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# Remarks on the boundary set of spectral equipartitions

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## Abstract

Given a bounded open set  $\Omega$  in  $\mathbb{R}^n$  (or a compact Riemannian manifold with boundary), and a partition of  $\Omega$  by  $k$  open sets  $\omega_j$ , we consider the quantity  $\max_j \lambda(\omega_j)$ , where  $\lambda(\omega_j)$  is the ground state energy of the Dirichlet realization of the Laplacian in  $\omega_j$ .

We denote by  $\mathfrak{L}_k(\Omega)$  the infimum of  $\max_j \lambda(\omega_j)$  over all  $k$ -partitions. A minimal  $k$ -partition is a partition which realizes the infimum.

The purpose of this paper is to revisit properties of nodal sets and to explore if they are also true for minimal partitions, or more generally for spectral equipartitions. We focus on the length of the boundary set of the partition in the 2-dimensional situation.

**MSC(2010):** 35P15, 35B05, 35R01, 35J05, 35P05.

**Keywords:** Spectral partitions, Nodal domains, Nodal lines.

## 1 Introduction

Given a bounded open set  $\Omega$  in  $\mathbb{R}^n$  (or in a Riemannian manifold), and a partition of  $\Omega$  by  $k$  open sets  $\omega_j$ , we consider the quantity  $\max_j \lambda(\omega_j)$ , where  $\lambda(\omega_j)$  is the ground state energy of the Dirichlet realization of the Laplacian in  $\omega_j$ .

We denote by  $\mathfrak{L}_k(\Omega)$  the infimum of  $\max_j \lambda(\omega_j)$  over all  $k$ -partitions. A minimal  $k$ -partition is a partition which realizes the infimum. Although the analysis of  $k$ -minimal partitions is rather standard when  $k = 2$  (we find the nodal domains of a second eigenfunction), the analysis for higher values of  $k$  becomes non trivial and quite interesting.

The purpose of this paper is to revisit various properties of nodal sets and to explore if they are also true for minimal partitions, or more generally for spectral equipartitions. We focus on the length of the boundary set of the partition in the 2-dimensional situation.

## 2 Definitions and notations

### 2.1 Spectral theory

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$ , or a compact Riemannian surface, possibly with boundary  $\partial\Omega$ , which we assume to be piecewise  $C^1$ . Let  $H(\Omega)$  be the realization of the Laplacian, or of the Laplace-Beltrami operator,  $-\Delta$  in  $\Omega$ , with Dirichlet boundary condition ( $u|_{\partial\Omega} = 0$ ). Let  $\{\lambda_j(\Omega)\}_{j \geq 1}$  be the increasing sequence of the eigenvalues of  $H(\Omega)$ , counted with multiplicity. The eigenspace associated with  $\lambda_k$  is denoted by  $E(\lambda_k)$ .

A groundstate  $u \in E(\lambda_1)$  does not vanish in  $\Omega$  and can be chosen to be positive. On the contrary, any eigenfunction  $u \in E(\lambda_k)$ ,  $k \geq 2$ , changes sign in  $\Omega$ , and hence has a nonempty zero set or *nodal set*,

$$N(u) = \overline{\{x \in \Omega \mid u(x) = 0\}}. \quad (2.1)$$

The connected components of  $\Omega \setminus N(u)$  are called the *nodal domains* of  $u$ . The number of nodal domains of  $u$  is denoted by  $\mu(u)$ .

Courant's nodal domain theorem says:

**Theorem 2.1 (Courant)** *Let  $k \geq 1$ , and let  $E(\lambda_k)$  be the eigenspace of  $H(\Omega)$  associated with the eigenvalue  $\lambda_k$ . Then,  $\forall u \in E(\lambda_k) \setminus \{0\}$ ,  $\mu(u) \leq k$ .*

Except in dimension 1, the inequality is strict in general. More precisely, we have:

**Theorem 2.2 (Pleijel)** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$ . There exists a constant  $k_0$  depending on  $\Omega$ , such that if  $k \geq k_0$ , then*

$$\mu(u) < k, \forall u \in E(\lambda_k) \setminus \{0\}.$$

Both theorems are proved in [P]. The main points in the proof of Pleijel's Theorem are the Faber-Krahn inequality and the Weyl asymptotic law. Faber-Krahn's inequality states that, for any bounded domain  $\omega$  in  $\mathbb{R}^2$ ,

$$\lambda_1(\omega) \geq \frac{\pi \mathbf{j}^2}{A(\omega)}, \quad (2.2)$$

where  $A(\omega)$  is the area of  $\omega$  and  $\mathbf{j}$  is the least positive zero of the Bessel function of order 0 ( $\mathbf{j} \sim 2.4$ ). Weyl's asymptotic law for the eigenvalues of  $H(\omega)$  states that

$$\lim_{k \rightarrow \infty} \frac{\lambda_k(\omega)}{k} = \frac{4\pi}{A(\omega)}. \quad (2.3)$$

Let  $\bar{\mu}(k)$  be the maximum value of  $\mu(u)$  when  $u \in E(\lambda_k) \setminus \{0\}$ . Combining the results of Faber-Krahn and Weyl, we obtain,

$$\limsup_{k \rightarrow +\infty} \frac{\bar{\mu}(k)}{k} \leq 4/\mathbf{j}^2 < 1. \quad (2.4)$$

**Remark 2.3** *Pleijel's Theorem extends to bounded domains in  $\mathbb{R}^n$ , and more generally to compact  $n$ -manifolds with boundary, with a constant  $\gamma(n) < 1$  replacing  $4/\mathbf{j}^2$  in the right-hand side of (2.4) (Peetre [Pe], Bérard-Meyer [BeMe]). It is also interesting to note that this constant is independent of the geometry.*

