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Stable continuous-state branching processes with immigration and Beta-Fleming-Viot processes with immigration.

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DRAFT VERSION

Abstract

Branching processes and Fleming-Viot processes are two main models in stochastic population theory. Incorporating an immigration in both models, we generalize the results of Shiga (1990) and Birkner *et al.* (2005) which respectively connect the Feller diffusion with the classical Fleming-Viot process and the α -stable continuous state branching process with the $Beta(2 - \alpha, \alpha)$ -generalized Fleming-Viot process. In a recent work, a new class of probability-measure valued processes, called M -generalized Fleming-Viot processes with immigration, has been set up in duality with the so-called M -coalescents. The purpose of this article is to investigate the links between this new class of processes and the continuous-state branching processes with immigration. In the specific case of the α -stable branching process conditioned to be never extinct, we get that its genealogy is given, up to a random time change, by a $Beta(2 - \alpha, \alpha - 1)$ -coalescent.

Key words. Measure-valued processes, Continuous-state branching processes, Fleming-Viot processes, Immigration, Beta-Coalescent, Generators, Random time change.

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1 Introduction

The connections between the Fleming-Viot processes and the continuous-state branching processes have been intensively studied. Shiga established in 1990 that a Fleming-Viot process may be recovered from the *ratio process* associated with a Feller diffusion up to a random time change, see [23]. This result has been generalized in 2005 by Birkner *et al* in [7] in the setting of Λ -generalized Fleming-Viot processes and continuous-state branching processes (CBs for short). In that paper they proved that the ratio process associated with an α -stable branching process is a time-changed $Beta(2 - \alpha, \alpha)$ -Fleming-Viot process for $\alpha \in (0, 2)$. The main goal of this article is to study such connections when immigration is incorporated in the underlying population. The continuous-state branching processes with immigration (CBIs for short) are a class of time-homogeneous Markov processes with values in \mathbb{R}_+ . They have been introduced by Kawazu and Watanabe in 1971, see [15], as limits of rescaled Galton-Watson processes with immigration. These processes are characterized by two functions Φ and Ψ respectively called the immigration mechanism and the branching mechanism. A new class of measure-valued processes with immigration has been recently set up in [13]. These processes, called M -generalized Fleming-Viot processes with immigration (M -GFVIs for short) are valued in the space of probability measures on $[0, 1]$. The notation M stands for a couple of finite measures (Λ_0, Λ_1) encoding respectively the rates of immigration and of reproduction. The genealogies of the M -GFVIs are given by the so-called M -coalescents. These processes are valued in the space of the partitions of \mathbb{Z}_+ , denoted by \mathcal{P}_∞^0 .

In the same manner as Birkner *et al.* in [7], Perkins in [21] and Shiga in [23], we shall establish some relations between continuous-state branching processes with immigration and M -GFVIs. A notion of continuous population with immigration may be defined using a flow of CBIs in the same spirit as Bertoin and Le Gall in [4]. This allows us to compare the two notions of continuous populations provided respectively by the CBIs and by the M -GFVIs. Using calculations of generators, we show that the following self-similar CBIs admit time-changed M -GFVIs for ratio processes:

- the Feller branching diffusion with branching rate σ^2 and immigration rate β (namely the CBI with $\Phi(q) = \beta q$ and $\Psi(q) = \frac{1}{2}\sigma^2 q^2$) which has for ratio process a time-changed M -Fleming-Viot process with immigration with $M = (\beta\delta_0, \sigma^2\delta_0)$,
- the CBI process with $\Phi(q) = d'\alpha q^{\alpha-1}$ and $\Psi(q) = dq^\alpha$ for some $d, d' \geq 0$, $\alpha \in (1, 2)$ which has for ratio process a time-changed M -generalized Fleming-Viot process with immigration with $M = (c'Beta(2 - \alpha, \alpha - 1), cBeta(2 - \alpha, \alpha))$, where $c' = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}d'$ and $c = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}d$.

We stress that the CBIs may reach 0 in which case the M -GFVIs involved describe the ratio process up to this hitting time only. When $d = d'$ or $\beta = \sigma^2$, the corresponding CBIs are respectively the α -stable branching process and the Feller branching diffusion *conditioned to be never extinct*. In that case, the M -coalescents are genuine Λ -coalescent viewed on \mathcal{P}_∞^0 . We get respectively a $Beta(2 - \alpha, \alpha - 1)$ -coalescent when $\alpha \in (1, 2)$ and a Kingman's coalescent for $\alpha = 2$. This differs from the α -stable branching process *without immigration* (already studied in [7]) for which the coalescent involved is a $Beta(2 - \alpha, \alpha)$ -coalescent.

Outline. The paper is organized as follows. In Section 2, we recall the definition of a continuous-state branching process with immigration and of an M -generalized Fleming-Viot process with immigration. We describe briefly how to define from a flow of CBIs a continuous population represented by a measure-valued process. We state in Section 3 the connections between the CBIs and M -GFVIs, mentioned in the Introduction, and study the random time change. Recalling the definition of an M -coalescent, we focus in Section 4 on the genealogy of the M -GFVIs involved. We establish that, when

the CBIs correspond with CB-processes conditioned to be never extinct, the M -coalescents involved are actually classical Λ -coalescents. We identify them and, as mentioned, the $Beta(2 - \alpha, \alpha - 1)$ -coalescent arises. In Section 5, we compare the generators of the M -GFVI and CBI processes and establish the main result.

2 A continuous population embedded in a flow of CBIs and the M -generalized Fleming-Viot with immigration

2.1 Background on continuous state branching processes with immigration

We will focus on critical continuous-state branching processes with immigration characterized by two functions of the variable $q \geq 0$:

$$\begin{aligned}\Psi(q) &= \frac{1}{2}\sigma^2 q^2 + \int_0^\infty (e^{-qu} - 1 + qu)\hat{\nu}_1(du) \\ \Phi(q) &= \beta q + \int_0^\infty (1 - e^{-qu})\hat{\nu}_0(du)\end{aligned}$$

where $\sigma^2, \beta \geq 0$ and $\hat{\nu}_0, \hat{\nu}_1$ are two Lévy measures such that $\int_0^\infty (1 \wedge u)\hat{\nu}_0(du) < \infty$ and $\int_0^\infty (u \wedge u^2)\hat{\nu}_1(du) < \infty$. The measure $\hat{\nu}_1$ is the Lévy measure of a spectrally positive Lévy process which characterizes the reproduction. The measure $\hat{\nu}_0$ characterizes the jumps of the subordinator that describes the arrival of immigrants in the population. The non-negative constants σ^2 and β correspond respectively to the continuous reproduction and the continuous immigration. Let \mathbb{P}_x be the law of a CBI $(Y_t, t \geq 0)$ started at x , and denote by \mathbb{E}_x the associated expectation. The law of $(Y_t, t \geq 0)$ can then be characterized by the Laplace transform of its marginal as follows: for every $q > 0$ and $x \in \mathbb{R}_+$,

$$\mathbb{E}_x[e^{-qY_t}] = \exp\left(-xv_t(q) - \int_0^t \Phi(v_s(q))ds\right)$$

where $\frac{\partial}{\partial t}v_t(q) = -\Psi(v_t(q))$, $v_0(q) = q$.

The pair (Ψ, Φ) is known as the branching-immigration mechanism. A CBI process $(Y_t, t \geq 0)$ is said to be conservative if for every $t > 0$ and $x \in [0, \infty[$, $\mathbb{P}_x[Y_t < \infty] = 1$. A result of Kawazu and Watanabe [15] states that $(Y_t, t \geq 0)$ is conservative if and only if for every $\epsilon > 0$

$$\int_0^\epsilon \frac{1}{|\Psi(q)|}dq = \infty.$$

We follow the seminal idea of Bertoin and Le Gall in [4] to define a genuine continuous population model with immigration on $[0, 1]$ associated with a CBI. Emphasizing the rôle of the initial value, we denote by $(Y_t(x), t \geq 0)$ a CBI started at $x \in \mathbb{R}_+$. The branching property ensures that $(Y_t(x+y), t \geq 0) \stackrel{\text{law}}{=} (Y_t(x) + X_t(y), t \geq 0)$ where $(X_t(y), t \geq 0)$ is a CBI $(\Psi, 0)$ starting from y (that is a CB-process without immigration and with branching mechanism Ψ) independent of $(Y_t(x), t \geq 0)$. The Kolmogorov's extension theorem allows one to construct a flow $(Y_t(x), t \geq 0, x \geq 0)$ such that for every $y \geq 0$, $(Y_t(x+y) - Y_t(x), t \geq 0)$ has the same law as $(X_t(y), t \geq 0)$ a CB-process started from y . We denote by $(M_t, t \geq 0)$ the Stieltjes-measure associated with the increasing process

$x \in [0, 1] \mapsto Y_t(x)$. Namely, define

$$\begin{aligned} M_t([x, y]) &:= Y_t(y) - Y_t(x), \quad 0 \leq x \leq y \leq 1. \\ M_t(\{0\}) &:= Y_t(0). \end{aligned}$$

The process $(Y_t(1), t \geq 0)$ is assumed to be conservative, therefore the process $(M_t, t \geq 0)$ is valued in the space \mathcal{M}_f of finite measures on $[0, 1]$. We stress that this space is locally compact, separable and metrizable. Therefore we may consider its one-point compactification (see [20] page 7). By a slight abuse of notation, we denote by $(Y_t, t \geq 0)$ the process $(Y_t(1), t \geq 0)$. The framework of measure-valued processes allows us to consider an infinitely many types model. Namely each individual has initially its own type (which lies in $[0, 1]$) and transmits it to its progeny. People issued from the immigration have a *distinguished* type fixed at 0. Since the types do not evolve in time, they allow us to track the ancestors at time 0. This model can be viewed as a superprocess without spatial motion (or without mutation in population genetics vocabulary).

