup_j (\partial D_j \cap \Omega)}. \quad (2.8)$$

The set  $\mathcal{R}_k(\Omega)$  of *regular  $k$ -partitions* is the subset of strong  $k$ -partitions in  $\mathfrak{D}_k(\Omega)$  whose boundary set  $N = N(\mathcal{D})$  satisfies the following properties.

- (i) The set  $N$  is locally a regular curve in  $\Omega$ , except possibly at finitely many points  $\{y_i\} \in N \cap \Omega$ , in the neighborhood of which  $N$  is the union of  $\nu(y_i)$  smooth semi-arcs at  $y_i$ ,  $\nu(y_i) \geq 3$ .
- (ii) The set  $N \cap \partial\Omega$  consists of finitely many points  $\{z_j\}$ . Near the point  $z_j$ , the set  $N$  is the union of  $\rho(z_j) \geq 1$  semi-arcs hitting  $\partial\Omega$  at  $z_j$ .
- (iii) The set  $N$  has the equal angle property. More precisely, at any *interior singular point*  $y_i$ , the semi-arcs meet with equal angles; at any *boundary singular point*  $z_j$ , the semi-arcs form equal angles together with the boundary  $\partial\Omega$ .

**Example.** A nodal partition  $\mathcal{D}(u)$  provides an example of a regular partition, and the boundary set  $N(\mathcal{D}(u))$  coincides with the nodal set  $N(u)$ . Note that for a regular partition, the number  $\nu(y_i)$  of semi-arcs at an interior singular point may be odd, whereas it is always even for a nodal partition.

Let us now introduce:

**Definition 2.6** We call spectral equipartition a strong  $k$ -partition  $\mathcal{D} = \{D_i\}$  such that  $\lambda(D_i) = \Lambda(\mathcal{D})$ , for  $i = 1, \dots, k$ . The number  $\Lambda(\mathcal{D})$  is called the energy of the equipartition.

**Example.** Nodal partitions provide examples of spectral equipartitions.

**Properties 2.7** Given an open bounded set  $\Omega$ ,

- (i) Minimal  $k$ -partitions exist ([CTV1, CTV2, CTV3]).
- (ii) Any minimal  $k$ -partition has a representative (modulo sets of capacity 0) which is a regular spectral equipartition ([HHOT1]).

### 2.3 Euler formula

Let  $\Omega \subset M$  be a bounded domain with piecewise smooth boundary  $\partial\Omega$ . Let  $N \subset \overline{\Omega}$  be a regular closed set (in the sense of Section 2.2, properties (i)-(iii)) such that the family  $\mathcal{D} = \{D_1, \dots, D_k\}$  of connected components of  $\Omega \setminus N$  is a regular, strong partition of  $\Omega$ . Recall that for a singular point  $y \in N \cap \Omega$ ,  $\nu(y)$  is the number of semi-arcs at  $y$ , and that for a singular point  $z \in N \cap \partial\Omega$ ,  $\rho(z)$  is the number of semi-arcs at  $z$ , not counting the two arcs contained in  $\partial\Omega$ . Let  $\mathcal{S}(\mathcal{D})$  denote the set of singular points of  $N(\mathcal{D})$ , both interior or boundary points, if any. We define the *index* of a point  $x \in \mathcal{S}(\mathcal{D})$  to be,

$$\iota(x) = \begin{cases} \nu(x) - 2, & \text{if } x \text{ is an interior singular point,} \\ \rho(x), & \text{if } x \text{ is a boundary singular point.} \end{cases} \quad (2.9)$$

We introduce the number  $\sigma(\mathcal{D})$  to be,

$$\sigma(\mathcal{D}) = \sum_{x \in \mathcal{S}(\mathcal{D})} \iota(x). \quad (2.10)$$

For a regular strong  $k$ -partition  $\mathcal{D} = \{D_j\}_{j=1}^k$  of  $\Omega$ , we have Euler's formula,

$$\chi(\Omega) + \frac{1}{2}\sigma(\mathcal{D}) = \sum_{j=1}^k \chi(D_j). \quad (2.11)$$

We refer to [HOMN] for a combinatorial proof of this formula in the case of an open set of  $\mathbb{R}^2$ . One can give a Riemannian proof using the global Gauss-Bonnet theorem. For a domain  $D$  with piecewise smooth boundary  $\partial D$  consisting of piecewise  $C^1$  simple closed curves  $\{C_i\}_{i=1}^n$ , with corners  $\{p_{i,j}\}$  ( $i = 1, \dots, n$  and  $j = 1, \dots, m_i$ ) and corresponding interior angles  $\theta_{i,j}$ , we have

$$2\pi\chi(D) = \int_D K + \sum_{i=1}^n \beta(C_i), \quad (2.12)$$

where

$$\beta(C_i) = \int_{C_i} \langle k, \nu_D \rangle + \sum_{j=1}^{m_i} (\pi - \theta_{i,j}).$$

In this formula,  $k$  is the geodesic curvature vector of the regular part of the curve  $C_i$ , and  $\nu_D$  is the unit normal to  $C_i$  pointing inside  $D$ .

To prove (2.11), it suffices to sum up the Gauss-Bonnet formulas relative to each domain  $D_j$  and to take into account the following facts:

- the integrals of the Gaussian curvature over the  $D_j$ 's add up to the integral of the Gaussian curvature over  $\Omega$ ,
- cancellations occur when adding the integrals of the geodesic curvature over the curves bounding two adjacent  $D_j$  (the unit normal vectors point in opposite directions), while they add up to give the integral of the geodesic curvature over the boundary of  $\Omega$ ,

- there are contributions coming from the angles associated with the singular points of  $N$  and, when summed up, these contributions yield the second term in the left-hand side of (2.11).

