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ASYMPTOTIC BEHAVIOR OF LOCAL PARTICLES NUMBERS IN BRANCHING RANDOM WALK

Ekaterina Vl. Bulinskaya^{1,2}

Abstract

Critical catalytic branching random walk on an integer lattice \mathbb{Z}^d is investigated for all $d \in \mathbb{N}$. The branching may occur at the origin only and the start point is arbitrary. The asymptotic behavior, as time grows to infinity, is determined for the mean local particles numbers. The same problem is solved for the probability of particles presence at a fixed lattice point. Moreover, the Yaglom type limit theorem is established for the local number of particles. Our analysis involves construction of an auxiliary Bellman-Harris branching process with six types of particles. The proofs employ the asymptotic properties of the (improper) c.d.f. of hitting times with taboo. The latter notion was recently introduced by the author for a non-branching random walk on \mathbb{Z}^d .

Keywords and phrases: critical branching random walk, Bellman-Harris process with particles of six types, Yaglom type conditional limit theorems, Kolmogorov's equations, random walk on integer lattice, hitting time with taboo.

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1 Introduction

Catalytic branching random walk (CBRW) on d -dimensional integer lattice is a model of particles population evolution. We recall its main features. Each particle independently of others may perform random walk on \mathbb{Z}^d and produce offsprings at the source of branching located w.l.g. at the origin. *Symmetric branching random walk* (SBRW) on \mathbb{Z}^d studied earlier, e.g., in [1], [2] and [21] is a particular case of CBRW on \mathbb{Z}^d (see [23]).

The model under consideration was proposed in [20] for $d = 1$ and studied for other $d \in \mathbb{N}$ in [4], [5], [13] and [23]. The analysis of CBRW in [13] and [23] has shown that similarly to many kinds of branching processes (see [12]) CBRW on \mathbb{Z}^d is classified as supercritical, critical or subcritical. According to [23], the *exponential growth* (as time tends to infinity) of total number of particles in population and local numbers of particles as well is characteristic for the supercritical CBRW on \mathbb{Z}^d . The term local refers to the (number of) particles located at a lattice point.

Quite different situation occurs for critical CBRW which is the main object of study in this paper. For example, for $d = 1$ or $d = 2$ the particles population degenerates with probability 1 but survives with strictly positive probability for $d \geq 3$ (see [5], [7], [13] and [20]). Moreover, the total number of particles conditioned on non-degeneracy has non-trivial discrete limit distribution, different for $d < 3$ and $d \geq 3$ (see [4], [5] and [20]). Thus, in the model of critical

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CBRW on \mathbb{Z}^d the asymptotic behavior (in time) of the total number of particles on the lattice depends on dimension d essentially and does not grow exponentially. As for the local particles numbers in the model of critical CBRW on \mathbb{Z}^d , earlier in [4]–[6], [11], [13], [14] and [20] there were only established the asymptotic properties of the number of particles located at the *source of branching*. In particular, it turns out that for all $d \in \mathbb{N}$ the probability of the presence of particles at the source of branching asymptotically vanishes. Notably, its asymptotic behavior as well as limit laws for properly normalized number of particles at the source of branching, conditioned on the presence of particles at the origin, have different forms for $d = 1, 2, 3, 4$ and $d \geq 5$. Among the arising limit distributions one can find exponential and discrete ones along with a mixture of such laws.

In the model of critical CBRW on \mathbb{Z}^d the behavior of number of particles located at an *arbitrary* point of the lattice remained unknown. The present work completes the picture. We study the asymptotic behavior in time of mean local particles numbers and that of probability of particles presence at a fixed point $y \neq \mathbf{0}$ where $\mathbf{0} = (0, 0, \dots, 0) \in \mathbb{Z}^d$. All the more, we obtain a conditional limit theorem for the properly normalized number of particles at such point y . It should be emphasized that in contrast to [4], [11], [13], [14] and [20] we admit the start of CBRW at an *arbitrary* point $x \in \mathbb{Z}^d$ and not only at the source of branching. Asymptotic properties of the number of particles at $\mathbf{0}$ for CBRW with an arbitrary start point were investigated in [6].

The structure of the rest of the paper is the following. In section 2 we describe the model in detail and formulate three main results. Theorem 1 is proved in Section 3. Section 4 is devoted to construction of the auxiliary Bellman-Harris branching process. Thereupon we establish Theorems 2 and 3 in Section 5.

2 Main results

Now we dwell on the definition of a critical CBRW on \mathbb{Z}^d . At the initial time $t = 0$ there is a single particle on the lattice located at a point $x \in \mathbb{Z}^d$. If $x \neq \mathbf{0}$, the particle performs a continuous time random walk until the time of the first hitting the origin. The random walk outside the origin is symmetric, homogeneous, irreducible (i.e. a particle passes from an arbitrary $u \in \mathbb{Z}^d$ to any $v \in \mathbb{Z}^d$ with positive probability within a finite time) and has a finite variance of jumps. Accordingly, we assume this random walk be specified by an infinitesimal matrix $A = (a(u, v))_{u, v \in \mathbb{Z}^d}$ such that

$$\begin{aligned} a(u, v) &= a(v, u), \quad a(u, v) = a(\mathbf{0}, v - u) := a(v - u), \quad u, v \in \mathbb{Z}^d, \\ \sum_{v \in \mathbb{Z}^d} a(v) &= 0 \text{ where } a(\mathbf{0}) < 0 \text{ and } a(v) \geq 0 \text{ if } v \neq \mathbf{0}, \quad \sum_{v \in \mathbb{Z}^d} \|v\|^2 a(v) < \infty. \end{aligned}$$

If $x = \mathbf{0}$ or the particle has just hit the origin it spends there an exponentially distributed time (with parameter 1). Afterwards, it either dies with probability $\alpha \in (0, 1)$ producing before the death a random number of offsprings ξ or leaves the source of branching with probability $1 - \alpha$. In the latter case the intensity of transition from the origin to a point $v \neq \mathbf{0}$ is given by

$$\bar{a}(\mathbf{0}, v) = -(1 - \alpha) \frac{a(v)}{a(\mathbf{0})}.$$

At the origin the branching is determined by a probability generating function

$$f(s) := \mathbf{E}s^\xi = \sum_{k=0}^{\infty} f_k s^k, \quad s \in [0, 1].$$

In [13] CBRW on \mathbb{Z}^d is called *critical* if the following relations hold

$$\alpha f'(1) + (1 - \alpha)(1 - h_d) = 1 \quad \text{and} \quad \sigma^2 := \alpha f''(1) < \infty. \quad (1)$$

Here h_d is the probability of the event that a particle leaving the origin will never return there. By the recurrence of a random walk on \mathbb{Z} and \mathbb{Z}^2 one has $h_1 = h_2 = 0$. It is well known that $h_d \in (0, 1)$ for $d \geq 3$.

Newborn particles are located at the origin at the birth moment. They evolve according to the scheme described above independently of each other as well as of the parent particles. The number of particles located at a point $y \in \mathbb{Z}^d$ at time $t \geq 0$ is denoted by $\mu(t; y)$.

The goal of the paper is three-fold. Firstly, we find the asymptotic behavior (as $t \rightarrow \infty$) of the mean number of particles $m(t; x, y) := \mathbf{E}_x \mu(t; y)$ located at a point $y \in \mathbb{Z}^d$, $y \neq \mathbf{0}$, at time $t \geq 0$ (everywhere the index x means that our CBRW starts at $x \in \mathbb{Z}^d$). Secondly, we retrieve the asymptotic behavior of the probability $q(t; x, y) := \mathbf{P}_x(\mu(t; y) > 0)$ of the presence of particles at the point y at time t . Thirdly, we establish a limit theorem for properly normalized local numbers $\mu(t; y)$ conditioned on $\mu(t; y) > 0$ as $t \rightarrow \infty$.

To formulate the main results of the paper we introduce some more notation. Let $p(t; x, y)$ be the transition probability from x to y within time $t \geq 0$ for a random walk on \mathbb{Z}^d generated by matrix A . Set

$$G_\lambda(x, y) := \int_0^\infty e^{-\lambda t} p(t; x, y) dt, \quad \lambda > 0, \quad x, y \in \mathbb{Z}^d.$$

Note that the Green's function $G_0(x, y) := \lim_{\lambda \rightarrow 0^+} G_\lambda(x, y)$ is well-defined and takes finite values for $d \geq 3$ by virtue of the transience of our random walk on \mathbb{Z}^d , $d \geq 3$. One can check (see [13]) that $h_d = (aG_0(\mathbf{0}, \mathbf{0}))^{-1}$, $d \in \mathbb{N}$, where $a := -a(\mathbf{0})$.

As shown in [22], Theorem 2.1.1 (see also [13]), for any fixed $x, y \in \mathbb{Z}^d$, one has

$$p(t; x, y) \sim \frac{\gamma_d}{t^{d/2}}, \quad p(t; \mathbf{0}, \mathbf{0}) - p(t; x, y) \sim \frac{\tilde{\gamma}_d(y - x)}{t^{1+d/2}}, \quad t \rightarrow \infty, \quad (2)$$

where $\gamma_d := ((2\pi)^d |\det \phi''_{\theta\theta}(\mathbf{0})|)^{-1/2}$, $\phi(\theta) := \sum_{z \in \mathbb{Z}^d} a(z, \mathbf{0}) \cos(z, \theta)$, $\theta \in [-\pi, \pi]^d$,

$$\phi''_{\theta\theta}(\mathbf{0}) = \left(\frac{\partial^2 \phi(\theta)}{\partial \theta_i \partial \theta_j} \Big|_{\theta=\mathbf{0}} \right)_{i,j \in \{1, \dots, d\}}, \quad \tilde{\gamma}_d(z) := \frac{1}{2(2\pi)^d} \int_{\mathbb{R}^d} (v, z)^2 e^{(\phi''_{\theta\theta}(\mathbf{0})v, v)/2} dv, \quad z \in \mathbb{Z}^d,$$

and (\cdot, \cdot) stands for the scalar product in \mathbb{R}^d . In particular, it follows that the value

$$m_d := 1 - (1 - \alpha)a^{-1} + 2(1 - \alpha)a^{-1}G_0^{-2}(\mathbf{0}, \mathbf{0}) \int_0^\infty t p(t; \mathbf{0}, \mathbf{0}) dt$$

is finite for $d \geq 5$. Set also $q(s, t; x, y) := 1 - \mathbf{E}_x s^{\mu(t; y)}$, $s \in [0, 1]$, $t \geq 0$, $x, y \in \mathbb{Z}^d$. For $d = 2$ we use the function

$$J(s; y) := \alpha \int_0^\infty (f(1 - q(s, u; \mathbf{0}, y)) - 1 + q(s, u; \mathbf{0}, y)) du, \quad s \in [0, 1], \quad y \in \mathbb{Z}^d.$$

The main results are contained in the following three theorems. For the sake of completeness their statements include the case $y = \mathbf{0}$ studied earlier in [4]–[6], [11], [13], [14] and [20].