**Remark 2.4** *It follows from Pleijel's Theorem that the equality  $\bar{\mu}(k) = k$  can only occur for finitely many values of  $k$ . The analysis of the equality case is very interesting. We refer to [HHOT1] for more details.*

**Remark 2.5** *In dimension 1, counting the nodal domains of an eigenfunction of a Dirichlet Sturm-Liouville problem in some interval  $[a, b]$  is the same as counting the number of zeroes of the eigenfunction. An analog in dimension 2 is to consider the length of the nodal set of eigenfunctions. We shall come back to this question in Section 3.*

## 2.2 Partitions

For this section, we refer to [HHOT1]. Let  $k$  be a positive integer. A (weak)  $k$ -partition of the open bounded set  $\Omega$  is<sup>1</sup> a family  $\mathcal{D} = \{D_j\}_{j=1}^k$  of pairwise disjoint sets such that  $\cup_{j=1}^k D_j \subset \Omega$ . We denote by  $\mathfrak{D}_k = \mathfrak{D}_k(\Omega)$  the set of  $k$ -partitions such that the domains  $D_j$  are open and connected.

Given  $\mathcal{D} \in \mathfrak{D}_k$ , we define the *energy*  $\Lambda(\mathcal{D})$  of the partition as,

$$\Lambda(\mathcal{D}) = \max_j \lambda(D_j), \quad (2.5)$$

where  $\lambda(D_j)$  is the groundstate energy of  $H(D_j)$ . We now define the number  $\mathfrak{L}_k(\Omega)$  as:

$$\mathfrak{L}_k(\Omega) = \inf_{\mathcal{D} \in \mathfrak{D}_k} \Lambda(\mathcal{D}). \quad (2.6)$$

A partition  $\mathcal{D}$  is called *minimal* if  $\mathfrak{L}_k(\Omega) = \Lambda(\mathcal{D})$ .

**Example.** The nodal domains of an eigenfunction  $u \in E(\lambda) \setminus \{0\}$  of  $H(\Omega)$  form a  $\mu(u)$ -partition of  $\Omega$  denoted by  $\mathcal{D}(u)$ . Such a partition is called a *nodal*

<sup>1</sup>Note that we start from a very weak notion of partition. We refer to [HHOT1] for a more precise definition of classes of  $k$ -partitions and for the notion of regular representatives.

partition.

It turns out that  $\mathfrak{L}_2(\Omega) = \lambda_2(\Omega)$  and that minimal 2-partitions are nodal partitions. The situation when  $k \geq 3$  is more complicated, and more interesting.

A partition  $\mathcal{D} = \{D_j\}_{j=1}^k \in \mathfrak{D}(\Omega)$  is called *strong* if,

$$\text{Int}(\overline{\cup_j D_j}) = \Omega. \quad (2.7)$$

The *boundary set*  $N(\mathcal{D})$  of a strong partition  $\mathcal{D} = \{D_j\}_{j=1}^k \in \mathfrak{D}(\Omega)$  is the closed set,

$$N(\mathcal{D}) = \overline{\cup_j (\partial D_j \cap \Omega)}. \quad (2.8)$$

The set  $\mathcal{R}(\Omega)$  of *regular partitions* is the subset of strong partitions in  $\mathfrak{D}(\Omega)$  whose boundary set  $N = N(\mathcal{D})$  satisfies the following properties:

- (i) The set  $N$  is locally a regular curve in  $\Omega$ , except possibly at finitely many points  $\{y_i\} \in N \cap \Omega$ , in the neighborhood of which  $N$  is the union of  $\nu(y_i)$  smooth semi-arcs at  $y_i$ ,  $\nu(y_i) \geq 3$ .
- (ii) The set  $N \cap \partial\Omega$  consists of finitely many points  $\{z_j\}$ . Near the point  $z_j$ , the set  $N$  is the union of  $\rho(z_j) \geq 1$  semi-arcs hitting  $\partial\Omega$  at  $z_j$ .
- (iii) The set  $N$  has the equal angle property. More precisely, at any *interior singular point*  $y_i$ , the semi-arcs meet with equal angles; at any *boundary singular point*  $z_j$ , the semi-arcs form equal angles together with the boundary  $\partial\Omega$ .

**Example.** A nodal partition  $\mathcal{D}(u)$  provides an example of a regular partition, and the boundary set  $N(\mathcal{D}(u))$  coincides with the nodal set  $N(u)$ . Note that for a regular partition, the number  $\nu(y_i)$  of semi-arcs at an interior singular point may be odd, whereas it is always even for a nodal partition.

## 2.3 Results on minimal partitions

The first important results are that minimal partitions exist ([CTV1, CTV2, CTV3]), and that any minimal partition has a representative (modulo sets of capacity 0) which is regular ([HHOT1]). Let us now introduce:

**Definition 2.6** We call *spectral equipartition* a strong  $k$ -partition  $\mathcal{D} = \{D_j\}_{j=1}^k$  such that  $\lambda(D_j) = \Lambda(\mathcal{D})$ , for  $j = 1, \dots, k$ . The number  $\Lambda(\mathcal{D})$  is called the *energy of the equipartition*.

