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Abstract. We consider the operator $H := \partial_t - \Delta + V$ in 2D or 3D waveguide. With an adapted global Carleman estimate with singular weight functions we give a stability result for the time dependent part of the potential for this particular geometry. Two cases are considered: the bounded waveguide with mixed Dirichlet and Neumann conditions and the open waveguide with Dirichlet boundary conditions.

1. Introduction

We first consider a bounded waveguide $\Omega = (-L, L) \times \mathcal{D}$ in $\mathbb{R}^d$ with $d = 2$ or $d = 3$. In the two-dimensional case, $\mathcal{D} = (0, h)$, where h is a fixed positive constant, while in the three-dimensional case, $\mathcal{D}$ is a connected, bounded and open domain of $\mathbb{R}^2$ with $\mathcal{C}^\infty$ boundary denoted Γ_1. We denote $\Gamma_1 = \Gamma_1^+ \cup \Gamma_1^-$. We will denote $x = (x_1, x_2)$ a generic point of Ω where $x_1 \in (-L, L)$ and $x_2 \in \mathcal{D}$. We consider the heat equation

$$\begin{cases} \partial_t u - \Delta u + V(t,x)u = 0, & \text{in } (0,T)\times\Omega, \\ u(t,x) = b(t,x), & \text{on } (0,T)\times[-L,L]\times\partial\mathcal{D}, \\ \partial_\nu u(t,\pm L, x_2) = k_\pm(t,x_2), & \text{on } (0,T)\times\mathcal{D}, \\ u(0,x) = u_0(x), & \text{on } \Omega, \end{cases} \tag{1.1}$$

where $V(t,x) = q(t,x_2)f(x_1)$ and $f(x_1) > 0$. The aim of this paper is to give a stability and uniqueness result for the time dependent part of the potential $q(t,x_2)$ using global Carleman estimates. We denote by ν the outward unit normal to Ω on $\partial\Omega$.
We shall use the following notations $Q = (0,T)\times\Omega$ and $\Sigma = (0,T)\times\partial\Omega$.

Our problem can be stated as follows:
Is it possible to determine the coefficient $q(t,x_2)$ from the measurement of $\partial_\nu(\partial_{x_1}u)$ on $(0,T)\times(-L,L)\times\Gamma_1^+$, where Γ_1^+ is a part of $\partial\mathcal{D}$?

Let u (resp. $\widetilde{u}$) be a solution of (1.1) associated with $(q,\ f,\ b,\ u_0)$ (resp. $(\widetilde{q},\ f,\ b,\ u_0)$).
Our main result is

$$\|q - \widetilde{q}\|^2_{L^2((\varepsilon,T-\varepsilon)\times\mathcal{D})} \leq C_\varepsilon \|\partial_\nu(\partial_{x_1}u) - \partial_\nu(\partial_{x_1}\widetilde{u})\|^2_{L^2((0,T)\times(-L,L)\times\Gamma_1^+)}, \quad 0 < \varepsilon < \frac{T}{2},$$

where C_ε is a positive constant which depends on $(\Omega, \Gamma_1^+, \varepsilon, T)$ and where the above norms are weighted Sobolev norms.

Using a method introduce in [AT], we first derive a global Carleman estimate with singular weight for the operator $H := \partial_t - \Delta$ with a boundary term on a part Γ of the boundary Γ of $\mathcal{D}$. Then using these estimate and following the method developed by Choulli and Yamamoto [CY06], we give a stability and uniqueness result for the time dependent part $q(t,x_2)$ of the potential $V(t,x)$.
For the first time, the method of Carleman estimates was introduced in the field of inverse problems in the work of Bukhgeim and Klibanov [BK]. A recent book by Klibanov and Timonov [KT] is devoted to the Carleman estimates applied to inverse coefficient problems.
The problem of recovering time independent coefficients has attracted considerable attention recently and many theoretical results exist. Regarding time dependent coefficient few results exist. In the case of source term, Canon and Esteva [CE] established uniqueness and *a priori* estimates for the heat conduction equation with

over specified data. Choulli and Yamamoto [CY06] obtained a stability result, in a restricted class, for the inverse problem of determining a source term $f(x,t)$ from Neumann boundary data for the heat equation in a bounded domain. In a recent work, Choulli and Yamamoto [CY11] considered the inverse problem of finding a control parameter $p(t)$ that reach a desired temperature $h(t)$ along a curve $\gamma(t)$ for a parabolic semi-linear equation with homogeneous Neumann boundary data and they established existence, uniqueness as well as Lipschitz stability. Using optic geometric solution, Choulli [Ch09] considered the inverse problem of determining a general time dependent coefficient of order zero for parabolic equations from Dirichlet to Neumann map and he proved uniqueness as well as stability. In [E07] and [E08], Eskin considered the same inverse problem for hyperbolic and the Schrödinger equations with time-dependent electric and magnetic potential and he established uniqueness by gauge invariance.
The idea introduce in [Y] allows us to take into account the particular geometry of our domain. Indeed, the x_1-derivative do not alter the Dirichlet condition.
In a second part, we will consider an open waveguide $\Omega = \mathbb{R} \times \mathcal{D}$ in $\mathbb{R}^d$. We will denote $x = (x_1, x_2)$ a generic point of Ω where $x_1 \in \mathbb{R}$ and $x_2 \in \mathcal{D}$. We consider the heat equation

