Localisation of small obstacles in Stokes flow

Fabien Caubet, Marc Dambrine

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Abstract

We want to detect small obstacles immersed in a fluid flowing in a larger bounded domain Ω in the three dimensional case. We assume that the fluid motion is governed by the steady-state Stokes equations. We make a measurement on a part of the exterior boundary $\partial\Omega$ and then have a Kohn-Vogelius approach to locate these obstacles. We use here the notion of topological derivative in order to determine the number of objects and their rough location. Thus, we first establish an asymptotic expansion of the solution of the Stokes equations in Ω when we add small obstacles inside. Then, we use it to find a topological asymptotic expansion of the considered Kohn-Vogelius functional, which give us a formula of its topological gradient. Finally, we make some numerical simulations, exploring the efficiency and the limits of this method.

Keywords: Geometric inverse problem, topological sensitivity analysis, topological gradient, Stokes equations, Kohn-Vogelius functional.

AMS Classification: 49Q10, 35R30, 74P15, 49Q12, 76D55.

1 Introduction

The detection of objects immersed in a fluid can be seen as a minimization of a shape functional using shape gradient (see [34, 35, 2, 14]). However this approach does not permit to modify the topology of the obstacles. In particular, we have to know how many objects are present to reconstruct them.

We make the assumption of small size of the objects to use asymptotic formulae which allows us to change of point of view (see the works of Ammari *et al.* [7, 6, 9, 8, 5]). Then, we use the notion of *topological gradient* in order to determine the number of objects and their rough location, *i.e.* to have an initialization for a shape gradient algorithm. The topological sensitivity analysis consists in studying the variation of a shape functional with respect to the modification of the topology of the domain. It was introduced by Schumacher in [43] and Sokolowski *et al.* in [44] for the compliance minimization in linear elasticity. Then, Masmoudi studied the Laplace equation case introducing a generalization of the adjoint method (see [23]) and the use of a truncation technique to give a topological sensitivity framework on a fixed functional space. By using this approach, the topological asymptotic expansion of a large class of shape functionals was given for the linear elasticity by Garreau *et al.* in [28] and for the Poisson and the Stokes equations by Guillaume *et al.* in [30, 31]. The Helmoltz equations was also studied by Samet *et al.* in [13, 42] and the quasi-Stokes problem by Hassine *et al.* in [33]. The topological sensitivity was also used to elastic-wave imaging of finite solid bodies containing cavities by Bonnet *et al.* in [19], or in the study of

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the Maxwell equations by Masmoudi *et al.* in [37] or even in the study of electrodynamic and acoustic inverse scattering in the time domain by Bonnet in [18].

This kind of inverse problems arises, for example, in moulds filling during which small gas bubbles can be created and trapped inside the material (as it is mentioned in [15]) or in the detection of mines. This last topic is treated in the electrical impedance tomography case by Brühl *et al.* in [21]. They use asymptotic analysis to design a direct (*i.e.* non-iterative) reconstruction algorithm thanks to an explicit characterization of the inclusions.

In this paper, we study the Stokes equations in the three dimensional case. We want to detect some obstacles immersed in a fluid flowing in a larger bounded domain Ω. We make a boundary measurement on the part of the exterior boundary $\partial\Omega$ and then study a Kohn-Vogelius type cost functional. The topological asymptotic expansion of this kind of functional has studied by Ben Abda *et al.* in [15] but they impose Neumann boundary conditions on the boundary of the objects. Here, we have to deal with Dirichlet boundary conditions on the inclusions boundaries and the Kohn-Vogelius approach leads to consider Dirichlet and mixed boundary conditions on the exterior boundary. These modifications lead additional difficulties.

The paper is organized as follows. First, we introduce the adopted notations. Then, in Section 2, we present in details the considered problem and give the main idea used to study it: we introduce some perturbed domains and the considered Kohn-Vogelius functional. Section 3 is devoted to the statement of the main results. We give the topological asymptotic expansion of this functional, in particular when we add spherical objects. Indeed, in this very particular case, we have an explicit formula for the topological gradient. In Section 4, we prove the asymptotic expansion of the solution of the considered Stokes problems when we add small obstacles inside. Then, we use the resulting estimates to prove the main results by splitting the functional in Section 5. Finally, we make numerical attempts in Section 6. We explore the efficiency of this method and point out these limits. Technical results needed to justify the expansions are postponed in appendices.

2 The problem setting

Introduction of the general notations. For a bounded Lipschitz open set $\Omega \subset \mathbb{R}^3$, we denote by $L^p(\Omega)$, $W^{m,p}(\Omega)$ and $H^s(\Omega)$ the usual Lebesgue and Sobolev spaces. We note in bold the vectorial functions and spaces: $\mathbf{L}^p(\Omega)$, $\mathbf{W}^{m,p}(\Omega)$, $\mathbf{H}^s(\Omega)$, etc. Moreover, we denote by $W_\alpha^{1,p}(\Omega)$ the weighted Sobolev spaces defined in Appendix B. For $k \in \mathbb{N}$, we notice $\|\cdot\|_{k,\Omega}$ the norm $\|\cdot\|_{H^k(\Omega)}$ and $|\cdot|_{1,\Omega}$ the semi-norm of $H^1(\Omega)$. We also use the notations $\|\cdot\|_{1/2,\partial\Omega}$ and $\|\cdot\|_{-1/2,\partial\Omega}$ to define respectively the norms $\|\cdot\|_{H^{1/2}(\partial\Omega)}$ and $\|\cdot\|_{H^{-1/2}(\partial\Omega)}$. We represent the duality product between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$ using the notation $\langle\cdot,\cdot\rangle_{\partial\Omega}$. Finally, $\mathbf{n}$ represents the external unit normal to $\partial\Omega$ and we define the space

$$L_0^2(\Omega) := \left\{ p \in L^2(\Omega), \int_\Omega p = 0 \right\}.$$

Framework. Let Ω be a bounded Lipschitz open set of $\mathbb{R}^3$ containing a Newtonian and incompressible fluid with coefficient of kinematic viscosity $\nu > 0$. Let $\omega \subset \mathbb{R}^3$ a fixed bounded Lipschitz domain containing the origin. For $z \in \Omega$ and $\varepsilon > 0$, we denote

$$\omega_{z,\varepsilon} := z + \varepsilon\omega.$$

The aim of this work is to detect some unknown objects included in Ω. We assume that a finite number m^* of obstacles $\omega_{z,\varepsilon}^* \subset \Omega$ have to be detected. Moreover, we assume that

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they are well separated and have the geometry form

$$\omega^*_{z_k,\varepsilon} = z^*_k + \varepsilon\omega^*_k, \qquad 1 \le k \le m^*,$$

where ε is the shared diameter and $\omega^*_k \subset \mathbb{R}^3$ are bounded Lipschitz domains containing the origin. The points $z^*_k \in \Omega$, $1 \le k \le m^*$, determine the location of the objects. Finally, we assume that, for all $1 \le k \le m^*$, $\omega^*_{z_k,\varepsilon}$ is far from the boundary $\partial\Omega$.

Let $\boldsymbol{f} \in \mathbf{H}^{1/2}(\partial\Omega)$ such that $\boldsymbol{f} \ne \boldsymbol{0}$ satisfying the compatibility condition

$$\int_{\partial\Omega} \boldsymbol{f} \cdot \mathbf{n} = 0. \tag{2.1}$$

In order to determine the location of the objects, we make a measurement $\boldsymbol{g} \in \mathbf{H}^{-1/2}(O)$ on a part O of the exterior boundary $\partial\Omega$ with $O \subsetneq \partial\Omega$. Then, we notice $\omega^*_\varepsilon := \bigcup_{k=1}^{m^*} \omega^*_{z_k,\varepsilon}$ and consider the following overdetermined Stokes problem

$$\begin{cases} -\nu\Delta\boldsymbol{u} + \nabla p &=& \boldsymbol{0} & \text{in } \Omega\backslash\overline{\omega^*_\varepsilon} \\ \operatorname{div}\boldsymbol{u} &=& 0 & \text{in } \Omega\backslash\overline{\omega^*_\varepsilon} \\ \boldsymbol{u} &=& \boldsymbol{f} & \text{on } \partial\Omega \\ \boldsymbol{u} &=& \boldsymbol{0} & \text{on } \partial\omega^*_\varepsilon \\ \sigma(\boldsymbol{u},p)\mathbf{n} &=& \boldsymbol{g} & \text{on } O \subset \partial\Omega. \end{cases} \tag{2.2}$$

Here $\sigma(\boldsymbol{u},p)$ represents the stress tensor defined by

$$\sigma(\boldsymbol{u},p) := \nu\left(\nabla\boldsymbol{u} + {}^t\nabla\boldsymbol{u}\right) - p\mathrm{I}.$$

Notice that, if $\operatorname{div}\boldsymbol{u} = 0$ in Ω, we have

$$-\nu\Delta\boldsymbol{u} + \nabla p = -\operatorname{div}\left(\nu\mathcal{D}(\boldsymbol{u})\right) + \nabla p = -\operatorname{div}\left(\sigma(\boldsymbol{u},p)\right) \quad \text{in } \Omega,$$

with $\mathcal{D}(\boldsymbol{u}) := \left(\nabla\boldsymbol{u} + {}^t\nabla\boldsymbol{u}\right)$. Therefore, we can consider two problems:

$$\begin{cases} \text{Find } (\boldsymbol{u}^\varepsilon_D, p^\varepsilon_D) \in \mathbf{H}^1(\Omega\backslash\overline{\omega_\varepsilon}) \times \mathrm{L}^2_0(\Omega\backslash\overline{\omega_\varepsilon}) \text{ such that} \\ \quad -\nu\Delta\boldsymbol{u}^\varepsilon_D + \nabla p^\varepsilon_D &=& \boldsymbol{0} & \text{in } \Omega\backslash\overline{\omega_\varepsilon} \\ \quad \operatorname{div}\boldsymbol{u}^\varepsilon_D &=& 0 & \text{in } \Omega\backslash\overline{\omega_\varepsilon} \\ \quad \boldsymbol{u}^\varepsilon_D &=& \boldsymbol{f} & \text{on } \partial\Omega \\ \quad \boldsymbol{u}^\varepsilon_D &=& \boldsymbol{0} & \text{on } \partial\omega_\varepsilon \end{cases} \tag{2.3}$$

and

$$\begin{cases} \text{Find } (\boldsymbol{u}^\varepsilon_N, p^\varepsilon_N) \in \mathbf{H}^1(\Omega\backslash\overline{\omega_\varepsilon}) \times \mathrm{L}^2(\Omega\backslash\overline{\omega_\varepsilon}) \text{ such that} \\ \quad -\nu\Delta\boldsymbol{u}^\varepsilon_N + \nabla p^\varepsilon_N &=& \boldsymbol{0} & \text{in } \Omega\backslash\overline{\omega_\varepsilon} \\ \quad \operatorname{div}\boldsymbol{u}^\varepsilon_N &=& 0 & \text{in } \Omega\backslash\overline{\omega_\varepsilon} \\ \quad \sigma(\boldsymbol{u}^\varepsilon_N, p^\varepsilon_N)\mathbf{n} &=& \boldsymbol{g} & \text{on } O \\ \quad \boldsymbol{u}^\varepsilon_N &=& \boldsymbol{f} & \text{on } \partial\Omega\backslash\overline{O} \\ \quad \boldsymbol{u}^\varepsilon_N &=& \boldsymbol{0} & \text{on } \partial\omega_\varepsilon, \end{cases} \tag{2.4}$$

where $\omega_\varepsilon := \bigcup_{k=1}^m \omega_{z_k,\varepsilon}$ for a finite number m of objects located in $z_1, \ldots, z_m$. These problems are classically well-defined. We refer to [20, 27] for the results of existence and uniqueness of $(\boldsymbol{u}^\varepsilon_D, p^\varepsilon_D)$. Notice that the compatibility condition (2.1) associated to Problem (2.3) is satisfied. The existence and the uniqueness of $(\boldsymbol{u}^\varepsilon_N, p^\varepsilon_N)$ is guaranteed by Theorem A.1 recalled in Appendix A.

One can remark that if ω_ε coincides with the actual domain ω_ε^*, then $\boldsymbol{u}_D^\varepsilon = \boldsymbol{u}_N^\varepsilon$ in $\Omega\backslash\overline{\omega_\varepsilon}$. According to this observation, we propose a resolution of the inverse problem of reconstructing ω_ε^* based on the minimization of the following Kohn-Vogelius functional

$$\mathcal{F}_\varepsilon^{KV}(\boldsymbol{u}_D^\varepsilon, \boldsymbol{u}_N^\varepsilon) := \frac{1}{2}\int_{\Omega\backslash\overline{\omega_\varepsilon}} \nu|\mathcal{D}(\boldsymbol{u}_D^\varepsilon) - \mathcal{D}(\boldsymbol{u}_N^\varepsilon)|^2.$$

We then define

$$\mathcal{J}_{KV}(\Omega\backslash\overline{\omega_\varepsilon}) := \mathcal{F}_\varepsilon^{KV}(\boldsymbol{u}_D^\varepsilon, \boldsymbol{u}_N^\varepsilon).$$

We assume that there exists ω_ε^* such that the overdetermined problem (2.2) has a solution. Hence, according to the identifiability result [4, Theorem 1.2] proved by Alvarez *et al.*, the domain ω_ε^* is unique. Notice that in [4], ω_ε^* is assumed to have a $C^{1,1}$ boundary but we can only assume that he has a Lipschitz boundary in the Stokes case (see [14, Theorem 2.1]). Hence, if we find ω_ε such that $\mathcal{J}(\Omega\backslash\overline{\omega_\varepsilon}) = 0$, then $\boldsymbol{u}_D^\varepsilon = \boldsymbol{u}_N^\varepsilon$ in $\Omega\backslash\overline{\omega_\varepsilon}$, *i.e.* $\boldsymbol{u}_D^\varepsilon$ satisfies (2.2) and thus $\omega_\varepsilon = \omega_\varepsilon^*$ is the real domain.

Remark 2.1. *In the following $\boldsymbol{u}_D^0$, $\boldsymbol{u}_N^0$, Ω_0, etc. will denote respectively the solutions, the domain, etc. when $\omega_\varepsilon = \emptyset$ and not $\omega_\varepsilon = \omega_0 = \bigcup_{k=1}^m \{z_k\}$.*

3 The main results

From now, we consider that we seek a single obstacle $\omega_{z,\varepsilon} := z + \varepsilon\omega$, located at a point $z \in \Omega$. Notice that in the case of several inclusions, we proceed by detecting the objects one by one. Thus, after detecting a first obstacle $\omega_{z_1,\varepsilon_1}$, we work replacing the whole domain Ω by $\Omega\backslash\overline{\omega_{z_1,\omega_1}}$ (and then we have $\partial\omega_{z_1,\varepsilon_1} \subset \partial(\Omega\backslash\overline{\omega_{z_1,\omega_1}})\backslash\overline{O}$) and the results presented below (in particular the topological derivative) are still valid for a new inclusion $\omega_{z,\varepsilon}$. Note that, the asymptotic expansion of the solution of elliptic boundary value problem in multiply perforated domains is studied in [16, 38].

3.1 Introduction of the needed functional tools

We recall that the topological sensitivity analysis consists in the study of the variations of a design functional $\mathcal{J}$ with respect to the insertion of a small *hole* (here an object) $\omega_{z,\varepsilon}$ at the point $z \in \Omega$. The aim is to obtain an asymptotic expansion of $\mathcal{J}$ of the form

$$\mathcal{J}(\Omega_{z,\varepsilon}) = \mathcal{J}(\Omega) + \xi(\varepsilon)\delta\mathcal{J}(z) + o(\xi(\varepsilon)) \qquad \forall z \in \Omega,$$

where $\varepsilon > 0$, where ξ is a positive scalar function which going to zero with ε and where

$$\Omega_{z,\varepsilon} := \Omega\backslash\overline{\omega_{z,\varepsilon}},$$

with $\omega_{z,\varepsilon} := z + \varepsilon\omega$. We summarize the notations concerning the domains in Figure 1.

The computation of the topological gradient $\delta\mathcal{J}$ exposed below is strongly influenced by the paper of Ben Abda *et al.* [15]. We were also inspired by the works of Guillaume *et al.* [31], of Abdelwahed *et al.* [1] and of Hassine [32]. In [31], they use the Masmoudi's approach mentioned in the introduction. Here, we do not use this truncation technique but we follow the simplified approach presented in [15] (see also [1] and [32]). However, contrary to the problem studied in [15], we impose Dirichlet boundary conditions on the interior boundary. Moreover, contrary to [1] and [32], we have here to consider mixed boundary conditions on the exterior boundary (due to our Kohn-Vogelius approach).
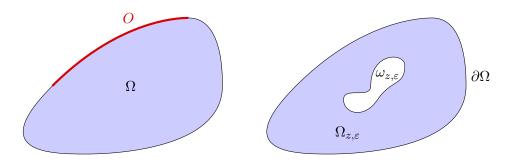
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Figure 1: The initial domain and the same domain after inclusion of an object

We recall the expression of the fundamental solution $(E, \boldsymbol{P})$ to the Stokes system in $\mathbb{R}^3$ given by

$$E(x) \;=\; \frac{1}{8\pi\nu\,\|x\|}\left(\mathrm{I} + \mathrm{e}_r{}^t\mathrm{e}_r\right), \qquad \boldsymbol{P}(x) \;=\; \frac{x}{4\pi\,\|x\|^3}, \tag{3.1}$$

with $\mathrm{e}_r = \dfrac{x}{\|x\|}$; that is $-\nu\Delta E_j + \nabla\boldsymbol{P}_j = \delta\mathbf{e}_j$, where E_j denotes the jth column of E, $(\mathbf{e}_j)_{j=1}^3$ is the canonical basis of $\mathbb{R}^3$ and δ is the Dirac distribution.