Note that the proof of (2.11) does not use the fact that the semi-arcs meet at the singular points of  $N$  with equal angles.

### 3 Lower bounds for the length of the boundary set of a regular spectral equipartition

#### 3.1 Introduction

Let  $\mathcal{D} = \{D_1, \dots, D_k\}$  be a regular spectral equipartition with energy  $\Lambda = \Lambda(\mathcal{D})$ . The boundary set  $N(\mathcal{D})$  of the partition consists of singular points  $\{y_i\}_{i=1}^a$  inside  $\Omega$ , of singular points  $\{z_i\}_{i=1}^b$  on  $\partial\Omega$ , of  $C^1$  arcs  $\{\gamma_i\}_{i=1}^c$  which bound two adjacent domains of the partition, and of arcs  $\{\delta_i\}_{i=1}^d$  contained in  $\partial\Omega$ . We define the length of the boundary set  $N(\mathcal{D})$  by the formula,

$$P(\mathcal{D}) := \sum_{i=1}^c \ell(\gamma_i) + \frac{1}{2} \ell(\partial\Omega), \quad (3.1)$$

where  $\ell$  denotes the length of the curves. Note that  $\ell(\partial\Omega) = \sum_{i=1}^d \ell(\delta_i)$  and that  $\sum_{i=1}^k \ell(\partial D_i) = 2P(\mathcal{D})$ .

In this section, we investigate lower bounds for  $P(\mathcal{D})$  in terms of the energy  $\Lambda(\mathcal{D})$  and the area  $A(\Omega)$ .

As a matter of fact, we show that the methods introduced in [BrGr, Br, Sa1] apply to regular spectral equipartitions, and hence to minimal partitions. We provide three estimates.

1. The first estimate holds for plane domains, and follows the method of [BrGr].
2. The second estimate applies to a compact Riemannian surface (with or without boundary), and follows the method of [Sa1].
3. The third estimate is a local estimate based on the method of [Br].

Let  $\mathcal{D} = \{D_i\}_{i=1}^k$  be a regular spectral equipartition with energy  $\Lambda = \Lambda(\mathcal{D})$ . Let  $R(D_i)$  be the inner radius of the set  $D_i$ . Recall that  $\mathbf{j}$  denotes the least positive zero of the Bessel function of order 0.

#### 3.2 The method of Brüning-Gromes

In this section,  $\Omega$  is a bounded domain in  $\mathbb{R}^2$ , with piecewise  $C^1$  boundary. We only sketch the method which relies on three inequalities.

1. The monotonicity of eigenvalues and the characterization of the ground state imply that

$$\forall i, 1 \leq i \leq k, \quad R(D_i) \leq \frac{\mathbf{j}}{\sqrt{\Lambda}}. \quad (3.2)$$

2. The Faber-Krahn inequality and the isoperimetric inequality imply that

$$\forall i, 1 \leq i \leq k, \quad \frac{2\pi\mathbf{j}}{\sqrt{\Lambda}} \leq \ell(\partial D_i). \quad (3.3)$$

3. The generalized F ej es-Toth isoperimetric inequality ([BrGr], Hilfssatz 2) asserts that, for  $1 \leq i \leq k$ ,

$$A(D_i) \leq R(D_i)\ell(\partial D_i) - \chi(D_i)\pi R^2(D_i). \quad (3.4)$$

Using that  $\chi(D_i) \leq 1$ , we immediately see that the function  $r \rightarrow r\ell(\partial D_i) - \chi(D_i)\pi r^2$  is non-decreasing for  $2\pi r \leq \ell(\partial D_i)$ . Using inequalities (3.2) and (3.3), it follows that one can substitute  $\frac{\mathbf{j}}{\sqrt{\Lambda}}$  to  $R(D_i)$  in (3.4) and obtain,

$$A(D_i) \leq \frac{\mathbf{j}}{\sqrt{\Lambda}}\ell(\partial D_i) - \chi(D_i)\pi\left(\frac{\mathbf{j}}{\sqrt{\Lambda}}\right)^2, \quad \text{for } 1 \leq i \leq k. \quad (3.5)$$

Summing up the inequalities (3.5), for  $1 \leq i \leq k$ , we obtain

$$A(\Omega) \leq \frac{\mathbf{j}}{\sqrt{\Lambda}} \sum_i \ell(\partial D_i) - \sum_i \chi(D_i)\pi \frac{\mathbf{j}^2}{\Lambda}.$$

Using Euler's formula (2.11), we conclude that

$$A(\Omega) \leq \frac{2\mathbf{j}}{\sqrt{\Lambda}}P(\mathcal{D}) - [\chi(\Omega) + \frac{1}{2}\sigma(\mathcal{D})]\pi \frac{\mathbf{j}^2}{\Lambda}.$$

We have proved,

**Proposition 3.1** *Let  $\Omega$  be a bounded open set in  $\mathbb{R}^2$ , and let  $\mathcal{D}$  be a regular spectral equipartition of  $\Omega$ . The length  $P(\mathcal{D})$  of the boundary set of  $\mathcal{D}$  is bounded from below in terms of the energy  $\Lambda(\mathcal{D})$ . More precisely,*

$$\frac{A(\Omega)}{2\mathbf{j}}\sqrt{\Lambda(\mathcal{D})} + \frac{\pi\mathbf{j}}{2\sqrt{\Lambda(\mathcal{D})}}[\chi(\Omega) + \frac{1}{2}\sigma(\mathcal{D})] \leq P(\mathcal{D}). \quad (3.6)$$

Note that (3.6) is actually slightly better than the estimate in [BrGr] which does not take into account the term  $\sigma(\mathcal{D})$  when  $\mathcal{D}$  is the nodal partition for an eigenfunction  $u$  associated with the eigenvalue  $\Lambda$ . This fact is suggested in [Sa1].