$$\begin{cases} \partial_t u - \Delta u + V(t,x)u = 0 \ \text{ in } \ (0,T) \times \Omega, \\ u(t,x) = b(t,x) \ \text{ on } \ (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x) \ \text{ on } \ \Omega, \end{cases} \tag{1.2}$$

where $V(t,x) = q(t,x_2) f(x_1)$ and $f(x_1) \geq c_{min} > 0$. Using an approach similar to the previous case we will give a stability and uniqueness result for the time dependent part of the potential $q(t,x_2)$ using global Carleman estimates. We denote by ν the outward unit normal to Ω on $\partial\Omega = \mathbb{R} \times \partial\mathcal{D}$.
This paper is organized as follows. In section 2, using an adapted global Carleman estimate for the operator H, we give a stability result for the coefficient q of problem (1.1). In section 3, we consider the open wave guide and we establish a stability result for the coefficient q of problem (1.2).

2. Stability result for a bounded waveguide

In this section we consider problem (1.1). We will establish a stability result and deduce a uniqueness for the coefficient q. We give the result for the bounded waveguide with mixed boundary conditions. The Carleman estimate will be the key ingredient in the proof of such a stability estimate.
From now on we set

$$\Omega = (-L,L) \times \mathcal{D}, \ Q = (0,T) \times \Omega, \ \Sigma_1 = (0,T) \times [-L,L] \times \partial\mathcal{D},$$

$$\Sigma_2' = (0,T) \times \{-L\} \times \mathcal{D}, \ \Sigma_2'' = (0,T) \times \{L\} \times \mathcal{D}, \ \Sigma_2 = \Sigma_2' \cup \Sigma_2''$$

and

$$\Sigma = (0,T) \times \partial\Omega.$$

2.1. Carleman estimate

Let us introduce the differential operator

$$P = \partial_t - \Delta_x + A(t,x)\cdot\nabla_x \cdot + B(t,x)$$

with $A \in L^\infty(Q,\mathbb{R}^d)$ and $B \in L^\infty(Q)$. Let Γ_1^+ be a closed subset of $\partial\mathcal{D}$, let $\alpha \in (-L,L)$ and let Γ^+ be defined by

$$\Gamma^+ = (-L,L)\times\Gamma_1^+.$$

Let ψ_2 be a $\mathcal{C}^4(\mathbb{R}^{d-1})$ function satisfying the following conditions:

(i) $\psi_2(x_2) > 0$ in $\overline{\mathcal{D}}$,

(ii) There exists $C_0 > 0$ such that $|\nabla\psi_2| \geq C_0 > 0 \quad \text{in} \quad \mathcal{D}$,

(iii) $\partial_\nu\psi_2 \leq 0 \;\text{ on }\; (\partial\mathcal{D}\setminus\Gamma_1^+)$.

For the proof of existence of a function satisfying these conditions, we refer to [FI] and [CIK]. Now let $\psi_1 \in \mathcal{C}^4(\mathbb{R})$ be a function satisfying the following conditions

(i) $\psi_1(x_1) > 0$ for $x \in (-L,L)$,

(ii) $\psi_1'(x_1) < 0$ for $x \in (-L,\alpha)$,

(iii) $\psi_1'(x_1) > 0$ for $x \in (\alpha,L)$,

(iv) $\psi_1'(-L) = \psi_1'(L) = 0$.

One can easily prove existence of a function satisfying these conditions. Choose $\psi(x_1,x_2) = \psi_1(x_1)\psi_2(x_2)$. Then, ψ is a $\mathcal{C}^4(\mathbb{R}^d)$ function satisfying the conditions

Assumption 2.1 *We have:*

- $\psi(x) > 0$ *in* $\overline{\Omega}$,
- *There exists* $C_0 > 0$ *such that* $|\nabla\psi| \geq C_0 > 0 \quad$ *in* $\quad\Omega$,
- $\partial_\nu\psi \leq 0$ *on* $(\partial\Omega\setminus\Gamma^+)$,
- $\partial_{x_1}\psi(x) < 0$ *for* $x \in (-L,\alpha)\times\mathcal{D}$,
- $\partial_{x_1}\psi(x) > 0$ *for* $x \in (\alpha,L)\times\mathcal{D}$.

Now, let us introduce the function

$$\eta(t,x) = g(t)\left(e^{2\lambda|\psi|_\infty} - e^{\lambda\psi(x)}\right), \quad \rho > 0 \tag{2.3}$$

with

$$g(t) = \frac{1}{t(T-t)}.$$

We consider the following Carleman estimate.