Theorem 1 Let $x, y \in \mathbb{Z}^d$. The following relations are valid being different for $y \neq \mathbf{0}$ and $y = \mathbf{0}$, namely, as $t \rightarrow \infty$,

$$\begin{aligned} m(t; x, y) &\sim \frac{\gamma_1}{\sqrt{t}}, & m(t; x, \mathbf{0}) &\sim \frac{\gamma_1 a}{(1-\alpha)\sqrt{t}}, & d = 1, \\ m(t; x, y) &\sim \frac{\gamma_2}{t}, & m(t; x, \mathbf{0}) &\sim \frac{\gamma_2 a}{(1-\alpha)t}, & d = 2, \\ m(t; x, y) &\sim \frac{G_0(x, \mathbf{0})G_0(\mathbf{0}, y)}{2\pi\gamma_3\sqrt{t}}, & m(t; x, \mathbf{0}) &\sim \frac{aG_0(x, \mathbf{0})G_0(\mathbf{0}, \mathbf{0})}{2\pi\gamma_3(1-\alpha)\sqrt{t}}, & d = 3, \\ m(t; x, y) &\sim \frac{G_0(x, \mathbf{0})G_0(\mathbf{0}, y)}{\gamma_4 \ln t}, & m(t; x, \mathbf{0}) &\sim \frac{aG_0(x, \mathbf{0})G_0(\mathbf{0}, \mathbf{0})}{\gamma_4(1-\alpha) \ln t}, & d = 4, \\ m(t; x, y) &\rightarrow \frac{(1-\alpha)G_0(x, \mathbf{0})G_0(\mathbf{0}, y)}{aG_0^2(\mathbf{0}, \mathbf{0})m_d}, & m(t; x, \mathbf{0}) &\rightarrow \frac{G_0(x, \mathbf{0})}{G_0(\mathbf{0}, \mathbf{0})m_d}, & d \geq 5. \end{aligned}$$

Theorem 2 For $x, y \in \mathbb{Z}^d$ and $t \rightarrow \infty$ the following formulae hold true

$$\begin{aligned} q(t; x, y) &\sim \frac{2(1-\alpha)}{\sigma^2\gamma_1 a\sqrt{t} \ln t}, & d = 1, \\ q(t; x, y) &\sim \frac{\gamma_2}{t} \left(1 - \frac{a}{1-\alpha} J(0; y) \right), & y \neq \mathbf{0}, & d = 2, \\ q(t; x, \mathbf{0}) &\sim \frac{\gamma_2 a}{(1-\alpha)t} (1 - J(0; \mathbf{0})), & d = 2, \\ q(t; x, y) &\sim \frac{4\pi\gamma_3(1-\alpha)G_0(x, \mathbf{0})}{\sigma^2 a G_0^3(\mathbf{0}, \mathbf{0})\sqrt{t} \ln t}, & d = 3, \\ q(t; x, y) &\sim \frac{3\gamma_4(1-\alpha)G_0(x, \mathbf{0}) \ln t}{\sigma^2 a G_0^3(\mathbf{0}, \mathbf{0}) t}, & d = 4, \\ q(t; x, y) &\sim \frac{2m_d G_0(x, \mathbf{0})}{\sigma^2 G_0(\mathbf{0}, \mathbf{0})t}, & d \geq 5, \end{aligned}$$

where for $d = 2$ and $s \in [0, 1]$ the strict inequalities $J(s; y) < (1-s)(1-\alpha)/a$, $y \neq \mathbf{0}$, and $J(s; \mathbf{0}) < 1-s$ are valid.

Theorem 3 Given $x, y \in \mathbb{Z}^d$, $\lambda \in [0, \infty)$ and $s \in [0, 1]$, one has, as $t \rightarrow \infty$,

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathbb{E}_x \left(\exp \left\{ -\frac{\lambda \mu(t; y)}{\mathbb{E}_x(\mu(t; y) | \mu(t; y) > 0)} \right\} \middle| \mu(t; y) > 0 \right) &= \frac{1}{\lambda + 1}, & d = 1, \quad d = 3 \quad \text{or} \quad d \geq 5, \\ \lim_{t \rightarrow \infty} \mathbb{E}_x (s^{\mu(t; y)} | \mu(t; y) > 0) &= \frac{(1-\alpha)s - a(J(0; y) - J(s; y))}{1-\alpha - aJ(0; y)}, & y \neq \mathbf{0}, & d = 2, \\ \lim_{t \rightarrow \infty} \mathbb{E}_x (s^{\mu(t; \mathbf{0})} | \mu(t; \mathbf{0}) > 0) &= \frac{s - (J(0; \mathbf{0}) - J(s; \mathbf{0}))}{1 - J(0; \mathbf{0})}, & d = 2, \\ \lim_{t \rightarrow \infty} \mathbb{E}_x \left(\exp \left\{ -\frac{\lambda \mu(t; y)}{\mathbb{E}_x(\mu(t; y) | \mu(t; y) > 0)} \right\} \middle| \mu(t; y) > 0 \right) &= \frac{1}{3} + \frac{2}{3} \cdot \frac{2}{2 + 3\lambda}, & d = 4. \end{aligned}$$

Observe that the normalizing factor $\mathbb{E}_x(\mu(t; y) | \mu(t; y) > 0)$ arising in Theorem 3 is exactly $m(t; x, y)/q(t; x, y)$ and the asymptotic behavior of the functions $m(t; x, y)$ and $q(t; x, y)$ is given by Theorems 1 and 2, respectively.

To establish Theorem 1 it is useful to invoke the forward and backward Kolmogorov's differential equations (considered in appropriate Banach spaces) for mean numbers of particles at different points of the lattice and also the resulting integral equations (see [22]). As for Theorems 2 and 3, note that for proving results in [4], [11], [13], [14] and [20] concerning the number of particles at the origin the method of introduction of an auxiliary Bellman-Harris branching process with particles of *two types* was efficient. However, for proving Theorems 2 and 3 we have to involve a Bellman-Harris branching process with particles of *six types*. To apply the latter method we attend to a new notion of the *hitting time with taboo* in the framework of a (non-branching) random walk on \mathbb{Z}^d . More precisely, we use our recent results (see [8]) on the asymptotic behavior of the tail of the (improper) cumulative distribution function of this time. Due to that one can employ the theorems by V.A.Vatutin for Bellman-Harris branching processes with particles of several types (see, e.g., [16]–[19]). Afterwards we have to deal with sophisticated analytic estimates of the solutions of the parametric integral equations (see, e.g., [11], [13], [14] and [20]).

3 Proof of Theorem 1

Let us recall some useful results employed within this section. According to [10], Ch.3, Sec.2, the transition probabilities $p(t; x, y)$, $t \geq 0$, $x, y \in \mathbb{Z}^d$, of the random walk generated by matrix A satisfy the *backward* Kolmogorov's equations

$$\frac{dp(t; x, y)}{dt} = (Ap(t; \cdot, y))(x), \quad p(0; x, y) = \delta_y(x). \quad (3)$$

Here $(Ap(t; \cdot, y))(x) = \sum_{z \in \mathbb{Z}^d} a(x, z)p(t; z, y)$ and $\delta_y(\cdot)$ is a column vector in the space $l_2(\mathbb{Z}^d)$ with zero components except for the component 1 indexed by y . In a similar way, the *backward* Kolmogorov's equations for $m(t; x, y)$, $t \geq 0$, $x, y \in \mathbb{Z}^d$, (see, e.g., Theorem 2.1 in [23]) take the form

$$\frac{dm(t; x, y)}{dt} = (\bar{A}m(t; \cdot, y))(x) + \bar{\beta}_c (\Delta_{\mathbf{0}}m(t; \cdot, y))(x), \quad m(0; x, y) = \delta_y(x), \quad (4)$$

where $\bar{A} = (\bar{a}(u, v))_{u, v \in \mathbb{Z}^d} := A + (a^{-1}(1 - \alpha) - 1) \Delta_{\mathbf{0}}A$, $\Delta_{\mathbf{0}} := \delta_{\mathbf{0}}\delta_{\mathbf{0}}^T$ (T stands for transposition) and $\bar{\beta}_c := (1 - \alpha)a^{-1}G_0^{-1}(\mathbf{0}, \mathbf{0})$. Here we follow the notation of [23].

Lemma 1 *For each $y \in \mathbb{Z}^d$, the function $m(t; y, y)$ is non-increasing in t .*

PROOF. The monotonicity of $m(\cdot; y, y)$ for SBRW on \mathbb{Z}^d was established in Lemma 3.3.5 of [22]. The key step of its proof was to use self-adjointness of the operator $H := A + \beta_c \Delta_{\mathbf{0}}$ where $\beta_c := G_0^{-1}(\mathbf{0}, \mathbf{0})$. For CBRW the analog of H is the non self-adjoint operator $\bar{H} := \bar{A} + \bar{\beta}_c \Delta_{\mathbf{0}}$. However, Lemma 3.1 in [23] permits to pass to (self-adjoint) *symmetrization* of \bar{H} and then apply Lemma 3.3.5 in [22]. Further argument is similar to the proof of Theorem in [24]. \square

Equation (4) was obtained by differentiating at $s = 1$ the following *backward* Kolmogorov's equation for the generating function $F(s, t; x, y) := \mathbf{E}_x s^{\mu(t; y)}$, $s \in [0, 1]$, $t \geq 0$, $x, y \in \mathbb{Z}^d$, (see [23])

$$\frac{\partial F(s, t; x, y)}{\partial t} = (\bar{A}F(s, t; \cdot, y))(x) + (\Delta_{\mathbf{0}}\bar{f}(F(s, t; \cdot, y)))(x), \quad F(s, 0; x, y) = s^{\delta_y(x)}. \quad (5)$$

Here $\bar{f}(s) := \alpha(f(s) - s)$, $s \in [0, 1]$, is an infinitesimal generating function of the number of offsprings of a parent particle. We will employ (5) in Section 5.