### 3.3 The method of Savo

In this section, we follow the method of Savo [Sa1], and keep the same notations and assumptions. We sketch the proof in the case with boundary as it is not detailed in [Sa1]. Here,  $\Omega$  is a compact Riemannian surface with boundary. We denote the Laplace-Beltrami operator by  $\Delta$  and the Gaussian curvature by  $K$ . We write  $K = K_+ - K_-$  (the negative and positive parts of the curvature).

We assume that numbers  $\alpha \geq 0$  and  $D$  are given such that:

$$K \geq -\alpha^2 \text{ and } \delta(\Omega) \leq D ,$$

where  $\delta(\Omega)$  is the diameter of  $\Omega$ . Finally, we define the numbers

$$B(\Omega) = \int_{\Omega} K_+ - 2\pi\chi(\Omega) ,$$

and

$$C(\alpha, D) = \sqrt{\pi^2 + \frac{1}{4}\alpha^2 D^2} .$$

We recall the following results from [Sa1].

**Lemma 3.2** *Let  $\Omega$  be a compact Riemannian surface with piecewise  $C^1$  boundary. Then*

$$\frac{2}{\pi}A(\Omega)\sqrt{\lambda(\Omega)} \leq \ell(\partial\Omega) + R(\Omega) \max\{B(\Omega), 0\} , \quad (3.7)$$

where  $\lambda(\Omega)$  is the ground state energy of the Dirichlet realization for the Laplacian in  $\Omega$ , and  $R(\Omega)$  the inner radius of  $\Omega$ .

This is Proposition 3 in [Sa1] (p. 137). Note that when  $M$  is flat and  $\Omega$  is simply or doubly connected, we recover Polya's inequality [P] which reads:

$$\frac{2}{\pi}A(\Omega)\sqrt{\lambda(\Omega)} \leq \ell(\partial\Omega) . \quad (3.8)$$

**Lemma 3.3** *Let  $\Omega$  be a compact Riemannian surface with piecewise  $C^1$  boundary. Then,*

$$R(\Omega)\sqrt{\lambda(\Omega)} \leq \min \left\{ C(\alpha, D), \sqrt{\pi^2 + \frac{\alpha^2 C^2(\alpha, D)}{4\lambda(\Omega)}} \right\} =: \psi(\alpha; D; \lambda(\Omega)) .$$

This is Lemma 10 in [Sa1] (p. 141) using  $\lambda(\Omega)$  instead of  $\lambda$ .

**Lemma 3.4** *Let  $\Omega$  be a compact Riemannian surface with piecewise  $C^1$  boundary. Assume that  $B(\Omega) < 0$ . Then,*

$$2|B(\Omega)| \leq \lambda(\Omega)A(\Omega) \leq \frac{\pi}{2}\sqrt{\lambda(\Omega)}\ell(\partial\Omega) .$$

This is Lemma 11 in [Sa1] (p. 141), which relies on Dong's paper [Dong]. Note that the second inequality follows from Lemma 3.2.

Let us now proceed with the lower estimate of  $P(\mathcal{D})$  when  $\Omega$  is a Riemannian surface with boundary.

**Proposition 3.5** *Let  $\Omega$  be a compact Riemannian surface with piecewise  $C^1$  boundary. The length  $P(\mathcal{D})$  of the boundary set of a regular spectral equipartition  $\mathcal{D}$ , with energy  $\Lambda(\mathcal{D}) = \Lambda$ , satisfies the inequality*

$$P(\mathcal{D}) \geq \frac{4A(\Omega)\sqrt{\Lambda}}{4\pi + \pi^2\psi(\alpha, D; \Lambda)} - \frac{2\pi\psi(\alpha, D; \Lambda)}{\sqrt{\Lambda}(4\pi + \pi^2\psi(\alpha, D; \Lambda))} (B(\Omega) - \pi\sigma(\mathcal{D})). \quad (3.9)$$

**Proof.** The proof follows the ideas in [Sa1] closely. Since Savo does not provide all the details for the case with boundary, we provide them here. Lemma 3.2 applied to each  $D_j$  gives,

$$\frac{2}{\pi}A(D_j)\sqrt{\lambda(D_j)} \leq \ell(\partial D_j) + R(D_j) \max\{B(D_j), 0\}.$$

Since  $\lambda(D_j) = \Lambda$  for all  $j$ , summing up in  $j$ , we find that

$$\frac{2}{\pi}A(\Omega)\sqrt{\Lambda} \leq 2P(\mathcal{D}) + \sum_{j=1}^k R(D_j) \max\{B(D_j), 0\}. \quad (3.10)$$

Call  $T$  the second term in the right-hand side of the preceding inequality and define the sets,

$$J_+ := \{j \mid 1 \leq j \leq k, B(D_j) > 0\}, \quad J_- := \{j \mid 1 \leq j \leq k, B(D_j) \leq 0\}.$$

By Lemma 3.3, we have

$$T = \sum_{j \in J_+} R(D_j)B(D_j) \leq \frac{\psi(\alpha, D; \Lambda)}{\sqrt{\Lambda}} \sum_{j \in J_+} B(D_j). \quad (3.11)$$