3.2 The results

The following theorem give us the expression of the topological gradient of the Kohn-Vogelius functional $\mathcal{J}_{KV}$:

Theorem 3.1. *For $z \in \Omega$, the functional $\mathcal{J}_{KV}$ admits the following topological asymptotic expansion*

$$\mathcal{J}_{KV}(\Omega_{z,\varepsilon}) - \mathcal{J}_{KV}(\Omega) = \varepsilon\left[-\left(\int_{\partial\omega}\boldsymbol{\eta}_D\right)\cdot\boldsymbol{u}_D^0(z) + \left(\int_{\partial\omega}\boldsymbol{\eta}_N\right)\cdot\boldsymbol{u}_N^0(z)\right] + o(\varepsilon),$$

where $\boldsymbol{u}_D^0 \in \mathbf{H}^1(\Omega)$ and $\boldsymbol{u}_N^0 \in \mathbf{H}^1(\Omega)$ solve respectively Problems (2.3) and (2.4) with $\omega_\varepsilon = \emptyset$ and where $\boldsymbol{\eta}_\natural \in \mathbf{H}^{-1/2}(\partial\omega)/\mathbb{R}\mathbf{n}$ (with the subscript $\natural = D$ and $\natural = N$ respectively) is the respective solution of

$$\int_{\partial\omega} E(y-x)\boldsymbol{\eta}_\natural(x)\mathrm{ds}(x) = -\boldsymbol{u}_\natural^0(y) \qquad \forall y \in \partial\omega. \tag{3.2}$$

In the case of spherical objects $\omega = B(0,1)$, we get a more explicit formula for the topological gradient of $\mathcal{J}_{KV}$. Indeed,

$$\int_{\partial\omega}\boldsymbol{\eta}_D = -6\pi\nu\boldsymbol{u}_D^0(z) \quad \text{and} \quad \int_{\partial\omega}\boldsymbol{\eta}_N = -6\pi\nu\boldsymbol{u}_N^0(z)$$

holds in this very particular case (see for example [31, Proof of Corollary 4.2]). Then, we obtain the following corollary:

Corollary 3.2. *If $\omega = B(0,1)$ then, for $z \in \Omega$, the functional $\mathcal{J}_{KV}$ admits the following topological asymptotic expansion*

$$\mathcal{J}_{KV}(\Omega_{z,\varepsilon}) - \mathcal{J}_{KV}(\Omega) = 6\pi\nu\varepsilon\left(\left|\boldsymbol{u}_D^0(z)\right|^2 - \left|\boldsymbol{u}_N^0(z)\right|^2\right) + o(\varepsilon), \tag{3.3}$$

where $\boldsymbol{u}_D^0 \in \mathbf{H}^1(\Omega)$ and $\boldsymbol{u}_N^0 \in \mathbf{H}^1(\Omega)$ solve respectively Problems (2.3) and (2.4) with $\omega_\varepsilon = \emptyset$.

4 Asymptotic expansion of the solution of the Stokes problem

In order to provide an asymptotic expansion of the Kohn-Vogelius functional $\mathcal{J}_{KV}$, we need first an asymptotic expansion of the solution of the Stokes problems (2.3) and (2.4). In this section, we will work with exterior Stokes problems for which the theoretical tools are recalled in Appendix B (in particular the definition of weighted Sobolev space $\mathbf{W}_0^{1,2}$).

We recall that we here focus on the detection of a single obstacle (see the beginning of Section 3). This section is devoted to the proof of the following proposition:

Proposition 4.1. *The respective solutions $\boldsymbol{u}_D^\varepsilon \in \mathbf{H}^1(\Omega_{z,\varepsilon})$ and $\boldsymbol{u}_N^\varepsilon \in \mathbf{H}^1(\Omega_{z,\varepsilon})$ of Problems 2.3 and 2.4 admit the following asymptotic expansion (with the subscript $\natural = D$ and $\natural = N$ respectively):*

$$\boldsymbol{u}_\natural^\varepsilon(x) = \boldsymbol{u}_\natural^0(x) + \boldsymbol{U}_\natural\left(\frac{x-z}{\varepsilon}\right) + O_{\mathbf{H}^1(\Omega_{z,\varepsilon})}(\varepsilon),$$

where $(\boldsymbol{U}_\natural, P_\natural) \in \mathbf{W}_0^{1,2}(\mathbb{R}^3\backslash\overline{\omega}) \times \mathrm{L}^2(\mathbb{R}^3\backslash\overline{\omega})$ solves the following Stokes exterior problem

$$\begin{cases} -\nu\Delta\boldsymbol{U}_\natural + \nabla P_\natural &= \boldsymbol{0} &\text{in } \mathbb{R}^3\backslash\overline{\omega} \\ \operatorname{div}\boldsymbol{U}_\natural &= 0 &\text{in } \mathbb{R}^3\backslash\overline{\omega} \\ \boldsymbol{U}_\natural &= -\boldsymbol{u}_\natural^0(z) &\text{on } \partial\omega. \end{cases} \tag{4.1}$$

The notation $O_{\mathbf{H}^1(\Omega_{z,\varepsilon})}(\varepsilon)$ means that there exist a constant $c > 0$ (independent of ε) and $\varepsilon_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$

$$\left\| \boldsymbol{u}_\natural^\varepsilon(x) - \boldsymbol{u}_\natural^0(x) - \boldsymbol{U}_\natural\left(\frac{x-z}{\varepsilon}\right) \right\|_{1,\Omega_{z,\varepsilon}} \le c\varepsilon.$$

We only detail the Neumann case $\natural = N$. The Dirichlet case $\natural = D$ is a direct adaptation of this Neumann case (we also refer to [32, Proposition 3.1] and [1, Proposition 3.1]). We use uniform *a priori* estimates as what is done in [31], [32] or [1]. An alternative method to obtain the same result consists to use *multi-scale expansion* and the notion of *profile* (see [17], [25] or [39]).

4.1 Some notations and preliminaries

By Cauchy-Schartz and Korn's inequalities (see for example [36, Lemma 5.4.4]), for $\boldsymbol{u} \in \mathbf{H}^1(\Omega)$, the norm

$$|||\boldsymbol{u}||| := \left(\int_\Omega |\mathcal{D}(\boldsymbol{u})|^2 + |\boldsymbol{u}|^2\right)^{1/2}$$

is equivalent to the norm $\|\boldsymbol{u}\|_{1,\Omega}$. Let us also note that, for $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{H}^1(\Omega)$,

$$\int_\Omega \mathcal{D}(\boldsymbol{u}) : \mathcal{D}(\boldsymbol{v}) = 2\int_\Omega \mathcal{D}(\boldsymbol{u}) : \nabla\boldsymbol{v}. \tag{4.2}$$

Moreover, for a given function $u \in \mathrm{H}^1(\Omega)$, we define the function $\widetilde{u}$ on $\widetilde{\Omega} := \Omega/\varepsilon$ by $\widetilde{u}(y) = u(x)$, $y = x/\varepsilon$. Using that $\nabla_x u(x) = (\nabla_y\widetilde{u}(y))/\varepsilon$, we obtain

$$|u|_{1,\Omega}^2 = \int_\Omega |\nabla_x u(x)|^2\,\mathrm{d}x = \varepsilon\int_{\widetilde{\Omega}} |\nabla_y\widetilde{u}(y)|^2\,\mathrm{d}y.$$

Hence,

$$|u|_{1,\Omega} = \varepsilon^{1/2}\,|\widetilde{u}|_{1,\widetilde{\Omega}}. \tag{4.3}$$

Similarly, we obtain

$$\|u\|_{0,\Omega} = \varepsilon^{3/2} \|\widetilde{u}\|_{0,\widetilde{\Omega}}. \tag{4.4}$$

By changing the origin, the same equalities hold with the change of variables $y = (x-z)/\varepsilon$, for $z \in \Omega$.

Finally, let us introduce some other domains. Let $R > 0$ be such that the closed ball $\overline{B(z,R)}$ is included in Ω and $\omega_{z,\varepsilon} \subset B(z,R)$. We define the domains

$$\Omega_R^z := \Omega \backslash \overline{B(z,R)} \quad \text{and} \quad D_\varepsilon^z := B(z,R) \backslash \overline{\omega_{z,\varepsilon}}$$

(see Figure 2). Thus, in particular, we denote $\Omega_R^0 := \Omega \backslash \overline{B(0,R)}$ and $D_\varepsilon^0 := B(0,R) \backslash \overline{\varepsilon\omega}$.
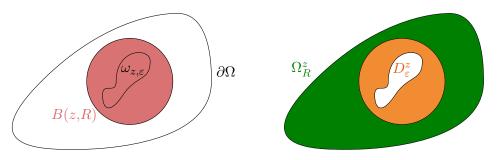

Figure 2: The truncated domain

4.2 Uniform *a priori* estimates

In order to prove Proposition 4.1, we will use uniform *a priori* estimates given by the two following lemmas.

This first lemma is quoted in [31, Lemma 6.2] (see also [32, Lemma 3.1] or [33, Lemma 7.1]) but we recall its proof for the reader's convenience:

Lemma 4.2 (Guillaume *et al.*, [31]). *Let $\boldsymbol{g} \in \mathbf{H}^{1/2}(\partial\omega)$ such that $\int_{\partial\omega} \boldsymbol{g} \cdot \mathbf{n} = 0$. We consider $(\boldsymbol{w}, s) \in \mathbf{W}_0^{1,2}(\mathbb{R}^3 \backslash \overline{\omega}) \times \mathrm{L}^2(\mathbb{R}^3 \backslash \overline{\omega})$ the solution of the Stokes exterior problem*

$$\begin{cases} -\nu\Delta\boldsymbol{w} + \nabla s &= \boldsymbol{0} & \text{in } \mathbb{R}^3 \backslash \overline{\omega} \\ \operatorname{div}\boldsymbol{w} &= 0 & \text{in } \mathbb{R}^3 \backslash \overline{\omega} \\ \boldsymbol{w} &= \boldsymbol{g} & \text{on } \partial\omega. \end{cases}$$

Then there exists a constant $c > 0$ (independent of ε and $\boldsymbol{g}$) and $\varepsilon_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$

$$\begin{aligned} \|\boldsymbol{w}\|_{0,D_\varepsilon^0/\varepsilon} &\leq c\varepsilon^{-1/2}\|\boldsymbol{g}\|_{1/2,\partial\omega}, & \|\boldsymbol{w}\|_{0,\Omega_R^0/\varepsilon} &\leq c\varepsilon^{-1/2}\|\boldsymbol{g}\|_{1/2,\partial\omega}, \\ |\boldsymbol{w}|_{1,D_\varepsilon^0/\varepsilon} &\leq c\|\boldsymbol{g}\|_{1/2,\partial\omega} & and \quad |\boldsymbol{w}|_{1,\Omega_R^0/\varepsilon} &\leq c\varepsilon^{1/2}\|\boldsymbol{g}\|_{1/2,\partial\omega}. \end{aligned}$$

Proof. According to the representation formula (see Theorem B.3), since $\int_{\partial\omega} \boldsymbol{g} \cdot \mathbf{n} = 0$, the velocity $\boldsymbol{w}$ can be written as

$$\boldsymbol{w}(y) = S\boldsymbol{\eta}(y) = \int_{\partial\omega} E(y-x)\boldsymbol{\eta}(x)\mathrm{ds}(x), \qquad y \in \mathbb{R}^3 \backslash \overline{\omega}, \tag{4.5}$$

where $\boldsymbol{\eta} \in \mathbf{H}^{-1/2}(\partial\omega)/\mathcal{R}$ is the unique solution of

$$S\boldsymbol{\eta}(y) = \boldsymbol{g}(y) \qquad \forall y \in \partial\omega \tag{4.6}$$

(the simple layer hydrodynamic potential operator S and the relation $\mathcal{R}$ being defined in Theorem B.3 and the Green function associated to the Stokes system E by (3.1)). Then, using a first order Taylor expansion of E, we have, for x in a bounded domain and y going to infinity,

$$E(y-x) = E(y) + O\left(\frac{1}{\|y\|^2}\right) \quad \text{and} \quad \nabla E(y-x) = \nabla E(y) + O\left(\frac{1}{\|y\|^3}\right)$$

and using (4.5),

$$\boldsymbol{w}(y) = E(y)\,\langle \boldsymbol{\eta}\,,\,1 \rangle_{-1/2,1/2,\partial\omega} + O\left(\frac{1}{\|y\|^2}\right)\langle \boldsymbol{\eta}\,,\,1\rangle_{-1/2,1/2,\partial\omega}$$

(with $O\left(\frac{1}{\|y\|^2}\right)$ uniform in x). Moreover, since S is an isomorphism from $\mathbf{H}^{-1/2}(\partial\omega)/\mathcal{R}$ into $\left\{\boldsymbol{g} \in \mathbf{H}^{1/2}(\partial\omega),\, \int_{\partial\omega} \boldsymbol{g}\cdot\mathbf{n} = 0\right\}$ (see Theorem B.3), we have, according to (4.6),

$$\|\boldsymbol{\eta}\|_{-1/2,\partial\omega} \leq c\,\|\boldsymbol{g}\|_{1/2,\partial\omega}\,.$$

Then, since $|E(y)| \leq \frac{c}{\|y\|}$, there exists $M > 0$ such that, for $\|y\| > M$,

$$|\boldsymbol{w}(y)| \leq \frac{c}{\|y\|}\,\|\boldsymbol{g}\|_{1/2,\partial\omega} + \frac{c}{\|y\|^2}\,\|\boldsymbol{g}\|_{1/2,\partial\omega}\,. \tag{4.7}$$

We differentiate under the sum sign in (4.5) and proceed as above using the fact that $|\nabla E(y)| \leq \frac{c}{\|y\|^2}$ to obtain that for $\|y\| > M$ (even if we have to increase M)

$$|\nabla \boldsymbol{w}(y)| \leq \frac{c}{\|y\|^2}\,\|\boldsymbol{g}\|_{1/2,\partial\omega} + \frac{c}{\|y\|^3}\,\|\boldsymbol{g}\|_{1/2,\partial\omega}\,. \tag{4.8}$$

Notice that since the upper bounds for $\boldsymbol{w}$ and its gradient (see (4.7) and (4.8)) are valid only for $\|y\| > M$, that is far away of $\partial\omega$, we can not obtain directly the estimates on $D_\varepsilon^0/\varepsilon$. We split them in two estimates: one on $B(0,M)\setminus\overline{\omega}$ and one on $B(0,\frac{R}{\varepsilon})\setminus\overline{B(0,M)}$ (where we can use the above inequalities).

Using the change of variables $x = \varepsilon y$, Inequalities (4.7) and (4.8) are valid for $\varepsilon < R/M$ and

$$\|\boldsymbol{w}\|_{0,B(0,\frac{R}{\varepsilon})\setminus\overline{B(0,M)}}$$

$$\leq c\left(\int_{B(0,R)\setminus\overline{B(0,\varepsilon M)}} \frac{\varepsilon^2}{\|x\|^2}\frac{1}{\varepsilon^3}\right)^{1/2}\|\boldsymbol{g}\|_{1/2,\partial\omega} + c\left(\int_{B(0,R)\setminus\overline{B(0,\varepsilon M)}} \frac{\varepsilon^4}{\|x\|^4}\frac{1}{\varepsilon^3}\right)^{1/2}\|\boldsymbol{g}\|_{1/2,\partial\omega}$$

$$\leq c\left(\int_{B(0,R)} \frac{\varepsilon^2}{\|x\|^2}\frac{1}{\varepsilon^3}\right)^{1/2}\|\boldsymbol{g}\|_{1/2,\partial\omega} + c\left(\int_{B(0,R)} \frac{\varepsilon^4}{\|x\|^4}\frac{1}{\varepsilon^3}\right)^{1/2}\|\boldsymbol{g}\|_{1/2,\partial\omega}$$

$$\leq c\,\varepsilon^{-1/2}\|\boldsymbol{g}\|_{1/2,\partial\omega} + c\,\varepsilon^{1/2}\|\boldsymbol{g}\|_{1/2,\partial\omega}\,. \tag{4.9}$$

Moreover, since the weight ρ defined in Appendix B is bounded from above and below in $B(0,M)\setminus\overline{\omega}$ by non-negative constants and according to the definition of the norm $\|\cdot\|_{\mathbf{W}_0^{1,2}}$, there exists a constant $c > 0$ such that $\|\boldsymbol{w}\|_{1,B(0,M)\setminus\overline{\omega}} \leq c\,\|\boldsymbol{w}\|_{\mathbf{W}_0^{1,2}(B(0,M)\setminus\overline{\omega})}$. Using the *a priori* estimate in Theorem B.2, we also obtain $\|\boldsymbol{w}\|_{\mathbf{W}_0^{1,2}(\mathbb{R}^3\setminus\overline{\omega})} \leq c\,\|\boldsymbol{g}\|_{1/2,\partial\omega}$. Then,

$$\|\boldsymbol{w}\|_{1,B(0,M)\setminus\overline{\omega}} \leq c\,\|\boldsymbol{g}\|_{1/2,\partial\omega}\,. \tag{4.10}$$

Gathering the estimates on each subdomains (4.9) and (4.10), we obtain the global estimate

$$\|\boldsymbol{w}\|_{0,\frac{D_\varepsilon^0}{\varepsilon}} \leq c\,\varepsilon^{-1/2}\,\|\boldsymbol{g}\|_{1/2,\partial\omega}\,.$$

Similarly, we obtain $|\boldsymbol{w}|_{1,B(0,\frac{R}{\varepsilon})\backslash\overline{B(0,M)}} \leq c\,\varepsilon^{1/2}\,\|\boldsymbol{g}\|_{1/2,\partial\omega} + c\,\varepsilon^{3/2}\,\|\boldsymbol{g}\|_{1/2,\partial\omega}$ and then, using (4.10),

$$|\boldsymbol{w}|_{1,\frac{D_\varepsilon^0}{\varepsilon}} \leq c\,\|\boldsymbol{g}\|_{1/2,\partial\omega}\,.$$

Finally, without splitting the norm in two, we obtain

$$\|\boldsymbol{w}\|_{0,\frac{\Omega_R^0}{\varepsilon}} \leq c\,\varepsilon^{-1/2}\,\|\boldsymbol{g}\|_{1/2,\partial\omega} + c\,\varepsilon^{1/2}\,\|\boldsymbol{g}\|_{1/2,\partial\omega}$$

$$|\boldsymbol{w}|_{1,\frac{\Omega_R^0}{\varepsilon}} \leq c\,\varepsilon^{1/2}\,\|\boldsymbol{g}\|_{1/2,\partial\omega} + c\,\varepsilon^{3/2}\,\|\boldsymbol{g}\|_{1/2,\partial\omega}\,.$$

$\square$

Next we want to prove the following lemma:

Lemma 4.3. *For $\boldsymbol{\varphi} \in \mathbf{H}^{-1/2}(O)$, $\boldsymbol{\Phi} \in \mathbf{H}^{1/2}(\partial\Omega\backslash\overline{O})$, $\boldsymbol{\Psi} \in \mathbf{H}^1(D_0)$ such that $\operatorname{div}\boldsymbol{\Psi} = 0$ and $\varepsilon > 0$, let $(\boldsymbol{v}_\varepsilon, q_\varepsilon) \in \mathbf{H}^1(\Omega_{z,\varepsilon}) \times \mathrm{L}^2(\Omega_{z,\varepsilon})$ be the solution of the Stokes problem*

$$(4.11) \quad \begin{cases} -\nu\Delta\boldsymbol{u}_\varepsilon + \nabla p_\varepsilon &= \boldsymbol{0} & \text{in } \Omega_{z,\varepsilon} \\ \operatorname{div}\boldsymbol{u}_\varepsilon &= 0 & \text{in } \Omega_{z,\varepsilon} \\ \sigma(\boldsymbol{u}_\varepsilon, p_\varepsilon)\mathbf{n} &= \boldsymbol{\varphi} & \text{on } O \\ \boldsymbol{u}_\varepsilon &= \boldsymbol{\Phi} & \text{on } \partial\Omega\backslash\overline{O} \\ \boldsymbol{u}_\varepsilon &= \boldsymbol{\Psi} & \text{on } \partial\omega_{z,\varepsilon}. \end{cases}$$

Then there exist a constant $c > 0$ (independent of ε and $\boldsymbol{g}$) and $\varepsilon_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$

$$\|\boldsymbol{u}_\varepsilon\|_{0,\Omega_{z,\varepsilon}} \leq c\left(\|\boldsymbol{\varphi}\|_{-1/2,O} + \|\boldsymbol{\Phi}\|_{1/2,\partial\Omega\backslash\overline{O}} + \varepsilon^{1/2}\,\|\boldsymbol{\Psi}(z+\varepsilon y)\|_{1/2,\partial\omega_{z,\varepsilon}}\right).$$

To prove this result, we split it in two lemmas. First, we prove the following result inspired by [33, Lemma 7.2] and [31, Lemma 6.3]:

Lemma 4.4. *Let $\varepsilon > 0$. For $\boldsymbol{\varphi} \in \mathbf{H}^{-1/2}(\mathcal{O})$ and $\boldsymbol{\Phi} \in \mathbf{H}^{1/2}(\partial\Omega\backslash\overline{\mathcal{O}})$, let $(\boldsymbol{v}_\varepsilon, q_\varepsilon) \in \mathbf{H}^1(\Omega_{z,\varepsilon}) \times \mathrm{L}^2(\Omega_{z,\varepsilon})$ be the solution of the Stokes problem*

$$(4.12) \quad \begin{cases} -\nu\Delta\boldsymbol{v}_\varepsilon + \nabla q_\varepsilon &= \boldsymbol{0} & \text{in } \Omega_{z,\varepsilon} \\ \operatorname{div}\boldsymbol{v}_\varepsilon &= 0 & \text{in } \Omega_{z,\varepsilon} \\ \sigma(\boldsymbol{v}_\varepsilon, q_\varepsilon)\mathbf{n} &= \boldsymbol{\varphi} & \text{on } O \\ \boldsymbol{v}_\varepsilon &= \boldsymbol{\Phi} & \text{on } \partial\Omega\backslash\overline{O} \\ \boldsymbol{v}_\varepsilon &= \boldsymbol{0} & \text{on } \partial\omega_{z,\varepsilon}. \end{cases}$$

Then there exist a constant $c > 0$ (independent of ε and $\boldsymbol{\varphi}$) and $\varepsilon_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$

$$\|\boldsymbol{v}_\varepsilon\|_{1,\Omega_{z,\varepsilon}} \leq c\left(\|\boldsymbol{\varphi}\|_{-1/2,O} + \|\boldsymbol{\Phi}\|_{1/2,\partial\Omega\backslash\overline{O}}\right).$$

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Proof. Let $\varepsilon > 0$ and $(\boldsymbol{v}_\varepsilon, q_\varepsilon) \in \mathbf{H}^1(\Omega_{z,\varepsilon}) \times \mathrm{L}^2(\Omega_{z,\varepsilon})$ be the solution of Problem (4.12). Let $(\boldsymbol{V}_\varepsilon, Q_\varepsilon) \in \mathbf{H}^1(\Omega_{z,\varepsilon}) \times \mathrm{L}^2(\Omega_{z,\varepsilon})$ be the solution of

$$
\begin{cases}
-\nu \Delta \boldsymbol{V}_\varepsilon + \nabla Q_\varepsilon &=\ \mathbf{0} \quad &\text{in } \Omega_{z,\varepsilon} \\
\operatorname{div} \boldsymbol{V}_\varepsilon &=\ 0 \quad &\text{in } \Omega_{z,\varepsilon} \\
\sigma(\boldsymbol{V}_\varepsilon, Q_\varepsilon)\mathbf{n} &=\ \mathbf{0} \quad &\text{on } O \\
\boldsymbol{V}_\varepsilon &=\ \boldsymbol{\Phi} \quad &\text{on } \partial\Omega \backslash \overline{O} \\
\boldsymbol{V}_\varepsilon &=\ \mathbf{0} \quad &\text{on } \partial\omega_{z,\varepsilon}.
\end{cases}
\tag{4.13}
$$

Let $\widetilde{\boldsymbol{v}_\varepsilon}$ and $\widetilde{\boldsymbol{V}_\varepsilon}$ the respective extensions of $\boldsymbol{v}_\varepsilon$ and $\boldsymbol{V}_\varepsilon$ to Ω by 0. Then, we have for all $\boldsymbol{\Psi} \in \left\{ \boldsymbol{\Psi} \in \mathbf{H}^1(\Omega_{z,\varepsilon}),\ \operatorname{div} \boldsymbol{\Psi} = 0,\ \boldsymbol{\Psi}_{|\partial\omega_{z,\varepsilon}} = \mathbf{0},\ \boldsymbol{\Psi}_{|\partial\Omega\backslash\overline{O}} = \mathbf{0} \right\}$

$$
\frac{1}{2}\nu \int_{\Omega_{z,\varepsilon}} \mathcal{D}(\boldsymbol{v}_\varepsilon - \boldsymbol{V}_\varepsilon) : \mathcal{D}(\boldsymbol{\Psi}) = \langle \boldsymbol{\varphi},\, \boldsymbol{\Psi} \rangle_O
$$

and then taking $\boldsymbol{\Psi} = \boldsymbol{v}_\varepsilon - \boldsymbol{V}_\varepsilon$

$$
\frac{1}{2}\nu \left\| \mathcal{D}(\widetilde{\boldsymbol{v}_\varepsilon} - \widetilde{\boldsymbol{V}_\varepsilon}) \right\|_{0,\Omega}^2 = \langle \boldsymbol{\varphi},\, \boldsymbol{v}_\varepsilon - \boldsymbol{V}_\varepsilon \rangle_O.
$$

Thus, there exists a constant (independent of ε) such that

$$
\left\| \mathcal{D}(\widetilde{\boldsymbol{v}_\varepsilon} - \widetilde{\boldsymbol{V}_\varepsilon}) \right\|_{0,\Omega}^2 \leq c \left\| \boldsymbol{\varphi} \right\|_{-1/2,O} \left\| \widetilde{\boldsymbol{v}_\varepsilon} - \widetilde{\boldsymbol{V}_\varepsilon} \right\|_{1,\Omega}.
$$

Moreover, since $\boldsymbol{v}_\varepsilon - \boldsymbol{V}_\varepsilon = \mathbf{0}$ on $\partial\Omega\backslash\overline{O}$, Korn's inequality (see for example [41, eq. (2.14) page 19]) leads

$$
\left\| \widetilde{\boldsymbol{v}_\varepsilon} - \widetilde{\boldsymbol{V}_\varepsilon} \right\|_{1,\Omega} \leq c \left\| \mathcal{D}(\widetilde{\boldsymbol{v}_\varepsilon} - \widetilde{\boldsymbol{V}_\varepsilon}) \right\|_{0,\Omega}
$$

(with a constant c independent of ε). Hence,

$$
\begin{aligned}
\left\| \boldsymbol{v}_\varepsilon - \boldsymbol{V}_\varepsilon \right\|_{1,\Omega_{z,\varepsilon}}^2 = \left\| \widetilde{\boldsymbol{v}_\varepsilon} - \widetilde{\boldsymbol{V}_\varepsilon} \right\|_{1,\Omega}^2 &\leq c \left\| \boldsymbol{\varphi} \right\|_{-1/2,O} \left\| \mathcal{D}(\widetilde{\boldsymbol{v}_\varepsilon} - \widetilde{\boldsymbol{V}_\varepsilon}) \right\|_{0,\Omega} \\
&\leq c \left\| \boldsymbol{\varphi} \right\|_{-1/2,O} \left\| \widetilde{\boldsymbol{v}_\varepsilon} - \widetilde{\boldsymbol{V}_\varepsilon} \right\|_{1,\Omega} \leq c \left\| \boldsymbol{\varphi} \right\|_{-1/2,O} \left\| \boldsymbol{v}_\varepsilon - \boldsymbol{V}_\varepsilon \right\|_{1,\Omega_{z,\varepsilon}}.
\end{aligned}
$$

Thus,

$$
\left\| \boldsymbol{v}_\varepsilon - \boldsymbol{V}_\varepsilon \right\|_{1,\Omega_{z,\varepsilon}} \leq c \left\| \boldsymbol{\varphi} \right\|_{-1/2,O}.
$$

Now, let us prove that $\left\| \boldsymbol{V}_\varepsilon \right\|_{1,\Omega_{z,\varepsilon}} \leq c \left\| \boldsymbol{\Phi} \right\|_{1/2,\partial\Omega\backslash\overline{O}}$. For a fixed $\varepsilon_0 > 0$, Problem (4.13) is well-posed and admits a unique solution $(\boldsymbol{V}_{\varepsilon_0}, Q_{\varepsilon_0}) \in \mathbf{H}^1(\Omega_{z,\varepsilon_0}) \times \mathrm{L}^2(\Omega_{z,\varepsilon_0})$ and there exists $c > 0$ such that

$$
\left\| \boldsymbol{V}_{\varepsilon_0} \right\|_{1,\Omega_{z,\varepsilon_0}} \leq c \left\| \boldsymbol{\Phi} \right\|_{1/2,\partial\Omega\backslash\overline{O}}.
$$

Let $0 < \varepsilon_1 < \varepsilon_0$ such that $\Omega_{z,\varepsilon_0} \subset \Omega_{z,\varepsilon}$ for all $0 < \varepsilon < \varepsilon_1$. Let $\widetilde{\boldsymbol{V}_{\varepsilon_0}}$ the extension of $\boldsymbol{V}_{\varepsilon_0}$ to Ω by 0. The solution $\boldsymbol{V}_\varepsilon$ of (4.13) can be considered as the solution of the following minimization problem: $\min\limits_{\boldsymbol{V} \in \mathcal{U}} \left\{ \nu \left| \boldsymbol{V} \right|_{1,\Omega_{z,\varepsilon}} \right\}$, where

$$
\mathcal{U} := \left\{ \boldsymbol{V} \in \mathbf{H}^1(\Omega_{z,\varepsilon}),\ \operatorname{div} \boldsymbol{V} = 0 \text{ in } \Omega_{z,\varepsilon},\ \boldsymbol{V} = \mathbf{0} \text{ on } \partial\omega_{z,\varepsilon},\ \boldsymbol{V} = \boldsymbol{\Phi} \text{ on } \partial\Omega\backslash\overline{O} \right\}.
$$

Hence, for all $0 < \varepsilon < \varepsilon_1$, we have

$$
\left| \boldsymbol{V}_\varepsilon \right|_{1,\Omega_{z,\varepsilon}} \leq c \left| \widetilde{\boldsymbol{V}_{\varepsilon_0}} \right|_{1,\Omega_{z,\varepsilon}} = c \left| \boldsymbol{V}_{\varepsilon_0} \right|_{1,\Omega_{z,\varepsilon_0}} \leq c \left\| \boldsymbol{V}_{\varepsilon_0} \right\|_{1,\Omega_{z,\varepsilon_0}} \leq c \left\| \boldsymbol{\Phi} \right\|_{1/2,\partial\Omega\backslash\overline{O}}.
$$

Notice that $\|\boldsymbol{V}_0\|_{1,\Omega} \le c \, \|\boldsymbol{\Phi}\|_{1/2,\partial\Omega\setminus\overline{O}}$. Hence, using Poincaré's inequality,

$$\|\boldsymbol{V}_\varepsilon\|_{0,\Omega_{z,\varepsilon}} = \left\|\widetilde{\boldsymbol{V}_\varepsilon}\right\|_{0,\Omega} \le \left\|\widetilde{\boldsymbol{V}_\varepsilon} - V_0\right\|_{0,\Omega} + \|\boldsymbol{V}_0\|_{0,\Omega} \le c \left|\widetilde{\boldsymbol{V}_\varepsilon} - V_0\right|_{1,\Omega} + \|\boldsymbol{V}_0\|_{0,\Omega}$$

$$\le c \left|\widetilde{\boldsymbol{V}_\varepsilon}\right|_{1,\Omega} + c \, \|\boldsymbol{V}_0\|_{1,\Omega} \le c \, |\boldsymbol{V}_\varepsilon|_{1,\Omega_{z,\varepsilon}} + c \, \|\boldsymbol{V}_0\|_{1,\Omega} \le c \, \|\boldsymbol{\Phi}\|_{1/2,\partial\Omega\setminus\overline{O}} \,.$$

Hence, we have the announced result. $\qquad\qquad\square$

Next we prove the following lemma which is an adaptation of [33, Lemma 7.3]:

Lemma 4.5. *Let $\varepsilon > 0$. For $\boldsymbol{\Psi} \in \mathbf{H}^1(\Omega)$ such that $\operatorname{div}\boldsymbol{\Psi} = 0$ in Ω, let $(\boldsymbol{u}_\varepsilon, p_\varepsilon) \in \mathbf{H}^1(\Omega_{z,\varepsilon}) \times \mathrm{L}^2(\Omega_{z,\varepsilon})$ be the solution of the Stokes boundary value problem*

$$\begin{cases} -\nu\Delta\boldsymbol{u}_\varepsilon + \nabla p_\varepsilon &= \boldsymbol{0} & \text{in } \Omega_{z,\varepsilon} \\ \operatorname{div}\boldsymbol{u}_\varepsilon &= 0 & \text{in } \Omega_{z,\varepsilon} \\ \sigma(\boldsymbol{u}_\varepsilon, p_\varepsilon)\mathbf{n} &= \boldsymbol{0} & \text{on } O \\ \boldsymbol{u}_\varepsilon &= \boldsymbol{0} & \text{on } \partial\Omega\setminus\overline{O} \\ \boldsymbol{u}_\varepsilon &= \boldsymbol{\Psi} & \text{on } \partial\omega_{z,\varepsilon}. \end{cases}$$

Then there exist a constant $c > 0$ (independent of ε and $\boldsymbol{g}$) and $\varepsilon_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$,

$$\begin{aligned} \|\boldsymbol{u}_\varepsilon\|_{0,\Omega_R^z} &\le c\varepsilon \, \|\boldsymbol{\Psi}(z+\varepsilon y)\|_{1/2,\partial\Omega}\,, & |\boldsymbol{u}_\varepsilon|_{1,\Omega_R^z} &\le c\varepsilon \, \|\boldsymbol{\Psi}(z+\varepsilon y)\|_{1/2,\partial\Omega}\,, \\ \|\boldsymbol{u}_\varepsilon\|_{0,D_\varepsilon^z} &\le c\varepsilon \, \|\boldsymbol{\Psi}(z+\varepsilon y)\|_{1/2,\partial\Omega} & \text{and} \quad |\boldsymbol{u}_\varepsilon|_{1,D_\varepsilon^z} &\le c\varepsilon^{1/2} \, \|\boldsymbol{\Psi}(z+\varepsilon y)\|_{1/2,\partial\Omega}\,. \end{aligned}$$

Proof. Let $(\boldsymbol{V}_\varepsilon, Q_\varepsilon) \in \mathbf{W}_0^{1,2}(\mathbb{R}^3\setminus\overline{\omega}) \times \mathrm{L}^2(\mathbb{R}^3\setminus\overline{\omega})$ the solution of the exterior Stokes problem

$$\begin{cases} -\nu\Delta\boldsymbol{V}_\varepsilon + \nabla Q_\varepsilon &= \boldsymbol{0} & \text{in } \mathbb{R}^3\setminus\overline{\omega} \\ \operatorname{div}\boldsymbol{V}_\varepsilon &= 0 & \text{in } \mathbb{R}^3\setminus\overline{\omega} \\ \boldsymbol{V}_\varepsilon &= \tfrac{1}{\varepsilon}\boldsymbol{\Psi}(z+\varepsilon y) & \text{on } \partial\omega. \end{cases}$$

We define $\boldsymbol{v}_\varepsilon := \varepsilon\boldsymbol{V}_\varepsilon\left(\frac{x-z}{\varepsilon}\right)$ and $q_\varepsilon := Q_\varepsilon\left(\frac{x-z}{\varepsilon}\right)$. Hence $(\boldsymbol{w}_\varepsilon := \boldsymbol{v}_\varepsilon - \boldsymbol{u}_\varepsilon, s_\varepsilon := q_\varepsilon - p_\varepsilon)$ solves

$$\begin{cases} -\nu\Delta\boldsymbol{w}_\varepsilon + \nabla s_\varepsilon &= \boldsymbol{0} & \text{in } \Omega_{z,\varepsilon} \\ \operatorname{div}\boldsymbol{w}_\varepsilon &= 0 & \text{in } \Omega_{z,\varepsilon} \\ \sigma(\boldsymbol{w}_\varepsilon, s_\varepsilon)\mathbf{n} &= \sigma(\boldsymbol{v}_\varepsilon, q_\varepsilon)\mathbf{n} & \text{on } O \\ \boldsymbol{w}_\varepsilon &= \boldsymbol{v}_\varepsilon & \text{on } \partial\Omega\setminus\overline{O} \\ \boldsymbol{w}_\varepsilon &= \boldsymbol{0} & \text{on } \partial\omega_{z,\varepsilon}. \end{cases}$$

By Lemma 4.4, there exists a constant $c > 0$ and $\varepsilon_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$

$$\|\boldsymbol{w}_\varepsilon\|_{1,\Omega_{z,\varepsilon}} \le c \left(\|\sigma(\boldsymbol{v}_\varepsilon, q_\varepsilon)\mathbf{n}\|_{-1/2,O} + \|\boldsymbol{v}_\varepsilon\|_{1/2,\partial\Omega\setminus\overline{O}} \right). \tag{4.14}$$