Theorem 2.2 (Theorem 3.4, [Ch09]) *Let Assumption 2.1 be fulfilled. Then, there exist three constants* s_0, C *and* λ *depending of* Ω, T, Γ^+, $|A|_{L^\infty(Q,\mathbb{R}^d)}$ *and* $|B|_{L^\infty(Q)}$ *such that*

$$\int_Q e^{-2s\eta}\left[(sg)^{-1}(\Delta u)^2 + (sg)^{-1}(\partial_t u)^2 + sg|\nabla u|^2 + (sg)^3u^2\right] dx\ dt \tag{2.4}$$

$$\leq C\left(\int_Q e^{-2s\eta}(Pu)^2\ dx\ dt + \int_{(0,T)\times\Gamma^+} e^{-2s\eta} sg(\partial_\nu u)^2 d\sigma\ dt\right)$$

for $s \geq s_0$ *and* $u \in \mathcal{C}^{2,1}(Q)$, $u = 0$ *on* $\overline{\Sigma}$.

2.2. Inverse Problem

Let u be solution of

$$\begin{cases} \partial_t u - \Delta u + q(t,x_2)f(x_1)u = 0 & \text{in} \quad Q, \\ u(t,x) = b(t,x) & \text{on} \quad \Sigma_1, \\ \partial_\nu u(t,-L,x_2) = k_-(t,x), \ \ \partial_\nu u(t,L,x_2) = k_+(t,x) & \text{on} \quad (0,T)\times\mathcal{D}, \\ u(0,x) = u_0(x) & \text{in} \quad \Omega, \end{cases}$$

and $\widetilde{u}$ be solution of

$$\begin{cases} \partial_t \widetilde{u} - \Delta \widetilde{u} + \widetilde{q}(t,x_2)f(x_1)\widetilde{u} = 0 & \text{in} \quad Q, \\ \widetilde{u}(t,x) = b(t,x) & \text{on} \quad \Sigma_1, \\ \partial_\nu \widetilde{u}(t,-L,x_2) = k_-(t,x_2), \ \ \partial_\nu \widetilde{u}(t,L,x_2) = k_+(t,x_2) & \text{on} \quad (0,T)\times\mathcal{D}, \\ \widetilde{u}(0,x) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{2.5}$$

Let us consider the following conditions.

Assumption 2.3 *Here we assume that*

- $q(t,x_2)f(x_1), \widetilde{q}(t,x_2)f(x_1) \in \mathcal{C}^{1+\alpha,\frac{\alpha}{2}}(\overline{Q})$,
- $b \in \mathcal{C}^{3+\alpha,1+\frac{\alpha}{2}}([0,T]\times[-L,L]\times\partial\mathcal{D})$,
- $u_0 \in \mathcal{C}^{3,\alpha}(\overline{\Omega})$,
- $k^\pm \in \mathcal{C}^{2+\alpha,1+\frac{\alpha}{2}}([0,T]\times\overline{\mathcal{D}})$,
- $\partial_t b(0,x) - \Delta_x u_0(x) + q(0,x_2)f(x_1)u_0(x) = 0, \quad x=(x_1,x_2)\in[-L,L]\times\partial\mathcal{D}$,
- $\partial_t b(0,x) - \Delta_x u_0(x) + \widetilde{q}(0,x_2)f(x_1)u_0(x) = 0, \quad x=(x_1,x_2)\in[-L,L]\times\partial\mathcal{D}$,
- $f>0,\ b>0,\ u_0>0,$
- $\widetilde{u}(t,-L,x_2)>0, \quad \widetilde{u}(t,L,x_2)>0, \quad (t,x_2)\in[0,T]\times\overline{\mathcal{D}}.$

Notice that Assumption 2.3 and the maximum principle implies that $\widetilde{u}(t,x)>0$ for $(t,x)\in\overline{Q}$. Moreover, according to [LSU, Chapter 4 Section 5], we have u, $\widetilde{u}$, $\partial_{x_1}u$, $\partial x_1\widetilde{u}\in\mathcal{C}^{2,1}(\overline{Q})$.

Remark 2.4 *Assume that $b(t,x)$ can be extended to a function $b_1\in\mathcal{C}^{3+\alpha,1+\frac{\alpha}{2}}(\overline{\Sigma})$ such that $b_1>0$ and the compatibility condition*

$$\partial_t b_1(0,x) - \Delta_x u_0(x) + \widetilde{q}(0,x_2)f(x_1)u_0(x) = 0, \quad x\in\Gamma$$

is fulfilled. Let w be the solution of

$$\begin{cases} \partial_t w - \Delta w + \widetilde{q}(t,x_2)f(x_1)w = 0 & \mathit{in} \quad Q, \\ w(t,x) = b_1(t,x) & \mathit{on} \quad \Sigma, \\ w(0,x) = u_0(x) & \mathit{in} \quad \Omega, \end{cases}$$

Choose $k_\pm$ such that

$$\partial_\nu w(t,\pm L,x_2) = k_\pm(t,x_2), \quad (t,x_2)\in[0,T]\times\mathcal{D}.$$

Then w will be a solution of (2.5) *and by uniqueness we obtain*

$$w(t,x) = \widetilde{u}(t,x), \quad (t,x) \in Q.$$

It follows that $\widetilde{u}_{|\overline{\Sigma}} = b_1 > 0$ and the last item of Assumption 2.3 can be removed.