Using the definition of  $B(D_j)$ , we find that

$$\begin{aligned} \sum_{j=1}^k B(D_j) &= \int_{\Omega} K_+ - 2\pi \sum_{j=1}^k \chi(D_j) \\ &= B(\Omega) + 2\pi\chi(\Omega) - 2\pi \sum_{j=1}^k \chi(D_j) \end{aligned}$$

and hence, using Euler's formula (2.11),

$$\sum_{j=1}^k B(D_j) = B(\Omega) - \pi\sigma(\mathcal{D}).$$

On the other hand, we have

$$\begin{aligned} \sum_{j \in J_+} B(D_j) &= \sum_{j=1}^k B(D_j) - \sum_{j \in J_-} B(D_j) \\ &= B(\Omega) - \pi\sigma(\mathcal{D}) + \sum_{j \in J_-} |B(D_j)|, \end{aligned}$$

and we can estimate the last term in the right-hand side using Lemma 3.4. Namely,

$$\sum_{j \in J_-} |B(D_j)| \leq \frac{\pi}{4} \sqrt{\Lambda} \sum_{j \in J_-} \ell(\partial D_j) \leq \frac{\pi}{2} \sqrt{\Lambda} P(\mathcal{D}).$$

Finally, we obtain the following estimate for  $T$ ,

$$T \leq \frac{\psi(\alpha, D; \Lambda)}{\sqrt{\Lambda}} \{B(\Omega) - \pi\sigma(\mathcal{D}) + \frac{\pi}{2} \sqrt{\Lambda} P(\mathcal{D})\}.$$

Using (3.10), it follows that

$$A(\Omega) \sqrt{\Lambda} \leq \left\{ \pi + \frac{\pi^2}{4} \psi(\alpha, D; \Lambda) \right\} P(\mathcal{D}) + \frac{\pi \psi(\alpha, D; \Lambda)}{2\sqrt{\Lambda}} \{B(\Omega) - \pi\sigma(\mathcal{D})\}. \quad (3.12)$$

This proves the proposition.  $\square$

### 3.4 A loose local lower estimate for $P(\mathcal{D})$

For simplicity, we now assume that  $\Omega$  is a bounded domain in  $\mathbb{R}^2$ , with piecewise  $C^1$  boundary. We also assume that we are given some point  $x_0 \in \Omega$ , some radius  $R$  and some positive number  $\rho$ , small with respect to  $R$ , such that  $B(x_0, R + \rho) \subset \Omega$ . Note that the ball  $B(x_0, R)$  could be replaced by any regular domain.

#### 3.4.1 A local estimate à la Brünig-Gromes : eigenvalues

**Lemma 3.6** *Let  $\lambda$  be an eigenvalue of  $H(\Omega)$ , and let  $u \in E(\lambda)$  be a non-zero eigenfunction associated with  $\lambda$ . If  $\lambda r^2 > \mathbf{j}^2$ , then any disk  $B(x, r) \subset \Omega$  contains at least a point of the nodal set  $N(u)$ .*

This follows immediately from the monotonicity of the Dirichlet eigenvalues with respect to domain inclusion.

**Lemma 3.7** *Let  $\lambda$  be an eigenvalue of  $H(\Omega)$ , assumed to be large enough. Let  $r > 0$  be such that  $0 < r \leq \rho < \frac{R}{10}$ , and  $\lambda r^2 > 4\mathbf{j}^2$ . Then, there exists a family of points  $\{x_1, \dots, x_N\}$  such that:*

- (1) For  $1 \leq j \leq N$ ,  $x_j \in N(u) \cap B(x_0, R - \frac{r}{2})$ .
- (2) The balls  $B(x_j, \frac{r}{2})$ ,  $1 \leq j \leq N$ , are pairwise disjoint and contained in  $B(x_0, R) \subset \Omega$ .
- (3) We have the inclusion  $B(x_0, R - r) \subset \cup_{j=1}^N B(x_j, 2r)$ .
- (4) The number  $N$  satisfies,  $r^2 N \geq 0.2 R^2$ .

*Proof.* (a) Consider the ball  $B(x_0, R - r)$  and take  $y_1, y_2$  to be the end points of a diameter of the closed ball. Because  $r \leq \rho < R/10$  and  $r^2 \lambda > 4\mathbf{j}^2$ , we have that  $B(y_i, \frac{r}{2}) \subset B(x_0, R - \frac{r}{2}) \subset \Omega$  and  $B(y_i, \frac{r}{2}) \cap N(u) \neq \emptyset$ . Choose  $x_i$  in

$B(y_i, \frac{r}{2}) \cap N(u)$ . Then,  $x_i \in N(u) \cap B(x_0, R - \frac{r}{2})$ ,  $B(x_1, \frac{r}{2}) \cap B(x_2, \frac{r}{2}) = \emptyset$  and  $B(x_i, \frac{r}{2}) \subset B(x_0, R) \subset \Omega$ .

(b) Take a maximal element  $\{x_1, \dots, x_N\}$  (with respect to inclusion) in the set

$$\mathcal{F} := \{(x_1, \dots, x_k) \mid x_i \in N(u) \cap B(x_0, R - \frac{r}{2}), B(x_i, \frac{r}{2}) \text{ pairwise disjoint}\},$$

so that the family  $\{x_1, \dots, x_N\}$  satisfies (1) and (2).

We claim that (3) holds. Indeed, otherwise we could find  $y \in B(x_0, R - r)$  with  $d(x_i, y) \geq 2r$ , for  $1 \leq i \leq N$ . Because  $B(y, \frac{r}{2}) \cap N(u) \neq \emptyset$ , we would find some  $z \in B(x_0, R - \frac{r}{2}) \cap N(u) \cap B(y, \frac{r}{2})$  such that  $B(z, \frac{r}{2}) \cap (\cup_{j=1}^N B(x_j, \frac{r}{2})) = \emptyset$ . This would contradict the maximality of the family.

(c) Assertion (3) implies that  $\pi(R - r)^2 \leq \sum_{j=1}^N A(B(x_j, 2r)) = 4\pi N r^2$  and since  $r \leq \rho < R/10$ , we get  $r^2 N \geq (0.9)^2 \frac{1}{4} R^2$ . The Lemma is proved.  $\square$

Recall that  $N(u)$  consists of finitely many points and finitely many  $C^1$  arcs with finite length.