In Lemma 2 we derive a counterpart of the forward Kolmogorov's equation for the function $F(s, t; x, y)$, $s \in [0, 1]$, $t \geq 0$, $x, y \in \mathbb{Z}^d$, and, as a consequence, the forward Kolmogorov's equation for $m(t; x, y)$. Recall that \bar{A}^* denotes an adjoint operator for \bar{A} and $(\bar{A}^* m(t; x, \cdot))(y) = \sum_{z \in \mathbb{Z}^d} m(t; x, z) \bar{a}(z, y)$.

Lemma 2 *For $s \in [0, 1]$, $t \geq 0$, $x, y \in \mathbb{Z}^d$, the following relation holds true*

$$\begin{aligned} \frac{\partial F(s, t; x, y)}{\partial t} &= (s - 1) \sum_{z \in \mathbb{Z}^d, z \neq y} \bar{a}(z, y) \mathbf{E}_x s^{\mu(t; y)} \mu(t; z) + (s - 1) \bar{a}(y, y) \mathbf{E}_x s^{\mu(t; y) - 1} \mu(t; y) \\ &+ \delta_{\mathbf{0}}(y) \bar{f}(s) \mathbf{E}_x s^{\mu(t; y) - 1} \mu(t; y), \quad F(s, 0; x, y) = s^{\delta_x(y)}. \end{aligned} \quad (6)$$

Moreover, one has

$$\frac{d m(t; x, y)}{d t} = (\bar{A}^* m(t; x, \cdot))(y) + \bar{\beta}_c (\Delta_{\mathbf{0}} m(t; x, \cdot))(y), \quad m(0; x, y) = \delta_x(y). \quad (7)$$

PROOF. As usual in derivation of forward Kolmogorov's equations, we consider all possible evolutions of the particles population within the time interval $[t, t + h]$ and let $h \rightarrow 0+$. To justify arising passages to the limit we involve the Lebesgue theorem on dominated convergence and useful estimates for transition probabilities (see proof of Lemma 3 in [10], Ch.3, Sec.2). We also benefit from finiteness of the mean total number of particles $M(t; x) := \mathbf{E}_x (\sum_{z \in \mathbb{Z}^d} \mu(t; z))$ for each $x \in \mathbb{Z}^d$ and $t \geq 0$. The latter observation is true since the last function belonging to $l_\infty(\mathbb{Z}^d)$ is a solution of the linear differential equation in (4) with the initial condition $M(0; x) = 1$ for all x (instead of $\delta_y(x)$ in (4)), see [23].

Equation (7) is an immediate consequence of (6) due to formula $m(t; x, y) = \partial_s F(s, t; x, y)|_{s=1}$. We also take into account that $\bar{f}'(1) = \bar{\beta}_c$ in view of (1). \square

Consider equations (4) and (7) as inhomogeneous ones for differential equation (3) in Banach space $l_\infty(\mathbb{Z}^d)$. Applying the variation of constant formula (see [9], Ch.2, Sec.1) we infer that

$$\begin{aligned} m(t; x, y) &= p(t; x, y) + \left(1 - \frac{a}{1 - \alpha}\right) \int_0^t p(t - u; x, \mathbf{0}) m'(u; \mathbf{0}, y) du \\ &+ \frac{a \bar{\beta}_c}{1 - \alpha} \int_0^t p(t - u; x, \mathbf{0}) m(u; \mathbf{0}, y) du, \end{aligned} \quad (8)$$

$$\begin{aligned} m(t; x, y) &= p(t; x, y) + \left(\frac{1 - \alpha}{a} - 1\right) \int_0^t m(t - u; x, \mathbf{0}) p'(u; \mathbf{0}, y) du \\ &+ \bar{\beta}_c \int_0^t m(t - u; x, \mathbf{0}) p(u; \mathbf{0}, y) du. \end{aligned} \quad (9)$$

An analogous result for SBRW on \mathbb{Z}^d can be found in [22], Theorem 1.4.1. Now we can give PROOF OF THEOREM 1. To find the asymptotic behavior of $m(t; x, y)$, $t \rightarrow \infty$, $x, y \in \mathbb{Z}^d$, $y \neq \mathbf{0}$, we estimate each of the summands in the right-hand sides of (8) and (9) when $x \neq \mathbf{0}$ and $x = \mathbf{0}$, respectively, as $t \rightarrow \infty$. Namely, we will show that, for $d = 1$ and $d = 2$, the main contribution to the asymptotic behavior of the right-hand side of (8), as well as of (9), is due

to the first summand. However, for $d \geq 3$, the asymptotic behavior of the right-hand sides of (8) and (9) is determined only by the third summands. It is worth mentioning that, for $d = 1$ and $d = 2$, the third summands in (8) and (9) vanish in view of equality $\bar{\beta}_c = 0$.

Let $x = \mathbf{0}$. The asymptotic behavior of the first summand in the right-hand side of (9) is given by (2). The estimate of the second summand could be obtained on account of Lemma 6 in [13] and, in particular, relation (20). However, to avoid verifying the bounded variation of the functions $p(t; x, y)$ and $m(t; x, y)$ in variable t we choose another approach consisting in direct estimation of the second summand. Recall that representation (2.1.15) in [22] entails the inequalities $p'(t; \mathbf{0}, \mathbf{0}) \leq 0$, $p'(t; \mathbf{0}, \mathbf{0}) \leq p'(t; \mathbf{0}, y)$, $p''(t; \mathbf{0}, \mathbf{0}) \geq 0$ and $p''(t; \mathbf{0}, \mathbf{0}) - p''(t; \mathbf{0}, y) \geq 0$, $t \geq 0$. Then by virtue of (2) as well as the classical results on differentiating the asymptotic formulae (see, e.g., [3], Ch.7, Sec.3), for $d \in \mathbb{N}$, one has

$$p'(t; \mathbf{0}, \mathbf{0}) \sim -\frac{d \gamma_d}{2 t^{d/2+1}}, \quad p'(t; \mathbf{0}, \mathbf{0}) - p'(t; \mathbf{0}, y) \sim -\frac{(d+2) \tilde{\gamma}_d(y)}{2 t^{d/2+2}}, \quad t \rightarrow \infty.$$

Whence taking into account Lemma 5.1.2 in [22] ("lemma on convolutions") and the already proved assertion of Theorem 1 for $x = \mathbf{0}$ and $y = \mathbf{0}$ we deduce that, as $t \rightarrow \infty$,

$$\begin{aligned} \int_0^t m(t-u; \mathbf{0}, \mathbf{0}) p'(u; \mathbf{0}, y) du &= \int_0^t m(t-u; \mathbf{0}, \mathbf{0}) (p'(u; \mathbf{0}, y) - p'(u; \mathbf{0}, \mathbf{0})) du \\ + \int_0^t m(t-u; \mathbf{0}, \mathbf{0}) p'(u; \mathbf{0}, \mathbf{0}) du &= m(t; \mathbf{0}, \mathbf{0}) - m(t; \mathbf{0}, \mathbf{0}) + o(m(t; \mathbf{0}, \mathbf{0})) = o(m(t; \mathbf{0}, \mathbf{0})). \end{aligned} \quad (10)$$

Combining relations (2), (9) and (10) we establish Theorem 1 for $d = 1$ and $d = 2$ when $x = \mathbf{0}$. The statement of Theorem 1 for $d \geq 3$ and $x = \mathbf{0}$ follows from formulae (2), (9) and (10) by Lemma 5.1.2 in [22] and in view of Theorem 1 for the known case $x = y = \mathbf{0}$.

Let $x \neq \mathbf{0}$. Similarly to the case $x = \mathbf{0}$ we see that

$$\int_0^t p(t-u; x, \mathbf{0}) m'(u; \mathbf{0}, y) du = \int_0^t m(t-u; \mathbf{0}, y) p'(u; x, \mathbf{0}) du = o(m(t; \mathbf{0}, y)), \quad t \rightarrow \infty. \quad (11)$$

Thus, the combination of (2), (8) and (11) proves Theorem 1 for $d = 1$ or $d = 2$ and $x \neq \mathbf{0}$. For $d \geq 3$ and $x \neq \mathbf{0}$ we estimate the third summand in (8) with the help of Lemma 5.1.2 in [22], relation (2) and the assertion of Theorem 1 for $d \geq 3$ and $x = \mathbf{0}$ established above. \square

4 Auxiliary Bellman-Harris branching process

Let us briefly describe a Bellman-Harris branching process with particles of six types. It is initiated by a single particle of type $i = 1, \dots, 6$. The parent particle has a random life-length with a cumulative distribution function (c.d.f.) $G_i(t)$, $t \geq 0$. When dying the particle produces offsprings according to a generating function $f_i(\vec{s})$, $\vec{s} = (s_1, \dots, s_6) \in [0, 1]^6$. The new particles of type $j = 1, \dots, 6$ evolve independently with the life-length distribution $G_j(t)$ and an offspring generating function $f_j(\vec{s})$. Let $M := (\partial_{s_j} f_i |_{\vec{s}=(1, \dots, 1)})_{i,j=1, \dots, 6}$ be the mean matrix of the process. The Bellman-Harris branching process is called *critical indecomposable* if the Perron root of M (i.e. eigenvalue having the maximal modulus) equals 1 and for some integer n all elements of M^n are positive (see, e.g., [12], Ch.4, Sec.6 and 7). Denote the number of particles of type j

existing at time t by $Z_j(t)$, $t \geq 0$, $j = 1, \dots, 6$. Set $F_i(t; \vec{s}) = \mathbb{E}_i \left(\prod_{j=1}^6 s_j^{Z_j(t)} \right)$, $i = 1, \dots, 6$, $t \geq 0$, $\vec{s} \in [0, 1]^6$, where the index i means that the parent particle is of type i . In other words, $F_i(t; \vec{s})$ is a generating function of the numbers of particles of all types existing at time t given that the process is initiated by a single particle of type i .

Before demonstrating how an auxiliary Bellman-Harris process can be constructed in the framework of CBRW on \mathbb{Z}^d we have to introduce some notation. Recall that in [8] a new notion of a hitting time with taboo was proposed for a (non-branching) random walk on \mathbb{Z}^d generated by matrix A . Namely, let $\tau_{y,z}^-$, $y, z \in \mathbb{Z}^d$, $y \neq z$, be the time spent by the particle (performing the random walk) after leaving the starting point until the first hitting y if particle's trajectory does not pass z . Otherwise (if particle's trajectory passes point z before the first hitting y), $\tau_{y,z}^- = \infty$. Denote by $H_{x,y,z}^-(t)$, $t \geq 0$, the improper c.d.f. of $\tau_{y,z}^-$ given that the starting point of the random walk is $x \in \mathbb{Z}^d$.