Let \mathcal{C} be the class of functions on \mathcal{M}_f of the form

$$F(\eta) := G(\langle f_1, \eta \rangle, \dots, \langle f_n, \eta \rangle),$$

where $\langle f, \eta \rangle := \int_{[0,1]} f(x)\eta(dx)$, $G \in C^2(\mathbb{R}^n)$ and f_1, \dots, f_n are bounded measurable functions on $[0, 1]$. Section 9.3 of Li's book [20] (see Theorem 9.18 p. 218) ensures that the following operator acting on the space \mathcal{M}_f is an extended generator of $(M_t, t \geq 0)$. For any $\eta \in \mathcal{M}_f$,

$$\mathcal{L}F(\eta) := \sigma^2/2 \int_0^1 \int_0^1 \eta(da)\delta_a(db)F''(\eta; a, b) \quad (1)$$

$$+ \beta F'(\eta; 0) \quad (2)$$

$$+ \int_0^1 \eta(da) \int_0^\infty \hat{\nu}_1(dh)[F(\eta + h\delta_a) - F(\eta) - hF'(\eta, a)] \quad (3)$$

$$+ \int_0^\infty \hat{\nu}_0(dh)[F(\eta + h\delta_0) - F(\eta)] \quad (4)$$

where $F'(\eta; a) := \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon}[F(\eta + \epsilon\delta_a) - F(\eta)]$ is the Gateaux derivative of F at η in direction δ_a , and $F''(\eta; a, b) := G''(\eta; b)$ with $G(\eta) = F'(\eta; a)$. The terms (1) and (3) correspond to the reproduction, see for instance Section 6.1 p106 of Dawson [8]. The terms (2) and (4) correspond to the immigration. We stress that in our model the immigration is concentrated on 0, contrary to other works which consider infinitely many types for the immigrants. For the interested reader, the operator \mathcal{L} corresponds with that given in equation (9.25) of Section 9 of Li [20] by setting $H(d\mu) = \int_0^\infty \hat{\nu}_0(dh)\delta_{h\delta_0}(d\mu)$ and $\eta = \beta\delta_0$.

For all $\eta \in \mathcal{M}_f$, we denote by $|\eta|$ the total mass $|\eta| := \eta([0, 1])$. If $(M_t, t \geq 0)$ is a Markov process with the above operator for generator, the process $(|M_t|, t \geq 0)$ is a CBI. Indeed, let ψ be a twice differentiable function on \mathbb{R}_+ and define $F : \eta \mapsto \psi(|\eta|)$, we find $\mathcal{L}F(\eta) = zG_B\psi(z) + G_I\psi(z)$ for $z = |\eta|$, where

$$G_B\psi(z) = \frac{\sigma^2}{2}\psi''(z) + \int_0^\infty [\psi(z+h) - \psi(z) - h\psi'(z)]\hat{\nu}_1(dh) \quad (5)$$

$$G_I\psi(z) = \beta\psi'(z) + \int_0^\infty [\psi(z+h) - \psi(z)]\hat{\nu}_0(dh). \quad (6)$$

2.2 Background on M -generalized Fleming-Viot processes with immigration

We denote by \mathcal{M}_1 the space of probability measures on $[0, 1]$. Let c_0, c_1 be two non-negative real numbers and ν_0, ν_1 be two measures on $[0, 1]$ such that $\int_0^1 x\nu_0(dx) < \infty$ and $\int_0^1 x^2\nu_1(dx) < \infty$. Following the notation of [13], we define the couple of finite measures $M = (\Lambda_0, \Lambda_1)$ such that

$$\Lambda_0(dx) = c_0\delta_0(dx) + x\nu_0(dx), \quad \Lambda_1(dx) = c_1\delta_0(dx) + x^2\nu_1(dx).$$

The M -generalized Fleming-Viot process with immigration describes a population with *constant size* which evolves by resampling. Let $(\rho_t, t \geq 0)$ be an M -generalized Fleming-Viot process with immigration. The evolution of this process is a superposition of a continuous evolution, and a discontinuous one. The continuous evolution can be described as follows: every couple of individuals is sampled at constant rate c_1 , in which case one of the two individuals gives its type to the other: this is a reproduction event. Furthermore, any individual is picked at constant rate c_0 , and its type replaced by the distinguished type 0 (the immigrant type): this is an immigration event. The discontinuous evolution is prescribed by two independent Poisson point measures N_0 and N_1 on $\mathbb{R}_+ \times [0, 1]$ with respective intensity $dt \otimes \nu_0(dx)$ and $dt \otimes \nu_1(dx)$. More precisely, if (t, x) is an atom of $N_0 + N_1$ then t is a jump time of the process $(\rho_t, t \geq 0)$ and the conditional law of ρ_t given ρ_{t-} is:

- $(1 - x)\rho_{t-} + x\delta_U$, if (t, x) is an atom of N_1 , where U is distributed according to ρ_{t-}
- $(1 - x)\rho_{t-} + x\delta_0$, if (t, x) is an atom of N_0 .

If (t, x) is an atom of N_1 , an individual is picked at random in the population at generation $t-$ and generates a proportion x of the population at time t : this is a reproduction event, as for the genuine generalized Fleming-Viot process (see [5] p278). If (t, x) is an atom of N_0 , the individual 0 at time $t-$ generates a proportion x of the population at time t : this is an immigration event. In both cases, the population at time $t-$ is reduced by a factor $1 - x$ so that, at time t , the total size is still 1. The genealogy of this population (which is identified as a probability measure on $[0, 1]$) is given by an M -coalescent (see Section 4 below). This description is purely heuristic (we stress for instance that the atoms of $N_0 + N_1$ may form an infinite dense set), to make a rigorous construction of such processes, we refer to the Section 5.2 of [13] (or alternatively Section 3.2 of [14]).

For any $p \in \mathbb{N}$ and any continuous function f on $[0, 1]^p$, we denote by G_f the map

$$\rho \in \mathcal{M}_1 \mapsto \langle f, \rho^{\otimes p} \rangle := \int_{[0,1]^p} f(x_1, \dots, x_p) \rho(dx_1) \dots \rho(dx_p).$$

Let $(\mathcal{F}, \mathcal{D})$ denote the generator of $(\rho_t, t \geq 0)$ and its domain. The vector space generated by the functionals of the type G_f forms a core of $(\mathcal{F}, \mathcal{D})$ and we have (see Lemma 5.2 in [13]):

$$\mathcal{F}G_f(\rho) = c_1 \sum_{1 \leq i < j \leq p} \int_{[0,1]^p} [f(x^{i,j}) - f(x)] \rho^{\otimes p}(dx) \quad (1')$$

$$+ c_0 \sum_{1 \leq i \leq p} \int_{[0,1]^p} [f(x^{0,i}) - f(x)] \rho^{\otimes p}(dx) \quad (2')$$

$$+ \int_0^1 \nu_1(dr) \int \rho(da) [G_f((1-r)\rho + r\delta_a) - G_f(\rho)] \quad (3')$$

$$+ \int_0^1 \nu_0(dr) [G_f((1-r)\rho + r\delta_0) - G_f(\rho)]. \quad (4')$$

where x denotes the vector (x_1, \dots, x_p) and

- the vector $x^{0,i}$ is defined by $x_k^{0,i} = x_k$, for all $k \neq i$ and $x_i^{0,i} = 0$,
- the vector $x^{i,j}$ is defined by $x_k^{i,j} = x_k$, for all $k \neq j$ and $x_j^{i,j} = x_i$.

3 Relations between CBIs and M -GFVIs

3.1 Forward results

The expressions of the generators of $(M_t, t \geq 0)$ and $(\rho_t, t \geq 0)$ lead us to specify the connections between CBIs and GFVIs. We add a cemetery point Δ to the space \mathcal{M}_1 and define $(R_t, t \geq 0) := (\frac{M_t}{|M_t|}, t \geq 0)$, the ratio process with lifetime $\tau := \inf\{t \geq 0; |M_t| = 0\}$. By convention, for all $t \geq \tau$, we set $R_t = \Delta$. As mentioned in the Introduction, we shall focus our study on the two following critical CBIs:

- (i) $(Y_t, t \geq 0)$ is CBI with parameters $\sigma^2, \beta \geq 0$ and $\hat{\nu}_0 = \hat{\nu}_1 = 0$.
- (ii) $(Y_t, t \geq 0)$ is a CBI with $\sigma^2 = \beta = 0$, $\hat{\nu}_0(dh) = c'h^{-\alpha}1_{h>0}dh$ and $\hat{\nu}_1(dh) = ch^{-1-\alpha}1_{h>0}dh$.

We first establish in the following proposition a dichotomy for the finiteness of the lifetime, depending on the ratio immigration over reproduction.