Now, if we set $v = u - \widetilde{u}$, then v satisfies

$$\begin{cases} \partial_t v - \Delta v + q(t,x_2)f(x_1)v = (\widetilde{q}(t,x_2) - q(t,x_2))f(x_1)\widetilde{u} & \text{in} \quad Q, \\ v(t,x) = 0 & \text{on} \quad \Sigma_1, \\ \partial_\nu v(t,-L,x_2) = 0, \;\; \partial_\nu v(t,L,x_2) = 0 & \text{on} \quad (0,T) \times \mathcal{D}, \\ v(0,x) = 0 & \text{in} \quad \Omega, \end{cases}$$

Thus with the change of function $w = \dfrac{v}{f\widetilde{u}}$, w is solution of the following system

$$\begin{cases} \partial_t w - \Delta w + \mathbb{A} \cdot \nabla w + aw = \widetilde{q}(t,x_2) - q(t,x_2) & \text{in} \quad Q, \\ w(t,x) = 0 & \text{on} \quad \Sigma_1, \\ \partial_\nu w(t,-L,x_2) = 0, \;\; \partial_\nu w(t,L,x_2) = 0 & \text{on} \quad (0,T) \times \mathcal{D}, \\ w(0,x) = 0 & \text{in} \quad \Omega, \end{cases}$$

where

$$\mathbb{A} = \frac{-2}{f\widetilde{u}} \nabla(f\widetilde{u}) \;\; \text{and} \;\; a = \frac{\partial_t(f\widetilde{u}) - \Delta(f\widetilde{u}) + V}{f\widetilde{u}}.$$

We consider the x_1-derivative of the previous system and we set $z := \partial_{x_1} w$. Let us observe that for all function $g \in \mathcal{C}^1(\overline{Q})$ we have

$$\partial_\nu g(t,L,x_2) = \partial_{x_1} g(t,L,x_2), \quad x_2 \in \mathcal{D} \tag{2.6}$$

and

$$\partial_\nu g(t,-L,x_2) = -\partial_{x_1} g(t,-L,x_2), \quad x_2 \in \mathcal{D}. \tag{2.7}$$

Moreover, if $g(t,x) = 0$ for $(t,x) \in (-L,L) \times \partial\mathcal{D}$, since ∂_{x_1} is a tangent derivative on $(-L,L) \times \partial D$, we have

$$\partial_{x_1} g(t,x_1,x_2) = 0, \quad (t,x_1,x_2) \in (0,T) \times (-L,L) \times \partial\mathcal{D}. \tag{2.8}$$

From (2.6), (2.7), (2.8) and the fact that $z \in \mathcal{C}^{2,1}(\overline{Q})$, we deduce that z is the solution of

$$\begin{cases} \partial_t z - \Delta z + \mathbb{A} \cdot \nabla z + az + B_1 z = B_2 \partial_{x_2} w + bw & \text{in} \quad Q, \\ z(t,x) = 0 & \text{on} \quad \Sigma, \\ z(0,x) = d(x) & \text{in} \quad \Omega \end{cases} \tag{2.9}$$

with

$$B_1 := -2\partial_{x_1}\left(\frac{\partial_{x_1}(f\widetilde{u})}{f\widetilde{u}}\right), \quad B_2 := 2\partial_{x_1}\left(\frac{\partial_{x_2}(f\widetilde{u})}{f\widetilde{u}}\right) \;\; \text{and} \;\; b = -\partial_{x_1} a.$$

Note that Assumption 2.3 implies that $\mathbb{A}$, B_1, B_2 and a, b are bounded.
Set

$$I_1(z) = \int_Q e^{-2s\eta}\left[(sg)^{-1}(\Delta z)^2 + (sz)^{-1}(\partial_t z)^2 + sg|\nabla z|^2 + (sg)^3 u^2\right] \, dx\, dt. \tag{2.10}$$

Applying the Carleman estimate (2.4) to $u = z$, we obtain

$$I_1(z) \le C \left[\int_{(0,T)\times\Gamma^+} e^{-2s\eta} sg(\partial_\nu z)^2 d\sigma dt \right.$$

$$\left. + \int_Q e^{-2s\eta} \left(|\nabla z|^2 + |z|^2\right) dx\ dt + \int_Q e^{-2s\eta} \left(|\nabla w|^2 + |w|^2\right) dx\ dt \right].$$

The second integral of the right hand side of the previous estimate is "absorbed" by the left hand side, for s sufficiently large. For the last integral, we need the following lemma proved in [Ch09]:

Lemma 2.5 *Let F be a function in $\mathcal{C}(\overline{Q})$). Then we have the following estimate:*

$$\int_Q \left| \int_\alpha^{x_1} F(t,\xi,x_2) d\xi \right|^2 e^{-2s\eta} dx_1\ dx_2\ dt \le C \int_Q |F(t,x)|^2 e^{-2s\eta} dx\ dt.$$