Return to CBRW on \mathbb{Z}^d . In this section we assume that CBRW may start at the origin or at a fixed point $y \neq \mathbf{0}$. We divide the particles population existing at time $t \geq 0$ into seven groups. The particles located at time t at the origin (respectively, at y) form the first (respectively, second) group having cardinality $\mu(t; \mathbf{0})$ (respectively, $\mu(t; y)$). Next consider at time t a family of particles labeled by a collection (u, v, w) of lattice points, its cardinality being $\mu_{u,v,w}(t)$. It consists of the particles which have left u at least once within time interval $[0, t]$, upon the last leaving u have yet reached neither v nor w but eventually will hit v before possible hitting w . Our third group corresponds to $(u, v, w) = (\mathbf{0}, y, \mathbf{0})$, the fourth to $(y, \mathbf{0}, y)$, the fifth to $(\mathbf{0}, \mathbf{0}, y)$ and the sixth to $(y, y, \mathbf{0})$. The seventh group comprises the rest of particles not included into the above six groups. Note that the last group consists of the particles having infinite life-length since after time t they will not hit the origin any more. So, after time t these particles will not produce any offsprings and have no influence on the numbers of particles in other six groups.

Now we can introduce an auxiliary Bellman-Harris process and use it for the study of CBRW on \mathbb{Z}^d . Consider a six-dimensional Bellman-Harris process having the following c.d.f. G_i and generating function f_i , $i = 1, \dots, 6$,

$$\begin{aligned} G_1(t) &= 1 - e^{-t}, & f_1(\vec{s}) &= \alpha f(s_1) + (1 - \alpha)H_{\mathbf{0},y,\mathbf{0}}^-(0)s_2 + (1 - \alpha)(H_{\mathbf{0},y,\mathbf{0}}^-(\infty) - H_{\mathbf{0},y,\mathbf{0}}^-(0))s_3 \\ & & &+ (1 - \alpha)H_{\mathbf{0},\mathbf{0},y}^-(\infty)s_5 + (1 - \alpha)(1 - H_{\mathbf{0},y,\mathbf{0}}^-(\infty) - H_{\mathbf{0},\mathbf{0},y}^-(\infty)), \\ G_2(t) &= 1 - e^{-at}, & f_2(\vec{s}) &= H_{y,\mathbf{0},y}^-(0)s_1 + (H_{y,\mathbf{0},y}^-(\infty) - H_{y,\mathbf{0},y}^-(0))s_4 \\ & & &+ H_{y,y,\mathbf{0}}^-(\infty)s_6 + (1 - H_{y,\mathbf{0},y}^-(\infty) - H_{y,y,\mathbf{0}}^-(\infty)), \\ G_3(t) &= \frac{H_{\mathbf{0},y,\mathbf{0}}^-(t) - H_{\mathbf{0},y,\mathbf{0}}^-(0)}{H_{\mathbf{0},y,\mathbf{0}}^-(\infty) - H_{\mathbf{0},y,\mathbf{0}}^-(0)}, & f_3(\vec{s}) &= s_2, & G_4(t) &= \frac{H_{y,\mathbf{0},y}^-(t) - H_{y,\mathbf{0},y}^-(0)}{H_{y,\mathbf{0},y}^-(\infty) - H_{y,\mathbf{0},y}^-(0)}, & f_4(\vec{s}) &= s_1, \\ G_5(t) &= \frac{H_{\mathbf{0},\mathbf{0},y}^-(t)}{H_{\mathbf{0},\mathbf{0},y}^-(\infty)}, & f_5(\vec{s}) &= s_1, & G_6(t) &= \frac{H_{y,y,\mathbf{0}}^-(t)}{H_{y,y,\mathbf{0}}^-(\infty)}, & f_6(\vec{s}) &= s_2, \end{aligned}$$

where $H_{x,y,z}^-(\infty) := \lim_{t \rightarrow \infty} H_{x,y,z}^-(t)$. The symmetry and homogeneity of the random walk generated by matrix A imply identities $H_{\mathbf{0},y,\mathbf{0}}^- \equiv H_{y,\mathbf{0},y}^-$ and $H_{\mathbf{0},\mathbf{0},y}^- \equiv H_{y,y,\mathbf{0}}^-$, whence $G_3 \equiv G_4$ and $G_5 \equiv G_6$. It is not difficult to see that for the branching process constructed in this way one has $(\mu(t; \mathbf{0}), \mu(t; y), \mu_{\mathbf{0},y,\mathbf{0}}(t), \mu_{y,\mathbf{0},y}(t), \mu_{\mathbf{0},\mathbf{0},y}(t), \mu_{y,y,\mathbf{0}}(t)) \stackrel{Law}{=} (Z_1(t), \dots, Z_6(t))$, $t \geq 0$.

Observe that the introduced Bellman-Harris branching process with particles of six types is critical indecomposable. Indeed, it is an easy computation task to check that all entries of M^6

are positive. Furthermore, if $H_{\mathbf{0},y,\mathbf{0}}^-(0) \neq 0$ (that is $a(\mathbf{0}, y) > 0$) then already all entries of M^4 are positive. Hence, the constructed process is indecomposable. To verify its criticality note that in view of Theorem 3 in [8] one can rewrite the first relation in (1) as follows

$$\alpha f'(1) = 1 - (1 - \alpha) \left(H_{\mathbf{0},\mathbf{0},y}^-(\infty) + \frac{(H_{\mathbf{0},y,\mathbf{0}}^-(\infty))^2}{1 - H_{\mathbf{0},\mathbf{0},y}^-(\infty)} \right). \quad (12)$$

Then by inspecting the explicit expression for the characteristic polynomial of the mean matrix M we deduce that it has the form

$$\det(M - \kappa I) = \kappa^2(\kappa - 1)R(\kappa)$$

where I is a unit matrix, $\kappa \in \mathbb{C}$ and

$$\begin{aligned} R(\kappa) := & \kappa^3 + \kappa^2(1 - \alpha f'(1)) + \kappa(1 - \alpha f'(1) - (2 - \alpha)H_{\mathbf{0},\mathbf{0},y}^-(\infty) - (1 - \alpha)(H_{\mathbf{0},y,\mathbf{0}}^-(0))^2) \\ & + (1 - \alpha)(H_{\mathbf{0},y,\mathbf{0}}^-(\infty) - H_{\mathbf{0},y,\mathbf{0}}^-(0))^2 - (1 - \alpha)(H_{\mathbf{0},\mathbf{0},y}^-(\infty))^2. \end{aligned}$$

The polynomial $R(\kappa)$ has no real roots greater than 1 because $R(1) > 0$ and $R'(\kappa) > 0$ for $\kappa \geq 1$. In fact, due to identity (12) we obtain the representation with strictly positive summands

$$\begin{aligned} R(1) = & (1 - H_{\mathbf{0},\mathbf{0},y}^-(\infty))((1 - \alpha)H_{\mathbf{0},\mathbf{0},y}^-(\infty) + 1) + (1 - \alpha)H_{\mathbf{0},y,\mathbf{0}}^-(\infty)(H_{\mathbf{0},y,\mathbf{0}}^-(\infty) - H_{\mathbf{0},y,\mathbf{0}}^-(0)) \\ & + \frac{(1 - \alpha)H_{\mathbf{0},y,\mathbf{0}}^-(\infty)(H_{\mathbf{0},y,\mathbf{0}}^-(\infty) - H_{\mathbf{0},y,\mathbf{0}}^-(0)(1 - H_{\mathbf{0},\mathbf{0},y}^-(\infty)))}{1 - H_{\mathbf{0},\mathbf{0},y}^-(\infty)} + \frac{(1 - \alpha)(H_{\mathbf{0},y,\mathbf{0}}^-(\infty))^2}{1 - H_{\mathbf{0},\mathbf{0},y}^-(\infty)}. \end{aligned}$$

Moreover, if $\kappa \geq 1$ then

$$\begin{aligned} R'(\kappa) = & 3\kappa^2 + 2\kappa(1 - \alpha f'(1)) + 1 - \alpha f'(1) - (2 - \alpha)H_{\mathbf{0},\mathbf{0},y}^-(\infty) - (1 - \alpha)(H_{\mathbf{0},y,\mathbf{0}}^-(0))^2 \\ & > 3 - 2H_{\mathbf{0},\mathbf{0},y}^-(\infty) - H_{\mathbf{0},y,\mathbf{0}}^-(0) > 0. \end{aligned}$$

Thus, the greatest positive real root of the characteristic polynomial of M is 1. Hence, by the Frobenius theorem (see, e.g., Theorem 2 in [12], Ch.4, Sec.5) 1 is the Perron root of M . So, the auxiliary Bellman-Harris process is critical.