Proposition 1 *Recall the notation $\tau = \inf\{t \geq 0, Y_t = 0\}$.*

- If $\frac{\beta}{\sigma^2} \geq \frac{1}{2}$ in case (i) or $\frac{c'}{c} \geq \frac{\alpha-1}{\alpha}$ in case (ii), then $\mathbb{P}[\tau = \infty] = 1$.
- If $\frac{\beta}{\sigma^2} < \frac{1}{2}$ in case (i) or $\frac{c'}{c} < \frac{\alpha-1}{\alpha}$ in case (ii), then $\mathbb{P}[\tau < \infty] = 1$

We then deal with the random change of time. In the case of a CB-process (that is a CBI process without immigration), Birkner *et al.* used the Lamperti representation and worked on the embedded stable spectrally positive Lévy process. We shall work directly on the CBI process instead.

Proposition 2 *In both cases (i) and (ii), taking $\alpha = 2$ in case (i), we have:*

$$\mathbb{P}\left(\int_0^\tau Y_s^{1-\alpha} ds = \infty\right) = 1.$$

In other words, the additive functional of $(Y_t, t \geq 0)$ defined by $C : t \mapsto \int_0^t Y_s^{1-\alpha} ds$ maps $[0, \tau[$ to $[0, \infty[$.

By convention, if τ is almost surely finite we set $C(t) = C(\tau) = \infty$ for all $t \geq \tau$. Denote by C^{-1} the right continuous inverse of the functional C . This maps $[0, \infty[$ to $[0, \tau[$. We stress that in most cases, $(R_t, t \geq 0)$ is not a Markov process. Nevertheless, in some cases, through a change of time, the process $(R_t, t \geq 0)$ may be changed into a Markov process. This shall be stated in the following Theorem where the functional C is central.

For every $x, y > 0$, denote by $Beta(x, y)(dr)$ the finite measure with density

$$r^{x-1}(1-r)^{y-1}1_{(0,1)}(r)dr,$$

and recall that its total mass is given by the Beta function $B(x, y)$.

Theorem 3 *Let $(M_t, t \geq 0)$ be the measure-valued process associated to a process $(Y_t(x), x \in [0, 1], t \geq 0)$ with characteristics $\sigma^2, \beta \geq 0$, and $\hat{\nu}_0, \hat{\nu}_1$ the Lévy measures.*

(i) If $\hat{\nu}_0 = \hat{\nu}_1 = 0$. Let $C(t) = \int_0^t Y_s^{-1} ds$, the process $(R_{C^{-1}(t)})_{t \geq 0}$ is a M -Fleming-Viot process with immigration with

$$\Lambda_0(dr) = \beta \delta_0(dr) \text{ and } \Lambda_1(dr) = \sigma^2 \delta_0(dr).$$

(ii) If $\sigma^2 = \beta = 0$ and for some $\alpha \in (1, 2)$ and some $c > 0, c' \geq 0$

$$\hat{\nu}_0(dh) = c' h^{-\alpha} 1_{h>0} dh \text{ and } \hat{\nu}_1(dh) = c h^{-1-\alpha} 1_{h>0} dh,$$

Let $C(t) = \int_0^t Y_s^{1-\alpha} ds$, then the process $(R_{C^{-1}(t)})_{t \geq 0}$ is a M -generalized Fleming-Viot process with immigration with

$$\Lambda_0(dr) = c' \text{Beta}(2 - \alpha, \alpha - 1)(dr) \text{ and } \Lambda_1(dr) = c \text{Beta}(2 - \alpha, \alpha)(dr).$$

In case (i), the process $(Y_t, t \geq 0)$ is a CBI with $\Psi(q) = \frac{\sigma^2}{2} q^2$ and $\Phi(q) = \beta q$, that is a Feller branching diffusion with continuous immigration. In case (ii), the process $(Y_t, t \geq 0)$ is a CBI with $\Psi(q) = dq^\alpha$ and $\Phi(q) = d' \alpha q^{\alpha-1}$ with $d' = \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)} c'$ and $d = \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)} c$. The proof requires rather technical arguments on the generators and is given in Section 5.

Remark 3.1 • The CBIs in the statement of Theorem 3 with $\sigma^2 = \beta$ in case (i) or $c = c'$ in case (ii), are also CBs conditioned on non extinction and are studied further in Section 4.

• Contrary to the case without immigration, see Theorem 1.1 in [7], we have to restrict ourselves to $\alpha \in (1, 2]$.

So far, we state that the ratio process $(R_t, t \geq 0)$ associated to $(M_t, t \geq 0)$, once time changed by C^{-1} , is a M -GFVI process. Conversely, starting from a M -GFVI process, we could wonder how to recover the measure-valued CBI process $(M_t, t \geq 0)$. This lead us to investigate the relation between the time changed ratio process $(R_{C^{-1}(t)}, t \geq 0)$ and the process $(Y_t, t \geq 0)$.

Proposition 4 In case (i) of Theorem 3, the additive functional $(C(t), t \geq 0)$ and $(R_{C^{-1}(t)}, 0 \leq t < \tau)$ are independent.

This proves that in case (i) we need additional randomness to reconstruct M from the M -GFVI process. On the contrary, in case (ii), the process $(Y_t, t \geq 0)$ is clearly not independent of the ratio process $(R_t, t \geq 0)$.

The proof of Propositions 1, 2 and 4 are given in the next Subsection.

3.2 Proofs of Propositions 1, 2 and 4

Proof of Proposition 1. Let $(X_t(x), t \geq 0)$ denote an α -stable branching process started at x (with $\alpha \in (1, 2]$). Denote ζ its absorption time, $\zeta := \inf\{t \geq 0; X_t(x) = 0\}$. The following construction of the process $(Y_t(0), t \geq 0)$ may be deduced from the expression of the Laplace transform of the CBI process. We write:

$$(Y_t(0), t \geq 0) = \left(\sum_{i \in \mathcal{I}} X_{(t-t_i)_+}^i, t \geq 0 \right) \quad (7)$$

with $\sum_i \delta_{(t_i, X^i)}$ a Poisson random measure on $\mathbb{R}_+ \times \mathcal{D}(\mathbb{R}_+, \mathbb{R}_+)$ with intensity $dt \otimes \mu$, where $\mathcal{D}(\mathbb{R}_+, \mathbb{R}_+)$ denotes the space of càdlàg functions, and μ is defined as follows:

- in case (ii), $\mu(dX) = \int \hat{\nu}_0(dx) \mathbb{P}_x(dX)$, where \mathbb{P}_x is the law of a CB(Ψ) with $\Psi(q) = dq^\alpha$. Formula (7) may be understood as follows: at the jump times t_i of a pure jump stable subordinator with Lévy measure $\hat{\nu}_0$, a new arrival of immigrants, of size X_0^i , occurs in the population. Each of these "packs", labelled by $i \in \mathcal{I}$, generates its own descendance $(X_t^i, t \geq 0)$, which is a CB(Ψ) process.
- in case (i), $\mu(dX) = \beta \mathbb{N}(dX)$, where \mathbb{N} is the canonical measure of the CB(Ψ) with $\Psi(q) = \frac{\sigma^2}{2} q^2$ (see Le Gall [19] for a definition of the canonical measure \mathbb{N}). This case may be seen as a limit case of the case (ii) when $\alpha \rightarrow 2$. The pure jump subordinator degenerates into a continuous subordinator equal to $(t \mapsto \beta t)$. The immigrants no more arrive by packs, but appear continuously. In other words, we have $\mathbb{N}(X_0 > 0) = 0$.

The process $(Y_t(0), t \geq 0)$ is a CBI(Ψ, Φ) started at 0. We call \mathcal{R} the set of zeros of $(Y_t(0), t > 0)$:

$$\mathcal{R} := \{t > 0; Y_t(0) = 0\}.$$

Denote $\zeta_i = \inf \{t > 0, X_t^i = 0\}$ the lifetime of the branching process X^i . The intervals $]t_i, t_i + \zeta_i[$ represent the time where X^i is alive. Therefore \mathcal{R} is the set of the positive real numbers left uncovered by the random intervals $]t_i, t_i + \zeta_i[$, that is:

$$\mathcal{R} := \mathbb{R}_+^* \setminus \bigcup_{i \in \mathcal{I}}]t_i, t_i + \zeta_i[.$$

The lengths ζ_i have law $\mu(\zeta \in dt)$ thanks to the Poisson construction of $Y(0)$. We now distinguish the two cases:

- Feller case: this corresponds to $\alpha = 2$. We have $\Psi(q) := \frac{\sigma^2}{2} q$ and $\Phi(q) := \beta q$, and thus

$$\mu[\zeta > t] = \beta \mathbb{N}[\zeta > t] = \frac{2\beta}{\sigma^2} \frac{1}{t}$$

see Li [20] p. 62. Using Example 1 p. 180 of Fitzsimmons et al. [12], we deduce that

$$\mathcal{R} = \emptyset \text{ if and only if } \frac{2\beta}{\sigma^2} \geq 1. \quad (8)$$

- Stable case: this corresponds to $\alpha \in (1, 2)$. Recall $\Psi(q) := dq^\alpha$, $\Phi(q) := d'\alpha q^{\alpha-1}$. In that case, we have

$$\mu[\zeta > t] = \int_0^\infty \hat{\nu}_0(dx) \mathbb{P}_x[\zeta > t] = \int_0^\infty \hat{\nu}_0(dx) [1 - \exp(-x\mathbb{N}(\zeta > t))] = \Phi(\mathbb{N}(\zeta > t))$$

and

$$\mathbb{N}(\zeta > t) = d^{-\frac{1}{\alpha-1}} [(\alpha-1)t]^{-\frac{1}{\alpha-1}},$$

see Li [20] again. Thus, $\mu[\zeta > t] = \frac{\alpha}{\alpha-1} \frac{d'}{d} \frac{1}{t}$. Recall that $\frac{d'}{d} = \frac{c'}{c}$. Therefore, using reference [12], we deduce that

$$\mathcal{R} = \emptyset \text{ if and only if } \frac{c'}{c} \geq \frac{\alpha-1}{\alpha}. \quad (9)$$

This allows us to establish the first point of Proposition 1: under the assumptions of the first point, we get $\mathcal{R} = \emptyset$, and the inequality $Y_t(1) \geq Y_t(0)$ for all t ensures that $\tau = \infty$.