**Lemma 3.8** *Let  $\{x_1, \dots, x_N\}$  be a maximal family as given by Lemma 3.7. Assume that  $r^2 \lambda < 16\mathbf{j}^2$ . Then there exists no nodal curve  $\gamma \subset N(u)$  which is simply closed and contained in any of the balls  $B(x_j, \frac{r}{4})$ ,  $1 \leq j \leq N$ .*

*Proof.* Indeed, otherwise, there would be a nodal domain contained in one of the balls  $B(x_j, \frac{r}{4})$  and hence we would have  $r^2 \lambda \geq 16\mathbf{j}^2$ .

We can now prove the following local estimate.

**Proposition 3.9** *Let  $\lambda$  be an eigenvalue of  $H(\Omega)$ , assumed to be large enough. Let  $u$  be a non-zero eigenfunction associated with  $\lambda$ . Then, the length of the nodal set  $N(u)$  inside  $B(x_0, R)$  is bounded from below by  $10^{-2} R^2 \sqrt{\lambda}$ .*

**Proof.** Choose  $(r, \lambda)$  so that  $4\mathbf{j}^2 < r^2 \lambda < 16\mathbf{j}^2$ , with  $r \leq \rho < R/10$ . By Lemma 3.7, the  $N$  balls  $B(x_j, \frac{r}{4})$  are pairwise disjoint with center on  $N(u)$ . By Lemma 3.8, the length of  $N(u) \cap B(x_j, \frac{r}{4})$  is at least  $\frac{r}{2}$ . It follows that

$$\ell(N(u) \cap B(x_0, R)) \geq \sum_j \ell(N(u) \cap B(x_j, \frac{r}{4})) \geq N \frac{r}{2}, \quad (3.13)$$

and the result follows in view of the estimates  $r^2 N \geq 0.2 r^2$  and  $r^2 \lambda < 16\mathbf{j}^2$ .

**Remark 3.10** *Proposition 3.9 can be generalized to the case of a compact Riemannian surface with or without boundary. In that case, one needs to consider balls with radii less than the injectivity radius of the surface, and replace the Faber-Krahn inequality by a local Faber-Krahn inequality, using the fact that the metric can be at small scale compared with a Euclidean metric (see [Br] for more details).*

### 3.4.2 A local estimate à la Brünig-Gromes : spectral equipartitions

The above proof applies to a regular spectral equipartition of energy  $\Lambda$ . It is enough in the statements to replace the nodal set  $N(u)$  of  $u$  by the boundary set  $N(\mathcal{D})$  of the partition  $\mathcal{D}$ . We just rewrite the first statement.

**Lemma 3.11** *Let  $\Lambda$  be the energy of a regular spectral equipartition. If  $\Lambda r^2 > \mathbf{j}^2$ , then any disk  $B(x, r) \subset \Omega$  contains at least one point of boundary set of the partition.*

This follows immediately from the monotonicity of the Dirichlet eigenvalues with respect to domain inclusion.

## 4 Estimates involving the cardinality of the partitions

Let  $\mathcal{D} = \{D_i\}_{i=1}^k$  be a partition of  $\Omega$ . We call the number  $k$  the *cardinality* of the partition, and we denote it by  $k = \#(\mathcal{D})$ .

### 4.1 Estimates on the energy $\Lambda(\mathcal{D})$ and on $\mathfrak{L}_k(\Omega)$

#### Proposition 4.1

(i) *Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^2$ . The energy  $\Lambda(\mathcal{D})$  of a partition  $\mathcal{D}$  of  $\Omega$  satisfies the inequality,*

$$\Lambda(\mathcal{D}) \geq \frac{\pi \mathbf{j}^2}{A(\Omega)} \#(\mathcal{D}). \quad (4.1)$$

*In particular, for any  $k \geq 1$ , we have the inequality,*

$$\mathfrak{L}_k(\Omega) \geq \frac{\pi \mathbf{j}^2}{A(\Omega)} k. \quad (4.2)$$

(ii) *Let  $\Omega$  be a bounded open subset on a compact Riemannian surface. Then,*

$$A(\Omega) \liminf_{k \rightarrow \infty} \frac{\mathcal{L}_k(\Omega)}{k} \geq \pi \mathbf{j}^2. \quad (4.3)$$

**Proof.** Assertion (i) is an immediate consequence of the Faber-Krahn inequality (2.2). To prove Assertion (ii), we use the fact that on a general compact surface  $M$ , we have the following asymptotic isoperimetric and Faber-Krahn inequalities (which actually hold in arbitrary dimension).

**Lemma 4.2** ([BeMe], Lemma II.15, p. 528) *Let  $(M, g)$  be a compact Riemannian surface. For any  $\epsilon > 0$ , there exists a positive number  $a(M, g, \epsilon)$  such that for any regular domain  $\omega \subset M$  with area  $A(\omega)$  less than or equal to  $a(M, g, \epsilon)$ ,*

$$\begin{cases} \ell(\partial\omega) & \geq (1 - \epsilon)\ell(\partial\omega^*), \\ \lambda(\omega) & \geq (1 - \epsilon)^2 \frac{\pi \mathbf{j}^2}{A(\omega)}, \end{cases}$$

where  $\omega^*$  is a Euclidean disk of area  $A(\omega)$ .