Denote by $\vec{v} = (v_1, \dots, v_6)$ and $\vec{u} = (u_1, \dots, u_6)$ the left and right positive eigenvectors corresponding to the Perron root of M such that $(\vec{u}, \vec{1}) = 1$ and $(\vec{v}, \vec{u}) = 1$ where $\vec{1} = (1, \dots, 1) \in \mathbb{R}^6$. Taking into account (12) we rewrite the components of \vec{u} and \vec{v} in the convenient form

$$u_1 = u_4 = u_5 = \frac{1 - H_{\mathbf{0},\mathbf{0},y}^-(\infty)}{U}, \quad u_2 = u_3 = u_6 = \frac{H_{\mathbf{0},y,\mathbf{0}}^-(\infty)}{U}, \quad (13)$$

$$v_1 = \frac{U}{V}, \quad v_2 = \frac{U(1 - \alpha)H_{\mathbf{0},y,\mathbf{0}}^-(\infty)}{V(1 - H_{\mathbf{0},\mathbf{0},y}^-(\infty))}, \quad v_3 = \frac{U(1 - \alpha)(H_{\mathbf{0},y,\mathbf{0}}^-(\infty) - H_{\mathbf{0},y,\mathbf{0}}^-(0))}{V}, \quad (14)$$

$$v_4 = v_2(H_{\mathbf{0},y,\mathbf{0}}^-(\infty) - H_{\mathbf{0},y,\mathbf{0}}^-(0)), \quad v_5 = \frac{U(1 - \alpha)H_{\mathbf{0},\mathbf{0},y}^-(\infty)}{V}, \quad v_6 = v_2 H_{\mathbf{0},\mathbf{0},y}^-(\infty) \quad (15)$$

where the auxiliary variables U and V are defined by way of

$$U := 3(1 - H_{\mathbf{0},\mathbf{0},y}^-(\infty) + H_{\mathbf{0},y,\mathbf{0}}^-(\infty)),$$

$$V := 3 - 2\alpha f'(1) - (2 - \alpha)H_{\mathbf{0},\mathbf{0},y}^-(\infty) + (1 - \alpha)((H_{\mathbf{0},y,\mathbf{0}}^-(\infty) - H_{\mathbf{0},y,\mathbf{0}}^-(0))^2 - (H_{\mathbf{0},y,\mathbf{0}}^-(0))^2 - (H_{\mathbf{0},\mathbf{0},y}^-(\infty))^2).$$

Using decomposition $f(1-x) = 1 - f'(1)x + f''(1)x^2/2 + o(x^2)$, $x \rightarrow 0+$, along with formulae (12)–(15) and the definition of $\vec{f}(\vec{s}) = (f_1(\vec{s}), \dots, f_6(\vec{s}))$, it is not difficult to verify by standard calculations that

$$x - \left(\vec{v}, \vec{1} - \vec{f}(\vec{1} - \vec{u}x) \right) \sim Bx^2, \quad x \rightarrow 0+, \quad \text{where} \quad B := \frac{\sigma^2 (1 - H_{\mathbf{0}, \mathbf{0}, y}^-(\infty))^2}{2UV}. \quad (16)$$

In the next two lemmas we apply theorems proved in papers [16]–[19] to the constructed six-dimensional Bellman-Harris branching process and then reformulate the obtained results for CBRW on \mathbb{Z}^d when $d \geq 5$. Common to these theorems are the conditions of criticality and indecomposability of the Bellman-Harris process which were established above. Another common condition on the behavior of the function $x - (\vec{v}, \vec{1} - \vec{f}(\vec{1} - \vec{u}x))$ is fulfilled due to (16). However, various Vatutin's theorems involve different assumptions on the order of asymptotic decrease of the tails of $G_k(\cdot)$, $k = 1, \dots, 6$. It is worth to mention that such asymptotic behavior was established in [8], Theorem 3. Namely, our result for $d \leq 5$ corresponds to condition of Theorem 1 in [16] whereas the cases $d = 6$ and $d \geq 7$ meet the respective conditions of Theorem 3 in [19] and Theorem 2 in [18].

Lemma 3 *Given $y \in \mathbb{Z}^5$, $y \neq \mathbf{0}$, for CBRW on \mathbb{Z}^5 one has*

$$q(t; \mathbf{0}, y) = o(t^{-3/4}), \quad q(t; y, y) = o(t^{-3/4}), \quad t \rightarrow \infty.$$

PROOF. To apply Theorem 1 in [16] to the six-dimensional Bellman-Harris process constructed above for CBRW on \mathbb{Z}^5 we verify the conditions of that theorem. According to the definition of $\vec{G}(\cdot) = (G_1(\cdot), \dots, G_6(\cdot))$ and by Theorem 3 in [8] for $d = 5$, the variable β in condition 2) of Theorem 1 in [16] is equal to $3/2$ whereas the function $L_1(t)$ in the same condition tends to a constant, as $t \rightarrow \infty$. The validity of condition 3) of Theorem 1 in [16] is implied by Theorem 1 in [17] (for our process the function $L_1(t)$ in this theorem tends to $1/B$, as $t \rightarrow \infty$, in view of (16)) combined with the definition of $\vec{G}(\cdot)$ and Theorem 3 in [8] for $d = 5$. Thus, we may employ Theorem 1 in [16]. Taking into account Theorem 3 in [8] for $d = 5$ and formulae (13)–(15) we deduce from Theorem 1 in [16] that $\lim_{t \rightarrow \infty} \mathbf{E}_i(s_2^{Z_2(t)} | \vec{Z}(t) \neq \vec{0}) = 1$ for each $s_2 \in [0, 1]$ and $i = 1, 2$ (as usual, $\vec{Z}(t) = (Z_1(t), \dots, Z_6(t))$ and $\vec{0} = (0, \dots, 0) \in \mathbb{R}^6$). Setting $s_2 = 0$ in the last relation one has $\mathbf{P}_i(Z_2(t) > 0) = o(\mathbf{P}_i(\vec{Z}(t) \neq \vec{0}))$, as $t \rightarrow \infty$. Moreover, examining the proof of Theorem 1 in [16] we can show that for our Bellman-Harris process the slowly varying function $L^*(x)$ in the assertion of that theorem turns equivalent to $1/\sqrt{B}$, as $x \rightarrow 0+$. Consequently, the indicated in [16] formula (0.4) can be sharpened in our case, namely, the function $\mathbf{P}_i(\vec{Z}(t) \neq \vec{0})$ has an order of decreasing $t^{-3/4}$, as $t \rightarrow \infty$. Whence by the connection between CBRW on \mathbb{Z}^5 and the auxiliary Bellman-Harris process we complete the proof. \square

Lemma 4 *In the framework of CBRW on \mathbb{Z}^d with $d \geq 6$ the following relations hold true for $y \in \mathbb{Z}^d$, $y \neq \mathbf{0}$,*

$$q(t; \mathbf{0}, y) \sim \frac{2m_d}{\sigma^2 t}, \quad q(t; y, y) \sim \frac{2m_d G_0(\mathbf{0}, y)}{\sigma^2 G_0(\mathbf{0}, \mathbf{0}) t}, \quad t \rightarrow \infty.$$

PROOF. Let us apply Theorem 3 in [19] to our Bellman-Harris branching process when $d = 6$. To this end we verify whether all the conditions of Theorem 3 in [19] are satisfied. In view of (16) relation (6) in [19] is valid for our process and the function $L_1(n)$ in (6) tends

to $1/B$, as $n \rightarrow \infty$. Equality (7) in [19] is also satisfied due to (6) in [19] in view of the definition of $\vec{G}(\cdot)$ and Theorem 3 in [8] for $d = 6$. Now we may apply Theorem 3 in [19]. In particular, it follows that for each $i = 1, 2$ the expressions $\lim_{t \rightarrow \infty} \mathbf{P}_i(Z_1(t) = 0 | \vec{Z}(t) \neq \vec{0})$ and $\lim_{t \rightarrow \infty} \mathbf{P}_i(Z_2(t) = 0 | \vec{Z}(t) \neq \vec{0})$ coincide, are positive and strictly less than 1. Consequently, $\mathbf{P}_i(Z_1(t) > 0) \sim \mathbf{P}_i(Z_2(t) > 0)$, as $t \rightarrow \infty$. The asymptotic behavior of $q(t; \mathbf{0}, \mathbf{0}) = \mathbf{P}_1(Z_1(t) > 0)$ and $q(t; y, \mathbf{0}) = \mathbf{P}_2(Z_1(t) > 0)$ can be found in [6], Lemmas 2 and 4. Thus, Lemma 4 is proved for $d = 6$.

For $d \geq 7$ we will employ Theorem 2 in [18]. Condition (6) of that theorem is valid due to Theorem 1 in [17] (for our process, the function $L_1(t)$ in this theorem tends to $1/B$, as $t \rightarrow \infty$) by virtue of the definition of $\vec{G}(\cdot)$ and Theorem 3 in [8] for $d \geq 7$. The definition of $G_k(\cdot)$ and Theorem 3 in [8] for $d \geq 7$ also imply that $\int_0^\infty t dG_k < \infty$ for each $k = 1, \dots, 6$. So, all the conditions of Theorem 2 in [18] are satisfied and it follows that $\lim_{t \rightarrow \infty} \mathbf{P}_i(Z_k(t) = 0 | \vec{Z}(t) \neq \vec{0}) = 0$ for each $k = 1, \dots, 6$ and $i = 1, 2$. Hence, $\mathbf{P}_i(Z_1(t) > 0) \sim \mathbf{P}_i(Z_2(t) > 0)$, $t \rightarrow \infty$. Notably, the asymptotic behavior of $q(t; \mathbf{0}, \mathbf{0}) = \mathbf{P}_1(Z_1(t) > 0)$ and $q(t; y, \mathbf{0}) = \mathbf{P}_2(Z_1(t) > 0)$ can be found in [6], Lemmas 2 and 4. Lemma 4 is proved for $d \geq 7$. \square

Concluding this section we derive an integral equation in function $q(\cdot; \mathbf{0}, y)$, $y \neq \mathbf{0}$, which is a counterpart of equation (2.6) in [11] for $q(\cdot; \mathbf{0}, \mathbf{0})$. Our integral equation will be essentially used for proving Theorem 2 when $d = 4$. Before formulating the corresponding statement we have to introduce some more notation. Let τ_z be the time spent by a particle performing a random walk generated by matrix A until the first hitting a point $z \in \mathbb{Z}^d$. In a similar way, τ_z^- is the time spent by the particle *after leaving the starting point* of the random walk until the first hitting the point z . If the starting point of the random walk is z then the first hitting z means the first return to z . Denote by $H_{x,z}(t)$ and $H_{x,z}^-(t)$, $t \geq 0$, the (improper) c.d.f. of τ_z and τ_z^- , respectively, given that the starting point of the random walk is $x \in \mathbb{Z}^d$. Obviously, $H_{x,z}(t) = G_2 * H_{x,z}^-(t)$ for $t \geq 0$ and $x, z \in \mathbb{Z}^d$. Set also $K(t) := \alpha f'(1)G_1(t) + (1 - \alpha)G_1 * H_{\mathbf{0},\mathbf{0}}^-(t)$ and $h(s) := \alpha(f(1-s) - 1 + f'(1)s)$, $s \in [0, 1]$. Note that the function $K(t)$ and the function $K_d(t)$, $d \in \mathbb{N}$, arising in [11] and [13], coincide for each $t \geq 0$. Thus, Lemma 2.3 in [11] and Lemma 11 in [13] in which the asymptotic properties of c.d.f. $K_d(t)$ and its density $k_d(t)$ are established, as $t \rightarrow \infty$, may be applied to our function K .