We deal now with the second point. Assume that $\frac{c'}{c} < \frac{\alpha-1}{\alpha}$ or $\frac{\beta}{\sigma^2} < \frac{1}{2}$. By assertions (8) and (9), we already know that $\mathcal{R} \neq \emptyset$. Using Corollary 4 (Equation 17 p 183) of [12], we show below that \mathcal{R} is not bounded. Recall that

$$\mu[\zeta > s] = \Phi(\mathbb{N}(\zeta > s)) = \frac{\kappa}{s}$$

with $\kappa = \frac{\alpha}{\alpha-1} \frac{d'}{d} = \frac{\alpha}{\alpha-1} \frac{c'}{c} < 1$. Thus $\int_1^u \mu[\zeta > s] ds = \kappa \ln(u)$ and we obtain

$$\exp\left(-\int_1^u \mu[\zeta > s] ds\right) = \left(\frac{1}{u}\right)^\kappa.$$

Therefore, since $\kappa < 1$,

$$\int_1^\infty \exp\left(-\int_1^u \mu[\zeta > s] ds\right) du = \infty,$$

which implies that \mathcal{R} is not bounded. Recall from Subsection 2.1 that we may write $Y_t(1) = Y_t(0) + X_t(1)$ for all $t \geq 0$ with $(X_t(1), t \geq 0)$ a CB-process independent of $(Y_t(0), t \geq 0)$. Let $\xi := \inf\{t \geq 0, X_t(1) = 0\}$ be the extinction time of $(X_t(1), t \geq 0)$. Since \mathcal{R} is not bounded, $\mathcal{R} \cap (\xi, \infty) \neq \emptyset$, and $\tau < \infty$ almost surely. The same argument applies for the Feller case with $\alpha = 2, d' = \frac{\beta}{2}$ and $d = \frac{\sigma^2}{2}$. \square

Proof of Proposition 2. Recall that $Y_t(x)$ is the value of the CBI started at x at time t . We will denote by $\tau^x(0) := \inf\{t > 0, Y_t(x) = 0\}$. The stopping time $\tau^1(0)$ is nothing but the variable τ studied above. In both cases (i) and (ii), the processes are self-similar, see Kyprianou & Pardo [16]. Namely, we have

$$(xY_{x^{1-\alpha}t}(1), t \geq 0) \stackrel{law}{=} (Y_t(x), t \geq 0),$$

where we take $\alpha = 2$ in case (i). Performing the change of variable $s = x^{1-\alpha}t$, we obtain

$$\int_0^{\tau^x(0)} dt Y_t(x)^{1-\alpha} \stackrel{law}{=} \int_0^{\tau^1(0)} ds Y_s(1)^{1-\alpha}. \quad (10)$$

We proved in Proposition 1 that, depending on the values of the parameters:

- Either $\mathbb{P}(\tau^x(0) < \infty) = 1$ for every x . In particular, denoting $\tau^x(1) = \inf\{t > 0, Y_t(x) \leq 1\}$, we have $\mathbb{P}(\tau^x(1) < \infty) = 1$ for every $x > 1$. Let $x > 1$.

$$\int_0^{\tau^x(0)} dt Y_t(x)^{1-\alpha} = \int_0^{\tau^x(1)} dt Y_t(x)^{1-\alpha} + \int_{\tau^x(1)}^{\tau^x(0)} dt Y_t(x)^{1-\alpha}$$

By the strong Markov property applied at the stopping time $\tau^x(1)$,

$$\int_{\tau^x(1)}^{\tau^x(0)} dt Y_t(x)^{1-\alpha} \stackrel{law}{=} \int_0^{\tau^1(0)} dt \tilde{Y}_t(1)^{1-\alpha},$$

with $(\tilde{Y}_t(1), t \geq 0)$ an independent copy started from 1. Since

$$\mathbb{P}\left(\int_0^{\tau^x(1)} dt Y_t(x)^{1-\alpha} > 0\right) = 1,$$

the equality (10) is impossible unless both sides of the equality are infinite almost surely. We thus get that $C(\tau) = \infty$ almost surely in that case.

- Either $\mathbb{P}(\tau^x(0) = \infty) = 1$ for every x , on which case we may rewrite (10) as follows:

$$\int_0^\infty dt Y_t(x)^{1-\alpha} \stackrel{\text{law}}{=} \int_0^\infty ds Y_s(1)^{1-\alpha}.$$

Since, for $x > 1$, the difference $(Y_t(x) - Y_t(1), t \geq 0)$ is an α -stable CB-process started at $x - 1 > 0$, we again face a contradiction, and $C(\tau) = \infty$ almost surely again.

This proves the statement \square

Remark 3.1 *The situation is quite different when the CBI process starts at 0, in which case the time change also diverges in the neighbourhood of 0. The same change of variables as in (10) yields, for all $0 < x < k$,*

$$\int_0^{\iota^x(k)} dt Y_t(x)^{1-\alpha} \stackrel{\text{law}}{=} \int_0^{\iota^1(k/x)} dt Y_t(1)^{1-\alpha},$$

with $\iota^x(k) = \inf\{t > 0, Y_t(x) \geq k\} \in [0, \infty]$. Letting x tend to 0, we get $\iota^1(k/x) \rightarrow \infty$ and the right hand side diverges to infinity. Thus, the left hand side also diverges, which implies that:

$$\mathbb{P}\left(\int_0^{\iota^0(k)} dt Y_t(0)^{1-\alpha} = \infty\right) = 1.$$

Proof of Proposition 4. Let $(Y_t)_{t \geq 0}$ be a Feller branching diffusion with continuous immigration with parameters (σ^2, β) . Consider an independent M -Fleming-Viot $(\rho_t, t \geq 0)$ with $M = (\beta\delta_0, \sigma^2\delta_0)$. We first establish that $(Y_t\rho_{C(t)}, 0 \leq t < \tau)$ has the same law as the measure-valued branching process $(M_t, 0 \leq t < \tau)$. Recall that \mathcal{L} denote the generator of $(M_t, t \geq 0)$ (here only the terms (1) and (2) are considered). Consider $F(\eta) := \psi(z)\langle\phi, \rho\rangle^m$ with $z = |\eta|$, ψ a twice differentiable map valued in \mathbb{R}_+ and ϕ a non-negative bounded measurable function. Note that the generator acting on such functions F characterizes the law of $(M_{t \wedge \tau}, t \geq 0)$. First we easily obtain that

$$\begin{aligned} F'(\eta; 0) &= \psi'(z)\langle\phi, \rho\rangle^m + \frac{m}{z}[\phi(0)\langle\phi, \rho\rangle^{m-1} - \langle\phi, \rho\rangle^m], \\ F''(\eta; a, b) &= \psi''(z)\langle\phi, \rho\rangle^m + m\frac{\psi'(z)}{z}[(\phi(b) + \phi(a))\langle\phi, \rho\rangle^{m-1} - 2\langle\phi, \rho\rangle^m] \\ &\quad + \frac{m(m-1)}{z^2}\phi(a)[\phi(b)\langle\phi, \rho\rangle^{m-2} - \langle\phi, \rho\rangle^{m-1}]. \end{aligned}$$

Simple calculations yield,

$$\begin{aligned} \mathcal{L}F(\eta) &= \left[z\left(\frac{\sigma^2}{2}\psi''(z)\right) + \beta\psi'(z)\right]\langle\phi, \rho\rangle^m \\ &\quad + \frac{1}{z}\left[\sigma^2\frac{m(m-1)}{2}(\langle\phi^2, \rho\rangle\langle\phi, \rho\rangle^{m-2} - \langle\phi, \rho\rangle^m) + \beta m(\phi(0)\langle\phi, \rho\rangle^{m-1} - \langle\phi, \rho\rangle^m)\right]. \end{aligned}$$

We recognize in the first line the generator of $(Y_t, t \geq 0)$ and in the second, $\frac{1}{z}\mathcal{F}G_f(\rho)$ with $f(x_1, \dots, x_m) = \prod_{i=1}^m \phi(x_i)$ and $c_0 = \beta, c_1 = \sigma^2$. We easily get that this is the generator of the Markov process $(Y_t\rho_{C(t)}, t \geq 0)$ with lifetime τ . We conclude that it has the same law as $(M_{t \wedge \tau}, t \geq 0)$. We rewrite this equality in law as follows:

$$(Y_t \rho_{C(t)}, 0 \leq t < \tau) \stackrel{\text{law}}{=} (|M_t| R_{C^{-1}(C(t))}, 0 \leq t < \tau), \quad (11)$$

with C defined by $C(t) = \int_0^t |M_s|^{-1} ds$ for $0 \leq t < \tau$ on the right hand side. Since $(C(t), t \geq 0)$ and $(\rho_t, t \geq 0)$ are independent on the left hand side and the decomposition in (11) is unique, we have also $(C(t), 0 \leq t < \tau)$ and $(R_{C^{-1}(t)}, 0 \leq t < \tau)$ independent on the right hand side.