We also have

$$\|\boldsymbol{v}_\varepsilon\|_{1/2,\partial\Omega\setminus\overline{O}} \le \|\boldsymbol{v}_\varepsilon\|_{1/2,\partial\Omega} \le \|\boldsymbol{v}_\varepsilon\|_{1,\Omega_R^z}. \tag{4.15}$$

Moreover,

$$\|\sigma(\boldsymbol{v}_\varepsilon, q_\varepsilon)\mathbf{n}\|_{-1/2,O} \le \|\boldsymbol{v}_\varepsilon\|_{1,\Omega_R^z}. \tag{4.16}$$

Indeed, for all $\boldsymbol{\varphi} \in \mathbf{H}^{1/2}(O)$ and all $\boldsymbol{\Phi} \in \mathbf{H}^1(\Omega_R^z)$ extension of $\boldsymbol{\varphi}$ such that $\boldsymbol{\Phi}_{|\partial\Omega\setminus\overline{O}} = \boldsymbol{0}$ and $\boldsymbol{\Phi}_{|\partial(B(0,R))} = \boldsymbol{0}$, we have

$$\langle\sigma(\boldsymbol{v}_\varepsilon, q_\varepsilon)\mathbf{n}, \boldsymbol{\varphi}\rangle_{-1/2,1/2,O} = \nu \int_{\Omega_R^z} \mathcal{D}(\boldsymbol{v}_\varepsilon) : \nabla(\boldsymbol{\Phi}) \le c \, \|\mathcal{D}(\boldsymbol{v}_\varepsilon)\|_{0,\Omega_R^z} \, \|\boldsymbol{\Phi}\|_{1,\Omega_R^z}$$

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and, choosing $\boldsymbol{\Phi}$ such that $\|\boldsymbol{\Phi}\|_{1,\Omega_R^z} = \|\boldsymbol{\varphi}\|_{1/2,O}$, we obtain that

$$\|\sigma(\boldsymbol{v}_\varepsilon, q_\varepsilon)\mathbf{n}\|_{-1/2,O} \leq c\,\|\mathcal{D}(\boldsymbol{v}_\varepsilon)\|_{0,\Omega_R^z}\,.$$

Finally, using the change of variables $x = z + \varepsilon y$ (see (4.3) and (4.4)) and Lemma 4.2, we obtain

$$
\begin{aligned}
\|\boldsymbol{v}_\varepsilon\|_{0,\Omega_R^z} &\leq& \varepsilon^{3/2}\,\|\varepsilon\boldsymbol{V}_\varepsilon\|_{0,\frac{\Omega_R^0}{\varepsilon}} &\leq& c\varepsilon\,\|\boldsymbol{\Psi}(z+\varepsilon y)\|_{1/2,\partial\omega} \\
|\boldsymbol{v}_\varepsilon|_{1,\Omega_R^z} &\leq& \varepsilon^{1/2}\,|\varepsilon\boldsymbol{V}_\varepsilon|_{1,\frac{\Omega_R^0}{\varepsilon}} &\leq& c\varepsilon\,\|\boldsymbol{\Psi}(z+\varepsilon y)\|_{1/2,\partial\omega}\,.
\end{aligned}
\tag{4.17}
$$

Hence, gathering inequalities (4.14), (4.15), (4.16) and (4.17),

$$\|\boldsymbol{w}_\varepsilon\|_{1,\Omega_{z,\varepsilon}} \leq c\varepsilon\,\|\boldsymbol{\Psi}(z+\varepsilon y)\|_{1/2,\partial\omega}\,.$$

Proceeding as above (using the change of variables $x = z+\varepsilon y$ and Lemma 4.2), we also have

$$\|\boldsymbol{v}_\varepsilon\|_{0,D_\varepsilon^z} \leq c\varepsilon\,\|\boldsymbol{\Psi}(z+\varepsilon y)\|_{1/2,\partial\omega} \quad \text{and} \quad |\boldsymbol{v}_\varepsilon|_{1,D_\varepsilon^z} \leq c\varepsilon^{1/2}\,\|\boldsymbol{\Psi}(z+\varepsilon y)\|_{1/2,\partial\omega}\,.$$

Therefore, we have

$$
\begin{aligned}
|\boldsymbol{u}_\varepsilon|_{1,\Omega_R^z} &\leq |\boldsymbol{v}_\varepsilon|_{1,\Omega_R^z} + |\boldsymbol{w}_\varepsilon|_{1,\Omega_R^z} \leq c\varepsilon\,\|\boldsymbol{\Psi}(z+\varepsilon y)\|_{1/2,\partial\omega} \\
\|\boldsymbol{u}_\varepsilon\|_{0,\Omega_R^z} &\leq \|\boldsymbol{v}_\varepsilon\|_{0,\Omega_R^z} + \|\boldsymbol{w}_\varepsilon\|_{0,\Omega_R^z} \leq c\varepsilon\,\|\boldsymbol{\Psi}(z+\varepsilon y)\|_{1/2,\partial\omega}\,,
\end{aligned}
$$

and similarly

$$\|\boldsymbol{u}_\varepsilon\|_{0,D_\varepsilon^z} \leq c\varepsilon\,\|\boldsymbol{\Psi}(z+\varepsilon y)\|_{1/2,\partial\omega} \quad \text{and} \quad |\boldsymbol{u}_\varepsilon|_{1,D_\varepsilon^z} \leq c\varepsilon^{1/2}\,\|\boldsymbol{\Psi}(z+\varepsilon y)\|_{1/2,\partial\omega}\,.$$

$\square$

Proof of Lemma 4.3. Lemma 4.3 is a direct consequence of the two previous results splitting Problem (4.11) into the two problems intervening in Lemmas 4.4 and 4.5. $\square$

4.3 Proof of Proposition 4.1

In order to simplify the notations, let us define

$$\boldsymbol{r}_N^\varepsilon(x) := \boldsymbol{u}_N^\varepsilon(x) - \boldsymbol{u}_N^0(x) - \boldsymbol{U}_N\left(\frac{x-z}{\varepsilon}\right) \quad \text{and} \quad s_N^\varepsilon(x) := p_N^\varepsilon(x) - p_N^0(x) - \frac{1}{\varepsilon}P_N\left(\frac{x-z}{\varepsilon}\right),$$

where $(\boldsymbol{U}_N, P_N) \in \mathbf{W}_0^{1,2}(\mathbb{R}^3\backslash\overline{\omega}) \times \mathrm{L}^2(\mathbb{R}^3\backslash\overline{\omega})$ solves (4.1) with $\natural = N$. Then the pair $(\boldsymbol{r}_N^\varepsilon, s_N^\varepsilon) \in \mathbf{H}^1(\Omega_{z,\varepsilon}) \times \mathrm{L}^2(\Omega_{z,\varepsilon})$ solves

$$
\begin{cases}
-\nu\Delta\boldsymbol{r}_N^\varepsilon + \nabla s_N^\varepsilon &=& \mathbf{0} & \text{in } \Omega_{z,\varepsilon} \\
\operatorname{div}\boldsymbol{r}_N^\varepsilon &=& 0 & \text{in } \Omega_{z,\varepsilon} \\
\sigma(\boldsymbol{r}_N^\varepsilon, s_N^\varepsilon)\mathbf{n} &=& -\frac{1}{\varepsilon}\sigma(\boldsymbol{U}_N, P_N)\left(\frac{x-z}{\varepsilon}\right)\mathbf{n} & \text{on } O \\
\boldsymbol{r}_N^\varepsilon &=& -\boldsymbol{U}_N\left(\frac{x-z}{\varepsilon}\right) & \text{on } \partial\Omega\backslash\overline{O} \\
\boldsymbol{r}_N^\varepsilon &=& -\boldsymbol{u}_N^0(x) + \boldsymbol{u}_N^0(z) & \text{on } \partial\omega_{z,\varepsilon}.
\end{cases}
$$

Then we know by Lemma 4.3 that there exists a constant $c > 0$ independent of ε such that

$$
\begin{aligned}
\|\boldsymbol{r}_N^\varepsilon\|_{1,\Omega_{z,\varepsilon}} \leq c\Bigg(&\frac{1}{\varepsilon}\left\|\sigma(\boldsymbol{U}_N, P_N)\left(\frac{x-z}{\varepsilon}\right)\mathbf{n}\right\|_{-1/2,O} \\
&+ \left\|\boldsymbol{U}_N\left(\frac{x-z}{\varepsilon}\right)\right\|_{1/2,\partial\Omega\backslash\overline{O}} + \varepsilon^{1/2}\left\|-\boldsymbol{u}_N^0(z+\varepsilon y) + \boldsymbol{u}_N^0(z)\right\|_{1/2,\partial\omega}\Bigg).
\end{aligned}
\tag{4.18}
$$

Mimicking the proof of (4.16), we have

$$\left\| \sigma(\boldsymbol{U}_N, P_N) \left(\frac{x-z}{\varepsilon} \right) \mathbf{n} \right\|_{-1/2,O} \leq c \left\| (\nabla \boldsymbol{U}_N) \left(\frac{x-z}{\varepsilon} \right) \right\|_{0,\Omega_R^z}.$$

Using the change of variables $y = x - z$, we obtain

$$\left\| \sigma(\boldsymbol{U}_N, P_N) \left(\frac{x-z}{\varepsilon} \right) \mathbf{n} \right\|_{-1/2,O} \leq c \left\| (\nabla \boldsymbol{U}_N) \left(\frac{y}{\varepsilon} \right) \right\|_{0,\Omega_R^0} = c \varepsilon \left\| \nabla \left(\boldsymbol{U}_N \left(\frac{y}{\varepsilon} \right) \right) \right\|_{0,\Omega_R^0}.$$

Then, using the change of variables $y = \varepsilon x$ (see (4.4)),

$$\left\| \sigma(\boldsymbol{U}_N, P_N) \left(\frac{x-z}{\varepsilon} \right) \mathbf{n} \right\|_{-1/2,O} \leq c \varepsilon^{5/2} \left\| \nabla \boldsymbol{U}_N \right\|_{0,\frac{\Omega_R^0}{\varepsilon}}.$$

Thus, Lemma 4.2 gives

$$\left\| \sigma(\boldsymbol{U}_N, P_N) \left(\frac{x-z}{\varepsilon} \right) \mathbf{n} \right\|_{-1/2,O} \leq c \varepsilon^3 \left\| \boldsymbol{u}_N^0(z) \right\|_{1/2,\partial\omega} \leq c \varepsilon^3. \tag{4.19}$$

Proceeding as for $\left\| \sigma(\boldsymbol{U}_N, P_N) \left(\frac{x-z}{\varepsilon} \right) \mathbf{n} \right\|_{-1/2,O}$, we have, using the change of variables $y = \frac{x-z}{\varepsilon}$ (see (4.3) and (4.4)),

$$
\begin{aligned}
\left\| \boldsymbol{U}_N \left(\frac{x-z}{\varepsilon} \right) \right\|_{1/2,\partial\Omega\backslash\overline{O}} &\leq c \left(\left\| \boldsymbol{U}_N \left(\frac{x-z}{\varepsilon} \right) \right\|_{0,\Omega_R^z} + \left\| \nabla (\boldsymbol{U}_N) \left(\frac{x-z}{\varepsilon} \right) \right\|_{0,\Omega_R^z} \right) \\
&\leq c \left(\varepsilon^{3/2} \left\| \boldsymbol{U}_N (x) \right\|_{0,\frac{\Omega_R^0}{\varepsilon}} + \varepsilon^{5/2} \left\| \nabla \boldsymbol{U}_N \right\|_{0,\frac{\Omega_R^0}{\varepsilon}} \right).
\end{aligned}
$$

Thus, according to Lemma 4.2, we obtain

$$\left\| \boldsymbol{U}_N \left(\frac{x-z}{\varepsilon} \right) \right\|_{1/2,\partial\Omega\backslash\overline{O}} \leq c \varepsilon \left\| \boldsymbol{u}_N^0(z) \right\|_{1/2,\partial\omega} \leq c \varepsilon. \tag{4.20}$$

Using a Taylor expansion, we obtain $\boldsymbol{u}_N^0(z + \varepsilon y) = \boldsymbol{u}_N^0(z) + \varepsilon \nabla \boldsymbol{u}_N^0(\xi_y) y$ with $\xi_y \in \omega_{z,\varepsilon}$. Since $\nabla \boldsymbol{u}_N^0$ is uniformly bounded in $\omega_{z,\varepsilon}$, we have

$$\left\| \boldsymbol{u}_N^0(z + \varepsilon y) - \boldsymbol{u}_N^0(z) \right\|_{1/2,\partial\omega} \leq c \varepsilon. \tag{4.21}$$

Hence, relations (4.18), (4.19), (4.20) and (4.21) implies

$$\left\| \boldsymbol{u}_N^\varepsilon(x) - \boldsymbol{u}_N^0(x) - \boldsymbol{U}_N \left(\frac{x-z}{\varepsilon} \right) \right\|_{1,\Omega_{z,\varepsilon}} \leq c \varepsilon.$$

5 Proof of Theorem 3.1

5.1 A preliminary lemma

First we need an estimate of the norm $\|\cdot\|_{1/2,\partial\omega_{z,\varepsilon}}$ of an uniformly bounded function. Here $\|\cdot\|_{1/2,\partial\omega_{z,\varepsilon}}$ has to be seen as the trace norm

$$\|\boldsymbol{f}\|_{1/2,\partial\omega_{z,\varepsilon}} := \inf \left\{ \|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega\backslash\overline{\omega_{z,\varepsilon}})}, \; \boldsymbol{u} \in \mathbf{H}^1(\Omega\backslash\overline{\omega_{z,\varepsilon}}), \boldsymbol{u}_{|\partial\omega_{z,\varepsilon}} = \boldsymbol{f} \right\}.$$

Lemma 5.1. *Let $\varepsilon \in (0, 1/2)$. If $\boldsymbol{u} \in \mathbf{H}^1(\Omega)$ is such that its restriction to $\overline{\omega_{z,1}}$ is C^1, then there exists a constant $c > 0$ independent of ε such that*

$$\|\boldsymbol{u}\|_{1/2, \partial\omega_{z,\varepsilon}} \le c\varepsilon^{1/2}.$$

Proof. From Theorem C.1, there exists a constant $c > 0$ independent of ε such that

$$\|\boldsymbol{u}\|_{1/2, \partial\omega_{z,\varepsilon}} \le c\varepsilon^{-1/2} \|\boldsymbol{u}\|_{\mathbf{L}^2(\partial\omega_{z,\varepsilon})} + c\left(\iint_{\partial\omega_{z,\varepsilon} \times \partial\omega_{z,\varepsilon}} \frac{|\boldsymbol{u}(x) - \boldsymbol{u}(y)|^2}{|x - y|^3} \mathrm{d}s(x)\mathrm{d}s(y)\right)^{1/2}.$$

Since $\boldsymbol{u}$ is uniformly bounded on $\partial\omega_{z,\varepsilon}$, we use the change of variables $y = z + \varepsilon x$ to prove that there exists a constant $c > 0$ independent of ε such that

$$\|\boldsymbol{u}\|_{\mathbf{L}^2(\partial\omega_{z,\varepsilon})} \le c\varepsilon.$$

Moreover, using the changes of variables $x = z + \varepsilon X$ and $y = z + \varepsilon Y$, the fact that $\boldsymbol{u}(z + \varepsilon X) = \boldsymbol{u}(z) + \varepsilon\nabla(\boldsymbol{u})(\xi_X)X$, $\xi_X \in \omega_{z,\varepsilon}$ and $\boldsymbol{u}(z + \varepsilon Y) = \boldsymbol{u}(z) + \varepsilon\nabla(\boldsymbol{u})(\xi_Y)Y$, $\xi_Y \in \omega_{z,\varepsilon}$, there exists $c > 0$ independent of ε such that

$$\left(\iint_{\partial\omega_{z,\varepsilon} \times \partial\omega_{z,\varepsilon}} \frac{|\boldsymbol{u}(x) - \boldsymbol{u}(y)|^2}{|x - y|^3} \mathrm{d}s(x)\mathrm{d}s(y)\right)^{1/2}$$
$$= \left(\iint_{\partial\omega \times \partial\omega} \varepsilon^4 \frac{|\varepsilon(\nabla(\boldsymbol{u})(\xi_X)X - \nabla(\boldsymbol{u})(\xi_Y)Y)|^2}{\varepsilon^3 |X - Y|^3} \mathrm{d}s(x)\mathrm{d}s(y)\right)^{1/2} \le c\varepsilon^{3/2}.$$

$\square$

5.2 Splitting the variations of the objective

Now, we turn our attention to the Kohn-Vogelius functional given by

$$\mathcal{J}_{KV}(\Omega_{z,\varepsilon}) = \frac{1}{2}\nu \int_{\Omega_{z,\varepsilon}} |\mathcal{D}(\boldsymbol{u}_D^\varepsilon) - \mathcal{D}(\boldsymbol{u}_N^\varepsilon)|^2.$$

Lemma 5.2. *We have*

$$\mathcal{J}_{KV}(\Omega_{z,\varepsilon}) - \mathcal{J}_{KV}(\Omega) = A_D + A_N, \tag{5.1}$$

where

$$A_D := \frac{1}{2}\nu \int_{\Omega_{z,\varepsilon}} \mathcal{D}(\boldsymbol{u}_D^\varepsilon - \boldsymbol{u}_D^0) : \mathcal{D}(\boldsymbol{u}_D^\varepsilon - \boldsymbol{u}_D^0)$$
$$+ \nu \int_{\Omega_{z,\varepsilon}} \mathcal{D}(\boldsymbol{u}_D^\varepsilon - \boldsymbol{u}_D^0) : \mathcal{D}(\boldsymbol{u}_D^0) - \frac{1}{2}\nu \int_{\omega_{z,\varepsilon}} |\mathcal{D}(\boldsymbol{u}_D^0)|^2$$

and

$$A_N := \int_{\partial\omega_{z,\varepsilon}} \left[\sigma(\boldsymbol{u}_N^\varepsilon - \boldsymbol{u}_N^0, p_N^\varepsilon - p_N^0)\mathbf{n}\right] \cdot \boldsymbol{u}_N^0 - \frac{1}{2}\nu \int_{\omega_{z,\varepsilon}} |\mathcal{D}(\boldsymbol{u}_N^0)|^2.$$