Lemma 5 For $y \in \mathbb{Z}^d$, $y \neq \mathbf{0}$, one has

$$q(t; \mathbf{0}, y) = (1 - \alpha)G_1 * (H_{\mathbf{0},y}^-(t) - H_{\mathbf{0},y}(t)) + q(\cdot; \mathbf{0}, y) * K(t) - h(q(\cdot; \mathbf{0}, y)) * G_1(t). \quad (17)$$

PROOF. Recall integral equations (see, e.g., [12], Ch.8, Sec.1) for probability generating functions $\vec{F}(t; \vec{s}) := (F_1(t; \vec{s}), \dots, F_6(t; \vec{s}))$ of a six-dimensional Bellman-Harris process

$$F_i(t; \vec{s}) = s_i(1 - G_i(t)) + \int_0^t f_i(\vec{F}(t-u; \vec{s})) dG_i(u), \quad t \geq 0, \quad s_i \in [0, 1], \quad i = 1, \dots, 6.$$

By setting here $\vec{s} = (1, 0, 1, 1, 1, 1)$ and substituting the explicit formulae for G_j , $j = 3, 4, 5, 6$, and f_i , $i = 1, \dots, 6$, we get six integral equations in functions $F_i(t) := F_i(t; (1, 0, 1, 1, 1, 1))$, $t \geq 0$, $i = 1, \dots, 6$. Substituting the fourth and the sixth ones into the second equation and solving the obtained renewal equation in $F_2(\cdot)$ we find

$$\begin{aligned} F_2(t) &= G_2 * \left(1 - H_{\mathbf{0},y,\mathbf{0}}^-(t) - H_{\mathbf{0},\mathbf{0},y}^- * \sum_{k=0}^\infty H_{\mathbf{0},\mathbf{0},y}^{*k}(t)\right) \\ &+ F_1 * G_2 * \left(H_{\mathbf{0},y,\mathbf{0}}^-(0) + (H_{\mathbf{0},y,\mathbf{0}}^-(\cdot) - H_{\mathbf{0},y,\mathbf{0}}^-(0)) * \sum_{k=0}^\infty H_{\mathbf{0},\mathbf{0},y}^{*k}(t)\right) \end{aligned}$$

where $H_{x,z,r}(t) := G_2 * H_{x,z,r}^-(t)$, $t \geq 0$, $x, z, r \in \mathbb{Z}^d$, $z \neq r$. Now we substitute the last equation as well as the third and the fifth equations in functions F_i into the first one. After some algebraic transformations we obtain the following non-linear integral equation in function F_1

$$\begin{aligned} F_1(t) &= 1 - \alpha G_1 * (1 - f(F_1(t))) - (1 - \alpha) G_1 * (1 - F_1(\cdot)) * H_{\mathbf{0},\mathbf{0}}^-(t) \\ &\quad - (1 - \alpha) G_1 * (H_{\mathbf{0},y}^-(t) - H_{\mathbf{0},y}(t)) \end{aligned} \quad (18)$$

provided that the following two equalities are valid

$$H_{\mathbf{0},\mathbf{0}}^-(t) = H_{\mathbf{0},\mathbf{0},y}^-(t) + \sum_{k=0}^{\infty} H_{\mathbf{0},y,\mathbf{0}}^- * H_{y,y,\mathbf{0}}^{*k} * H_{y,\mathbf{0},y}(t), \quad H_{\mathbf{0},y}^-(t) = H_{\mathbf{0},y,\mathbf{0}}^- * \sum_{k=0}^{\infty} H_{\mathbf{0},\mathbf{0},y}^{*k}(t),$$

for each $t \geq 0$. The first of them is true since any trajectory from $\mathbf{0}$ to $\mathbf{0}$ of a particle performing a random walk on \mathbb{Z}^d either passes y exactly k times, $k = 1, 2, \dots$, or does not hit y until the first returning to $\mathbf{0}$. Similar argument justifies the second equality as well. Recall that due to the connection between the CBRW on \mathbb{Z}^d and the constructed Bellman-Harris process one has $q(t; \mathbf{0}, y) = \mathbb{P}_1(Z_2(t) > 0) = 1 - F_1(t)$. Hence, rewriting (18) as an equation in $q(t; \mathbf{0}, y)$ we come to (17). \square

5 Proofs of Theorems 2 and 3

First of all, we derive some integral equations to be treated in this section. Consider equation (5) as inhomogeneous one for differential equation (4) in Banach space $l_\infty(\mathbb{Z}^d)$. By the variation of constant formula we infer (for a similar deduction see [5]) that

$$q(s, t; x, y) = (1 - s)m(t; x, y) - \int_0^t m(t - u; x, \mathbf{0})h(q(s, u; \mathbf{0}, y)) du \quad (19)$$

where $q(s, t; x, y) = 1 - F(s, t; x, y)$, $s \in [0, 1]$, $t \geq 0$, $x, y \in \mathbb{Z}^d$. Substituting $x = \mathbf{0}$ in the last equation we come to an integral equation in function $q(s, t; \mathbf{0}, y)$

$$q(s, t; \mathbf{0}, y) = (1 - s)m(t; \mathbf{0}, y) - \int_0^t m(t - u; \mathbf{0}, \mathbf{0})h(q(s, u; \mathbf{0}, y)) du. \quad (20)$$

Note that $q(0, t; x, y)$ is equal to $q(t; x, y)$. Thus, on account of (19) one has

$$q(t; x, y) = m(t; x, y) - \int_0^t m(t - u; x, \mathbf{0})h(q(u; \mathbf{0}, y)) du. \quad (21)$$

Substituting $x = \mathbf{0}$ in (21) we derive an integral equation in function $q(t; \mathbf{0}, y)$

$$q(t; \mathbf{0}, y) = m(t; \mathbf{0}, y) - \int_0^t m(t - u; \mathbf{0}, \mathbf{0})h(q(u; \mathbf{0}, y)) du. \quad (22)$$

Now let us prove Theorems 2 and 3 for $x = \mathbf{0}$. Since their proofs depend on $d \in \mathbb{N}$ essentially, we have to consider the cases $d = 1$, $d = 2$, $d = 3$, $d = 4$ and $d \geq 5$ separately. Evidently, Theorem 2 for $x = \mathbf{0}$ and $d \geq 6$ is implied by Lemma 4. Due to Lemmas 1–3 and equation (22) the proof of Theorem 2 for $x = \mathbf{0}$ in the respective cases $d = 1$, $d = 2$, $d = 3$ and $d = 5$ mainly follows the scheme proving, respectively, Theorem 2 in [20], Theorem 2 in [4], Theorem 4 in

[13] (item 3) and Theorem 4 in [13] (item 4). Moreover, by virtue of Lemma 5 the proof of Theorem 2 for $x = \mathbf{0}$ and $d = 4$ is similar to that of Theorem 1.1 in [11]. So, we give only a few comments on the proof of Theorem 2 for $x = \mathbf{0}$ and $d \leq 5$.

If $d = 1$ then the equality $\int_0^\infty h(q(u; \mathbf{0}, y)) du = (1 - \alpha)a^{-1}$ is valid. Furthermore, in view of (2), (8), (9) and Theorem 5 in [13] one gets the useful estimate

$$m(t; \mathbf{0}, y) - (1 - \alpha)a^{-1}m(t; \mathbf{0}, \mathbf{0}) = O(t^{-3/2}), \quad t \rightarrow \infty.$$

When $d = 2$ one can check the strict inequality $J(0; y) = \int_0^\infty h(q(u; \mathbf{0}, y)) du < (1 - \alpha)a^{-1}$. However, if $d = 3$ then $\int_0^\infty h(q(u; \mathbf{0}, y)) du = (1 - \alpha)a^{-1}G_0(\mathbf{0}, y)G_0^{-1}(\mathbf{0}, \mathbf{0})$ and

$$m(t; \mathbf{0}, y) - (1 - \alpha)a^{-1}G_0(\mathbf{0}, y)G_0^{-1}(\mathbf{0}, \mathbf{0})m(t; \mathbf{0}, \mathbf{0}) = O(t^{-1}), \quad t \rightarrow \infty,$$

by virtue of (2), (9) and Theorem 5 in [13]. For $d = 4$ the first summand in (17) is $o(t^{-1})$, $t \rightarrow \infty$, by Lemma 3 in [6] and it does not contribute to the (main term of) asymptotic behavior of $q(t; \mathbf{0}, y)$. As for $d = 5$, one has $\int_0^\infty h(q(u; \mathbf{0}, y)) du = (1 - \alpha)a^{-1}G_0(\mathbf{0}, y)G_0^{-1}(\mathbf{0}, \mathbf{0})$ and

$$m(t; \mathbf{0}, y) - (1 - \alpha)a^{-1}G_0(\mathbf{0}, y)G_0^{-1}(\mathbf{0}, \mathbf{0})m(t; \mathbf{0}, \mathbf{0}) = O(t^{-3/2}), \quad t \rightarrow \infty,$$

in view of (2), (9), Theorem 5 and Corollary 1 in [13]. Thus, Theorem 2 is proved for $x = \mathbf{0}$.

Turn to Theorem 3 when $x = \mathbf{0}$. The proof of Theorem 3 for $x = \mathbf{0}$ is similar to those of Theorem 4 in [14], Theorem 2 in [4] and Theorem 4 in [6] for $d = 1, 3$, $d = 2$ and $d \geq 5$, respectively. Note only that the constant c^* arising in the proof of Theorem 3 for $x = \mathbf{0}$ in contrast to its counterpart in Theorem 4 in [14] is equal to $\sigma^2\gamma_1^2a/(2(1-\alpha))$ and $\sigma^2aG_0^3(\mathbf{0}, \mathbf{0})G_0(\mathbf{0}, y)/(8\pi^2\gamma_3^2(1-\alpha))$ when $d = 1$ and $d = 3$, respectively. At last, the constant c_d^* appearing in Theorem 4 in [6] equals $(1 - \alpha)G_0(\mathbf{0}, y)\sigma^2/(2aG_0(\mathbf{0}, \mathbf{0})m_d^2)$ in the case of Theorem 3 for $x = \mathbf{0}$ and $d \geq 5$. Since the limit theorem for $\mu(t; \mathbf{0})$ when $d = 4$ was established by another approach, namely the moment method, we give the detailed proof of the limit theorem for $\mu(t; y)$ when $d = 4$. So, to complete the proof of Theorem 3 for $x = \mathbf{0}$ we dwell on the case $d = 4$ in detail.