It is easy to see that minimal  $k$ -partitions are spectral equipartitions of energy  $\mathfrak{L}_k$ . Note that for a  $k$ -equipartition of energy  $\Lambda$ , we deduce from the Faber-Krahn inequality that

$$\Lambda \geq \frac{\pi \mathbf{j}^2}{A(\Omega)} k. \quad (2.9)$$

In particular, for minimal  $k$ -partitions we have (see also Section 4)

$$\mathfrak{L}_k(\Omega) \geq \frac{\pi \mathbf{j}^2}{A(\Omega)} k. \quad (2.10)$$

## 2.4 Euler formula

Let  $\Omega \subset M$  be a bounded domain with piecewise smooth boundary  $\partial\Omega$ . Let  $N \subset \overline{\Omega}$  be a regular closed set (in the sense of Section 2.2, properties (i)-(iii)) such that the family  $\mathcal{D} = \{D_1, \dots, D_k\}$  of connected components of  $\Omega \setminus N$  is a regular, strong partition of  $\Omega$ . Recall that for a singular point  $y \in N \cap \Omega$ ,  $\nu(y)$  is the number of semi-arcs at  $y$ , and that for a singular point  $z \in N \cap \partial\Omega$ ,  $\rho(z)$  is the number of semi-arcs at  $z$ , not counting the two arcs contained in  $\partial\Omega$ . Let  $\mathcal{S}(\mathcal{D})$  denote the set of singular points of  $N(\mathcal{D})$ , both interior or boundary points, if any. We define the *index* of a point  $x \in \mathcal{S}(\mathcal{D})$  to be,

$$\iota(x) = \begin{cases} \nu(x) - 2, & \text{if } x \text{ is an interior singular point,} \\ \rho(x), & \text{if } x \text{ is a boundary singular point.} \end{cases} \quad (2.11)$$

We introduce the number  $\sigma(\mathcal{D})$  to be,

$$\sigma(\mathcal{D}) = \sum_{x \in \mathcal{S}(\mathcal{D})} \iota(x). \quad (2.12)$$

For a regular strong  $k$ -partition  $\mathcal{D} = \{D_j\}_{j=1}^k$  of  $\Omega$ , we have Euler's formula,

$$\chi(\Omega) + \frac{1}{2}\sigma(\mathcal{D}) = \sum_{j=1}^k \chi(D_j). \quad (2.13)$$

We refer to [HOMN] for a combinatorial proof of this formula in the case of an open set of  $\mathbb{R}^2$ . One can give a Riemannian proof using the global Gauss-Bonnet theorem. For a domain  $D$  with piecewise smooth boundary  $\partial D$  consisting of piecewise  $C^1$  simple closed curves  $\{C_i\}_{i=1}^n$ , with corners  $\{p_{i,j}\}$  ( $i = 1, \dots, n$  and  $j = 1, \dots, m_i$ ) and corresponding interior angles  $\theta_{i,j}$ , we have

$$2\pi\chi(D) = \int_D K + \sum_{i=1}^n \beta(C_i), \quad (2.14)$$

where

$$\beta(C_i) = \int_{C_i} \langle k, \nu_D \rangle + \sum_{j=1}^{m_i} (\pi - \theta_{i,j}).$$

In this formula,  $k$  is the geodesic curvature vector of the regular part of the curve  $C_i$ , and  $\nu_D$  is the unit normal to  $C_i$  pointing inside  $D$ .

To prove (2.13), it suffices to sum up the Gauss-Bonnet formulas relative to each domain  $D_j$  and to notice that

- the integrals of the Gaussian curvature over the  $D_j$ 's add up to the integral of the Gaussian curvature over  $\Omega$ ,
- cancellations occur when adding the integrals of the geodesic curvature over the curves bounding two adjacent  $D_j$  (the unit normal vectors point in opposite directions), while they add up to give the integral of the geodesic curvature over the boundary of  $\Omega$ ,
- there are contributions coming from the angles associated with the singular points of  $N$  and, when summed up, these contributions yield the second term in the left-hand side of (2.13).

Note that the proof of (2.13) does not use the fact that the semi-arcs meet at the singular points of  $N$  with equal angles.

### 3 Lower bounds for the length of the boundary set of a regular equipartition

#### 3.1 Introduction

Let  $\mathcal{D} = \{D_1, \dots, D_k\}$  be a regular equipartition with energy  $\Lambda = \Lambda(\mathcal{D})$ . The boundary set  $N(\mathcal{D})$  of the partition consists of singular points  $\{y_i\}_{i=1}^a$  inside  $\Omega$ , of singular points  $\{z_i\}_{i=1}^b$  on  $\partial\Omega$ , of  $C^1$  arcs  $\{\gamma_i\}_{i=1}^c$  which bound two adjacent domains of the partition, and of arcs  $\{\delta_i\}_{i=1}^d$  contained in  $\partial\Omega$ . We define the length of the boundary set  $N(\mathcal{D})$  by the formula,

$$P(\mathcal{D}) := \sum_{i=1}^c \ell(\gamma_i) + \frac{1}{2} \ell(\partial\Omega), \quad (3.1)$$

where  $\ell$  denotes the length of the curves. Note that  $\ell(\partial\Omega) = \sum_{i=1}^d \ell(\delta_i)$ .

In this section, we investigate lower bounds for  $P(\mathcal{D})$  in terms of the energy  $\Lambda(\mathcal{D})$  and the area  $A(\Omega)$ .

As a matter of fact, we show that the methods introduced in [BrGr, Br, Sa1] apply in the above context of a regular equipartition, and hence to minimal partitions. We provide three estimates.

1. The first estimate holds for plane domains, and follows the method of [BrGr].
2. The second estimate applies to a compact Riemannian surface (with or without boundary), and follows the method of [Sa1].
3. The third estimate is a local estimate based on the method of [Br].

When applied to minimal  $k$ -partitions, one may consider the asymptotic behaviour of the estimates as  $k$  tends to infinity. The estimates deduced from

[BrGr] are sharper (but restricted to domains in the Euclidean plane). The third estimate is quite loose, but it is valuable for its local character. It is not clear how to apply the isoperimetric methods of [BrGr, Sa1] to obtain a sharper local estimate.