Proof. The functional $\mathcal{J}_{KV}$ can be expanded in

$$\mathcal{J}_{KV}(\Omega_{z,\varepsilon}) = \underbrace{\mathcal{J}_{DD}(\Omega_{z,\varepsilon})}_{:= \frac{1}{2}\nu \int_{\Omega_{z,\varepsilon}} |\mathcal{D}(\boldsymbol{u}_D^\varepsilon)|^2} + \underbrace{\mathcal{J}_{DN}(\Omega_{z,\varepsilon})}_{:= -\nu \int_{\Omega_{z,\varepsilon}} \mathcal{D}(\boldsymbol{u}_D^\varepsilon) : \mathcal{D}(\boldsymbol{u}_N^\varepsilon)} + \underbrace{\mathcal{J}_{NN}(\Omega_{z,\varepsilon})}_{:= \frac{1}{2}\nu \int_{\Omega_{z,\varepsilon}} |\mathcal{D}(\boldsymbol{u}_N^\varepsilon)|^2}.$$

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Variation of $\mathcal{J}_{DD}$. We have

$$
\begin{aligned}
\mathcal{J}_{DD}(\Omega_{z,\varepsilon}) - \mathcal{J}_{DD}(\Omega) &= \frac{1}{2}\nu \int_{\Omega_{z,\varepsilon}} |\mathcal{D}(\boldsymbol{u}_D^\varepsilon)|^2 - \frac{1}{2}\nu \int_{\Omega} |\mathcal{D}(\boldsymbol{u}_D^0)|^2 \\
&= \frac{1}{2}\nu \int_{\Omega_{z,\varepsilon}} \mathcal{D}(\boldsymbol{u}_D^\varepsilon - \boldsymbol{u}_D^0) : \mathcal{D}(\boldsymbol{u}_D^\varepsilon - \boldsymbol{u}_D^0) \\
&\quad + \nu \int_{\Omega_{z,\varepsilon}} \mathcal{D}(\boldsymbol{u}_D^\varepsilon - \boldsymbol{u}_D^0) : \mathcal{D}(\boldsymbol{u}_D^0) - \frac{1}{2}\nu \int_{\omega_{z,\varepsilon}} |\mathcal{D}(\boldsymbol{u}_D^0)|^2. \quad (5.2)
\end{aligned}
$$

Variation of $\mathcal{J}_{DN}$. We have, using Green's formula on (2.4) and (4.2),

$$
\begin{aligned}
\mathcal{J}_{DN}(\Omega_{z,\varepsilon}) - \mathcal{J}_{DN}(\Omega) &= -\nu \int_{\Omega_{z,\varepsilon}} \mathcal{D}(\boldsymbol{u}_N^\varepsilon) : \mathcal{D}(\boldsymbol{u}_D^\varepsilon) + \nu \int_{\Omega} \mathcal{D}(\boldsymbol{u}_N^0) : \mathcal{D}(\boldsymbol{u}_D^0) \\
&= 2 \int_{\partial\Omega} \left[\sigma(\boldsymbol{u}_N^0 - \boldsymbol{u}_N^\varepsilon, p_N^0 - p_N^\varepsilon) \mathbf{n} \right] \cdot \boldsymbol{f}. \qquad (5.3)
\end{aligned}
$$

Variation of $\mathcal{J}_{NN}$. We have

$$
\begin{aligned}
\mathcal{J}_{NN}(\Omega_{z,\varepsilon}) - \mathcal{J}_{NN}(\Omega) &= \frac{1}{2}\nu \int_{\Omega_{z,\varepsilon}} |\mathcal{D}(\boldsymbol{u}_N^\varepsilon)|^2 - \frac{1}{2}\nu \int_{\Omega} |\mathcal{D}(\boldsymbol{u}_N^0)|^2 \\
&= \frac{1}{2}\nu \int_{\Omega_{z,\varepsilon}} \mathcal{D}(\boldsymbol{u}_N^\varepsilon - \boldsymbol{u}_N^0) : \mathcal{D}(\boldsymbol{u}_N^\varepsilon) + \frac{1}{2}\nu \int_{\Omega_{z,\varepsilon}} \mathcal{D}(\boldsymbol{u}_N^\varepsilon - \boldsymbol{u}_N^0) : \mathcal{D}(\boldsymbol{u}_N^0) \\
&\quad - \frac{1}{2}\nu \int_{\omega_{z,\varepsilon}} |\mathcal{D}(\boldsymbol{u}_N^0)|^2.
\end{aligned}
$$

Then, using Green's formula and (4.2) in the two first integrals, we obtain

$$
\begin{aligned}
\mathcal{J}_{NN}(\Omega_{z,\varepsilon}) - \mathcal{J}_{NN}(\Omega) &= \int_{\partial\Omega} \left[\sigma(\boldsymbol{u}_N^\varepsilon - \boldsymbol{u}_N^0, p_N^\varepsilon - p_N^0) \mathbf{n} \right] \cdot \boldsymbol{u}_N^\varepsilon \\
&\quad + \int_{\partial\Omega} \left[\sigma(\boldsymbol{u}_N^\varepsilon - \boldsymbol{u}_N^0, p_N^\varepsilon - p_N^0) \mathbf{n} \right] \cdot \boldsymbol{u}_N^0 + \int_{\partial\omega_{z,\varepsilon}} \left[\sigma(\boldsymbol{u}_N^\varepsilon - \boldsymbol{u}_N^0, p_N^\varepsilon - p_N^0) \mathbf{n} \right] \cdot \boldsymbol{u}_N^0 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \frac{1}{2}\nu \int_{\omega_{z,\varepsilon}} |\mathcal{D}(\boldsymbol{u}_N^0)|^2. \quad (5.4)
\end{aligned}
$$

Variation of $\mathcal{J}_{KV}$. In order to obtain the variation $\mathcal{J}_{KV}$, we will sum the variations of $\mathcal{J}_{DD}$, $\mathcal{J}_{DN}$ and $\mathcal{J}_{NN}$. First, we remark that $\boldsymbol{f} - \boldsymbol{u}_N^0$ and $\boldsymbol{f} - \boldsymbol{u}_N^\varepsilon$ belong to $\mathbf{H}_{00}^{1/2}(O)$ since they belong to $\mathbf{H}^{1/2}(\partial\Omega)$ and vanish on $\partial\Omega \backslash \overline{O}$. Then

$$
\begin{aligned}
&2 \int_{\partial\Omega} \left[\sigma(\boldsymbol{u}_N^\varepsilon - \boldsymbol{u}_N^0, p_N^\varepsilon - p_N^0) \mathbf{n} \right] \cdot \boldsymbol{f} \\
&\quad - \int_{\partial\Omega} \left[\sigma(\boldsymbol{u}_N^\varepsilon - \boldsymbol{u}_N^0, p_N^\varepsilon - p_N^0) \mathbf{n} \right] \cdot \boldsymbol{u}_N^0 - \int_{\partial\omega_{z,\varepsilon}} \left[\sigma(\boldsymbol{u}_N^\varepsilon - \boldsymbol{u}_N^0, p_N^\varepsilon - p_N^0) \mathbf{n} \right] \cdot \boldsymbol{u}_N^\varepsilon \\
&= \int_{\partial\Omega} \left[\sigma(\boldsymbol{u}_N^\varepsilon - \boldsymbol{u}_N^0, p_N^\varepsilon - p_N^0) \mathbf{n} \right] \cdot (\boldsymbol{f} - \boldsymbol{u}_N^0) + \int_{\partial\Omega} \left[\sigma(\boldsymbol{u}_N^\varepsilon - \boldsymbol{u}_N^0, p_N^\varepsilon - p_N^0) \mathbf{n} \right] \cdot (\boldsymbol{f} - \boldsymbol{u}_N^\varepsilon) \\
&= \left\langle \sigma(\boldsymbol{u}_N^\varepsilon - \boldsymbol{u}_N^0, p_N^\varepsilon - p_N^0)\mathbf{n} ,\, \boldsymbol{f} - \boldsymbol{u}_N^0 \right\rangle_O + \left\langle \sigma(\boldsymbol{u}_N^\varepsilon - \boldsymbol{u}_N^0, p_N^\varepsilon - p_N^0)\mathbf{n} ,\, \boldsymbol{f} - \boldsymbol{u}_N^\varepsilon \right\rangle_O \\
&\quad + \int_{\partial\Omega\backslash\overline{O}} \left[\sigma(\boldsymbol{u}_N^\varepsilon - \boldsymbol{u}_N^0, p_N^\varepsilon - p_N^0) \mathbf{n} \right] \cdot (\boldsymbol{f} - \boldsymbol{u}_N^0) + \int_{\partial\Omega\backslash\overline{O}} \left[\sigma(\boldsymbol{u}_N^\varepsilon - \boldsymbol{u}_N^0, p_N^\varepsilon - p_N^0) \mathbf{n} \right] \cdot (\boldsymbol{f} - \boldsymbol{u}_N^\varepsilon).
\end{aligned}
$$

Since $\sigma(\boldsymbol{u}_N^\varepsilon, p_N^\varepsilon)\mathbf{n} = \sigma(\boldsymbol{u}_N^0, p_N^0)\mathbf{n} = \boldsymbol{g}$ on O and $\boldsymbol{u}_N^0 = \boldsymbol{u}_N^\varepsilon = \boldsymbol{f}$ on $\partial\Omega\backslash\overline{O}$, we prove that this expression is null. Finally, we obtain the expansion (5.1) using the variations of $\mathcal{J}_{DD}$, $\mathcal{J}_{DN}$ and $\mathcal{J}_{NN}$ (see (5.2), (5.3) and (5.4)). $\qquad\square$

5.3 Asymptotic expansion of A_N

We know using elliptic regularity that $\nabla \boldsymbol{u}_N^0$ is uniformly bounded on $\omega_{z,\varepsilon}$. Thus

$$-\frac{1}{2}\nu \int_{\omega_{z,\varepsilon}} |\mathcal{D}(\boldsymbol{u}_N^0)|^2 \le c \int_{\omega} \varepsilon^3 = O(\varepsilon^3). \tag{5.5}$$

Let us recall that

$$\boldsymbol{r}_N^\varepsilon(x) := \boldsymbol{u}_N^\varepsilon(x) - \boldsymbol{u}_N^0(x) - \boldsymbol{U}_N\left(\frac{x-z}{\varepsilon}\right) \quad \text{and} \quad s_N^\varepsilon(x) := p_N^\varepsilon(x) - p_N^0(x) - \frac{1}{\varepsilon}P_N\left(\frac{x-z}{\varepsilon}\right),$$

where $(\boldsymbol{U}_N, P_N) \in \mathbf{W}_0^{1,2}(\mathbb{R}^3\backslash\overline{\omega}) \times \mathrm{L}^2(\mathbb{R}^3\backslash\overline{\omega})$ solves (4.1) with $\natural = N$. Then the following equality holds

$$\int_{\partial\omega_{z,\varepsilon}} \left[\sigma(\boldsymbol{u}_N^\varepsilon - \boldsymbol{u}_N^0, p_N^\varepsilon - p_N^0)\mathbf{n}\right] \cdot \boldsymbol{u}_N^0 =$$

$$\int_{\partial\omega_{z,\varepsilon}} [\sigma(\boldsymbol{r}_N^\varepsilon, s_N^\varepsilon)\mathbf{n}] \cdot \boldsymbol{u}_N^0 + \frac{1}{\varepsilon}\int_{\partial\omega_{z,\varepsilon}} \left[\sigma(\boldsymbol{U}_N, P_N)\left(\frac{x-z}{\varepsilon}\right)\mathbf{n}\right] \cdot \boldsymbol{u}_N^0. \tag{5.6}$$

Let us first focus on the first term in the right-hand side of (5.6). We recall that for all $\boldsymbol{\varphi} \in \mathbf{H}^{1/2}(\partial\omega_{z,\varepsilon})$ and all $\boldsymbol{\Phi} \in \mathbf{H}^1(\Omega_{z,\varepsilon})$ extension of $\boldsymbol{\varphi}$ such that $\boldsymbol{\Phi}_{|\partial\Omega} = \mathbf{0}$, we have

$$\langle \sigma(\boldsymbol{r}_N^\varepsilon, s_N^\varepsilon)\mathbf{n}\,, \boldsymbol{\varphi} \rangle_{-1/2,1/2,\partial\omega_{z,\varepsilon}} = \frac{1}{2}\nu \int_{\Omega_{z,\varepsilon}} \mathcal{D}(\boldsymbol{r}_N^\varepsilon) : \mathcal{D}(\boldsymbol{\Phi}) \le c \|\mathcal{D}(\boldsymbol{r}_N^\varepsilon)\|_{0,\Omega_{z,\varepsilon}} \|\boldsymbol{\Phi}\|_{1,\Omega_{z,\varepsilon}}.$$

Hence, choosing $\boldsymbol{\Phi}$ such that $\|\boldsymbol{\Phi}\|_{1,\Omega_{z,\varepsilon}} = \|\boldsymbol{\varphi}\|_{1,\partial\Omega_{z,\varepsilon}}$, we obtain that

$$\|\sigma(\boldsymbol{r}_N^\varepsilon, s_N^\varepsilon)\mathbf{n}\|_{-1/2,\partial\omega_{z,\varepsilon}} \le c \|\mathcal{D}(\boldsymbol{r}_N^\varepsilon)\|_{0,\Omega_{z,\varepsilon}}. \tag{5.7}$$

Therefore, using the explicit upper bound of $\|\boldsymbol{u}_N^0\|_{1/2,\partial\omega_{z,\varepsilon}}$ given by Lemma 5.1, we have

$$\left|\int_{\partial\omega_{z,\varepsilon}} [\sigma(\boldsymbol{r}_N^\varepsilon, s_N^\varepsilon)\mathbf{n}] \cdot \boldsymbol{u}_N^0\right| \le \|\sigma(\boldsymbol{r}_N^\varepsilon, s_N^\varepsilon)\mathbf{n}\|_{-1/2,\partial\omega_{z,\varepsilon}} \|\boldsymbol{u}_N^0\|_{1/2,\partial\omega_{z,\varepsilon}} \le c\,\varepsilon^{1/2} \|\boldsymbol{r}_N^\varepsilon\|_{1,\Omega_{z,\varepsilon}}.$$

Then, using the explicit upper bound of $\|\boldsymbol{r}_N^\varepsilon\|_{1,\Omega_{z,\varepsilon}}$ given by Proposition 4.1, we obtain

$$\left|\int_{\partial\omega_{z,\varepsilon}} [\sigma(\boldsymbol{r}_N^\varepsilon, s_N^\varepsilon)\mathbf{n}] \cdot \boldsymbol{u}_N^0\right| \le c\,\varepsilon^{3/2} = O(\varepsilon^{3/2}). \tag{5.8}$$

Let us now focus on the second term in the right-hand side of (5.6). Using the change of variables $x = z + \varepsilon\,y$, we have

$$\int_{\partial\omega_{z,\varepsilon}} \left[\sigma(\boldsymbol{U}_N, P_N)\right]\left(\frac{x-z}{\varepsilon}\right)\mathbf{n} \cdot \boldsymbol{u}_N^0 \,\mathrm{d}s(x) \quad = \quad \varepsilon^2 \int_{\partial\omega} [\sigma(\boldsymbol{U}_N, P_N)(y)\mathbf{n}] \cdot \boldsymbol{u}_N^0(z + \varepsilon y)\,\mathrm{d}s(y).$$

Since $\boldsymbol{u}_N^0(z+\varepsilon y) = \boldsymbol{u}_N^0(z) + \varepsilon\nabla(\boldsymbol{u}_N^0)(\xi_y)y$, $\xi_y \in \omega_{z,\varepsilon}$ and since $\mathcal{D}(\boldsymbol{u}_N^0)$ is uniformly bounded in $\omega_{z,\varepsilon}$, we have

$$\varepsilon^2 \int_{\partial\omega} [\sigma(\boldsymbol{U}_N, P_N)(y)\mathbf{n}] \cdot \boldsymbol{u}_N^0(z + \varepsilon y) = \varepsilon^2 \left(\int_{\partial\omega} \sigma(\boldsymbol{U}_N, P_N)(y)\mathbf{n}\right) \cdot \boldsymbol{u}_N^0(z) + O(\varepsilon^3).$$

Taking into account that $\mathbf{n}$ is oriented inside ω, we use the jump condition of the single layer potential (see [36, Lemma 5.6.5]) to obtain that

$$-\sigma(\boldsymbol{U}_N, P_N)\mathbf{n} + \sigma(\boldsymbol{W}_N, Q_N)\mathbf{n} = -\boldsymbol{\eta}_N,$$

where $\boldsymbol{\eta}_N \in \mathbf{H}^{-1/2}(\partial\omega)/\mathbb{R}\mathbf{n}$ is defined by (3.2) and $(\boldsymbol{W}_N, Q_N) \in \mathbf{H}^1(\omega) \times \mathrm{L}^2(\omega)$ solves

$$\begin{cases} -\nu\Delta\boldsymbol{W}_N + \nabla Q_N &= \mathbf{0} & \text{in } \omega \\ \operatorname{div}\boldsymbol{W}_N &= 0 & \text{in } \omega \\ \boldsymbol{W}_N &= \boldsymbol{U}_N & \text{on } \partial\omega. \end{cases}$$

Since $-\operatorname{div}\sigma(\boldsymbol{W}_N, Q_N) = -\nu\Delta\boldsymbol{W}_N + \nabla Q_N = \mathbf{0}$ in ω, we have $\int_{\partial\omega}\sigma(\boldsymbol{W}_N, Q_N)\mathbf{n} = 0$ and

$$\int_{\partial\omega_{z,\varepsilon}}\left[\sigma(\boldsymbol{U}_N, P_N)\right]\left(\frac{x-z}{\varepsilon}\right)\mathbf{n}\right] \cdot \boldsymbol{u}_N^0 = \varepsilon^2\left(\int_{\partial\omega}\boldsymbol{\eta}_N\right) \cdot \boldsymbol{u}_N^0(z) + O(\varepsilon^3). \qquad (5.9)$$

Gathering (5.5), (5.8) and (5.9), we obtain

$$A_N = \varepsilon\left(\int_{\partial\omega}\boldsymbol{\eta}_N\right)\boldsymbol{u}_N^0(z) + o(\varepsilon). \qquad (5.10)$$