Set $s(t) := s(t; \lambda) = \exp\{-\lambda \ln^2 t/(c^*t)\}$ where $c^* := \sigma^2aG_0^3(\mathbf{0}, \mathbf{0})G_0(\mathbf{0}, y)/(3\gamma_4^2(1-\alpha))$, $t > 0$ and $\lambda \geq 0$. By Theorems 1 and 2 for $x = \mathbf{0}$ and $d = 4$ we see that

$$\mathbf{E}_0(\mu(t; y) | \mu(t; y) > 0) = \frac{m(t; \mathbf{0}, y)}{q(t; \mathbf{0}, y)} \sim \frac{c^*t}{\ln^2 t}, \quad t \rightarrow \infty. \quad (23)$$

The inequality $1 - e^{-z} \leq z$ for $z \geq 0$ yields

$$q(s(t), u; \mathbf{0}, y) = \mathbf{E}_0\left(1 - \exp\left\{-\frac{\lambda \ln^2 t \mu(u; y)}{c^*t}\right\}\right) \leq \frac{\lambda \ln^2 t}{c^*t} \mathbf{E}_0\mu(u; y) = \frac{\lambda \ln^2 t}{c^*t} m(u; \mathbf{0}, y)$$

where $u \geq 0$ and $t > 0$. By virtue of this estimate combined with Theorem 1 and the inequality $h(z) \leq \sigma^2 z^2$ (being true for $z \geq 0$ small enough) one has for t large enough

$$\begin{aligned} & \int_0^{t/\ln^3 t} m(t-u; \mathbf{0}, \mathbf{0})h(q(s(t), u; \mathbf{0}, y))du \\ & \leq \frac{\sigma^2 \lambda^2 \ln^4 t}{c^{*2}t^2} \int_0^{t/\ln^3 t} m^2(u; \mathbf{0}, y)m(t-u; \mathbf{0}, \mathbf{0})du = \frac{\rho_1(t; \lambda) \ln t}{t}. \end{aligned} \quad (24)$$

Here $\rho_1 \in \mathcal{U}$ and \mathcal{U} is the class of all bounded functions $\rho(t; \lambda)$ vanishing as $t \rightarrow \infty$ uniformly in $\lambda \in [0, b]$, whatever positive b is taken. In a similar way, we obtain

$$\begin{aligned} & \int_{t-t/\ln^2 t}^t m(t-u; \mathbf{0}, \mathbf{0}) h(q(s(t), u; \mathbf{0}, y)) du \\ & \leq \frac{\sigma^2 \lambda^2 \ln^4 t}{c^{*2} t^2} \int_{t-t/\ln^2 t}^t m^2(u; \mathbf{0}, y) m(t-u; \mathbf{0}, \mathbf{0}) du = \frac{\rho_2(t; \lambda) \ln t}{t} \end{aligned} \quad (25)$$

for $\rho_2 \in \mathcal{U}$. It is not difficult to show that uniformly in $u \in [t/\ln^3 t, t-t/\ln^2 t]$

$$\ln u \sim \ln t, \quad \ln(t-u) \sim \ln t, \quad t \rightarrow \infty. \quad (26)$$

These facts, Theorem 1 and the relation $h(z) \sim \sigma^2 z^2/2$, $z \rightarrow 0$, allow us to claim that

$$\begin{aligned} I(t; \lambda) & : = \int_{t/\ln^3 t}^{t-t/\ln^2 t} h(q(s(t), u; \mathbf{0}, y)) m(t-u; \mathbf{0}, \mathbf{0}) du \\ & = \frac{\sigma^2 m(t; \mathbf{0}, \mathbf{0})}{2} \int_{t/\ln^3 t}^{t-t/\ln^2 t} q^2(s(t), u; \mathbf{0}, y) du (1 + \rho_3(t; \lambda)) \end{aligned}$$

where $\rho_3 \in \mathcal{U}$. After changing the variable $u = tv$ and using Theorems 1 and 2 for $x = \mathbf{0}$ and $d = 4$ we get

$$I(t; \lambda) = \frac{3}{2q(t; \mathbf{0}, y)} \int_{1/\ln^3 t}^{1-1/\ln^2 t} q^2(s(t), tv; \mathbf{0}, y) dv (1 + \rho_4(t; \lambda)), \quad \rho_4 \in \mathcal{U}. \quad (27)$$

In the last integral the function $q(s(t; \lambda), tv; \mathbf{0}, y)$ can be replaced by $q(s(tv; \lambda v), tv; \mathbf{0}, y)$. Indeed, as $1 - e^{-z} \leq z$ for $z \geq 0$, we have

$$\begin{aligned} & |q(s(t; \lambda), tv; \mathbf{0}, y) - q(s(tv; \lambda v), tv; \mathbf{0}, y)| \\ & = \mathbf{E}_0 \left(\exp \left\{ -\frac{\lambda v \ln^2(tv)}{c^* tv} \mu(tv; y) \right\} - \exp \left\{ -\frac{\lambda \ln^2 t}{c^* t} \mu(tv; y) \right\} \right) \\ & \leq \mathbf{E}_0 \left(1 - \exp \left\{ -\frac{\lambda(-2 \ln t \ln v - \ln^2 v)}{c^* t} \mu(tv; y) \right\} \right) \leq \frac{\lambda(-2 \ln t \ln v - \ln^2 v)}{c^* t} m(tv; \mathbf{0}, y). \end{aligned}$$

Since functions $z \ln z$ and $z \ln^2 z$ are bounded for $z \in (0, 1)$, by virtue of Theorems 1 and 2 for $x = \mathbf{0}$ along with relation (26) we see that uniformly in $v \in [1/\ln^3 t, 1-1/\ln^2 t]$ and $0 \leq \lambda \leq \Lambda$ with an arbitrary positive Λ

$$\frac{q(s(t; \lambda), tv; \mathbf{0}, y)}{q(tv; \mathbf{0}, y)} - \frac{q(s(tv; \lambda v), tv; \mathbf{0}, y)}{q(tv; \mathbf{0}, y)} \rightarrow 0, \quad t \rightarrow \infty. \quad (28)$$

Set $\varphi(t; \lambda) := q(s(t; \lambda), t; \mathbf{0}, y)/(\lambda q(t; \mathbf{0}, y))$, $t > 0$, $\lambda \geq 0$. Then dividing both sides of (20) by $\lambda q(t; \mathbf{0}, y)$ and using (24)–(28) along with Theorem 2 for $x = 0$ and relation $1 - e^{-z} \sim z$, $z \rightarrow 0$, we obtain

$$\varphi(t; \lambda) = 1 + \rho_5(t; \lambda) - \frac{3\lambda}{2} \int_{1/\ln^3 t}^{1-1/\ln^2 t} \varphi^2(tv; \lambda v) dv, \quad \rho_5 \in \mathcal{U}.$$

Changing the variable $w = \lambda v$ leads to the following relation

$$\varphi(t; \lambda) = 1 + \rho_5(t; \lambda) - \frac{3}{2} \int_{\lambda/\ln^3 t}^{\lambda(1-1/\ln^2 t)} \varphi^2\left(\frac{tw}{\lambda}; w\right) dw.$$

The argument similar to the proof of Theorem 4 in [14] establishes that

$$\lim_{t \rightarrow \infty} \varphi(t; \lambda) = \varphi(\lambda) = \frac{2}{3\lambda + 2}, \quad 0 < \lambda \leq \Lambda_0, \quad (29)$$

where Λ_0 is some positive number and $\varphi(\lambda)$ is the unique solution of the equation

$$\varphi(\lambda) = 1 - \frac{3}{2} \int_0^\lambda \varphi^2(w) dw, \quad \lambda \geq 0.$$

Invoking the definition of $\varphi(t; \lambda)$ we rewrite relation (29) by way of

$$\lim_{t \rightarrow \infty} \mathbf{E}_0 \left\{ \exp \left\{ -\frac{\lambda \ln^2 t \mu(t; y)}{c^* t} \right\} \middle| \mu(t; y) > 0 \right\} = 1 - \lambda \lim_{t \rightarrow \infty} \varphi(t; \lambda) = \frac{1}{3} + \frac{2}{3} \cdot \frac{2}{3\lambda + 2} \quad (30)$$

for $0 < \lambda \leq \Lambda_0$. Since both the Laplace transform of a non-negative random variable and the function $1/3 + 2/3 \cdot 2/(3\lambda + 2)$ are analytic and bounded in the domain $\{\lambda : \operatorname{Re} \lambda > 0\} \subset \mathbb{C}$, by the uniqueness theorem for analytic functions relation (30) is valid for each λ with $\operatorname{Re} \lambda > 0$ (for an analogous deduction see, e.g., [15]). Combining (23) and (30) we complete the proof of Theorem 3 for $x = 0$ and $d = 4$. Thus, Theorem 3 is proved for $x = \mathbf{0}$.

Next we prove Theorems 2 and 3 when $x \neq \mathbf{0}$. As a preliminary we derive some more integral equations. In the framework of CBRW on \mathbb{Z}^d , the parent particle can either hit the point $\mathbf{0}$ or not within time interval $[0, t]$. In the latter case at time t there is a single particle on \mathbb{Z}^d located at the point y or outside it. Consequently,

$$\begin{aligned} \mathbf{E}_x s^{\mu(t; y)} &= \mathbf{E}_x s^{\mu(t; y)} \mathbb{I}(\tau_{\mathbf{0}} \leq t) + \mathbf{E}_x s^{\mu(t; y)} \mathbb{I}(\tau_{\mathbf{0}} > t, \mu(t; y) = 1) + \mathbf{E}_x s^{\mu(t; y)} \mathbb{I}(\tau_{\mathbf{0}} > t, \mu(t; y) = 0) \\ &= \mathbf{E}_x s^{\mu(t; y)} \mathbb{I}(\tau_{\mathbf{0}} \leq t) + s \mathbf{P}_x(\tau_{\mathbf{0}} > t, \mu(t; y) = 1) + \mathbf{P}_x(\tau_{\mathbf{0}} > t, \mu(t; y) = 0) \end{aligned} \quad (31)$$

where $\mathbb{I}(\cdot)$ stands for the indicator of a set. Evidently, the first summand in (31) can be rewritten in the form

$$\begin{aligned} \mathbf{E}_x s^{\mu(t; y)} \mathbb{I}(\tau_{\mathbf{0}} \leq t) &= \int_{\{\tau_{\mathbf{0}} \leq t\}} s^{\mu(t; y)} d\mathbf{P}_x = \int_{\{\tau_{\mathbf{0}} \leq t\}} \mathbf{E}_x (s^{\mu(t; y)} | \tau_{\mathbf{0}}) d\mathbf{P}_x \\ &= \int_0^t \mathbf{E}_x (s^{\mu(t; y)} | \tau_{\mathbf{0}} = u) dH_{x, \mathbf{0}}(u) = \int_0^t \mathbf{E}_0 s^{\mu(t-u; y)} dH_{x, \mathbf{0}}(u). \end{aligned} \quad (32)$$