Proof

We recall here the proof of this lemma. An application of the Cauchy-Schwarz inequality yields

$$\int_Q \left| \int_\alpha^{x_1} F(t,\xi,x_2) d\xi \right|^2 e^{-2s\eta} dx_1\ dx_2\ dt \le C \int_Q \int_\alpha^{x_1} |F(t,\xi,x_2)|^2 d\xi e^{-2s\eta} dx_1\ dx_2\ dt. \tag{2.11}$$

Let $r(x_1,\xi)$ be defined by $r(x_1,\xi) = e^{-2s[\phi(t,x_1,x_2)-\phi(t,\xi,x_2)]}$. Note that

$$\partial_\xi r(x_1,\xi) = -2s\lambda \frac{\partial \psi}{\partial x_1}(\xi,x_2) g(t) e^{\lambda\psi(\xi,x_2)} r(x_1,\xi).$$

Thus, Assumption 2.1 implies that $\partial_\xi r(x_1,\xi) < 0$ for $\alpha < \xi < x_1 < L$ and $\partial_\xi r(x_1,\xi) > 0$ for $-L < x_1 < \xi < \alpha$. Then, in the region

$$\{(x_1,\xi):\ \alpha \le \xi \le x_1 \le L\} \cup \{(x_1,\xi):\ -L \le x_1 \le \xi \le \alpha\}$$

we have

$$r(x_1,\xi) \le r(x_1,x_1) = 1. \tag{2.12}$$

Applying estimate (2.11) , we get

$$\int_Q \left| \int_\alpha^{x_1} F(t,\xi,x_2) d\xi \right|^2 e^{-2s\eta} dx\ dt$$

$$\le C \left[\int_0^T \int_{\mathcal{D}} \int_{-L}^{\alpha} \int_{x_1}^{\alpha} |F(t,\xi,x_2)|^2 e^{-2s\eta(t,\xi,x_2)} r(x_1,\xi) d\xi\ dx_1\ dx_2\ dt \right.$$

$$\left. + \int_0^T \int_{\mathcal{D}} \int_{\alpha}^{L} \int_{\alpha}^{x_1} |F(t,\xi,x_2)|^2 e^{-2s\eta(t,\xi,x_2)} r(x_1,\xi) d\xi\ dx_1\ dx_2\ dt \right]. \tag{2.13}$$

For the first term on the right hand side of (2.13), formula (2.12) implies

$$\int_0^T \int_{\mathcal{D}} \int_{-L}^{\alpha} \int_{x_1}^{\alpha} |F(t,\xi,x_2)|^2 e^{-2s\eta(t,\xi,x_2)} r(x_1,\xi) d\xi\ dx_1\ dx_2\ dt$$

$$\le \int_0^T \int_{\mathcal{D}} \int_{-L}^{\alpha} \int_{x_1}^{\alpha} |F(t,\xi,x_2)|^2 e^{-2s\eta(t,\xi,x_2)} d\xi\ dx_1\ dx_2\ dt$$

and for the second term on the right hand side of (2.13) we obtain

$$\int_0^T \int_{\mathcal{D}} \int_\alpha^L \int_\alpha^{x_1} |F(t,\xi,x_2)|^2 e^{-2s\eta(t,\xi,x_2)} r(x_1,\xi) d\xi \ dx_1 \ dx_2 \ dt$$
$$\leq \int_0^T \int_{\mathcal{D}} \int_\alpha^L \int_\alpha^{x_1} |F(t,\xi,x_2)|^2 e^{-2s\eta(t,\xi,x_2)} d\xi \ dx_1 \ dx_2 \ dt.$$

We deduce easily Lemma 2.5 from these estimates.

■

Now let use return to the stability result.

2.3. Stability result

In this subsection we consider u $\widetilde{u}$, v, w and z introduced in the previous subsection. Our goal is to use the Carleman estimate (2.4). For this purpose, we will exploit the fact that in the bounded wave guide Ω a derivation with respect to x_1 does not alter the Dirichlet condition on Σ_1 and that the Neumann condition on Σ_2 becomes a Dirichlet condition. The main result of this subsection is the following stability estimate.

Theorem 2.6 *Let Assumptions 2.3 be fulfilled. Let $r > 0$ be such that*

$$r \geq max\left(\ |q \ |_{L^2((0,T)\times\mathcal{D})}, \ |\widetilde{q} \ |_{L^2((0,T)\times\mathcal{D})}\right).$$

Then, for any $0 < \varepsilon < \frac{T}{2}$, there exists a constant $C_\varepsilon > 0$ depending of ε, b, u_0, $k^\pm$ and r such that

$$\|q-\widetilde{q}\|^2_{L^2((\varepsilon,T-\varepsilon)\times\mathcal{D})} \leq C_\varepsilon \Big[\|\partial_\nu\partial_{x_1}\widetilde{u} - \partial_\nu\partial_{x_1}u\|^2_{L^2((0,T)\times\Gamma^+)}$$
$$+\|\widetilde{u}(\cdot,\alpha,\cdot) - u(\cdot,\alpha,\cdot)\|^2_{H^1_t\left(0,T, \ H^2_{x_2}(\mathcal{D})\right)}\Big]$$