Let  $\mathcal{D} = \{D_i\}_{i=1}^k$  be a regular equipartition with energy  $\Lambda$ . Let  $R(D_i)$  be the inner radius of the set  $D_i$ . Recall that  $\mathbf{j}$  denotes the least positive zero of the Bessel function of order 0.

### 3.2 The method of Brüning-Gromes

In this section,  $\Omega$  is a bounded domain in  $\mathbb{R}^2$ , with piecewise  $C^1$  boundary. We only sketch the method which relies on three inequalities.

1. The monotonicity of eigenvalues and the characterization of the ground state imply that

$$\forall i, 1 \leq i \leq k, \quad R(D_i) \leq \frac{\mathbf{j}}{\sqrt{\Lambda}}. \quad (3.2)$$

2. The Faber-Krahn inequality and the isoperimetric inequality imply that

$$\forall i, 1 \leq i \leq k, \quad \frac{2\pi\mathbf{j}}{\sqrt{\Lambda}} \leq \ell(\partial D_i). \quad (3.3)$$

3. The generalized F ej es-Toth isoperimetric inequality ([BrGr], Hilfssatz 2) asserts that, for  $1 \leq i \leq k$ ,

$$A(D_i) \leq R(D_i)\ell(\partial D_i) - \chi(D_i)\pi R^2(D_i). \quad (3.4)$$

Using that  $\chi(D_i) \leq 1$ , we immediately see that the function  $r \rightarrow r\ell(\partial D_i) - \chi(D_i)\pi r^2$  is non-decreasing for  $2\pi r \leq \ell(\partial D_i)$ .

Using inequalities (3.2) and (3.3), it follows that one can substitute  $\frac{\mathbf{j}}{\sqrt{\Lambda}}$  to  $R(D_i)$  in (3.4) and obtain,

$$A(D_i) \leq \frac{\mathbf{j}}{\sqrt{\Lambda}}\ell(\partial D_i) - \chi(D_i)\pi\left(\frac{\mathbf{j}}{\sqrt{\Lambda}}\right)^2, \quad \text{for } 1 \leq i \leq k. \quad (3.5)$$

Summing up the inequalities (3.5), for  $1 \leq i \leq k$ , we obtain

$$A(\Omega) \leq \frac{\mathbf{j}}{\sqrt{\Lambda}} \sum_i \ell(\partial D_i) - \sum_i \chi(D_i)\pi \frac{\mathbf{j}^2}{\Lambda}.$$

Using Euler's formula (2.13), we conclude that

$$A(\Omega) \leq \frac{2\mathbf{j}}{\sqrt{\Lambda}}P(\mathcal{D}) - [\chi(\Omega) + \frac{1}{2}\sigma(\mathcal{D})]\pi \frac{\mathbf{j}^2}{\Lambda}.$$



We finally conclude with the following estimate from below for the length  $P(\mathcal{D})$  of the boundary set of the partition  $\mathcal{D}$ .

$$\frac{A(\Omega)}{2j} \sqrt{\Lambda} + \frac{\pi j}{2\sqrt{\Lambda}} [\chi(\Omega) + \frac{1}{2}\sigma(\mathcal{D})] \leq P(\mathcal{D}). \quad (3.6)$$

Note that (3.6) is actually slightly better than the estimate in [BrGr] which does not take into account the term  $\sigma(\mathcal{D})$  when  $\mathcal{D}$  is the nodal partition for an eigenfunction  $u$  associated with the eigenvalue  $\Lambda$ . This fact is suggested in [Sa1].

### 3.3 The method of Savo

In this section, we follow the method of Savo [Sa1], and keep the same notations and assumptions. We sketch the proof in the case with boundary as it is not detailed in [Sa1]. Here,  $\Omega$  is a compact Riemannian surface with boundary. We denote the Laplace-Beltrami operator by  $\Delta$  and the Gaussian curvature by  $K$ . We write  $K = K_+ - K_-$  (the negative and positive parts of the curvature). We assume that  $\alpha \geq 0$  and  $D$  are given such that:

$$K \geq -\alpha^2,$$

and the diameter  $\delta(\Omega)$  of  $\Omega$  satisfies

$$\delta(\Omega) \leq D.$$

Finally, we define the numbers

$$B(\Omega) = \int_{\Omega} K_+ - 2\pi\chi(\Omega)$$

and

$$C(\alpha, D) = \sqrt{\pi^2 + \frac{1}{4}\alpha^2 D^2}.$$

We recall the following results from [Sa1].

**Lemma 3.1** *Let  $\Omega$  be a compact Riemannian surface with piecewise  $C^1$  boundary. Then*

$$\frac{2}{\pi} A(\Omega) \sqrt{\lambda(\Omega)} \leq \ell(\partial\Omega) + R(\Omega) \max\{B(\Omega), 0\}, \quad (3.7)$$

where  $\lambda(\Omega)$  is the ground state energy of the Dirichlet realization of the Laplacian in  $\Omega$ , and  $R(\Omega)$  the inner radius of  $\Omega$ .

This is Proposition 3 in [Sa1] (p. 137). Note that when  $M$  is flat and  $\Omega$  is simply or doubly connected, we recover Polyá's inequality [P] which reads:

$$\frac{2}{\pi} A(\Omega) \sqrt{\lambda(\Omega)} \leq \ell(\partial\Omega). \quad (3.8)$$

**Lemma 3.2** *Let  $\Omega$  be a compact Riemannian surface with piecewise  $C^1$  boundary. Then,*

$$R(\Omega)\sqrt{\lambda(\Omega)} \leq \min \left\{ C(\alpha, D), \sqrt{\pi^2 + \frac{\alpha^2 C^2(\alpha, D)}{4\lambda(\Omega)}} \right\} =: \psi(\alpha; D; \lambda(\Omega)).$$

This is Lemma 10 in [Sa1] (p. 141) using  $\lambda(\Omega)$  instead of  $\lambda$ .