5.4 Asymptotic expansion of A_D

We recall that

$$A_D = \frac{1}{2}\nu\int_{\Omega_{z,\varepsilon}}\mathcal{D}(\boldsymbol{u}_D^\varepsilon - \boldsymbol{u}_D^0):\mathcal{D}(\boldsymbol{u}_D^\varepsilon - \boldsymbol{u}_D^0) + \nu\int_{\Omega_{z,\varepsilon}}\mathcal{D}(\boldsymbol{u}_D^\varepsilon - \boldsymbol{u}_D^0):\mathcal{D}(\boldsymbol{u}_D^0) - \frac{1}{2}\nu\int_{\omega_{z,\varepsilon}}|\mathcal{D}(\boldsymbol{u}_D^0)|^2.$$

Proceeding as in the previous section 5.3, we prove that

$$-\frac{1}{2}\nu\int_{\omega_{z,\varepsilon}}|\mathcal{D}(\boldsymbol{u}_D^0)|^2 = o(\varepsilon).$$

Moreover, using Green's formula, we have

$$\nu\int_{\Omega_{z,\varepsilon}}\mathcal{D}(\boldsymbol{u}_D^\varepsilon - \boldsymbol{u}_D^0):\mathcal{D}(\boldsymbol{u}_D^0) = 2\int_{\partial\omega_{z,\varepsilon}}\left(\sigma(\boldsymbol{u}_D^0, p_D^0)\mathbf{n}\right) \cdot \left(\boldsymbol{u}_D^\varepsilon - \boldsymbol{u}_D^0\right)$$

$$= -2\int_{\partial\omega_{z,\varepsilon}}\left(\sigma(\boldsymbol{u}_D^0, p_D^0)\mathbf{n}\right) \cdot \boldsymbol{u}_D^0 = -\nu\int_{\omega_{z,\varepsilon}}|\mathcal{D}(\boldsymbol{u}_D^0)|^2 = o(\varepsilon).$$

Now, let us study $\frac{1}{2}\nu\int_{\Omega_{z,\varepsilon}}\mathcal{D}(\boldsymbol{u}_D^\varepsilon - \boldsymbol{u}_D^0):\mathcal{D}(\boldsymbol{u}_D^\varepsilon - \boldsymbol{u}_D^0)$. We define

$$\boldsymbol{r}_D^\varepsilon(x) := \boldsymbol{u}_D^\varepsilon(x) - \boldsymbol{u}_D^0(x) - \boldsymbol{U}_D\left(\frac{x-z}{\varepsilon}\right) \quad\text{and}\quad s_D^\varepsilon(x) := p_D^\varepsilon(x) - p_D^0(x) - \frac{1}{\varepsilon}P_D\left(\frac{x-z}{\varepsilon}\right),$$

where $(\boldsymbol{U}_D, P_D) \in \mathbf{W}_0^{1,2}(\mathbb{R}^3\backslash\overline{\omega}) \times \mathrm{L}^2(\mathbb{R}^3\backslash\overline{\omega})$ solves (4.1) with $\natural = D$. Using Green's formula

$$\nu\int_{\Omega_{z,\varepsilon}}\left|\mathcal{D}(\boldsymbol{u}_D^\varepsilon - \boldsymbol{u}_D^0)\right|^2 = 2\int_{\partial\omega_{z,\varepsilon}}\left[\sigma(\boldsymbol{u}_D^\varepsilon - \boldsymbol{u}_D^0, p_D^\varepsilon - p_D^0)\mathbf{n}\right] \cdot \left(\boldsymbol{u}_D^\varepsilon - \boldsymbol{u}_D^0\right)$$

$$= -2\int_{\partial\omega_{z,\varepsilon}}\left[\sigma(\boldsymbol{r}_D^\varepsilon, s_D^\varepsilon)\mathbf{n}\right] \cdot \boldsymbol{u}_D^0 - \frac{2}{\varepsilon}\int_{\partial\omega_{z,\varepsilon}}\left[\sigma(\boldsymbol{U}_D, P_D)\left(\frac{x-z}{\varepsilon}\right)\mathbf{n}\right] \cdot \boldsymbol{u}_D^0.$$

Proceeding as in the previous section 5.3 (see inequality (5.8)), we use an inequality similar to (5.7), the asymptotic expansion of $\boldsymbol{u}_D^\varepsilon$ given by Proposition 4.1 and Lemma 5.1 to obtain

$$\left| \int_{\partial \omega_{z,\varepsilon}} [\sigma(\boldsymbol{r}_D^\varepsilon, s_D^\varepsilon)\mathbf{n}] \cdot \boldsymbol{u}_D^0 \right| \leq c\,\varepsilon^{1/2}\,\|\boldsymbol{r}_D^\varepsilon\|_{1,\Omega_{z,\varepsilon}} = O(\varepsilon^{3/2}).$$

Moreover, using the change of variables $x = z + \varepsilon y$, we have

$$\int_{\partial \omega_{z,\varepsilon}} \left[\sigma(\boldsymbol{U}_D, P_D) \left(\frac{x-z}{\varepsilon} \right) \mathbf{n} \right] \cdot \boldsymbol{u}_D^0 = \varepsilon^2 \int_{\partial\omega} [\sigma(\boldsymbol{U}_D, P_D)(y)\,\mathbf{n}] \cdot \boldsymbol{u}_D^0(z + \varepsilon y).$$

Since $\boldsymbol{u}_D^0(z + \varepsilon y) = \boldsymbol{u}_D^0(z) + \varepsilon \nabla \boldsymbol{u}_D^0(\xi_y)y$, $\xi_y \in \omega_{z,\varepsilon}$ and since $\nabla \boldsymbol{u}_D^0$ is uniformly bounded in $\omega_{z,\varepsilon}$, we have

$$\varepsilon^2 \int_{\partial\omega} [\sigma(\boldsymbol{U}_D, P_D)(y)\,\mathbf{n}] \cdot \boldsymbol{u}_D^0(z + \varepsilon y)\mathrm{ds}(y) = \varepsilon^2 \left(\int_{\partial\omega} \sigma(\boldsymbol{U}_D, P_D)(y)\,\mathbf{n} \right) \boldsymbol{u}_D^0(z) + O(\varepsilon^3).$$

Taking into account that $\mathbf{n}$ is oriented inside ω, we use the jump condition of the single layer potential (see [36, Lemma 5.6.5]) to obtain that

$$-\sigma(\boldsymbol{U}_D, P_D) + \sigma(\boldsymbol{W}_D, Q_D) = -\boldsymbol{\eta}_D,$$

where $\boldsymbol{\eta}_D \in \mathbf{H}^{-1/2}(\partial\omega)/\mathbb{R}\mathbf{n}$ is defined by (3.2) and $(\boldsymbol{W}_D, Q_D) \in \mathbf{H}^1(\omega) \times \mathrm{L}^2(\omega)$ solves

$$\begin{cases} -\nu\Delta \boldsymbol{W}_D + \nabla Q_D &= \ \mathbf{0} &\text{in } \omega \\ \operatorname{div} \boldsymbol{W}_D &= \ 0 &\text{in } \omega \\ \boldsymbol{W}_D &= \ \boldsymbol{U}_D &\text{on } \partial\omega. \end{cases}$$

Since $\operatorname{div}\sigma(\boldsymbol{W}_D, Q_D) = -\nu\Delta\boldsymbol{W}_D + \nabla Q_D = 0$ in ω, we have $\int_{\partial\omega} \sigma(\boldsymbol{W}_D, Q_D)\mathbf{n} = 0$ and

$$\int_{\partial\omega_{z,\varepsilon}} \left[\sigma(\boldsymbol{U}_D, P_D) \left(\frac{x-z}{\varepsilon} \right) \mathbf{n} \right] \cdot \boldsymbol{u}_D^0 = \varepsilon^2 \left(\int_{\partial\omega} \boldsymbol{\eta}_D \right) \boldsymbol{u}_D^0(z) + O(\varepsilon^3).$$

Hence, we obtain

$$\nu \int_{\Omega_{z,\varepsilon}} \left| \mathcal{D}(\boldsymbol{u}_D^\varepsilon - \boldsymbol{u}_D^0) \right|^2 = -2\varepsilon \left(\int_{\partial\omega} \boldsymbol{\eta}_D \right) \boldsymbol{u}_D^0(z) + o(\varepsilon).$$

Therefore

$$A_D = -\varepsilon \left(\int_{\partial\omega} \boldsymbol{\eta}_D \right) \boldsymbol{u}_D^0(z) + o(\varepsilon). \tag{5.11}$$

5.5 Conclusion of the proof: asymptotic expansion of $\mathcal{J}_{KN}$

Using (5.1), (5.10) and (5.11), we conclude the proof of Theorem 3.1:

$$\mathcal{J}_{KV}(\Omega_{z,\varepsilon}) - \mathcal{J}_{KV}(\Omega) = -\varepsilon \left(\int_{\partial\omega} \boldsymbol{\eta}_D \right) \boldsymbol{u}_D^0(z) + \varepsilon \left(\int_{\partial\omega} \boldsymbol{\eta}_N \right) \boldsymbol{u}_N^0(z) + o(\varepsilon).$$

6 Numerical Simulations

6.1 Framework of the numerical simulations

The use of the topological derivative aims to give us the number of inclusions and their rough location. Then it can provide initial shapes for an optimization method based on the boundary variation method for which we have to know the number of connected objects we have to reconstruct.

To make the numerical simulations presented here, we use a P1b-P1 finite elements discretization to solve the Stokes equations (2.3) and (2.4). The framework is the following: we work with the water kinematic viscosity $\nu = 10^{-6}$ and the exterior boundary is assumed to be the parallelepiped rectangle $[-0.5, 0.5] \times [-0.25, 0.25] \times [-0.25, 0.25]$. Except when mentioned, the measurement is assumed to be made on all the faces except on the one given by $x = -0.5$. We consider the exterior Dirichlet boundary condition

$$\boldsymbol{f} := \begin{pmatrix} -(x^2 + xy + 4\cos(3x) + 8y + 2xz + \sin(z^2)) \\ -(-\frac{1}{2}y^2 - 2xy + x^2 + 12y\sin(3x) + y\cos(3z) + z^3) \\ -(x^2 + xy + y^3 - z^2 - \frac{1}{3}\sin(3z) + \cos(xy)) \end{pmatrix}.$$

In order to have a suitable pair (*measure* $\boldsymbol{g}$, *domain* ω^*), we use a synthetic data: we fix a shape ω^* (more precisely a finite number of obstacles $\omega_1^*, \ldots, \omega_m^*$), solve the Stokes problem (2.3) in $\Omega \backslash \overline{\omega^*}$ using another finite elements method (here a P2-P1 finite elements discretization) and extract the measurement $\boldsymbol{g}$ by computing $\sigma(\boldsymbol{u}, p)\mathbf{n}$ on O.

In the practical simulations that we present, we add spherical objects. Indeed, in this very particular case, we have an explicit formula for the topological gradient (see Corollary 3.2). In order to determine the radius of these spheres, we use a thresholding method. For an iteration k, it consists in determining the minimum argument P^* of the topological gradient $\delta \mathcal{J}_{KV}$ in $\Omega \backslash \left(\bigcup_{j=1}^{k} \overline{\omega_j} \right)$ and in defining the set $\mathscr{P}$ of the points $P \in \Omega \backslash \left(\bigcup_{j=1}^{k} \overline{\omega_j} \right)$ such that

$$\delta \mathcal{J}_{KV}(P) = \delta \mathcal{J}_{KV}(P^*) + 0.25 * |\delta \mathcal{J}_{KV}(P^*)|.$$

Then we fix a minimum radius $r_{\min} := 0.01$ and we define the radius of the kth sphere by

$$r_k := \max \left(r_{\min}, \min_{P \in \mathscr{P}} (|x_P - x_{P^*}|, |y_P - y_{P^*}|, |z_P - z_{P^*}|) \right). \qquad (6.1)$$

We use the classical topological gradient algorithm (see for example [24], [31], [33], [12]) that we recall here for reader's convenience:

Algorithm
1. fix an initial shape $\omega_0 = \emptyset$, a maximum number of iterations M and set $i = 1$ and $k = 0$,
2. solve Problems (2.3) and (2.4) in $\Omega \backslash \left(\bigcup_{j=0}^{k} \overline{\omega_j} \right)$,
3. compute the topological gradient $\delta \mathcal{J}_{KV}$ using Formula (3.3), *i.e.*

$$\delta \mathcal{J}_{KV}(P) = 6\pi\nu \left(\left| \boldsymbol{u}_N^0(z) \right|^2 - \left| \boldsymbol{u}_D^0(z) \right| \right) \qquad \forall P \in \Omega \backslash \left(\bigcup_{j=0}^{k} \overline{\omega_j} \right),$$

4. seek $P_{k+1}^* := \operatorname{argmin} \left(\delta \mathcal{J}_{KV}(P), P \in \Omega \backslash \left(\bigcup_{j=0}^{k} \overline{\omega_j} \right) \right)$,

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5. if $\left\|P_{k+1}^* - P_{j_0}\right\| < r_{k+1} + r_{j_0} + 0.01$ for $j_0 \in \{1, \ldots, k\}$, where r_{j_0} is the radius of ω_{j_0} and r_{k+1} is defined by (6.1), then $r_{j_0} = 1.1 * r_{j_0}$, get back to the step 2. and $i \leftarrow i + 1$ while $i \leq M$,

6. set $\omega_{k+1} = B(P_{k+1}^*, r_{k+1})$, where r_{k+1} is defined by (6.1),

7. while $i \leq M$, get back to the step 2, $i \leftarrow i + 1$ and $k \leftarrow k + 1$.

We add to this algorithm a stop test (in addition of the maximum number of iterations). In every iteration, we compute the functional $\mathcal{J}_{KV}$. This non-negative functional has to decrease at each iteration. Thus, we stop our implementation when it is not the case, *i.e.* when $\mathcal{J}_{KV}\left(\Omega \backslash \left(\bigcup_{j=0}^{k+1} \overline{\omega_j}\right)\right) > \mathcal{J}_{KV}\left(\Omega \backslash \left(\bigcup_{j=0}^{k} \overline{\omega_j}\right)\right)$.

Notice that with this algorithm, we add only one object at each iteration. This method can be slower than the one proposed by Carpio *et al.* in [22]: by seeking local minima of $\delta \mathcal{J}_{KV}$ and not a global minimum, they can add several objects simultaneously. However, in our case, it seems to be more appropriate because otherwise objects can be added wrongly. Moreover, notice that step 5 comes to the assumption that the objects are well separated.Finally, since we assumed that the obstacles are *far away from the exterior boundary*, we have to take away the added objects on it. Then, if the minimum of the topological gradient is on the exterior boundary, we push the added inclusion inside with a depth 0.005.

Concerning the mesh, we impose a fixed number of discretization points for the exterior boundary $\partial\Omega$, that is 20 points for $[-1, 1]$ and 10 points for $[-0.5, 0.5]$ (in order to have a uniform mesh). Concerning the added objects, we fix the characteristic size of a cell, that is $\frac{r}{2}$, where r is the radius of the sphere, in order to have approximatively 100 triangles to mesh the surface of each added sphere.

6.2 First simulations

First we want to detect three spheres ω_1^*, ω_2^* and ω_3^* centered respectively in $(0.425, 0, 0.225)$, $(-0.430, -0.225, 0.1)$ and $(-0.15, 0.222, -0.222)$ (*i.e.* near from the exterior boundary) with shared radius $r^* = 0.013$. The detection is quite efficient (see Figure 3). Indeed we
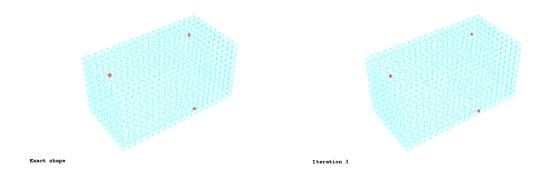

Figure 3: Detection of ω_1^*, ω_2^* and ω_3^*

detect three objects with shared radius $r = 0.01$ that we summarized in Table 1 Here, we stop the algorithm because of the functional increases as we can see in Figure 4.

In this first simulation, the objects are very far away from each other. But *what happens when the obstacles are closed from each other?* Figure 5 shows that the detection of closed objects is quite efficient too. Indeed, we want to detect three spheres ω_4^*, ω_5^* and ω_6^* centered respectively in $(0, 0.237, -0.237)$, $(0.04, 0.237, -0.237)$ and $(0.08, 0.237, -0.237)$

Table 1: Detection of ω_1^*, ω_2^* and ω_3^*

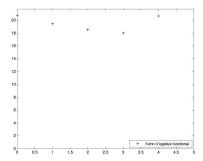| actual objects	(0.425,0,0.225)	(-0.430,-0.225,0.1)	(-0.15,0.222,-0.222)
approximate objects	(0.435,0,0.235)	(-0.430,-0.235,0.102)	(-0.149,0.235,-0.235)

Figure 4: Evolution of the functional $\mathcal{J}_{KV}$ during the detection of ω_1^*, ω_2^* and ω_3^*
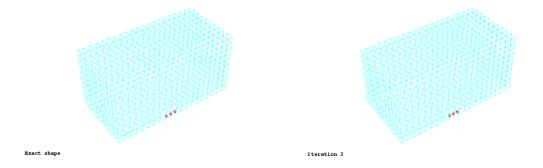

Figure 5: Detection of ω_4^*, ω_5^* and ω_6^*

with shared radius $r^* = 0.01$. We obtain three spheres with shared radius $r = 0.01$ as summarized in Table 2

Now the question we asked is: *can we detect other shapes than spheres?* Thus, we want to detect objects with different shapes: a sphere ω_7^* centered in $(0.2, -0.237, 0.237)$ with radius $r^* = 0.01$ and a cube ω_8^* given by $[0.08, 0.1] \times [0.227, 0.247] \times [-0.247, -0.227]$. We obtain Figure 6: a sphere centered in $(0.096, 0.235, -0.235)$ and one centered in $(0.202, -0.235, 0.235)$ with shared radius $r = 0.01$. Here again, we stop our algorithm because of the functional increases.

In conclusion of these first simulations, as we expected, this method permits to give us the number of objects we have to determine and their rough location. Moreover, it is efficient for different types of shapes, including shapes with corners.