Let  $\mathcal{D} = \{D_i\}$  be a partition of  $\Omega$ . Let

$$J_\epsilon = \{i \in \{1, \dots, k\} \mid A(D_i) > a(M, g, \epsilon)\}.$$

The number of elements of this set is bounded by

$$\#(J_\epsilon) \leq \frac{A(\Omega)}{a(M, g, \epsilon)}. \quad (4.4)$$

For any  $i \notin J_\epsilon$ , we can write,

$$\lambda(D_i) \geq (1 - \epsilon)^2 \frac{\pi \mathbf{j}^2}{A(D_i)}$$

and hence, provided that  $\#(\mathcal{D})$  is large enough,

$$\Lambda(\mathcal{D})A(\Omega) \geq (1 - \epsilon)^2 (\#(\mathcal{D}) - \frac{A(\Omega)}{a(M, g, \epsilon)}) \pi \mathbf{j}^2.$$

As a consequence, we obtain that

$$A(\Omega) \liminf_{k \rightarrow \infty} \frac{\mathcal{L}_k(\Omega)}{k} \geq (1 - \epsilon)^2 \pi \mathbf{j}^2.$$

We can now let  $\epsilon$  tend to zero to get the estimate (4.3).  $\square$

**Remarks.**

(1) We point out that the lower bounds in the Proposition only depend on the area of  $\Omega$ , not on its geometry.

(2) Similar inequalities on  $\mathcal{L}_k(\Omega)$  can also be deduced from [Pe], when  $\Omega$  is a bounded domain in a simply-connected surface  $M$  with Gaussian curvature  $K$ , such that  $\Omega \subset \Omega_0$ , a simply-connected domain satisfying  $A(\Omega_0) \sup_{\Omega_0} K^+ \leq \pi$ . Let us mention two particular cases.

(a) If  $M$  is a simply-connected surface with non-positive curvature, then according to [Pe],  $\lambda(\Omega)A(\Omega) \geq \pi \mathbf{j}^2$  for any bounded domain  $\Omega$  and we conclude that

$$A(\Omega) \frac{\mathcal{L}_k(\Omega)}{k} \geq \pi \mathbf{j}^2$$

for all  $k \geq 1$ , as in the Euclidean case.

(b) If  $M$  is the standard sphere, then according to [Pe],

$$\lambda(D)A(D) \geq \pi \mathbf{j}^2 \left(1 - \frac{A(D)}{4\pi}\right)$$

for any domain  $D$ , and one can conclude that, for any domain  $\Omega$ ,

$$A(\Omega) \frac{\mathcal{L}_k(\Omega)}{k} \geq \pi \mathbf{j}^2 - \frac{\mathbf{j}^2}{4k} A(\Omega),$$

for all  $k \geq 1$ .

**Remark 4.3** Given a  $k$ -partition  $\mathcal{D} = \{D_i\}_{i=1}^k$  of  $\Omega$ , one can also introduce the energy  $\Lambda_1(\mathcal{D}) := \frac{1}{k} \sum_{i=1}^k \lambda(D_i)$ , and define the number  $\mathfrak{L}_{k,1}(\Omega)$  by taking the infimum of the  $\Lambda_1$ -energy over all  $k$ -partitions. An easy convexity argument shows that the above inequalities (4.1), (4.2) and (4.3) hold with  $\Lambda(\mathcal{D})$  and  $\mathfrak{L}_k(\Omega)$  replaced by  $\Lambda_1(\mathcal{D})$  and  $\mathfrak{L}_{k,1}(\Omega)$  respectively. For example, we have the inequalities,

$$\begin{aligned} \Lambda(\mathcal{D}) &\geq \Lambda_1(\mathcal{D}) \geq \frac{\pi \mathbf{j}^2}{A(\Omega)} \sharp(\mathcal{D}), \quad \text{and,} \\ \mathfrak{L}_k(\Omega) &\geq \mathfrak{L}_{k,1}(\Omega) \geq \frac{\pi \mathbf{j}^2}{A(\Omega)} k. \end{aligned}$$

For bounded domains in  $\mathbb{R}^2$ , one can also give an asymptotic upper bound for  $\mathfrak{L}_k$ . More precisely,

**Property 4.4** For any regular bounded open subset of  $\mathbb{R}^2$ ,

$$\limsup_{k \rightarrow +\infty} \frac{\mathfrak{L}_k(\Omega)}{k} \leq \frac{\lambda(\text{Hexa}_1)}{A(\Omega)}, \quad (4.5)$$

where  $\text{Hexa}_1$  is the regular hexagon in  $\mathbb{R}^2$ , with area 1.

This can be seen by considering the hexagonal tiling in the plane, with hexagons of area  $a$  and the partition of  $\Omega$  given by taking the union of the hexagons contained in  $\Omega$ , whose number is asymptotically  $\frac{A(\Omega)}{a}$  when  $a$  tends to zero.

The inequalities (4.5) and (4.3) motivate the following two conjectures<sup>2</sup> for bounded domains in  $\mathbb{R}^2$ . They were proposed and analyzed in the recent years (see [BHV, BBO, CL, HHOT1, Hel]). The first one is that

**Conjecture 4.5** The limit of  $\mathfrak{L}_k(\Omega)/k$  as  $k \rightarrow +\infty$  exists.

The second one is that this limit is more explicitly given by

**Conjecture 4.6**

$$A(\Omega) \lim_{k \rightarrow +\infty} \frac{\mathfrak{L}_k(\Omega)}{k} = \lambda(\text{Hexa}_1).$$