**Lemma 3.3** *Let  $\Omega$  be a compact Riemannian surface with piecewise  $C^1$  boundary. Assume that  $B(\Omega) < 0$ . Then,*

$$2|B(\Omega)| \leq \lambda(\Omega)A(\Omega) \leq \frac{\pi}{2}\sqrt{\lambda(\Omega)}\ell(\partial\Omega).$$

This is Lemma 11 in [Sa1] (p. 141), which relies on Dong's paper [Dong]. Note that the second inequality follows from Lemma 3.1.

Let us now proceed with the lower estimate when  $\Omega$  is a Riemannian surface with boundary.

**Proposition 3.4** *Let  $\Omega$  be a compact Riemannian surface with piecewise  $C^1$  boundary. The length  $P(\mathcal{D})$  of the boundary set of a regular equipartition  $\mathcal{D}$ , with energy  $\Lambda$ , satisfies the inequality*

$$P(\mathcal{D}) \geq \frac{4A(\Omega)\sqrt{\Lambda}}{4\pi + \pi^2\psi(\alpha, D; \Lambda)} - \frac{2\pi\psi(\alpha, D; \Lambda)}{\sqrt{\Lambda}(4\pi + \pi^2\psi(\alpha, D; \Lambda))} (B(\Omega) - \pi\sigma(\mathcal{D})). \quad (3.9)$$

**Proof.** The proof follows the ideas in [Sa1] closely. Since Savo does not provide all the details for the case with boundary, we provide them here. Lemma 3.1 applied to each  $D_j$  gives,

$$\frac{2}{\pi}A(D_j)\sqrt{\lambda(D_j)} \leq \ell(\partial D_j) + R(D_j) \max\{B(D_j), 0\}.$$

Since  $\lambda(D_j) = \Lambda$  for all  $j$ , summing up in  $j$ , we find that

$$\frac{2}{\pi}A(\Omega)\sqrt{\Lambda} \leq 2P(\mathcal{D}) + \sum_{j=1}^k R(D_j) \max\{B(D_j), 0\}. \quad (3.10)$$

Call  $T$  the second term in the right-hand side of the preceding inequality and define the sets,

$$J_+ := \{j \mid 1 \leq j \leq k, B(D_j) > 0\}, \quad J_- := \{j \mid 1 \leq j \leq k, B(D_j) \leq 0\}.$$

By Lemma 3.2, we have

$$T = \sum_{j \in J_+} R(D_j)B(D_j) \leq \frac{\psi(\alpha, D; \Lambda)}{\sqrt{\Lambda}} \sum_{j \in J_+} B(D_j). \quad (3.11)$$

Using the definition of  $B(D_j)$ , we find that

$$\begin{aligned}\sum_{j=1}^k B(D_j) &= \int_{\Omega} K_+ - 2\pi \sum_{j=1}^k \chi(D_j) \\ &= B(\Omega) + 2\pi\chi(\Omega) - 2\pi \sum_{j=1}^k \chi(D_j)\end{aligned}$$

and hence, using Euler's formula (2.13),

$$\sum_{j=1}^k B(D_j) = B(\Omega) - \pi\sigma(\mathcal{D}).$$

On the other hand, we have

$$\begin{aligned}\sum_{j \in J_+} B(D_j) &= \sum_{j=1}^k B(D_j) - \sum_{j \in J_-} B(D_j) \\ &= B(\Omega) - \pi\sigma(\mathcal{D}) + \sum_{j \in J_-} |B(D_j)|,\end{aligned}$$

and we can estimate the last term in the right-hand side using Lemma 3.3. Namely,

$$\sum_{j \in J_-} |B(D_j)| \leq \frac{\pi}{4} \sqrt{\Lambda} \sum_{j \in J_-} \ell(\partial D_j) \leq \frac{\pi}{2} \sqrt{\Lambda} P(\mathcal{D}).$$

Finally, we obtain the following estimate for  $T$ ,

$$T \leq \frac{\psi(\alpha, D; \Lambda)}{\sqrt{\Lambda}} \left\{ B(\Omega) - \pi\sigma(\mathcal{D}) + \frac{\pi}{2} \sqrt{\Lambda} P(\mathcal{D}) \right\}.$$

Using (3.10), it follows that

$$A(\Omega)\sqrt{\Lambda} \leq \left\{ \pi + \frac{\pi^2}{4} \psi(\alpha, D; \Lambda) \right\} P(\mathcal{D}) + \frac{\pi\psi(\alpha, D; \Lambda)}{2\sqrt{\Lambda}} \left\{ B(\Omega) - \pi\sigma(\mathcal{D}) \right\}. \quad (3.12)$$

This proves the proposition.  $\square$

### 3.4 A loose local lower estimate for $P(\mathcal{D})$

For simplicity, we now assume that  $\Omega$  is a bounded domain in  $\mathbb{R}^2$ , with piecewise  $C^1$  boundary. We also assume that we are given some point  $x_0 \in \Omega$ , some radius  $R$  and some positive number  $\rho$ , small with respect to  $R$ , such that  $B(x_0, R+\rho) \subset \Omega$ . Note that the ball  $B(x_0, R)$  could be replaced by any regular domain.

#### 3.4.1 A local estimate à la Brüning-Gromes : eigenvalues

**Lemma 3.5** *Let  $\lambda$  be an eigenvalue of  $H(\Omega)$ , and let  $u \in E(\lambda)$  be a non-zero eigenfunction associated with  $\lambda$ . If  $\lambda r^2 > \mathbf{j}^2$ , then any disk  $B(x, r) \subset \Omega$  contains at least a point of the nodal set  $N(u)$ .*

This follows immediately from the monotonicity of the Dirichlet eigenvalues with respect to domain inclusion.