Concerning the case (ii) of Theorem 3, we easily observe that the presence of jumps implies that such a decomposition of the generator cannot hold. See for instance Equation (2.7) of [7] p344. The processes $(R_{C^{-1}(t)}, t \geq 0)$ and $(Y_t, t \geq 0)$ are not independent. \square

4 Genealogy of the Beta-Fleming-Viot processes with immigration

To describe the genealogy associated with stable CBs, Bertoin and Le Gall [6] and Birkner et al. [7] used partition-valued processes called Beta-coalescents. These processes form a subclass of Λ -coalescents, introduced independently by Pitman and Sagitov in 1999. A Λ -coalescent is an exchangeable process in the sense that its law is invariant under the action of any permutation. In words, there is no distinction between the individuals. Although these processes arise as models of genealogy for a wide range of stochastic populations, they are not in general adapted to describe the genealogy of a population with immigration. Recently, a larger class of processes called M -coalescents has been defined in [13] (see Section 5). These processes are precisely those describing the genealogy of M -GFVIs.

Remark 4.1 *We mention that the use of the lookdown construction in Birkner et al. [7] may be easily adapted to our framework and yields a genealogy for any conservative CBI. Moreover, other genealogies, based on continuous trees, have been investigated by Lambert [17] and Duquesne [9].*

4.1 Background on M -coalescents

Before focusing on the M -coalescents involved in the context of Theorem 3, we recall their general definition and the duality with the M -GFVIs. Contrary to the Λ -coalescents, the M -coalescents are only invariant by permutations letting 0 at 0. The individual 0 represents the immigrant lineage and is distinguished from the others. We denote by \mathcal{P}_∞^0 the space of partitions of $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. We recall that by convention the blocks of a partition are enumerated in the order of their smallest element. Let $[n]$ denote the set $\{0, \dots, n\}$ and \mathcal{P}_n^0 the space of partitions of $[n]$. Moreover the partition of $[n]$ into singletons is denoted by $0_{[n]}$. As in Section 2.2, the notation M stands for a pair of finite measures (Λ_0, Λ_1) such that:

$$\Lambda_0(dx) = c_0 \delta_0(dx) + x \nu_0(dx), \quad \Lambda_1(dx) = c_1 \delta_0(dx) + x^2 \nu_1(dx).$$

where c_0, c_1 are two non-negative real numbers and ν_0, ν_1 are two measures on $[0, 1]$ subject to the same conditions as in Section 2.2. Let N_0 and N_1 be two Poisson point measures with intensity respectively $dt \otimes \nu_0$ and $dt \otimes \nu_1$. An M -coalescent is a Feller process $(\Pi(t), t \geq 0)$ valued in \mathcal{P}_∞^0 with the following dynamics.

- At an atom (t, x) of N_1 , flip a coin with probability of "heads" x for each block not containing 0. All blocks flipping "heads" are merged immediately in one block. At time t , a proportion x share a common parent in the population.
- At an atom (t, x) of N_0 , flip a coin with probability of "heads" x for each block not containing 0. All blocks flipping "heads" coagulate immediately with the distinguished block. At time t , a proportion x of the population is children of immigrant.

In order to take into account the parameters c_0 and c_1 , imagine that at constant rate c_1 , two blocks (not containing 0) merge *continuously* in time, and at constant rate c_0 , one block (not containing 0) merged with the distinguished one. We refer to Section 4.2 of [13] for a rigorous definition. Let $\pi \in \mathcal{P}_n^0$, the jump rate of an M -coalescent from $0_{[n]}$ to π , denoted by q_π , is given as follows:

- If π has one block not containing 0 with k elements and $2 \leq k \leq n$, then

$$q_\pi = \lambda_{n,k} := \int_0^1 x^{k-2}(1-x)^{n-k} \Lambda_1(dx).$$

- If the distinguished block of π has $k+1$ elements (counting 0) and $1 \leq k \leq n$ then

$$q_\pi = r_{n,k} := \int_0^1 x^{k-1}(1-x)^{n-k} \Lambda_0(dx).$$

The next duality property is a key result and links the M -GFVIs to the M -coalescents. For any π in \mathcal{P}_∞^0 , define

$$\alpha_\pi : k \mapsto \text{the index of the block of } \pi \text{ containing } k.$$

We have the duality relation (see Lemma 4 in [14]): for any $p \geq 1$ and $f \in C([0, 1]^p)$,

$$\mathbb{E} \left[\int_{[0,1]^{p+1}} f(x_{\alpha_{\Pi(t)}(1)}, \dots, x_{\alpha_{\Pi(t)}(p)}) \delta_0(dx_0) dx_1 \dots dx_p \right] = \mathbb{E} \left[\int_{[0,1]^p} f(x_1, \dots, x_p) \rho_t(dx_1) \dots \rho_t(dx_p) \right],$$

where $(\rho_t, t \geq 0)$ is a M -GFVI started from the Lebesgue measure on $[0, 1]$. We establish a useful lemma relating genuine Λ -coalescents and M -coalescents. Consider a Λ -coalescent taking values in the set \mathcal{P}_∞^0 ; this differs from the usual convention, according to which they are valued in the set \mathcal{P}_∞ of the partitions of \mathbb{N} (see Chapters 1 and 3 of [2] for a complete introduction to these processes). In that framework, Λ -coalescents appear as a subclass of M -coalescents and the integer 0 may be viewed as a typical individual.

Lemma 5 *A M -coalescent, with $M = (\Lambda_0, \Lambda_1)$ is also a Λ -coalescent on \mathcal{P}_∞^0 if and only if*

$$(1-x)\Lambda_0(dx) = \Lambda_1(dx).$$

In that case $\Lambda = \Lambda_0$.

4.2 The $Beta(2-\alpha, \alpha-1)$ -coalescent

The aim of this Section is to show how a $Beta(2-\alpha, \alpha-1)$ -coalescent is embedded in the genealogy of an α -stable CB-process conditioned to be never extinct. Along the way, we also derive the fixed time genealogy of the Feller CBI.

We first state the following straightforward corollary of Theorem 3, which gives the genealogy of the ratio process at the random time $C^{-1}(t)$:

Corollary 6 *Let $(R_t, t \geq 0)$ be the ratio process of a CBI in case (i) or (ii). Using Theorem 3, we get for all $t \geq 0$:*

$$\mathbb{E} \left[\int_{[0,1]^{p+1}} f(x_{\alpha_{\Pi(t)}(1)}, \dots, x_{\alpha_{\Pi(t)}(p)}) \delta_0(dx_0) dx_1 \dots dx_p \right] = \mathbb{E} \left[\int_{[0,1]^p} f(x_1, \dots, x_p) R_{C^{-1}(t)}(dx_1) \dots R_{C^{-1}(t)}(dx_p) \right],$$

- In case (i), $(\Pi(t), t \geq 0)$ is a M -coalescent with $M = (\beta\delta_0, \sigma^2\delta_0)$,
- In case (ii), $(\Pi(t), t \geq 0)$ is a M -coalescent with $M = (c' \text{Beta}(2 - \alpha, \alpha - 1), c \text{Beta}(2 - \alpha, \alpha))$.

In general, we cannot set the random quantity $C(t)$ instead of t in the equation of Corollary 6. Nevertheless, using the independence property proved in Proposition 4, we get the following Corollary.

Corollary 7 *In case (i) of Theorem 3, assume $\frac{\beta}{\sigma^2} \geq \frac{1}{2}$, then for all $t \geq 0$,*

$$\mathbb{E} \left[\int_{[0,1]^{p+1}} f(x_{\Pi(C(t))^{(1)}}, \dots, x_{\Pi(C(t))^{(p)}}) \delta_0(dx_0) dx_1 \dots dx_p \right] = \mathbb{E} \left[\int_{[0,1]^p} f(x_1, \dots, x_p) R_t(dx_1) \dots R_t(dx_p) \right],$$

where $(\Pi(t), t \geq 0)$ is a M -coalescent with $M = (\beta\delta_0, \sigma^2\delta_0)$, $(C(t), t \geq 0) = \left(\int_0^t \frac{1}{Y_s} ds, t \geq 0 \right)$ and $(Y_t, t \geq 0)$ is independent of $(\Pi(t), t \geq 0)$.

We stress on a fundamental difference between Corollaries 6 and 7: whereas the first gives the genealogy of the ratio process R at the random time $C^{-1}(t)$, the second gives the genealogy of the ratio process R at a fixed time t . Notice that we impose the additional assumption that $\frac{\beta}{\sigma^2} \geq \frac{1}{2}$ in Corollary 7 for ensuring that the lifetime is infinite. Therefore, $R_t \neq \Delta$ for all $t \geq 0$, and we may consider its genealogy. Lemma 5 will allow us to establish our last result.