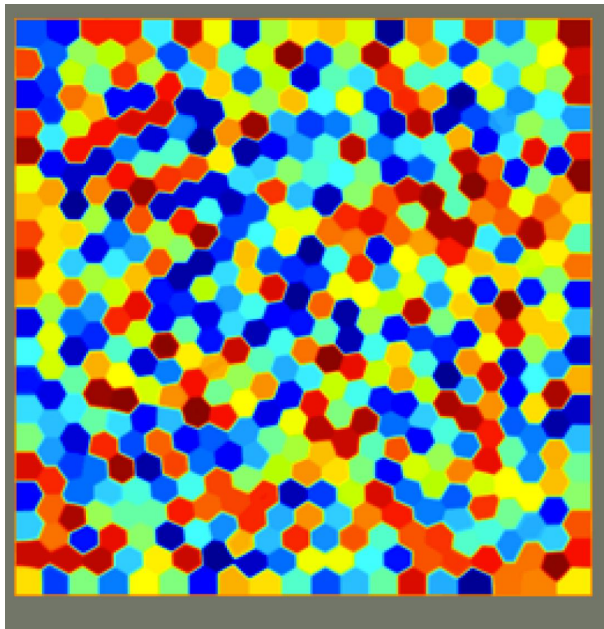


Figure 1: Computations of Bourdin-Bucur-Oudet for the periodic square. (Minimization of the sum)

The second conjecture says in particular that the limit only depends on the area of  $\Omega$ , not on its geometry (provided  $\Omega$  is a regular domain).

It is explored numerically in [BHV] why the second conjecture looks reasonable. Note that Caffarelli and Lin [CL], mention Conjecture 4.6 in relation with  $\mathfrak{L}_{k,1}(\Omega)$ . From this point of view, the recent numerical computations by Bourdin-Bucur-Oudet [BBO] for the asymptotic structure of the minimal partitions for  $\mathfrak{L}_{k,1}(\Omega)$  are very enlightening, see Figure 4.1. Remark 4.3 shows that (4.3) should be a strict inequality.

## 4.2 Asymptotics of the length of the boundary set of minimal regular $k$ -equipartitions for $k$ large.

In this Section, we only consider bounded open domains  $\Omega$  in  $\mathbb{R}^2$ . In this case, Conjecture 4.6 leads to a natural “hexagonal conjecture” for the length of the boundary set, namely

### Conjecture 4.7

$$\lim_{k \rightarrow +\infty} (P(\mathcal{D}_k)/\sqrt{k}) = \frac{1}{2} \ell(\text{Hexa}_1) \sqrt{A(\Omega)}, \quad (4.6)$$

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<sup>2</sup>The second author was informed of these conjectures by M. Van den Berg.



where  $\ell(\text{Hexa}_1)$  is the length of the boundary of the hexagon of area 1,

$$\ell(\text{Hexa}_1) = 2\sqrt{2\sqrt{3}} = 2(12)^{\frac{1}{4}}.$$

For regular spectral equipartitions  $\mathcal{D}$  of the domain  $\Omega$ , inequalities (3.6) and (4.1) yield,

$$\liminf_{\#\mathcal{D} \rightarrow \infty} \frac{P(\mathcal{D})}{\sqrt{\#\mathcal{D}}} \geq \frac{\sqrt{\pi}}{2} \sqrt{A(\Omega)}. \quad (4.7)$$

Assuming that  $\chi(\Omega) \geq 0$ , we have the uniform lower bound,

$$\frac{P(\mathcal{D})}{\sqrt{\#\mathcal{D}}} \geq \frac{\sqrt{\pi}}{2} \sqrt{A(\Omega)}. \quad (4.8)$$

**Remark.** Assume that  $\chi(\Omega) \geq 0$ , and that all the sub-domains  $D_i$  in the regular equipartition  $\mathcal{D}$  satisfy  $\chi(D_i) \geq 0$  as well. Then, Polya's inequality (3.8) yields the sharper inequality,

$$\frac{P(\mathcal{D})}{\sqrt{\#\mathcal{D}}} \geq \frac{\mathbf{j}}{\sqrt{\pi}} \sqrt{A(\Omega)}. \quad (4.9)$$

The following statement is a particular case of Theorem 1-B established by T.C. Hales [Ha] in his proof of Lord Kelvin's honeycomb conjecture (see also [Mor]).

**Theorem 4.8** *Let  $\Omega$  be a relatively compact open set in  $\mathbb{R}^2$ , and let  $\mathcal{D} = \{D_i\}$  be a regular finite partition of  $\Omega$ . Then,*

$$P(\mathcal{D}) + \frac{1}{2}\ell(\partial\Omega) \geq (12)^{\frac{1}{4}} \sum_{i=1}^{\#\mathcal{D}} \min(1, A(D_i)). \quad (4.10)$$

In order to optimize the use of this theorem, we consider  $\min_i A(D_i)$ , and apply the theorem to a dilated partition. If we dilate by  $t$  the length is multiplied by  $t$  and the area by  $t^2$ . So we take  $t = (\min_i A(D_i))^{-\frac{1}{2}}$ , and we obtain,

**Corollary 4.9** *For any regular partition  $\mathcal{D}$  of a bounded open subset  $\Omega$  of  $\mathbb{R}^2$ ,*

$$P(\mathcal{D}) + \frac{1}{2}\ell(\partial\Omega) \geq (12)^{\frac{1}{4}} (\min_i A(D_i))^{\frac{1}{2}} \#\mathcal{D}. \quad (4.11)$$

**Proposition 4.10** *Let  $\Omega$  be a regular bounded domain in  $\mathbb{R}^2$ .*

(i) *For  $k \geq 1$ , let  $\mathcal{D}_k$  be a minimal regular  $k$ -equipartition of  $\Omega$ . Then,*

$$\liminf_{k \rightarrow +\infty} \frac{P(\mathcal{D}_k)}{\sqrt{k}} \geq (12)^{\frac{1}{4}} \left( \frac{\pi \mathbf{j}^2}{\lambda(\text{Hexa}_1)} \right)^{\frac{1}{2}} A(\Omega)^{\frac{1}{2}}. \quad (4.12)$$