Proof

According to [LSU] and the maximum principle, Assumptions 2.3 implies that $u, \widetilde{u}, \partial_{x_2}u, \partial_{x_2}\widetilde{u} \in \mathcal{C}^1(Q)$ and $f\widetilde{u} \geq c_1 > 0$. Thus, w, $\partial_{x_2}w \in \mathcal{C}^1(Q)$ and one can write

$$w(t,x) = \int_\alpha^{x_1} z(t,x',x_2) \ dx' + w(t,\alpha,x_2), \quad (t,x_1,x_2) \in Q, \tag{2.14}$$

$$\partial_{x_2}w(t,x) = \int_\alpha^{x_1} \partial_{x_2}z(t,x',x_2) \ dx' + \partial_{x_2}w(t,\alpha,x_2), \quad (t,x_1,x_2) \in Q. \tag{2.15}$$

Let us consider the source term $B_2\partial_{x_2}w + bw$ of (2.9). Using representations (2.14) and (2.15), we get

$$\int_Q e^{-2s\eta}(B_2\partial_{x_2}w + bw)^2 \ dx \ dt \leq 4\int_Q e^{-2s\eta}\left(B_2 \int_\alpha^{x_1} \partial_{x_2}z(t,x',x_2) \ dx'\right)^2 \ dx \ dt$$
$$+4\int_Q e^{-2s\eta}\left(B_2\partial_{x_2}w(t,\alpha,x_2)\right)^2 \ dx \ dt + 4\int_Q e^{-2s\eta}\left(b\int_\alpha^{x_1} z(t,x',x_2) \ dx'\right)^2 \ dx \ dt$$
$$+4\int_Q e^{-2s\eta}\left(bw(t,\alpha,x_2)\right)^2 \ dx \ \ dt.$$

Then, applying Lemma 2.5, we obtain

$$\int_Q e^{-2s\eta}(B_2\partial_{x_2}w+bw)^2\ dx\ dt \leq C\left(\int_Q e^{-2s\eta}\left(|\partial_{x_2}(z)|^2+|z|^2\right)\ dx\ dt\right)$$

$$+C\left(\int_Q e^{-2s\eta}\left(|w(t,\alpha,x_2)|^2+|\partial_{x_2}w(t,\alpha,x_2)|^2\right)\ dx\ dt\right). \tag{2.16}$$

Note that Assumption 2.1 implies

$$\left(\int_Q e^{-2s\eta}\left(|w(t,\alpha,x_2)|^2+|\partial_{x_2}w(t,\alpha,x_2)|^2\right)\ dx\ dt\right)$$

$$\leq C\int_0^T\int_{\mathcal{D}}(|w(t,\alpha,x_2)|^2+|\partial_{x_2}w(t,\alpha,x_2)|^2)\ dx_2\ dt.$$

Combining this estimate with (2.16), we obtain

$$\int_Q e^{-2s\eta}(B_2\partial_{x_2}w+bw)^2\ dx\ dt \leq C\left(\int_Q e^{-2s\eta}\left(|\partial_{x_2}(z)|^2+|z|^2\right)\ dx\ dt\right.$$

$$\left.+\int_0^T\int_{\mathcal{D}}\left(|w(t,\alpha,x_2)|^2+|\partial_{x_2}w(t,\alpha,x_2)|^2\right)\ dx_2\ dt\right). \tag{2.17}$$

An application of the Carleman estimate (2.4) to z yields

$$\int_Q e^{-2s\eta}\left[(sg)^{-1}(\Delta z)^2+(sg)^{-1}(\partial_t z)^2+sg\ |\nabla z|^2+(sg)^3z^2\right]\ dx\ dt$$
$$\leq C\left(\int_Q e^{-2s\eta}(B_2\partial_{x_2}w+bw)^2\ dx\ dt+\int_{(0,T)\times\Gamma^+}e^{-2s\eta}sg(\partial_\nu z)^2\ d\sigma\ dt\right).$$

Combining this estimate with (2.17), we obtain

$$\int_Q e^{-2s\eta}\left[(sg)^{-1}(\Delta z)^2+(sg)^{-1}(\partial_t z)^2+sg|\nabla z|^2+(sg)^3z^2\right]\ dx\ dt$$
$$\leq C\left(\int_Q e^{-2s\eta}\left(|\partial_{x_2}(z)|^2+|z|^2\right)\ dx\ dt+\int_0^T\int_{\mathcal{D}}\left(|w(t,\alpha,x_2)|^2+|\partial_{x_2}w(t,\alpha,x_2)|^2\right)\ dx_2\ dt\right.$$
$$\left.+\int_{(0,T)\times\Gamma^+}e^{-2s\eta}sg(\partial_\nu z)^2\ d\sigma\ dt\right).$$

Then, for s sufficiently large, we get

$$\int_{Q_\varepsilon}\left(|\partial_t z|^2+|\Delta z|^2+|\nabla z|^2+|z|^2\right)\ dx\ dt \tag{2.18}$$
$$\leq C_\varepsilon\left(\int_0^T\int_{\Gamma^+}|\partial_\nu z|^2\ \ d\sigma\ dt+\int_0^T\int_{\mathcal{D}}\left(|w(t,\alpha,x_2)|^2+|\partial_{x_2}w(t,\alpha,x_2)|^2\right)\ dx_2\ dt\right).$$

with $Q_\varepsilon=(\varepsilon,T-\varepsilon)\times(-L,L)\times\mathcal{D}$. Now, note that

$$\widetilde{q}(t,x_2)-q(t,x_2)=\partial_t w-\Delta w+\mathbb{A}\cdot\nabla w+aw=Pw.$$