Theorem 8 (i) *If the process $(Y_t, t \geq 0)$ is a CBI such that $\sigma^2 = \beta$, $\hat{\nu}_1 = \hat{\nu}_0 = 0$, then the process $(\Pi(t/\sigma^2), t \geq 0)$ defined in Corollary 6 is a Kingman's coalescent valued in \mathcal{P}_∞^0 .*

(ii) *If the process $(Y_t, t \geq 0)$ is a CBI such that $\sigma^2 = \beta = 0$ and $\hat{\nu}_0(dh) = ch^{-\alpha}dh$, $\hat{\nu}_1(dh) = ch^{-\alpha-1}dh$ for some constant $c > 0$ then the process $(\Pi(t/c), t \geq 0)$ defined in Corollary 6 is a $\text{Beta}(2 - \alpha, \alpha - 1)$ -coalescent valued in \mathcal{P}_∞^0 .*

In both cases, the process $(Y_t, t \geq 0)$ involved in that Theorem may be interpreted as a CB-process $(X_t, t \geq 0)$ without immigration ($\beta = 0$ or $c' = 0$) conditioned on non-extinction, see Lambert [18]. We then notice that both the genealogies of the time changed Feller diffusion and of the time changed Feller diffusion conditioned on non extinction are given by the same Kingman's coalescent. On the contrary, the genealogy of the time changed α -stable CB-process is a $\text{Beta}(2 - \alpha, \alpha)$ -coalescent, whereas the genealogy of the time changed α -stable CB-process conditioned on non-extinction is a $\text{Beta}(2 - \alpha, \alpha - 1)$ -coalescent. We stress that for any $\alpha \in (1, 2)$ and any borelian B of $[0, 1]$, we have $\text{Beta}(2 - \alpha, \alpha - 1)(B) \geq \text{Beta}(2 - \alpha, \alpha)(B)$. This may be interpreted as the additional reproduction events needed for the process to never extinct.

4.3 Proofs.

Proof of Corollary 7. We first write:

$$\mathbb{E} \left[\int_{[0,1]^p} f(x_1, \dots, x_p) R_t(dx_1) \dots R_t(dx_p) \right] = \mathbb{E} \left[\int_{[0,1]^p} f(x_1, \dots, x_p) R_{C^{-1}(C(t))}(dx_1) \dots R_{C^{-1}(C(t))}(dx_p) \right].$$

Then, using the independence between $R_{C^{-1}}$ and C , the right hand side above is also equal to

$$\int \mathbb{P}(C(t) \in ds) \mathbb{E} \left[\int_{[0,1]^p} f(x_1, \dots, x_p) R_{C^{-1}(s)}(dx_1) \dots R_{C^{-1}(s)}(dx_p) \right].$$

And using Corollary 6, we have

$$\begin{aligned}
& \int \mathbb{P}(C(t) \in ds) \mathbb{E} \left[\int_{[0,1]^p} f(x_1, \dots, x_p) R_{C^{-1}(s)}(dx_1) \dots R_{C^{-1}(s)}(dx_p) \right] \\
&= \int \mathbb{P}(C(t) \in ds) \mathbb{E} \left[\int_{[0,1]^{p+1}} f(x_{\alpha_{\Pi(s)}(1)}, \dots, x_{\alpha_{\Pi(s)}(p)}) \delta_0(dx_0) dx_1 \dots dx_p \right] \\
&= \mathbb{E} \left[\int_{[0,1]^{p+1}} f(x_{\alpha_{\Pi(C(t))}(1)}, \dots, x_{\alpha_{\Pi(C(t))}(p)}) \delta_0(dx_0) dx_1 \dots dx_p \right].
\end{aligned}$$

The last equality holds by independence of $(\Pi(t), t \geq 0)$ and $(C(t), t \geq 0)$. \square

Remark 4.2 Notice the crucial rôle of the independence in order to establish Corollary 7. When this property fails, as in the stable case, we can no more perform the same manipulations and the question of describing the fixed time genealogy of the α -stable CB or CBI remains open. We refer to the discussion in Section 2.2 of Berestycki *et. al* [1].

The proof of Theorem 8 is a straightforward application of Lemma 5. Indeed, we easily check that the M -coalescents for which $M = (\sigma^2 \delta_0, \sigma^2 \delta_0)$ and $M = (c\text{Beta}(2 - \alpha, \alpha - 1), c\text{Beta}(2 - \alpha, \alpha))$ fulfill the conditions of Lemma 5. Therefore, we only have to establish this lemma.

Proof of Lemma 5. Let $(\Pi'(t), t \geq 0)$ be a Λ -coalescent on ∞ . Let $n \geq 1$, we may express the jump rate of $(\Pi'_{[n]}(t), t \geq 0)$ from $0_{[n]}$ to π by

$$q'_\pi = \begin{cases} 0 & \text{if } \pi \text{ has more than one non-trivial block} \\ \int_{[0,1]} x^k (1-x)^{n+1-k} x^{-2} \Lambda(dx) & \text{if the non trivial block has } k \text{ elements.} \end{cases}$$

Consider now a M -coalescent, denoting by q_π the jump rate from $0_{[n]}$ to π , we have

$$q_\pi = \begin{cases} 0 & \text{if } \pi \text{ has more than one non-trivial block} \\ \int_{[0,1]} x^k (1-x)^{n-k} x^{-2} \Lambda_1(dx) & \text{if } \pi_0 = \{0\} \text{ and the non trivial block has } k \text{ elements} \\ \int_{[0,1]} x^{k-1} (1-x)^{n+1-k} x^{-1} \Lambda_0(dx) & \text{if } \#\pi_0 = k. \end{cases}$$

Since the law of a Λ -coalescent is entirely described by the family of the jump rates of its restriction on $[n]$ from $0_{[n]}$ to π for π belonging to \mathcal{P}_n^0 (see Section 4.2 of [3]), the processes Π and Π' have the same law if and only if for all $n \geq 0$ and $\pi \in \mathcal{P}_n^0$, we have $q_\pi = q'_\pi$, that is if and only if $(1-x)\Lambda_0(dx) = \Lambda_1(dx)$. \square

5 Proof of Theorem 3

Recall the statement of Theorem 3. In order to get the connection between the two measure-valued processes $(R_t, t \geq 0)$ and $(M_t, t \geq 0)$, we may follow the ideas of Birkner *et al.* [7] and rewrite the generator of the process $(M_t, t \geq 0)$ using the "polar coordinates": for any $\eta \in \mathcal{M}_f$, we define

$$z := |\eta| \text{ and } \rho := \frac{\eta}{|\eta|}.$$

The proof relies on four lemmas. Lemmas 9, 10 and 12 allow us to study the generator \mathcal{L} on the class of functions of the type $F : \eta \mapsto G_f(\rho)$. We recall that this class forms a core for the M -GFVI

processes. Lemma 11 (lifted from Lemma 3.5 of [7]) relates stable Lévy-measures and Beta-measures. We end the proof using results on time change by the inverse of an additive functional. We conclude thanks to a result due to Volkonskiĭ in [24] about the generator of a time-changed process.

Lemma 9 *Assume that $\hat{\nu}_0 = \hat{\nu}_1 = 0$ and $b = 0$, the generator \mathcal{L} of $(M_t, t \geq 0)$ is reduced to the expressions (1) and (2):*

$$\mathcal{L}F(\eta) = \sigma^2/2 \int_0^1 \int_0^1 \eta(da)\delta_a(db)F''(\eta; a, b) + \beta F'(\eta; 0)$$

Let f be a continuous function on $[0, 1]^m$ and F be the map $\eta \mapsto G_f(\rho) := \langle f, \rho^{\otimes m} \rangle$. We have the following identity

$$|\eta|\mathcal{L}F(\eta) = \mathcal{F}G_f(\rho),$$

where \mathcal{F} is the generator of a Fleming-Viot process with immigration with reproduction rate $c_1 = \sigma^2$ and immigration rate $c_0 = \beta$, see expressions (1') and (2').

Proof. By the calculations in Section 4.3 of Etheridge [10] (but in a non-spatial setting, see also the proof of Theorem 2.1 p. 249 of Shiga [23]), we get:

$$\begin{aligned} \frac{\sigma^2}{2} \int_0^1 \int_0^1 \eta(da)\delta_a(db)F''(\eta; a, b) &= |\eta|^{-1} \frac{\sigma^2}{2} \int_0^1 \int_0^1 \frac{\partial^2 G_f}{\partial \rho(a) \partial \rho(b)}(\rho) [\delta_a(db) - \rho(db)] \rho(da) \\ &= |\eta|^{-1} \sigma^2 \sum_{1 \leq i < j \leq m} \int_{[0,1]^p} [f(x^{i,j}) - f(x)] \rho^{\otimes m}(dx). \end{aligned}$$

We focus now on the immigration part. We take f a function of the form $f : (x_1, \dots, x_m) \mapsto \prod_{i=1}^m \phi(x_i)$ for some function ϕ , and consider $F(\eta) := G_f(\rho) = \langle f, \rho^{\otimes m} \rangle$. We may compute:

$$\begin{aligned} F(\eta + h\delta_a) - F(\eta) &= \left\langle \phi, \frac{\eta + h\delta_a}{z + h} \right\rangle^m - \langle \phi, \rho \rangle^m \\ &= \sum_{j=2}^m \binom{m}{j} \left(\frac{z}{z+h} \right)^{m-j} \left(\frac{h}{z+h} \right)^j [\langle \phi, \rho \rangle^{m-j} \phi(a)^j - \langle \phi, \rho \rangle^m] \quad (\star) \\ &\quad + m \left(\frac{z}{z+h} \right)^{m-1} \left(\frac{h}{z+h} \right) [\langle \phi, \rho \rangle^{m-1} \phi(a) - \langle \phi, \rho \rangle^m] \quad (\star\star). \end{aligned}$$