6.3 Influence of the distance to the location of measurements

Now, we want to now how far we can detect an inclusion by try to recover a sphere increasingly pushed away from the boundary: precisely, the sphere with radius $r^* = 0.01$ centered in $(0, y^*, 0.1)$ with $y^* = 0.232, 0.225, 0.2, 0.1, 0$. In the three first tests, the inclusion is

Table 2: Detection of ω_4^*, ω_5^* and ω_6^*

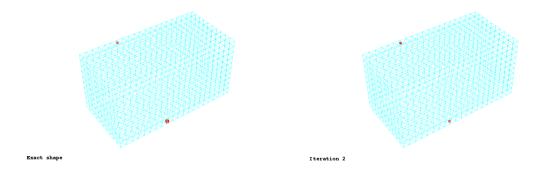| actual objects	(0,0.237,-0.237)	(0.04,0.237,-0.237)	(0.08,0.237,-0.237)
approximate objects	(0.0088,0.235,-0.235)	(0.044,0.235,-0.235)	(0.079,0.235,-0.235)

Figure 6: Detection of ω_7^* and ω_8^*

approximatively detected. However, the method fails when $y^* = 0.1$ and $y^* = 0$: the functional increases when we add the first object (which means that we add it wrongly).

In order to be more precise, let us define d the distance between the object and the exterior boundary given by $y = 0.25$ (*i.e.* $d = 0.25 - y^* + r^*$). Let us define the non-dimensional distance $\eta := \frac{d}{2r^*}$. We want to study the error between $P^* := (x^*, y^*, z^*)$ and $P(\eta) := (x(\eta), y(\eta), z(\eta))$ which are the respective coordinates of the center of the real object and its approximation. Then, we define the two following errors

$$\mathrm{Err}_1(\eta) := \frac{\|P^* - P(\eta)\|}{d} \quad \text{and} \quad \mathrm{Err}_2(\eta) := \frac{\|P^* - P(\eta)\|}{2r^*},$$

where $\|\cdot\|$ represents the Euclidean norm. Notice that Err_1 is the classical relative error. In Figure 7, we represent Err_1 and Err_2 versus the non-dimensional distance η. The result
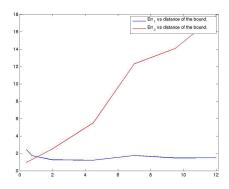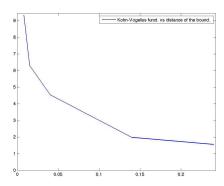

Figure 7: The Euclidean norm of the error between the exact and the approximate coordinates versus the non-dimensional distance η

obtained is similar to the one obtained by Ben Abda *et al.* in [15, Fig. 4.2.(b)] and shows that the error dramatically increases when the object is far away of the exterior boundary.
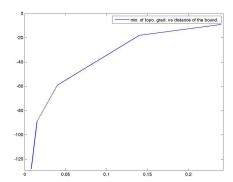
Figure 8: The initial Kohn-Vogelius functional and minimum of the topological gradient versus the distance of the obstacle from the exterior boundary d

Finally, Figure 8 shows that the more the obstacle is far away from the exterior boundary, the more the initial value of the functional $\mathcal{J}_{KV}$ is small and the initial value of the minimum of the topological gradient $\delta\mathcal{J}_{KV}(P_1^*)$ is big: they seem to tend to 0. This means that the sensitivity of the functional and its topological gradient decreases when the distance d increases. This explains the encountered difficulties.

Hence, these simulations show that the detection of obstacles far away from the exterior boundary is difficult. This phenomenon can be explained by the regularizing behavior of the Stokes equations. We emphasized this difficulty of detection pointing out that the functional $\mathcal{J}_{KV}$ and its topological gradient are less sensitive to the addition of obstacles when they are far away from the exterior boundary. This lack of sensitivity with respect to the object far away from the measurements is maybe due to the static (here stationary) physical context. Indeed, the sensitivity is better in a dynamical or waves context (see [22]).

Remark 6.1. *An intuitive solution may be to cumulate several experiments (with different choices of $\boldsymbol{f}$) in order to improve the detection. We then sum the topological gradients for each experiment. However, this manipulation does not lead to efficient results and even poor results.*

6.4 Influence of the size and the shape of the objects

Now, we want to study the influence of the volume of the objects on the efficiency of the detection. Thus, we want to detect a sphere with radius $r^* = 0.015, 0.02, 0.03, 0.04, 0.05$ and which is, for each test, taken away from the boundary of $d = 0.003$. The fourth first tests are efficient and permit to detect the object decreasing the Kohn-Vogelius functional $\mathcal{J}_{KV}$. Moreover, once the object detected, the radius of the sphere increases. We recapitulate the radius r obtained in each case in Table 3. However, for $r = 0.05$, the object is badly

Table 3: Radius of the approximate obstacles versus real radius of the object.

r^*	0.015	0.02	0.03	0.04
r	0.012	0.025	0.035	0.052

positioned as we can see in Figure 9 and the functional increases immediately: the real sphere is centered in $(0, 0.197, 0.1)$ and we obtain a sphere centered in $(0.395, 0.235, 0.235)$.
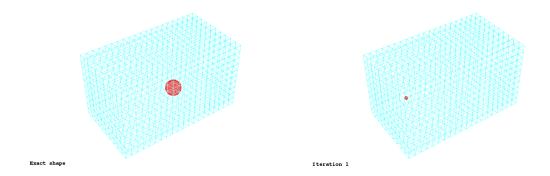
Figure 9: Detection of a big obstacle

Then, we want to know the influence of the size of the objects when several objects have to be to detected. We make the simulations detecting two spheres with radius r_1 and r_2 when $r_1 = 2r_2$, $r_1 = 3r_2$, $r_1 = 4r_2$ and $r_1 = 5r_2$. The simulations shows that the first object detected is the biggest sphere (even if we interchange the position of the objects). However, when an object is too big, we have the same conclusion than above: we add an obstacle badly positioned and the functional increases.

Finally, we wonder if the shape of the objects has an influence on the detection. We have seen in Section 6.2 that cubes can be detected. Are cubes more difficult or easier to detect than spheres? In the simulation represented in Figure 6, the first obstacle detected is the cube. Note that this is unchanged by interchanging the position of the objects. Then, we want to detect a *big* sphere and a *small* cube: the sphere ω_9^* centered in $(0.05, -0.222, 0.222)$ with radius $r^* = 0.025$ and the cube ω_{10}^* given by $[0.08, 0.1] \times [0.227, 0.247] \times [-0.247, -0.227]$. Notice that they are equally far away of the boundary. The results are presented in Figure 10. The first object detected is again the
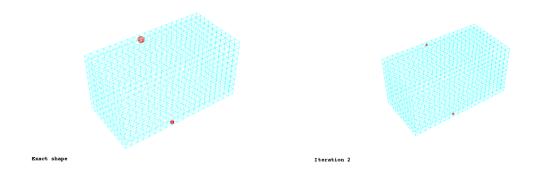

Figure 10: Detection of ω_9^* and ω_{10}^*

cube, even if it is the smaller. Finally, when we have to detect a *big* cube and a *small* sphere, the simulations shows that only the cube is well-detected.

In conclusion, the volume of the objects seems to have a major importance in the detection of them. It seems that we can only detect small obstacles. This is reasonable since the theoretical construction started with that assumption. Moreover, the shape of the obstacles seems to be important too: domain with geometrical singularities seems to be more easily imaged also with the topological gradient approach.

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6.5 Influence of the size of the domain where measurements are made

At this point, the measurements are made on all the faces of the parallelepiped rectangle except on the face $x = -0.5$. Now, we want to study the efficiency of the detection if the domain of measurement $O \subset \partial\Omega$ is reduced. To do this, we want to detect the objets ω_1^*, ω_2^* and ω_3^* presented in Section 6.2 (see Figure 3). One can note that here we have to refine the mesh to obtain efficient results: we use 32 discretization points for $[-1, 1]$ and 16 for $[-0.5, 0.5]$ and the characterized size of a cell of a sphere of radius r is now $\frac{r}{2.3}$.

We have seen that when $\partial\Omega \backslash \overline{O}$ is the face $x = -0.5$, the three objects are detected. However, if $\partial\Omega \backslash \overline{O}$ is the faces $x = -0.5$ and $y = -0.25$ (*i.e.* the faces near ω_2^*), then we only detect two objects: one centered in $(-0.165, 0.235, -0.235)$ and one centered in $(0.431, 0.05, 0.235)$. Hence, we do not detect ω_2^* (see Figure 11 on the left). Moreover, if $\partial\Omega \backslash \overline{O}$ is the faces $x = 0.5$ and $z = 0.25$ (*i.e.* the faces near ω_1^*), then we detect only two objects again: ω_1^* is not located in this case. Finally, if we make the measurement only on the face $y = 0.25$, then we only detect an object centered in $(-0.159, 0.235, -0.217)$, *i.e* the object ω_3^* which is near this face (see Figure 11 on the right).
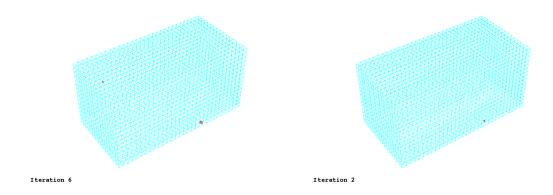

Iteration 6   Iteration 2

Figure 11: Detection of ω_1^*, ω_2^* and ω_3^* for different measurement domains

In conclusion, as we expected, it seems that we can only detect objects which are near from a measurement part of the exterior boundary.

7 Conclusion

Using a Kohn-Vogelius approach, we have detected the number of potential objects immersed in a fluid and their rough location. We have computed the topological gradient of the Kohn-Vogelius functional considered using an asymptotic expansion of the solution of the Stokes equations in the whole domain when we add small obstacles inside. We have made some numerical attempts adding spherical objects because of an explicit formula of the topological derivative holds in this very particular case. The simulations have shown that only small obstacles close to the part of the boundary where we make the measurements can be detected. Once these restrictions are satisfied, the detection is quite efficient, even for objects with corners. Our intuition to explain these numerical results is the penetration depth of the approach is poor in the stationary case. In the elastic case, this depth increases in a dynamical (or waves) setting.

A Some results on the Stokes problem with mixed boundary conditions

We recall classical results about the Stokes problem with mixed boundary conditions: a theorem of existence and uniqueness of the solution and a local regularity result.

We note C a generic positive constant, only depending on the geometry of the domain and on the dimension, which may change from line to line.

First, let us introduce some notations: for Ω an open set of $\mathbb{R}^N$ an open subset $\omega \subset\subset \Omega$ and a part O of the exterior boundary $\partial\Omega$, we define

$$\boldsymbol{V}_O(\Omega\backslash\overline{\omega}) := \left\{ \boldsymbol{u} \in \mathbf{H}^1(\Omega\backslash\overline{\omega}); \ \operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega\backslash\overline{\omega}, \ \boldsymbol{u} = \boldsymbol{0} \text{ on } \partial\omega \cup (\partial\Omega\backslash\overline{O}) \right\}.$$

Moreover, we denote respectively by $\langle \cdot, \cdot \rangle_{\Omega\backslash\overline{\omega}}$ and $\langle \cdot, \cdot \rangle_{\partial\Omega}$ (or $\langle \cdot, \cdot \rangle_{\partial\omega}$) the duality product between $\left[\mathbf{H}^1(\Omega\backslash\overline{\omega}) \right]'$ and $\mathbf{H}^1(\Omega\backslash\overline{\omega})$ and between $\mathbf{H}^{-1/2}(\partial\Omega)$ and $\mathbf{H}^{1/2}(\partial\Omega)$.

Theorem A.1 (Existence and uniqueness of the solution)**.** *Let Ω be a bounded Lipschitz open set of $\mathbb{R}^N$ ($N \in \mathbb{N}^*$) and let $\omega \subset\subset \Omega$ be a Lipschitz open subset of Ω such that $\Omega\backslash\overline{\omega}$ is connected. Let $O \subset \partial\Omega$ be a part of the exterior boundary and $\nu > 0$. Let $\boldsymbol{f} \in \left[\mathbf{H}^1(\Omega\backslash\overline{\omega}) \right]'$, $\boldsymbol{h_{ext}} \in \mathbf{H}^{1/2}(\partial\Omega\backslash\overline{O})$, $\boldsymbol{h_O} \in \mathbf{H}^{-1/2}(O)$ and $\boldsymbol{h_{int}} \in \mathbf{H}^{1/2}(\partial\omega)$. Then, the problem*

$$\begin{cases} -\nu\Delta\boldsymbol{u} + \nabla p &=& \boldsymbol{f} & \text{in } \Omega\backslash\overline{\omega} \\ \operatorname{div}\boldsymbol{u} &=& 0 & \text{in } \Omega\backslash\overline{\omega} \\ \sigma(\boldsymbol{u}, p)\mathbf{n} &=& \boldsymbol{h_O} & \text{on } O \\ \boldsymbol{u} &=& \boldsymbol{h_{ext}} & \text{on } \partial\Omega\backslash\overline{O} \\ \boldsymbol{u} &=& \boldsymbol{h_{int}} & \text{on } \partial\omega \end{cases} \tag{A.1}$$

admits a unique solution $(\boldsymbol{u}, p) \in \mathbf{H}^1(\Omega\backslash\overline{\omega}) \times \mathrm{L}^2(\Omega\backslash\overline{\omega})$ and the following estimate holds:

$$\|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega\backslash\overline{\omega})} + \|p\|_{\mathrm{L}^2(\Omega\backslash\overline{\omega})}$$
$$\leq C \left(\|\boldsymbol{f}\|_{[\mathbf{H}^1(\Omega\backslash\overline{\omega})]'} + \|\boldsymbol{h_{ext}}\|_{\mathbf{H}^{1/2}(\partial\Omega\backslash\overline{O})} + \|\boldsymbol{h_O}\|_{\mathbf{H}^{-1/2}(O)} + \|\boldsymbol{h_{int}}\|_{\mathbf{H}^{1/2}(\partial\omega)} \right).$$

Proof. Step 1: existence and uniqueness. According to [11, Lemma 3.3], let us consider $\boldsymbol{H} \in \mathbf{H}^1(\Omega\backslash\overline{\omega})$ such that $\operatorname{div}\boldsymbol{H} = 0$ in $\Omega\backslash\overline{\omega}$, $\boldsymbol{H} = \boldsymbol{h_{int}}$ on $\partial\omega$, $\boldsymbol{H} = \boldsymbol{h_{ext}}$ on $\partial\Omega\backslash\overline{O}$ such that $\displaystyle\int_{\partial\Omega\cup\partial\omega} \boldsymbol{H} \cdot \mathbf{n} = 0$ and satisfying

$$\|\boldsymbol{H}\|_{\mathbf{H}^1(\Omega\backslash\overline{\omega})} \leq C \left(\|\boldsymbol{h_{int}}\|_{\mathbf{H}^{1/2}(\partial\omega)} + \|\boldsymbol{h_{ext}}\|_{\mathbf{H}^{1/2}(\partial\Omega\backslash\overline{O})} \right). \tag{A.2}$$

Then the couple $(\boldsymbol{U} := \boldsymbol{u} - \boldsymbol{H}, p) \in \mathbf{H}^1(\Omega\backslash\overline{\omega}) \times \mathrm{L}^2(\Omega\backslash\overline{\omega})$ satisfies

$$\begin{cases} -\nu\Delta\boldsymbol{U} + \nabla p &=& \boldsymbol{f} + \nu\Delta\boldsymbol{H} & \text{in } \Omega\backslash\overline{\omega} \\ \operatorname{div}\boldsymbol{U} &=& 0 & \text{in } \Omega\backslash\overline{\omega} \\ \sigma(\boldsymbol{U}, P)\mathbf{n} &=& \boldsymbol{h_O} + \nu(\nabla\boldsymbol{H} + {}^t\nabla\boldsymbol{H})\mathbf{n} & \text{on } O \\ \boldsymbol{U} &=& \boldsymbol{0} & \text{on } \partial\Omega\backslash\overline{O} \\ \boldsymbol{U} &=& \boldsymbol{0} & \text{on } \partial\omega. \end{cases}$$

According to Lax-Milgram's theorem, there exists a unique $\boldsymbol{U} \in \boldsymbol{V}_O(\Omega\backslash\overline{\omega})$ such that for all $\boldsymbol{v} \in \boldsymbol{V}_O(\Omega\backslash\overline{\omega})$

$$\nu \int_{\Omega\backslash\overline{\omega}} \nabla\boldsymbol{U} : \nabla\boldsymbol{v} = \langle\boldsymbol{f},\boldsymbol{v}\rangle_{\Omega\backslash\overline{\omega}} - \nu\int_{\Omega\backslash\overline{\omega}} \nabla\boldsymbol{H} : \nabla\boldsymbol{v} - \langle\boldsymbol{h_O} + \nu(\nabla\boldsymbol{H} + {}^t\nabla\boldsymbol{H})\mathbf{n},\boldsymbol{v}\rangle_O \quad \text{(A.3)}$$

and we have, using (A.2),

$$\|\boldsymbol{U}\|_{\mathbf{H}^1(\Omega\backslash\overline{\omega})} \le C\left(\|\boldsymbol{f}\|_{[\mathbf{H}^1(\Omega\backslash\overline{\omega})]'} + \|\boldsymbol{h_{int}}\|_{\mathbf{H}^{1/2}(\partial\omega)} + \|\boldsymbol{h_{ext}}\|_{\mathbf{H}^{1/2}(\partial\Omega\backslash\overline{O})} + \|\boldsymbol{h_O}\|_{\mathbf{H}^{-1/2}(O)}\right). \tag{A.4}$$

In particular (A.3) is true for all $\boldsymbol{v} \in \boldsymbol{V}_O(\Omega\backslash\overline{\omega}) \cap \mathbf{H}_0^1(\Omega\backslash\overline{\omega})$. Then using De Rham's theorem (see for example [10, Lemma 2.7]), there exists $p \in \mathrm{L}^2(\Omega\backslash\overline{\omega})$, up to an additive constant, such that for all $\boldsymbol{v} \in \mathbf{H}_0^1(\Omega\backslash\overline{\omega})$

$$\nu \int_{\Omega\backslash\overline{\omega}} \nabla\boldsymbol{U} : \nabla\boldsymbol{v} - \int_{\Omega\backslash\overline{\omega}} p\,\mathrm{div}\,\boldsymbol{v} = \left\langle\boldsymbol{f}|_{\mathbf{H}_0^1(\Omega\backslash\overline{\omega})},\boldsymbol{v}\right\rangle_{\mathbf{H}^{-1}(\Omega\backslash\overline{\omega}),\mathbf{H}_0^1(\Omega\backslash\overline{\omega})} - \nu\int_{\Omega\backslash\overline{\omega}} \nabla\boldsymbol{H} : \nabla\boldsymbol{v}. \tag{A.5}$$