**Lemma 3.6** *Let  $\lambda$  be an eigenvalue of  $H(\Omega)$ , and let  $r > 0$  be such that  $0 < r \leq \rho < \frac{R}{10}$ , and  $\lambda r^2 > 4\mathbf{j}^2$ . Then there exists a family of points  $\{x_1, \dots, x_\ell\}$  such that*

- (1) *For  $1 \leq j \leq N$ ,  $x_j \in N(u) \cap B(x_0, R - \frac{r}{2})$ .*
- (2) *The balls  $B(x_j, \frac{r}{2}), 1 \leq j \leq N$ , are pairwise disjoint and contained in  $B(x_0, R) \subset \Omega$ .*
- (3) *We have the inclusion  $B(x_0, R - r) \subset \cup_{j=1}^N B(x_j, 2r)$ .*
- (4) *The number  $N$  satisfies,  $r^2 N \geq 0.2 R^2$ .*

*Proof.* (a) Consider the ball  $B(x_0, R - r)$  and take  $y_1, y_2$  to be the end points of a diameter of the closed ball. Because  $r \leq \rho < R/10$  and  $r^2 \lambda > 4\mathbf{j}^2$ , we have that  $B(y_i, \frac{r}{2}) \subset B(x_0, R - \frac{r}{2}) \subset \Omega$  and  $B(y_i, \frac{r}{2}) \cap N(u) \neq \emptyset$ . Choose  $x_i$  in  $B(y_i, \frac{r}{2}) \cap N(u)$ . Then,  $x_i \in N(u) \cap B(x_0, R - \frac{r}{2})$ ,  $B(x_1, \frac{r}{2}) \cap B(x_2, \frac{r}{2}) = \emptyset$  and  $B(x_i, \frac{r}{2}) \subset B(x_0, R) \subset \Omega$ .

(b) Take a maximal element  $\{x_1, \dots, x_N\}$  (with respect to inclusion) in the set

$$\mathcal{F} := \left\{ (x_1, \dots, x_k) \mid x_i \in N(u) \cap B(x_0, R - \frac{r}{2}), B(x_i, \frac{r}{2}) \text{ pairwise disjoint} \right\},$$

so that the family  $\{x_1, \dots, x_N\}$  satisfies (1) and (2).

*We claim that (3) holds.* Indeed, otherwise we could find  $y \in B(x_0, R - r)$  with  $d(x_i, y) \geq 2r$ , for  $1 \leq i \leq N$ . Because  $B(y, \frac{r}{2}) \cap N(u) \neq \emptyset$ , we would find some  $z \in B(x_0, R - \frac{r}{2}) \cap N(u) \cap B(y, \frac{r}{2})$  such that  $B(z, \frac{r}{2}) \cap (\cup_{j=1}^N B(x_j, \frac{r}{2})) = \emptyset$ . This would contradict the maximality of the family.

(c) Assertion (3) implies that  $\pi(R - r)^2 \leq \sum_{j=1}^N A(B(x_j, 2r)) = 4\pi N r^2$  and since  $r \leq \rho < R/10$ , we get  $r^2 N \geq (0.9)^2 \frac{1}{4} R^2$ . The Lemma is proved.  $\square$

Recall that  $N(u)$  consists of finitely many points and finitely many  $C^1$  arcs with finite length.

**Lemma 3.7** *Let  $\{x_1, \dots, x_N\}$  be a maximal family as given by Lemma 3.6. Assume that  $r^2 \lambda < 16\mathbf{j}^2$ . Then there exists no nodal curve  $\gamma \subset N(u)$  which is simply closed and contained in any of the balls  $B(x_j, \frac{r}{4}), 1 \leq j \leq N$ .*

*Proof.* Indeed, otherwise, there would be a nodal domain contained in one of the balls  $B(x_j, \frac{r}{4})$  and hence we would have  $\lambda \geq \frac{16\mathbf{j}^2}{r^2}$ .

We can now prove the following local estimate.

**Proposition 3.8** *Let  $\lambda$  be an eigenvalue of  $H(\Omega)$ , and let  $u$  be a non-zero eigenfunction associated with  $\lambda$ . Then, the length of the nodal set  $N(u)$  inside  $B(x_0, R)$  is bounded from below by  $10^{-2} R^2 \sqrt{\lambda}$ .*

**Proof.** Choose  $(r, \lambda)$  so that  $4\mathbf{j}^2 < r^2\lambda < 16\mathbf{j}^2$ , with  $r \leq \rho < R/10$ . By Lemma 3.6, the  $N$  balls  $B(x_j, \frac{r}{4})$  are pairwise disjoint with center on  $N(u)$ . By Lemma 3.7, the length of  $N(u) \cap B(x_j, \frac{r}{4})$  is at least  $\frac{r}{2}$ . It follows that

$$\ell(N(u) \cap B(x_0, R)) \geq \sum_j \ell(N(u) \cap B(x_j, \frac{r}{4})) \geq N \frac{r}{2}, \quad (3.13)$$

and the result follows in view of the estimates  $r^2N \geq 0.2r^2$  and  $r^2\lambda < 16\mathbf{j}^2$ .

**Remark 3.9** *Proposition 3.8 can be generalized to the case of a compact Riemannian surface with or without boundary. In that case, one needs to consider balls with radii less than the injectivity radius of the surface, and replace the Faber-Krahn inequality by a local Faber-Krahn inequality, using the fact that the metric can be at small scale compared with a Euclidean metric (see [Br] for more details).*

### 3.4.2 A local estimate à la Brüning-Gromes : spectral equipartitions

The above proof applies to a regular equipartition of energy  $\lambda$ . It is enough in the statements to replace the nodal set  $N(u)$  of  $u$  by the boundary set  $N(\mathcal{D})$  of the partition  $\mathcal{D}$ . We just rewrite the first statement.