We get that:

$$F'(\eta; a) = \frac{m}{z} [\phi(a) \langle \phi, \rho \rangle^{m-1} - \langle \phi, \rho \rangle^m].$$

Thus,

$$F'(\eta; 0) = |\eta|^{-1} \sum_{1 \leq i \leq m} \int_{[0,1]^p} [f(x^{0,i}) - f(x)] \rho^{\otimes m}(dx) \text{ and } \int F'(\eta; a) \eta(da) = 0 \quad (12)$$

for such function f . Now, we observe that the map $f \mapsto G_f$ is linear. Therefore, equation (12) holds true for linear combinations of these functions f . But, from the Stone-Weierstrass theorem, any symmetric continuous map from $[0, 1]^m$ to \mathbb{R} can be uniformly approximated by linear combinations of these functions. We now take f symmetric and continuous, and let f_k be an approximating sequence. We want to prove that (12) holds for f , knowing it holds for each f_k . We begin with the first identity of (12). The following computation identifies the limit of the right hand side:

$$\sum_{1 \leq i \leq m} \int_{[0,1]^m} [f_k(x^{0,i}) - f_k(x)] \rho^{\otimes m}(dx) \xrightarrow{k \rightarrow \infty} \sum_{1 \leq i \leq m} \int_{[0,1]^m} [f(x^{0,i}) - f(x)] \rho^{\otimes m}(dx).$$

For dealing with the left hand side, we introduce:

$$\Delta_h^a F := \frac{1}{h} [F(\eta + h\delta_a) - F(\eta)].$$

Since the function $f_k - f$ is symmetric, we may expand the m -fold product $\left\langle f_k - f, \left(\frac{\eta + h\delta_a}{z+h}\right)^{\otimes m} \right\rangle$ as follows:

$$\begin{aligned} \left\langle f_k - f, \left(\frac{\eta + h\delta_a}{z+h}\right)^{\otimes m} \right\rangle &= \sum_{i=0}^m \binom{m}{i} \left(\frac{h}{z+h}\right)^i \left\langle f_k - f, \left(\frac{\eta}{z+h}\right)^{\otimes m-i} \otimes \delta_a^{\otimes i} \right\rangle \\ &= \left\langle f_k - f, \left(\frac{\eta}{z+h}\right)^{\otimes m} \right\rangle + \sum_{i=1}^m \binom{m}{i} \left(\frac{h}{z+h}\right)^i \left\langle f_k - f, \left(\frac{\eta}{z+h}\right)^{\otimes m-i} \otimes \delta_a^{\otimes i} \right\rangle. \end{aligned}$$

We use here the notation

$$\langle g, \mu^{\otimes m-i} \otimes \delta_a^{\otimes i} \rangle := \int g(x_1, \dots, x_{m-i}, \underbrace{a, \dots, a}_{i \text{ terms}}) \mu(dx_1) \dots \mu(dx_{m-i}).$$

An easy calculation yields

$$\Delta_h^a F_k - \Delta_h^a F = \frac{1}{h} \left[\left\langle f_k - f, \left(\frac{\eta + h\delta_a}{z+h}\right)^{\otimes m} \right\rangle - \langle f_k - f, \rho^{\otimes m} \rangle \right].$$

For any continuous map f on $[0, 1]^m$, we denote by $\|f\|_\infty$ its supremum. On the one hand, we have

$$\begin{aligned} \left| \left\langle f_k - f, \left(\frac{\eta}{z+h}\right)^{\otimes m} \right\rangle - \langle f_k - f, \rho^{\otimes m} \rangle \right| &= |[(z+h)^{-m} - z^{-m}] \langle f_k - f, \eta^{\otimes m} \rangle| \\ &\leq \frac{h}{z^m} |\langle f_k - f, \eta^{\otimes m} \rangle| \leq h \|f_k - f\|_\infty. \end{aligned}$$

On the other hand, we have

$$\left| \sum_{i=1}^m \binom{m}{i} \left(\frac{h}{z+h}\right)^i \left\langle f_k - f, \left(\frac{\eta}{z+h}\right)^{\otimes m-i} \otimes \delta_a^{\otimes i} \right\rangle \right| \leq \sum_{i=1}^m \binom{m}{i} h^i \|f_k - f\|_\infty.$$

Dividing both expressions by $h \in (0, 1]$, we get:

$$|\Delta_h^a F_k - \Delta_h^a F| \leq \|f_k - f\|_\infty + \sum_{i=1}^m \binom{m}{i} h^{i-1} \|f_k - f\|_\infty \leq 2^m \|f_k - f\|_\infty$$

and thus,

$$\sup_{h \in (0, 1]} |\Delta_h^a F_k - \Delta_h^a F| \xrightarrow[k \rightarrow \infty]{} 0.$$

We may therefore exchange the limits when h tends to 0 and k goes to ∞ and we finally identifies the limit of the right hand side as: $\lim_{k \rightarrow \infty} F'_k(\eta; a) = F'(\eta; a)$. We have that the first identity of (12) holds for any symmetric continuous function. The quantity $F'_k(\eta; a)$ is bounded, and appealing to the Lebesgue Theorem, we get that the second identity of (12) also holds for any symmetric continuous function.

Now, for any continuous function f , we have $G_f = G_{\tilde{f}}$ where \tilde{f} is the symmetrised version of f . We thus get (12) for the continuous functions also. Recalling (1') and (2'), we then have the factorization

$$\mathcal{L}F(\eta) = |\eta|^{-1} \mathcal{F}G_f(\rho)$$

where \mathcal{F} is the generator of a Fleming-Viot process with immigration with reproduction rate $c_1 = \sigma^2$ and immigration rate $c_0 = \beta$. This proves the Lemma. \square

This first lemma will allow us to prove the case (i) of Theorem 3. We now focus on the case (ii). Assuming that $\sigma^2 = \beta = 0$, the generator of $(M_t, t \geq 0)$ reduces to

$$\mathcal{L}F(\eta) = \mathcal{L}_0F(\eta) + \mathcal{L}_1F(\eta) \quad (13)$$

where, as in equations (3) and (4) of Subsection 2.1,

$$\begin{aligned} \mathcal{L}_0F(\eta) &= \int_0^\infty \hat{\nu}_0(dh) [F(\eta + h\delta_0) - F(\eta)] \\ \mathcal{L}_1F(\eta) &= \int_0^1 \eta(da) \int_0^\infty \hat{\nu}_1(dh) [F(\eta + h\delta_a) - F(\eta) - hF'(\eta, a)]. \end{aligned}$$

The following lemma is a first step to compute the generator of $(R_t, t \geq 0)$ in the purely discontinuous case.

Lemma 10 *Let f be a continuous function on $[0, 1]^m$ and F be the map $\eta \mapsto G_f(\rho) := \langle f, \rho^{\otimes m} \rangle$. Recall the notation $\rho := \eta/|\eta|$ and $z = |\eta|$. We have the identities:*

$$\begin{aligned} \mathcal{L}_0F(\eta) &= \int_0^\infty \hat{\nu}_0(dh) \left[G_f \left(\left[1 - \frac{h}{z+h} \right] \rho + \frac{h}{z+h} \delta_0 \right) - G_f(\rho) \right] \\ \mathcal{L}_1F(\eta) &= z \int_0^\infty \hat{\nu}_1(dh) \int_0^1 \rho(da) \left[G_f \left(\left[1 - \frac{h}{z+h} \right] \rho + \frac{h}{z+h} \delta_a \right) - G_f(\rho) \right]. \end{aligned}$$

Proof. The identity for \mathcal{L}_0 is plain, we thus focus on \mathcal{L}_1 . In the same manner as in Lemma 9, we begin by considering a function f of the form $\prod_{i=1}^m \phi(x_i)$. As checked in Lemma 2 for $F(\eta) = G_f(\rho)$, we have $F'(\eta; a) = \frac{m}{z} [\langle \phi, \rho \rangle^{m-1} \phi(a) - \langle \phi, \rho \rangle^m]$ and then

$$\int \rho(da) \left[m \left(\frac{z}{z+h} \right)^{m-1} \left(\frac{h}{z+h} \right) [\langle \phi, \rho \rangle^{m-1} \phi(a) - \langle \phi, \rho \rangle^m] - hF'(\eta; a) \right] = 0.$$

The map $h \mapsto \int_0^1 \rho(da) [F(\eta + h\delta_a) - F(\eta) - hF'(\eta, a)]$ is then integrable with respect to the measure $\hat{\nu}_1$. This allows us to interchange the integrals and yields:

$$\mathcal{L}_1F(\eta) = z \int_0^\infty \hat{\nu}_1(dh) \int_0^1 \rho(da) \left[G_f \left(\frac{\eta + h\delta_a}{z+h} \right) - G_f(\rho) \right]. \quad (14)$$

for this function f . Once again, we may extend the validity of (14) to linear combinations of such functions f . We thus take f symmetric and continuous and approach it uniformly through such linear combinations f_k (thanks to Stone-Weierstrass theorem) for which equality (14) holds:

$$\mathcal{L}_1F_k(\eta) = z \int_0^\infty \hat{\nu}_1(dh) \int_0^1 \rho(da) \left[G_{f_k} \left(\frac{\eta + h\delta_a}{z+h} \right) - G_{f_k}(\rho) \right], \quad (15)$$

with the notation $F_k = G_{f_k}$. We first identify the limit in the right hand side of (15). We note that, from the uniform convergence of (f_k) to f ,

$$\lim_{k \rightarrow \infty} \int_0^1 \rho(da) \left[G_{f_k} \left(\frac{\eta + h\delta_a}{z + h} \right) - G_{f_k}(\rho) \right] = \int_0^1 \rho(da) \left[G_f \left(\frac{\eta + h\delta_a}{z + h} \right) - G_f(\rho) \right].$$