According to [11, Lemma 3.3] or [27, Theorem 3.2], we define $\boldsymbol{\varphi}_N \in \mathbf{H}^1(\Omega\backslash\overline{\omega})$ such that $\mathrm{div}\,\boldsymbol{\varphi}_N = 1$ in $\Omega\backslash\overline{\omega}$, $\boldsymbol{\varphi}_N = \mathbf{0}$ on $\partial\Omega\backslash\overline{O}$ and $\boldsymbol{\varphi}_N = \mathbf{0}$ on $\partial\omega$ with $\int_O \boldsymbol{\varphi}_N \cdot \mathbf{n} \ne 0$. Let $\boldsymbol{v} \in \mathbf{H}^1(\Omega\backslash\overline{\omega})$ such that $\boldsymbol{v} = \mathbf{0}$ on $\partial\Omega\backslash\overline{O}$, $\boldsymbol{v} = \mathbf{0}$ on $\partial\omega$ and define

$$c_b(\boldsymbol{v}) = \frac{1}{\int_{\partial(\Omega\backslash\omega)} \boldsymbol{\varphi}_N \cdot \mathbf{n}} \int_{\partial(\Omega\backslash\overline{\omega})} \boldsymbol{v} \cdot \mathbf{n}.$$

Using again [11, Lemma 3.3] or [27, Theorem 3.2], we define $\boldsymbol{v}_2 \in \boldsymbol{V}_O(\Omega\backslash\overline{\omega})$ in such a way that $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2 + c_b(\boldsymbol{v})\boldsymbol{\varphi}_N$, where $\boldsymbol{v}_1 \in \mathbf{H}_0^1(\Omega\backslash\overline{\omega})$ satisfies $\mathrm{div}\,\boldsymbol{v}_1 = \mathrm{div}\,(\boldsymbol{v} - c_b(\boldsymbol{v})\boldsymbol{\varphi}_N)$. Using (A.3) and (A.5), we then obtain

$$\int_{\Omega\backslash\overline{\omega}} \nu\nabla\boldsymbol{U} : \nabla\boldsymbol{v} - \int_{\Omega\backslash\overline{\omega}} p\,\mathrm{div}\,\boldsymbol{v} = \langle\boldsymbol{f},\boldsymbol{v}\rangle_{\Omega\backslash\overline{\omega}} - \nu\int_{\Omega\backslash\overline{\omega}} \nabla\boldsymbol{H} : \nabla\boldsymbol{v}$$

$$- \langle\boldsymbol{h_O} - \nu(\nabla\boldsymbol{H} + {}^t\nabla\boldsymbol{H})\mathbf{n},\boldsymbol{v}\rangle_O + \int_{\Omega\backslash\overline{\omega}} \nu\nabla\boldsymbol{U} : \nabla(c_b(\boldsymbol{v})\boldsymbol{\varphi}_N) - \int_{\Omega\backslash\overline{\omega}} p\,\mathrm{div}\,(c_b(\boldsymbol{v})\boldsymbol{\varphi}_N)$$

$$- \langle\boldsymbol{f},c_b(\boldsymbol{v})\boldsymbol{\varphi}_N\rangle_{\Omega\backslash\overline{\omega}} + \nu\int_{\Omega\backslash\overline{\omega}} \nabla\boldsymbol{H} : \nabla(c_b(\boldsymbol{v})\boldsymbol{\varphi}_N) + \langle\boldsymbol{h_O} - \nu(\nabla\boldsymbol{H} + {}^t\nabla\boldsymbol{H})\mathbf{n},c_b(\boldsymbol{v})\boldsymbol{\varphi}_N\rangle_O.$$

Therefore, choosing the additive constant for p such that

$$\int_{\Omega\backslash\overline{\omega}} p = \nu \int_{\Omega\backslash\overline{\omega}} \nabla\boldsymbol{U} : \nabla\boldsymbol{\varphi}_N$$

$$- \langle\boldsymbol{f},c_b(\boldsymbol{v})\boldsymbol{\varphi}_N\rangle_{\Omega\backslash\overline{\omega}} + \nu\int_{\Omega\backslash\overline{\omega}} \nabla\boldsymbol{H} : \nabla(c_b(\boldsymbol{v})\boldsymbol{\varphi}_N) + \langle\boldsymbol{h_O} - \nu(\nabla\boldsymbol{H} + {}^t\nabla\boldsymbol{H})\mathbf{n},c_b(\boldsymbol{v})\boldsymbol{\varphi}_N\rangle_O,$$

we prove that there exists a unique pair $(\boldsymbol{U},p) \in \boldsymbol{V}_O(\Omega\backslash\overline{\omega}) \times \mathrm{L}^2(\Omega\backslash\overline{\omega})$ such that for all $\boldsymbol{v} \in \mathbf{H}^1(\Omega\backslash\overline{\omega})$ with $\boldsymbol{v} = \mathbf{0}$ on $\partial\Omega\backslash\overline{O}$ and $\boldsymbol{v} = \mathbf{0}$ on $\partial\omega$,

$$\int_{\Omega\backslash\overline{\omega}} \nu\nabla\boldsymbol{U} : \nabla\boldsymbol{v} - \int_{\Omega\backslash\overline{\omega}} p\,\mathrm{div}\,\boldsymbol{v} = \langle\boldsymbol{f},\boldsymbol{v}\rangle_{\Omega\backslash\overline{\omega}} - \nu\int_{\Omega\backslash\overline{\omega}} \nabla\boldsymbol{H} : \nabla\boldsymbol{v} - \langle\boldsymbol{h_O} - \nu(\nabla\boldsymbol{H} + {}^t\nabla\boldsymbol{H})\mathbf{n},\boldsymbol{v}\rangle_{\partial\Omega}. \tag{A.6}$$

Step 2: estimate. Let $\boldsymbol{v} := \widetilde{\boldsymbol{v}} + c(p)\boldsymbol{\varphi}_N$, where

$$c(p) := \frac{1}{|\Omega \backslash \overline{\omega}|} \int_{\Omega \backslash \overline{\omega}} p$$

and $\widetilde{\boldsymbol{v}} \in \mathbf{H}_0^1(\Omega \backslash \overline{\omega})$ is such that $\operatorname{div} \widetilde{\boldsymbol{v}} = p - c(p)$ and $\|\widetilde{\boldsymbol{v}}\|_{\mathbf{H}_0^1(\Omega \backslash \overline{\omega})} \leq C \|p\|_{\mathrm{L}^2(\Omega \backslash \overline{\omega})}$ (see [11, Lemma 3.3]). Using $\boldsymbol{v}$ in (A.6), and according to (A.4), we obtain

$$\|\boldsymbol{U}\|_{\mathbf{H}^1(\Omega \backslash \overline{\omega})} + \|p\|_{\mathrm{L}^2(\Omega \backslash \overline{\omega})}$$
$$\leq C \left(\|\boldsymbol{f}\|_{[\mathbf{H}^1(\Omega \backslash \overline{\omega})]'} + \|\boldsymbol{h_{int}}\|_{\mathbf{H}^{1/2}(\partial \omega)} + \|\boldsymbol{h_{ext}}\|_{\mathbf{H}^{1/2}(\partial \Omega \backslash \overline{O})} + \|\boldsymbol{h_O}\|_{\mathbf{H}^{-1/2}(O)} \right)$$

and hence

$$\|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega \backslash \overline{\omega})} + \|p\|_{\mathrm{L}^2(\Omega \backslash \overline{\omega})}$$
$$\leq C \left(\|\boldsymbol{f}\|_{[\mathbf{H}^1(\Omega \backslash \overline{\omega})]'} + \|\boldsymbol{h_{int}}\|_{\mathbf{H}^{1/2}(\partial \omega)} + \|\boldsymbol{h_{ext}}\|_{\mathbf{H}^{1/2}(\partial \Omega \backslash \overline{O})} + \|\boldsymbol{h_O}\|_{\mathbf{H}^{-1/2}(O)} \right).$$

$\square$

B Some results on the exterior Stokes problem

First, we recall the definition of the weighted Sobolev spaces. We introduce the weight function $\rho(x) := (2 + |x|^2)^{1/2}$ and the following Sobolev spaces (for more details, see [3]):

Definition B.1. *Let $1 < p < \infty$. For each real number α and each open set $\mathcal{O} \subset \mathbb{R}^d$, we set*

$$\mathrm{L}_\alpha^p(\mathcal{O}) := \left\{ u \in \mathcal{D}'(\mathcal{O}), \rho^\alpha u \in \mathrm{L}^p(\mathcal{O}) \right\},$$
$$\mathrm{W}_\alpha^{1,p}(\mathcal{O}) := \begin{cases} \left\{ u \in \mathcal{D}'(\mathcal{O}), u \in \mathrm{L}_{\alpha-1}^p(\mathcal{O}), \nabla u \in \mathbf{L}_\alpha^p(\mathcal{O}) \right\} & \text{if } \frac{d}{p} + \alpha \neq 1, \\ \left\{ u \in \mathcal{D}'(\mathcal{O}), (\ln(\rho))^{-1} u \in \mathrm{L}_{\alpha-1}^p(\mathcal{O}), \nabla u \in \mathbf{L}_\alpha^p(\mathcal{O}) \right\} & \text{if } \frac{d}{p} + \alpha = 1. \end{cases}$$

Consider now the space $\overset{\circ}{\mathrm{W}}{}_\alpha^{1,p}(\mathcal{O}) := \overline{\mathcal{D}(\mathcal{O})}^{\|\cdot\|_{\mathrm{W}_\alpha^{1,p}(\mathcal{O})}}$. It is standard to check that

$$\overset{\circ}{\mathrm{W}}{}_\alpha^{1,p}(\mathcal{O}) = \left\{ v \in \mathrm{W}_\alpha^{1,p}(\mathcal{O}), v_{|\partial \mathcal{O}} = 0 \right\}.$$

The dual space of $\overset{\circ}{\mathrm{W}}{}_\alpha^{1,p}(\mathcal{O})$ is denoted by $\mathrm{W}_{-\alpha}^{-1,p'}(\mathcal{O})$, where p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$ (it is a subspace of $\mathcal{D}'(\mathcal{O})$).

Notice that these spaces are reflexive Banach spaces with respect to the norms:

$$\|u\|_{\mathrm{L}_\alpha^p(\mathcal{O})} := \|\rho^\alpha u\|_{\mathrm{L}^p(\mathcal{O})},$$
$$\|u\|_{\mathrm{W}_\alpha^{1,p}(\mathcal{O})} := \begin{cases} \left(\|u\|_{\mathrm{L}_{\alpha-1}^p(\mathcal{O})}^p + \|\nabla u\|_{\mathbf{L}_\alpha^p(\mathcal{O})}^p \right)^{1/p} & \text{if } \frac{d}{p} + \alpha \neq 1, \\ \left(\left\| \frac{u}{\ln(\rho)} \right\|_{\mathrm{L}_{\alpha-1}^p(\mathcal{O})}^p + \|\nabla u\|_{\mathbf{L}_\alpha^p(\mathcal{O})}^p \right)^{1/p} & \text{if } \frac{d}{p} + \alpha = 1. \end{cases}$$

We also recall the following theorem extracted from [29, Theorem 3.4 and Remark 3.4] which guarantees the existence and uniqueness of the solution of an exterior Stokes problem with Dirichlet boundary conditions:

Theorem B.2 (Girault *et al.*, [29])**.** *Let $\mathcal{O}$ be a Lipschitz-continuous exterior domain. For each $\boldsymbol{f} \in \mathbf{W}_0^{-1,2}(\mathcal{O})$, $g \in \mathrm{L}^2(\mathcal{O})$ and for each $\boldsymbol{\varphi} \in \mathbf{H}^{1/2}(\partial\mathcal{O})$, the problem*

$$
\begin{cases}
-\nu\Delta\boldsymbol{u} + \nabla\pi &= \boldsymbol{f} &\text{in } \mathcal{O} \\
\operatorname{div}\boldsymbol{u} &= g &\text{in } \mathcal{O} \\
\boldsymbol{u} &= \boldsymbol{\varphi} &\text{on } \partial\mathcal{O}
\end{cases}
$$

has a unique solution $(\boldsymbol{u}, \pi) \in \mathbf{W}_0^{1,2}(\mathcal{O}) \times \mathrm{L}^2(\mathcal{O})$. Moreover, there exists a positive constant $C = C(d, \mathcal{O}, \nu)$ such that

$$
\|\boldsymbol{u}\|_{\mathbf{W}_0^{1,2}(\mathcal{O})} + \|\pi\|_{\mathrm{L}^2(\mathcal{O})} \leq C \left(\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,2}(\mathcal{O})} + \|g\|_{\mathrm{L}^2(\mathcal{O})} + \|\boldsymbol{\varphi}\|_{\mathbf{H}^{1/2}(\mathcal{O})} \right).
$$

Finally we recall that $E(y) = \dfrac{1}{8\pi\nu\,\|y\|}(\mathrm{I} + \mathrm{e}_r{}^t\mathrm{e}_r)$ with $\mathrm{e}_r := \dfrac{y}{\|y\|}$ is the Green function associated to the Stokes system (see (3.1)). For an integrable vector $\boldsymbol{\varphi}$ on $\partial\omega$, we introduce the following integral operator (called simple layer hydrodynamic potential operator) for all $y \notin \partial\omega$:

$$
S\boldsymbol{\varphi}(y) := \int_{\partial\omega} E(y-x)\boldsymbol{\varphi}(x)\mathrm{d}s(x).
$$

Now we claim a theorem extracted from [26, §XI.B.5, Theorem 1, page 697–698]:

Theorem B.3. *Let ω an open bounded open set of $\mathbb{R}^N$ ($N = 2, 3$). Let $\boldsymbol{g} \in \mathbf{H}^{1/2}(\partial\omega)$ such that $\displaystyle\int_{\partial\omega} \boldsymbol{g}\cdot\mathbf{n} = 0$. We consider $(\boldsymbol{w}, s) \in \mathbf{W}_0^{1,2}(\mathbb{R}^3\backslash\overline{\omega}) \times \mathrm{L}^2(\mathbb{R}^3\backslash\overline{\omega})$ the solution of the Stokes exterior problem*

$$
\begin{cases}
-\nu\Delta\boldsymbol{w} + \nabla s &= \boldsymbol{0} &\text{in } \mathbb{R}^3\backslash\overline{\omega} \\
\operatorname{div}\boldsymbol{w} &= 0 &\text{in } \mathbb{R}^3\backslash\overline{\omega} \\
\boldsymbol{w} &= \boldsymbol{g} &\text{on } \partial\omega.
\end{cases}
$$

Hence the velocity $\boldsymbol{w}$ can be written thanks to the representation formula with hydrodynamical potential

$$
\boldsymbol{w}(y) = S\boldsymbol{\eta}(y) = \int_{\partial\omega} E(y-x)\boldsymbol{\eta}(x)\mathrm{d}s(x), \qquad y \in \mathbb{R}^3\backslash\overline{\omega},
$$

where $\boldsymbol{\eta} \in \mathbf{H}^{-1/2}(\partial\omega)^3/\mathcal{R}$ is the unique solution of

$$
S\boldsymbol{\eta}(y) = \boldsymbol{g}(y) \qquad \forall y \in \partial\omega.
$$

Here $\mathcal{R}$ is the equivalence relation $t \sim t'$ if $t - t' = \lambda\mathbf{n}$, $\lambda \in \mathbb{R}$.

Moreover S is an isomorphism from $\mathbf{H}^{-1/2}(\partial\omega)/\mathcal{R}$ into $\left\{ \boldsymbol{g} \in \mathbf{H}^{1/2}(\partial\omega), \displaystyle\int_{\partial\omega} \boldsymbol{g}\cdot\mathbf{n} = 0 \right\}$.

C A result concerning the space of traces

Here we recall a result used in the paper concerning the boundary values of functions, in particular when domains depend on a parameter (see [40, Chapter 4]):

Theorem C.1 ([40] Section 4.1.3. page 214)**.** *Let Ω and ω be two bounded simply connected domains of $\mathbb{R}^N$ ($N \geq 2$) of class $C^{0,1}$. Let $p \in (1, +\infty)$, $\varepsilon \in (0, 1/2)$ and $\omega_\varepsilon := \varepsilon\omega$. Let us assume that $\overline{\omega_\varepsilon} \subset \Omega$ and that there exists a constant $c > 0$ depending only of N, p, ω and Ω such that $\mathrm{d}(\omega_\varepsilon, \partial\Omega) > c\varepsilon$. Then*

$$
\langle\cdot\rangle_{p,\partial\omega_\varepsilon} \sim a(\varepsilon)\,\|\cdot\|_{\mathrm{L}^p(\partial\omega_\varepsilon)} + [\cdot]_{p,\partial\omega_\varepsilon}
$$

where

$$\langle f \rangle_{p, \partial \omega_\varepsilon} \quad := \quad \inf \left\{ \|u\|_{\mathrm{W}^{1,p}(\Omega \setminus \overline{\omega_\varepsilon})}, \, u \in \mathrm{W}^{1,p}(\Omega \setminus \overline{\omega_\varepsilon}), u_{|\partial \omega_\varepsilon} = f \right\},$$

$$a(\varepsilon) \quad := \quad \begin{cases} \varepsilon^{\frac{1-N}{p}} \min(1, \varepsilon^{\frac{N}{p}-1}), & \text{for } p < N \\ \varepsilon^{\frac{1-N}{p}} \min(1, |\log \varepsilon|^{\frac{1-p}{p}}), & \text{for } p = N \\ \varepsilon^{\frac{1-N}{p}}, & \text{for } p > N, \end{cases}$$

and

$$[f]_{1, \partial \omega_\varepsilon} \quad := \quad |\partial \omega_\varepsilon|^{-1} \iint_{\partial \omega_\varepsilon \times \partial \omega_\varepsilon} |f(x) - f(y)| \, \mathrm{ds}(x) \mathrm{ds}(y)$$

$$[f]_{p, \partial \omega_\varepsilon} \quad := \quad \left(\iint_{\partial \omega_\varepsilon \times \partial \omega_\varepsilon} \frac{|f(x) - f(y)|^p}{|x-y|^{N+p-2}} \mathrm{ds}(x) \mathrm{ds}(y) \right)^{1/p} \quad \text{for } p \in (1, +\infty).$$

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