**Lemma 3.10** *Let  $\lambda$  be the energy of a regular equipartition. If  $\lambda r^2 > \mathbf{j}^2$ , then any disk  $B(x, r) \subset \Omega$  contains at least one point of boundary set of the partition.*

This follows immediately from the monotonicity of the Dirichlet eigenvalues with respect to domain inclusion.

## 4 The problem for $k$ large

Recall Weyl's asymptotic estimate for the eigenvalues of  $H(\Omega)$ ,

$$A(\Omega) \lim_{k \rightarrow \infty} \frac{\lambda_k(\Omega)}{k} = 4\pi.$$

In view of this result, it is natural to investigate the behaviour of  $\mathfrak{L}_k(\Omega)$  when  $k$  tends to infinity.

### 4.1 On the asymptotics of $\mathfrak{L}_k(\Omega)$

We recall two conjectures<sup>2</sup> for domains in  $\mathbb{R}^2$  which were proposed and analyzed in the recent years (see [BHV, BBO, CL, HHOT1, Hel]). The first one is that

**Conjecture 4.1** *The limit of  $\mathfrak{L}_k(\Omega)/k$  as  $k \rightarrow +\infty$  exists.*

<sup>2</sup>The second author was informed of these conjectures by M. Van den Berg.

The second one is that this limit is more explicitly given by

**Conjecture 4.2**

$$A(\Omega) \lim_{k \rightarrow +\infty} \frac{\mathfrak{L}_k(\Omega)}{k} = \lambda(Hexa_1),$$

where  $Hexa_1$  is the regular hexagon in  $\mathbb{R}^2$ , with area 1.

This last conjecture says in particular that the limit is independent of the geometry of  $\Omega$  if  $\Omega$  is a regular domain.

Of course the optimality of the regular hexagonal tiling appears in various contexts in Physics. It is easy to show the upper bound in the second conjecture, and Faber-Krahn's inequality gives a weak lower bound involving the first eigenvalue of the disk. More precisely, we have

$$\frac{\pi \mathbf{j}^2}{A(\Omega)} \leq \liminf_{k \rightarrow +\infty} \frac{\mathfrak{L}_k(\Omega)}{k} \leq \limsup_{k \rightarrow +\infty} \frac{\mathfrak{L}_k(\Omega)}{k} \leq \frac{\lambda(Hexa_1)}{A(\Omega)}. \quad (4.1)$$

It is explored numerically in [BHV] why the second conjecture above looks reasonable.

One can also consider the spectral quantities

$$\mathfrak{L}_{k,1}(\Omega) = \frac{1}{k} \inf_{\mathcal{D} \in \mathfrak{D}_k} \sum_{j=1}^k \lambda(D_j).$$

It is clear that  $\mathfrak{L}_{k,1}(\Omega) \leq \mathfrak{L}_k(\Omega)$  for any  $k$ . The Faber-Krahn inequalities yields lower bounds for  $\mathfrak{L}_{k,1}(\Omega)$ . In particular, for domains in  $\mathbb{R}^2$ , one obtains

$$\frac{\pi \mathbf{j}^2}{A(\Omega)} \leq \mathfrak{L}_{k,1}(\Omega),$$

which improves (4.1) to

$$\frac{\pi \mathbf{j}^2}{A(\Omega)} \leq \liminf_{k \rightarrow +\infty} \frac{\mathfrak{L}_{k,1}(\Omega)}{k} \leq \liminf_{k \rightarrow +\infty} \frac{\mathfrak{L}_k(\Omega)}{k}.$$

Note that Caffarelli and Lin [CL], mention Conjecture 4.2 in relation with  $\mathfrak{L}_{k,1}(\Omega)$ .

The recent numerical computations by Bourdin-Bucur-Oudet [BBO] for the asymptotic structure of the minimal partitions for  $\mathfrak{L}_{k,1}(\Omega)$  are very enlightning, see Figure 4.1.

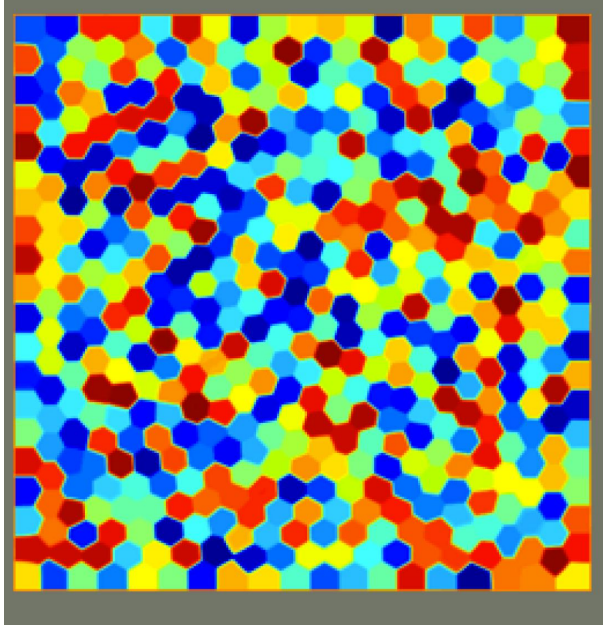


Figure 4.1: Computations of Bourdin-Bucur-Oudet for the periodic square. (Minimization of the sum)

The lower and upper bounds (4.1) are proved for a bounded domain in  $\mathbb{R}^2$ . As a matter of fact, the lower bound holds for a general compact surface  $\Omega$  with smooth boundary. This follows from the following asymptotic isoperimetric and Faber-Krahn inequalities (which actually hold in arbitrary dimension).