(ii) If  $\chi(\Omega) \geq 0$ , then for any regular spectral equipartition, we have the universal estimate

$$P(\mathcal{D}) + \frac{1}{2}\ell(\partial\Omega) \geq 12^{\frac{1}{8}} \left(\frac{\pi}{4}\right)^{\frac{1}{4}} A(\Omega)^{\frac{1}{2}} (\#\mathcal{D})^{\frac{1}{2}}. \quad (4.13)$$

**Remarks.**

- (a) Recall that  $\lambda(\text{Hexa}_1) \sim 18,5901$  and that  $\lambda(\text{Disk}_1) = \pi\mathbf{j}^2 \sim 18,1680$ . It follows that  $\left(\frac{\pi\mathbf{j}^2}{\lambda(\text{Hexa}_1)}\right)^{\frac{1}{2}} \sim 0,989$ , so that the right-hand side of (4.12) is very close to the right-hand side of (4.6), the hexagonal conjecture for the length.
- (b) Asymptotically, when we consider minimal regular  $k$ -equipartition  $\mathcal{D}_k$ , inequality (4.13) is weaker than (4.12) but it is universal, and independent of the asymptotics of the energy of the partition.

**Proof of the Proposition.**

(i) Let  $\mathcal{D} = \{D_i\}$  be a regular equipartition of  $\Omega$ . Faber-Krahn's inequality (2.2) gives,

$$\Lambda(\mathcal{D}) = \lambda(D_i) \geq \frac{\pi\mathbf{j}^2}{A(D_i)},$$

hence, combining with (4.11),

$$P(\mathcal{D}) + \frac{1}{2}\ell(\partial\Omega) \geq (12)^{\frac{1}{4}} (\pi\mathbf{j}^2)^{\frac{1}{2}} \frac{\#\mathcal{D}}{\sqrt{\Lambda(\mathcal{D})}}.$$

Let  $\mathcal{D}_k$  be a minimal regular  $k$ -equipartition of  $\Omega$ . Applying (4.2), we obtain

$$\frac{P(\mathcal{D}_k) + \frac{1}{2}\ell(\partial\Omega)}{\sqrt{k}} \geq (12)^{\frac{1}{4}} (\pi\mathbf{j}^2)^{\frac{1}{2}} \left(\frac{\mathfrak{L}_k(\Omega)}{k}\right)^{-\frac{1}{2}}.$$

Using the upper bound for  $\limsup_{k \rightarrow +\infty} \frac{\mathfrak{L}_k(\Omega)}{k}$  given by (4.5), we obtain the following asymptotic inequality for the length of a minimal regular  $k$ -equipartition,

$$\liminf_{k \rightarrow +\infty} \frac{P(\mathcal{D}_k)}{\sqrt{k}} \geq (12)^{\frac{1}{4}} \left(\frac{\pi\mathbf{j}^2}{\lambda(\text{Hexa}_1)}\right)^{\frac{1}{2}} A(\Omega)^{\frac{1}{2}}.$$

(ii) Assume that  $\chi(\Omega) \geq 0$ . Indeed, by (3.6), we have

$$P(\mathcal{D}) \geq \frac{A(\Omega)}{2\mathbf{j}} \sqrt{\Lambda(\mathcal{D})},$$

and hence,

$$\Lambda(\mathcal{D})^{-\frac{1}{2}} \geq \frac{A(\Omega)}{2\mathbf{j}P(\mathcal{D})} \geq \frac{A(\Omega)}{2\mathbf{j}(P(\mathcal{D}) + \frac{1}{2}\ell(\partial\Omega))}.$$

The same proof as for Assertion (i) gives, for any regular spectral equipartition,

$$P(\mathcal{D}) + \frac{1}{2}\ell(\partial\Omega) \geq (12)^{\frac{1}{4}} (\pi\mathbf{j}^2)^{\frac{1}{2}} \Lambda(\mathcal{D})^{-\frac{1}{2}}.$$

Inequality (4.13) follows.  $\square$

**Remark.** Assume now that  $\mathcal{D}(u_k)$  is the nodal partition of some  $k$ -th eigenfunction  $u_k$  of  $H(\Omega)$ . Assume furthermore that  $\chi(\Omega) \geq 0$ . The same reasoning as above gives,

$$P(\mathcal{D}) + \frac{1}{2}\ell(\partial\Omega) \geq (12)^{\frac{1}{4}}\mu(u_k)(\pi\mathbf{j}^2)^{\frac{1}{2}}(\lambda_k(\Omega))^{-\frac{1}{2}}.$$

Hence

$$\liminf_{k \rightarrow +\infty} \frac{P(\mathcal{D}(u_k))}{\sqrt{k}} \geq (12)^{\frac{1}{4}} \left( \liminf_{k \rightarrow +\infty} \frac{\mu(u_k)}{k} \right) (\mathbf{j}^2/4)^{\frac{1}{2}} A(\Omega)^{\frac{1}{2}}.$$

This inequality is less convincing, because we have no lower bound for the right-hand side. Indeed, on the round sphere  $\mathbb{S}^2$ , any eigenspace of the Laplacian contains eigenfunctions with only either 2 or 3 nodal domains, [LEW].

We may however go back to the initial inequality,

$$P(\mathcal{D}(u_k)) \geq \frac{A(\Omega)}{2\mathbf{j}} \sqrt{\lambda_k(\Omega)}.$$

Assuming Polya's conjecture [P1] which says that for bounded domains in  $\mathbb{R}^2$ ,  $\lambda_k(\Omega) \geq \frac{4\pi}{A(\Omega)}$ , for  $k \geq 1$ , we find that

$$P(\mathcal{D}(u_k)) \geq \frac{\sqrt{\pi}}{\mathbf{j}} \sqrt{A(\Omega)} \sqrt{k},$$

which should be compared to (4.8).

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