Thus, applying (2.14), one get

$$\widetilde{q}(t,x_2)-q(t,x_2)=P\int_{\alpha}^{x_1} z(t,x',x_2)\,dx'+Pw(t,\alpha,x_2).$$

Then, this representation and estimate (2.18) imply

$$\begin{aligned}&\int_{\varepsilon}^{T-\varepsilon}\int_{\mathcal{D}}\left(\widetilde{q}(t,x_2)-q(t,x_2)\right)^2\ dx_2\ dt\\ &\leq C\left(\int_{Q_\varepsilon}\left(|\partial_t z|^2+|\Delta z|^2+|\nabla z|^2+|z|^2\right)\ dx\ dt+\|w(\cdot,\alpha,\cdot)\ |^2_{H^1_t\left(0,T,H^2_{x_2}(\mathcal{D})\right)}\right)\\ &\leq C_\varepsilon\left(\int_0^T\int_{\Gamma^+}|\partial_\nu z|^2\ \ d\sigma\ dt+\ |w(\cdot,\alpha,\cdot)\ |^2_{H^1_t\left(0,T,H^2_{x_2}(\mathcal{D})\right)}\right).\end{aligned}$$

This completes the proof.

■

3. Stability for an unbounded waveguide

In this section we consider problem (1.2). We will establish a stability result and deduce a uniqueness for the coefficient q. We give the result for the open waveguide with Dirichlet boundary conditions. The Carleman estimate for unbounded domain will be the key ingredient in the proof of such a stability estimate.
From now on, we set

$$\Omega=\mathbb{R}\times\mathcal{D},\ Q=(0,T)\times\Omega,\ \Sigma=(0,T)\times\partial\Omega.$$

3.1. Global Carleman Estimate

Let $f(x_1)$ be a bounded positive function in $\mathcal{C}^2(\mathbb{R})$ such that $f(x_1)\geq c_{min}>0$, f and all its derivatives up to order two are bounded by a positive constant $\tilde{C}_0$.
Let $u=u(t,x)$ be a function equals to zero on $(0,T)\times\partial\Omega$ and solution of the Heat equation

$$\partial_t u-\Delta u=F.$$

We prove here a global Carleman-type estimate for u with a single observation acting on a part Γ_1^+ of the boundary Γ of $\mathcal{D}$ in the right-hand side of the estimate. Let ψ_2 be the function defined in subsection 2.1 and let ψ be a $\mathcal{C}^4(\mathbb{R}^d)$ function defined by

$$\psi(x_1,x_2)=e^{x_1}\psi_2(x_2).$$

Then, the function ψ satisfies the conditions:

Assumption 3.1

- $\psi(x)>0$ *in* $\overline{\Omega}$,
- *There exists* $C_0>0$ *such that* $|\nabla\psi|\geq C_0>0$ *in* Ω,
- $\partial_\nu\psi\leq 0$ *on* $(0,T)\times\mathbb{R}\times\Gamma_1^-$,
- $\inf_{x\in\overline{\Omega}}\partial_{x_1}\psi(x)>0$,

- $\lim_{x_1\to\pm\infty} \inf_{x_2\in\overline{D}} \left|\dfrac{\psi(x_1,x_2)}{x_1}\right| = +\infty.$

Now let us introduce the function

$$\varphi(t,x) = g(t)e^{\lambda\psi(x)}, \quad \lambda > 0 \ \text{ with } \ g(t) = \frac{1}{t(t-T)}.$$

Let H be the operator defined by

$$Hu := \partial_t u - \Delta u \ \text{ in } \ Q = \Omega \times (0,T). \tag{3.1}$$

We set $w = e^{-s\varphi}u$, $Mw = e^{-s\varphi}H(e^{s\varphi}w)$ for $s > 0$ and we introduce the following operators

$$M_1 w := -\Delta w - s^2|\nabla\varphi|^2 w - s\partial_t\varphi w, \tag{3.2}$$

$$M_2 w := \partial_t w + 2s\nabla\varphi\cdot\nabla w + s\Delta\varphi w. \tag{3.3}$$

Then the following result holds.