**Lemma 4.3** ([BeMe], **Lemma II.15, p. 528**) *Let  $(\Omega, g)$  be a compact Riemannian surface. For any  $\epsilon > 0$ , there exists a positive number  $a(M, g, \epsilon)$  such that for any regular domain  $\omega \subset \Omega$  with area  $A(\omega)$  less than or equal to  $a(M, g, \epsilon)$ ,*

$$\begin{cases} \ell(\partial\omega) & \geq (1 - \epsilon)\ell(\partial\omega^*), \\ \lambda(\omega) & \geq (1 - \epsilon)^2 \frac{\pi j^2}{A(\omega)}, \end{cases}$$

where  $\omega^*$  is a Euclidean disk of area  $A(\omega)$ .

Let  $\mathcal{D} = \{D_1, \dots, D_k\}$  be a  $k$ -equipartition of  $\Omega$ . Let

$$J_\epsilon = \{i \in \{1, \dots, k\} \mid A(D_i) > a(M, g, \epsilon)\}.$$

The number of elements of this set is bounded by

$$\#(J_\epsilon) \leq \frac{A(\Omega)}{a(M, g, \epsilon)}. \quad (4.2)$$

For any  $i \notin J_\epsilon$ , we can write,

$$\lambda(D_i) \geq (1 - \epsilon)^2 \frac{\pi \mathbf{j}^2}{A(D_i)}$$

and hence,

$$\Lambda(\mathcal{D})A(\Omega) \geq (1 - \epsilon)^2 \left( k - \frac{A(\Omega)}{a(M, g, \epsilon)} \right) \pi \mathbf{j}^2.$$

Finally, we obtain that

$$A(\Omega) \liminf_{k \rightarrow \infty} \frac{\mathcal{L}_k(\Omega)}{k} \geq (1 - \epsilon)^2 \pi \mathbf{j}^2.$$

We can now let  $\epsilon$  tend to zero to get the estimate,

$$A(\Omega) \liminf_{k \rightarrow \infty} \frac{\mathcal{L}_k(\Omega)}{k} \geq \pi \mathbf{j}^2. \quad (4.3)$$

We point out that the lower bound does not depend on the geometry of  $\Omega$ .

Inequality (4.3) can also be deduced from [Pe] when  $\Omega$  is a bounded domain in a simply-connected surface  $M$ , with Gaussian curvature  $K$ , such that  $\Omega \subset \Omega_0$ , a simply-connected domain satisfying  $A(\Omega_0) \sup_{\Omega_0} K^+ \leq \pi$ . Let us mention two particular cases.

1. If  $M$  is a simply-connected surface with non-positive curvature, then according to [Pe],  $\lambda(\Omega)A(\Omega) \geq \pi \mathbf{j}^2$  for any bounded domain  $\Omega$  and we conclude that

$$A(\Omega) \frac{\mathcal{L}_k(\Omega)}{k} \geq \pi \mathbf{j}^2$$

for all  $k \geq 1$ , as in the Euclidean case.

2. If  $M$  is the standard sphere, then according to [Pe],

$$\lambda(D)A(D) \geq \pi \mathbf{j}^2 \left( 1 - \frac{A(D)}{4\pi} \right)$$

for any domain  $D$ , and one can conclude that, for any domain  $\Omega$ ,

$$A(\Omega) \frac{\mathcal{L}_k(\Omega)}{k} \geq \pi \mathbf{j}^2 - \frac{\mathbf{j}^2}{4k} A(\Omega),$$

for all  $k \geq 1$ .

Note that these estimates actually hold for  $\mathfrak{L}_{k,1}(\Omega)$ .



## 4.2 Asymptotics of the length of the boundary set of minimal $k$ -partitions for $k$ large.

Of course the hexagonal conjecture leads to a natural conjecture for the length. The “hexagonal conjecture” for the length will be

$$\lim_{k \rightarrow +\infty} (P(\mathcal{D}_k)/\sqrt{k}) = \frac{1}{2} \ell(\text{Hexa}1) \sqrt{A(\Omega)}, \quad (4.4)$$

where  $\ell(\text{Hexa}1)$  is the length of the boundary of the hexagon of area 1:

$$\ell(\text{Hexa}1) = 2\sqrt{2\sqrt{3}}.$$

But we can at least get an asymptotic lower bound for the length in the following way. Knowing that  $\mathfrak{L}_k(\Omega) \rightarrow +\infty$  as  $k \rightarrow +\infty$  (a consequence of Faber-Krahn’s inequality), we deduce from (3.6) that,

$$\liminf_{k \rightarrow +\infty} (P(\mathcal{D}_k)/\sqrt{k}) \geq \frac{1}{2\mathbf{j}} \sqrt{\liminf_{k \rightarrow +\infty} \left( \frac{\mathcal{L}_k(\Omega)}{k} \right)}. \quad (4.5)$$

Together with Faber-Krahn’s inequality, this gives:

$$\liminf_{k \rightarrow +\infty} (P(\mathcal{D}_k)/\sqrt{k}) \geq \frac{\sqrt{\pi}}{2} \sqrt{A(\Omega)}. \quad (4.6)$$

Assuming that the elements of the minimal partitions have no hole, we could apply (3.8) and would get instead the sharper estimate

$$\liminf_{k \rightarrow +\infty} (P(\mathcal{D}_k)/\sqrt{k}) \geq \frac{\mathbf{j}}{\sqrt{\pi}} \sqrt{A(\Omega)}. \quad (4.7)$$

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