Now, the term (\star) , which may be found in the computation of Lemma 2, yields the following bound:

$$\int_{[0,1]} \rho(da) \left[G_{f_k} \left(\frac{\eta + h\delta_a}{z + h} \right) - G_{f_k}(\rho) \right] \leq \left(\frac{h}{h + z} \right)^2 H(f_k)$$

where $H(f_k) := 2^m (\|f_k\|_\infty - \langle f_k, \rho^{\otimes m} \rangle)^+$ converges to $H(f)$, and is thus bounded. Right hand side of the last inequality is then integrable with respect to the measure $\hat{\nu}_1$. This allows us to apply Lebesgue's theorem:

$$\lim_{k \rightarrow \infty} z \int_0^\infty \hat{\nu}_1(dh) \int_0^1 \rho(da) \left[G_{f_k} \left(\frac{\eta + h\delta_a}{z + h} \right) - G_{f_k}(\rho) \right] = z \int_0^\infty \hat{\nu}_1(dh) \int_0^1 \rho(da) \left[G_f \left(\frac{\eta + h\delta_a}{z + h} \right) - G_f(\rho) \right].$$

It remains to identify the limit in the left hand side of (15). Recall equation (3). Uniform convergence of f_k towards f together with the convergence of $F'_k(\eta, a)$ to $F'(\eta, a)$ ensure pointwise convergence inside the integral. We now look for a domination in order to apply Lebesgue's theorem. In the integral with respect to the measure $\hat{\nu}_1$, the term $(\star\star)$ in $F_k(\eta + h\delta_a) - F_k(\eta)$ is compensated by $hF'_k(\eta, a)$, namely their difference is bounded by $C \frac{h^2}{z(z+h)}$ for some constant $C > 0$. Moreover, the term (\star) is bounded by $C \left(\frac{h}{z+h} \right)^2$. Since both quantities are bounded by $C(h^2 \wedge h)$ with a constant C independent of k , we finally get that:

$$\lim_{k \rightarrow \infty} \mathcal{L}_1 F_k = \mathcal{L}_1 F.$$

Uniqueness of the limit allows us to conclude that (14) holds for symmetric and continuous function, and the same argument as in Lemma 2 finally yields (14) for every continuous function. \square

The previous lemma leads us to study the images of the measures $\hat{\nu}_0$ and $\hat{\nu}_1$ by the map $\phi_z : h \mapsto r := \frac{h}{h+z}$, for every $z > 0$. Denote $\lambda_z^0(dr) = \hat{\nu}_0 \circ \phi_z^{-1}$ and $\lambda_z^1(dr) = \hat{\nu}_1 \circ \phi_z^{-1}$. The following lemma is lifted from Lemma 3.5 of [7]:

Lemma 11 *There exist two measures ν_0, ν_1 such that $\lambda_z^0(dr) = s_0(z)\nu_0(dr)$ and $\lambda_z^1(dr) = s_1(z)\nu_1(dr)$ for some maps s_0, s_1 from \mathbb{R}_+ to \mathbb{R} if and only if for some $\alpha \in (0, 2), \beta \in (0, 1)$ and $c, c' > 0$:*

$$\hat{\nu}_1(dx) = cx^{-1-\alpha}dx, \quad \hat{\nu}_0(dx) = c'x^{-1-\beta}dx.$$

In this case:

$$s_1(z) = z^{-\alpha}, \quad \nu_1(dr) = r^{-2}c\text{Beta}(2 - \alpha, \alpha)(dr)$$

and

$$s_0(z) = z^{-\beta}, \quad \nu_0(dr) = r^{-1}c'\text{Beta}(1 - \beta, \beta)(dr).$$

Proof. The necessary part is given by the same arguments as in Lemma 3.5 of [7]. We focus on the sufficient part. Assuming that $\hat{\nu}_0, \hat{\nu}_1$ are as above, we have

- $\lambda_z^1(dr) = cz^{-\alpha}r^{-1-\alpha}(1-r)^{-1+\alpha}dr = z^{-\alpha}r^{-2}c\text{Beta}(2 - \alpha, \alpha)(dr)$, and thus $s_1(z) = z^{-\alpha}$.

- $\lambda_z^0(dr) = c'z^{-\beta}r^{-1-\beta}(1-r)^{-1+\beta}dr = z^{-\beta}r^{-1}c'Beta(1-\beta, \beta)(dr)$ and thus $s_0(z) = z^{-\beta}$. \square

The next lemma allows us to deal with the second statement of Theorem 3.

Lemma 12 *Assume that $\sigma^2 = \beta = 0$, $\hat{\nu}_0(dh) = ch^{-\alpha}1_{h>0}dh$ and $\hat{\nu}_1(dh) = ch^{-1-\alpha}1_{h>0}dh$. Let f be function on $[0, 1]^m$, and F be the map $\eta \mapsto G_f(\rho)$. We have*

$$|\eta|^{\alpha-1}\mathcal{L}F(\eta) = \mathcal{F}G_f(\rho),$$

where \mathcal{F} is the generator of a M -Fleming-Viot process with immigration, with $M = (c'Beta(2-\alpha, \alpha-1), cBeta(2-\alpha, \alpha))$, see expressions (3'), (4').

Proof. Recall equation (13):

$$\mathcal{L}F(\eta) = \mathcal{L}_0F(\eta) + \mathcal{L}_1F(\eta)$$

Recall from Lemma 9 that we have $\int_0^1 F'(\eta; a)\eta(da) = 0$ for $F(\eta) = G_f(\rho)$. Applying Lemma 10 and Lemma 11, we get that in the case $\sigma^2 = \beta = 0$ and $\hat{\nu}_1(dx) = cx^{-1-\alpha}dx$, $\hat{\nu}_0(dx) = c'x^{-1-\beta}dx$:

$$\begin{aligned} \mathcal{L}F(\eta) = \mathcal{L}G_f(\rho) = & s_0(z) \int_0^1 r^{-1}c'Beta(1-\beta, \beta)(dr)[G_f((1-r)\rho + r\delta_0) - G_f(\rho)] \\ & + zs_1(z) \int_0^1 r^{-2}cBeta(2-\alpha, \alpha)(dr) \int_0^1 \rho(da)[G_f((1-r)\rho + r\delta_a) - G_f(\rho)]. \end{aligned}$$

Recalling the expressions (3'), (4'), the factorization $h(z)\mathcal{L}F(\eta) = \mathcal{F}G(\rho)$ holds for some function h if

$$s_0(z) = zs_1(z),$$

if $\beta = \alpha - 1$. In that case, $h(z) = z^{\alpha-1}$. \square

To treat the case (i), replace α by 2 in the sequel. The process $(Y_t, R_t)_{t \geq 0}$ with lifetime τ has the Markov property. The additive functional $C(t) = \int_0^t \frac{1}{Y_s^{\alpha-1}} ds$ maps $[0, \tau)$ to $[0, \infty)$. From Theorem 65.9 of [22] and Proposition 2, the process $(Y_{C^{-1}(t)}, R_{C^{-1}(t)})_{t \geq 0}$ is a strong Markov process with infinite lifetime. Denote by \mathcal{U} the generator of $(Y_t, R_t)_{t \geq 0}$. As explained in Birkner et al. [7] (Equation (2.6) p314), the law of $(Y_t, R_t)_{t \geq 0}$ is characterized by \mathcal{U} acting on the following class of test functions:

$$(z, \rho) \in \mathbb{R}_+ \times \mathcal{M}_1 \mapsto F(z, \rho) := \psi(z)G_f(\rho)$$

for f a continuous function on $[0, 1]^m$, $m \geq 1$ and ψ a twice differentiable non-negative map. Theorem 3 of Volkonskii, see [24] (or Theorem 1.4 Chapter 6 of [11]) states that the Markov process with generator

$$\tilde{\mathcal{U}}F(z, \rho) := z^{\alpha-1}\mathcal{U}F(z, \rho)$$

coincides with $(Y_{C^{-1}(t)}, R_{C^{-1}(t)})_{t \geq 0}$. We establish now that $(R_{C^{-1}(t)}, t \geq 0)$ is a Markov process with the same generator as the Fleming-Viot processes involved in Theorem 3. Let $G(z, \rho) = G_f(\rho)$. In both cases (i) and (ii) of Theorem 3, we have:

$$\begin{aligned} z^{\alpha-1}\mathcal{U}G(z, \rho) &= z^{\alpha-1}\mathcal{L}F(\eta) \text{ with } F : \eta \mapsto G_f(\rho) \\ &= \mathcal{F}G_f(\rho). \end{aligned}$$

First equality holds since we took $\psi \equiv 1$ and the second uses Lemma 9 and Lemma 12. Since it does not depend on z , the process $(R_{C^{-1}(t)}, t \geq 0)$ is a Markov process, moreover it is a generalized Fleming-Viot process with immigration with parameters as stated. \square

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