It is easily seen that the probability at the second summand in (31) can be represented as follows

$$\mathbf{P}_x(\tau_{\mathbf{0}} > t, \mu(t; y) = 1) = H_{x, y, \mathbf{0}} * \sum_{k=0}^{\infty} H_{y, y, \mathbf{0}}^{*k} * (1 - G_2(t)) \quad \text{when } x \neq y, \quad (33)$$

$$\mathbf{P}_y(\tau_{\mathbf{0}} > t, \mu(t; y) = 1) = \sum_{k=0}^{\infty} H_{y, y, \mathbf{0}}^{*k} * (1 - G_2(t)). \quad (34)$$

It also turns convenient to write the third summand in (31) in the form

$$P_x(\tau_{\mathbf{0}} > t, \mu(t; y) = 0) = 1 - H_{x, \mathbf{0}}(t) - H_{x, y, \mathbf{0}} * \sum_{k=0}^{\infty} H_{y, y, \mathbf{0}}^{*k} * (1 - G_2(t)) \quad \text{if } x \neq y, \quad (35)$$

$$P_y(\tau_{\mathbf{0}} > t, \mu(t; y) = 0) = 1 - H_{y, \mathbf{0}}(t) - \sum_{k=0}^{\infty} H_{y, y, \mathbf{0}}^{*k} * (1 - G_2(t)). \quad (36)$$

Combining relations (31)–(36) we come to the desired integral equations

$$q(s, t; x, y) = (1 - s)H_{x, y, \mathbf{0}} * \sum_{k=0}^{\infty} H_{y, y, \mathbf{0}}^{*k} * (1 - G_2(t)) + \int_0^t q(s, t - u; \mathbf{0}, y) dH_{x, \mathbf{0}}(u) \quad \text{if } x \neq y,$$

$$q(s, t; y, y) = (1 - s) \sum_{k=0}^{\infty} H_{y, y, \mathbf{0}}^{*k} * (1 - G_2(t)) + \int_0^t q(s, t - u; \mathbf{0}, y) dH_{y, \mathbf{0}}(u).$$

In particular, for $s = 0$ one has

$$q(t; x, y) = H_{x, y, \mathbf{0}} * \sum_{k=0}^{\infty} H_{y, y, \mathbf{0}}^{*k} * (1 - G_2(t)) + \int_0^t q(t - u; \mathbf{0}, y) dH_{x, \mathbf{0}}(u) \quad \text{when } x \neq y, \quad (37)$$

$$q(t; y, y) = \sum_{k=0}^{\infty} H_{y, y, \mathbf{0}}^{*k} * (1 - G_2(t)) + \int_0^t q(t - u; \mathbf{0}, y) dH_{y, \mathbf{0}}(u). \quad (38)$$

Now we have the tools for proving Theorems 2 and 3 for $x \neq \mathbf{0}$. To establish Theorem 2 for $x \neq \mathbf{0}$ and $d \neq 2$ we employ equations (37) and (38). It is not difficult to see that the first summands in the right side of (37) and (38) are equal to $p(t; x, y) - \int_0^t p(t - u; \mathbf{0}, y) dH_{x, \mathbf{0}}(u)$ for $x \neq y$ and $x = y$, respectively. The latter expression can be rewritten as follows

$$\begin{aligned} & p(t; x, y) - \int_0^t p(t - u; \mathbf{0}, y) dH_{x, \mathbf{0}}(u) \\ &= p(t; x, y) - p(t; x, \mathbf{0}) + \int_0^t (p(t - u; \mathbf{0}, \mathbf{0}) - p(t - u; \mathbf{0}, y)) dH_{x, \mathbf{0}}(u) \end{aligned} \quad (39)$$

due to the obvious relation $p(t; x, \mathbf{0}) = \int_0^t p(t - u; \mathbf{0}, \mathbf{0}) dH_{x, \mathbf{0}}(u)$. The asymptotic behavior of the first summand at the right-hand side of (39) is given by formula (2) whereas the asymptotic behavior of the second summand in (39) can be found with the help of relation (2), Lemma 3 in [6] and Lemma 5.1.2 in [22]. Finally, the first summands in (37) and (38) are $O(t^{-3/2})$ when $d = 1$ and $O(t^{-d/2})$ when $d \geq 3$. Hence, the first summands in (37) and (38) are $o(q(t; \mathbf{0}, y))$, as $t \rightarrow \infty$, by Theorem 2 for $x = \mathbf{0}$. Moreover, on account of Lemma 3 in [6] and Lemma 5.1.2 in [22] we reveal that the last summands in (37) and (38) are equivalent to $q(t; \mathbf{0}, y)$ and $q(t; \mathbf{0}, y) G_0(x, \mathbf{0}) G_0^{-1}(\mathbf{0}, \mathbf{0})$ for $d = 1$ and $d \geq 3$, respectively, as $t \rightarrow \infty$. Hence Theorem 2 is proved for $x \neq \mathbf{0}$ and $d \neq 2$. As for Theorem 2 when $x \neq \mathbf{0}$ and $d = 2$ as well as Theorem 3 for $x \neq \mathbf{0}$, we only note that their proofs bear on analysis of equations (19) and (21). Since the proofs are similar to that of Theorem 5 in [6], they are omitted. So, Theorems 2 and 3 are proved completely. \square

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References

- [1] Albeverio S., Bogachev L.V. and Yarovaya E.B. *Asymptotics of branching symmetric random walk on the lattice with a single source*. Comptes Rendus Acad. Sci. Paris, Sér. I, Math., t.326 (1998), pp. 975-980.
- [2] Bogachev L.V. and Yarovaya E.B. *The moment analysis of a branching random walk on a lattice with a single source*. Dokl. Math., vol. 58 (1998), no. 3, pp. 403-406.
- [3] Bruijn N.G. *Asymptotic methods in analysis*. New York, Courier Dover Publications, 1981.
- [4] Bulinskaya E.VI. *Catalytic Branching Random Walk on a Two-Dimensional Lattice*. Theory Probab. Appl., vol. 55 (2011), no. 1, pp. 120-126.
- [5] Bulinskaya E.VI. *Catalytic Branching Random Walk on Three-Dimensional Lattice*. Theory Stoch. Proc., vol. 16 (2010), no. 2, 23-32.
- [6] Bulinskaya E.VI. *Limit Distributions Arising in Branching Random Walks on Integer Lattices*. Lithuan. Math. J., vol.51 (2011), no.3, pp.310-321, DOI: 10.1007/s10986-011-9128-2.
- [7] Bulinskaya E.VI. *Limit Distributions for the Number of Particles in Branching Random Walks*. Math. Notes, vol. 90 (2011), no. 6, 103-116.
- [8] Bulinskaya E.VI. *The Hitting Times with Taboo for a Random Walk on an Integer Lattice*. Prepublication de LPMA UPMC (univ. Paris-VI) no. 1456, 2011, 25 pp.; arXiv:1107.1074v1 [math.PR].
- [9] Daletsky Yu.L. and Krein M.G. *Stability of Solution of Differential Equations in Banach Space*. Nauka, Moscow, 1970 (in Russian).
- [10] Gikhman I.I. and Skorokhod A.V. *The Theory of Stochastic Processes II*. Springer, 2004.
- [11] Hu Y., Vatutin V. and Topchii V. *Branching random walk in \mathbb{Z}^4 with branching at the origin only*. arXiv:1006.4769v1 [math.PR] 24 Jun 2010.
- [12] Sewastianow B.A. *Verzweigungsprozesse*. Akademie-Verlag, Berlin, 1974, 326 s.
- [13] Topchii V.A. and Vatutin V.A. *Catalytic Branching Random Walk in \mathbb{Z}^d with branching at the origin only*. Sib. Adv. Math., vol. 14 (2011), no. 2, pp. 28-72.
- [14] Topchii V. and Vatutin V. *Individuals at the origin in the critical catalytic branching random walk*. Discrete Math. and Theor. Computer Sci. (2003), no. 6, pp. 325-332, <http://dmtcs.loria.fr/proceedings/html/dmAC7130.abs.html>.
- [15] Vatutin V.A.: *Critical Bellman-Harris Branching Processes Starting with a Large Number of Particles*. Math. Notes 40, 803-811 (1986).
- [16] Vatutin V.A. *Discrete Limit Distributions of the Number of Particles in a Multitype Age-Dependent Branching Processes*. Theory of Probab. and Appl., vol. 24 (1980), no. 3, pp. 509-520.

- [17] Vatutin V.A. *Limit Theorems for Critical Markov Branching Processes with Several Types of Particles and Infinite Second Moments*. Math. USSR-Sb., vol. 32(1977), no. 2, pp. 215-225.
- [18] Vatutin V.A. *Limit Theorem for a Critical Multitype Bellman-Harris Branching Process with Infinite Second Moments*. Theory of Probab. and Appl., vol. 23 (1979), no. 4, pp.776-788.
- [19] Vatutin V.A. *On a Class of Critical Bellman-Harris Branching Processes with Several Types of Particles*. Theory of Probab. and Appl., vol. 25 (1980), no. 4, pp. 760-771.
- [20] Vatutin V.A., Topchii V.A. and Yarovaya E.B. *Catalytic branching random walk and queueing systems with random number of independent servers*. Theory Probab. and Math. Statist., no. 69 (2004), pp. 1-15.
- [21] Yarovaya E.B. *A limit theorem for critical branching random walk on \mathbb{Z}^d with a single source*. Russ. Math. Surv., vol. 60 (2005), no. 1, pp. 173-175.
- [22] Yarovaya E.B. *Branching Random Walks in Inhomogeneous Medium*. Moscow, MSU, 2007, 104 pp. (in Russian).
- [23] Yarovaya E.B. *Criteria of the exponential growth of particles numbers in models of branching random walks*. Theory Probab. Appl., vol. 55 (2011), no. 4, pp. 705-731.
- [24] Yarovaya E.B. *The monotonicity of the probability of return into the source in models of branching random walks*. Mosc. Univ. Math. Bull., vol. 65 (2010), no. 2, pp. 78-80.

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