Theorem 3.2 *Let H, M_1, M_2 be the operators defined respectively by (3.1), (3.2), (3.3). We assume that Assumptions 3.1 are satisfied. Then there exist $\lambda_0 > 0$, $s_0 > 0$ and a positive constant $C = C(\Omega,\Gamma,T)$ such that, for any $\lambda \geq \lambda_0$ and any $s \geq s_0$, the next inequality holds:*

$$\begin{aligned} &s^3\lambda^4\int_Q e^{-2s\varphi}\varphi^3|u|^2\,dx\,dt + s\lambda\int_Q e^{-2s\varphi}\varphi|\nabla u|^2\,dx\,dt + \|M_1(e^{-s\varphi}u)\|^2_{L^2(Q)} \\ &+\|M_2(e^{-s\varphi}u)\|^2_{L^2(Q)} \leq C\left[s\lambda\int_0^T\int_{\mathbb{R}_{x_1}}\int_{\Gamma_1^+} e^{-2s\varphi}\varphi|\partial_\nu u|^2\,\partial_\nu\psi\,d\sigma\,dt\right. \\ &\left.+\int_Q e^{-2s\varphi}\,|Hu|^2\,dx\,dt\right], \end{aligned} \tag{3.4}$$

for all u satisfying $Hu \in L^2(\Omega\times(0,T))$, $u \in L^2(0,T;H_0^1(\Omega))$, $\partial_\nu u \in L^2(0,T;L^2(\Gamma))$.

Note that this theorem has already been proved in [FI], [I] and [T]. In the inequality (3.4) , we can also have an estimate of $\partial_t u$ and Δu (see [FI]).

3.2. Inverse Problem

In this subsection, we establish a stability result and deduce a uniqueness result for the coefficient q.
The Carleman estimate (3.4) will be the key ingredient in the proof of such a stability estimate.
Let u be solution of

$$\begin{cases} \partial_t u - \Delta u + q(t,x_2)f(x_1)u = 0 & \text{in} \quad Q, \\ u(t,x) = b(t,x) & \text{on} \quad \Sigma, \\ u(0,x) = u_0(x) & \text{in} \quad \Omega, \end{cases}$$

and $\widetilde{u}$ be solution of

$$\begin{cases} \partial_t \widetilde{u} - \Delta \widetilde{u} + \widetilde{q}(t,x_2) f(x_1) \widetilde{u} = 0 & \text{in} \quad Q, \\ \widetilde{u}(t,x) = b(t,x) & \text{on} \quad \Sigma, \\ \widetilde{u}(0,x) = u_0(x) & \text{in} \quad \Omega, \end{cases}$$

Assumption 3.3 *Here we assume that:*

- $q(t,x_2)f(x_1), \widetilde{q}(t,x_2)f(x_1) \in \mathcal{C}^{1+\alpha,1+\frac{\alpha}{2}}(\overline{Q}) \cap L^\infty(Q)$,
- $b \in \mathcal{C}^{3+\alpha,1+\frac{\alpha}{2}}(\overline{\Sigma}) \cap L^2((0,T); H^{\frac{5}{2}}(\partial\Omega)) \cap H^{\frac{3}{4}}((0,T); H^1(\partial\Omega))$,
- $u_0 \in \mathcal{C}^{3,\alpha}(\overline{\Omega}) \cap H^3(\Omega)$,
- $\partial_t b(0,x) - \Delta_x u_0(x) + q(0,x_2) f(x_1) u_0(x) = 0, \quad x = (x_1,x_2) \in \partial\Omega$,
- $\partial_t b(0,x) - \Delta_x u_0(x) + \widetilde{q}(0,x_2) f(x_1) u_0(x) = 0, \quad x = (x_1,x_2) \in \partial\Omega$,
- *There exists* $r > 0$ *such that* $b \geq r$ *and* $u_0 \geq r$.

If we set $v = u - \widetilde{u}$, then v satisfies

$$\begin{cases} \partial_t v - \Delta v + q(t,x_2) f(x_1) v = (\widetilde{q}(t,x_2) - q(t,x_2)) f(x_1) \widetilde{u} & \text{in} \quad Q, \\ v(t,x) = 0 & \text{on} \quad \Sigma, \\ v(0,x) = 0 & \text{in} \quad \Omega. \end{cases}$$

Then with the change of function $w = \dfrac{v}{f\widetilde{u}}$, w is solution of the following system

$$\begin{cases} \partial_t w - \Delta w + \mathbb{A} \cdot \nabla w + a w = \widetilde{q}(t,x_2) - q(t,x_2) & \text{in} \quad Q, \\ w(t,x) = 0 & \text{on} \quad \Sigma, \\ w(0,x) = 0 & \text{in} \quad \Omega. \end{cases}$$

where

$$\mathbb{A} = \frac{-2}{f\widetilde{u}} \nabla(f\widetilde{u}) \quad \text{and} \quad a = \frac{\partial_t(f\widetilde{u}) - \Delta(f\widetilde{u}) + V}{f\widetilde{u}}.$$

We consider the x_1-derivative of the previous system and we set $z := \partial_{x_1} w$, then z is solution of

$$\begin{cases} \partial_t z - \Delta z + \mathbb{A} \cdot \nabla z + a z + B_1 z + B_2 \partial_{x_2} w + b w = 0 & \text{in} \quad Q, \\ z(t,x) = 0 & \text{on} \quad \Sigma, \\ z(0,x) = d(x) & \text{in} \quad \Omega \end{cases} \tag{3.5}$$

with

$$B_1 := -2\partial_{x_1} \left(\frac{\partial_{x_1}(f\widetilde{u})}{f\widetilde{u}} \right), \quad B_2 := -2\partial_{x_1} \left(\frac{\partial_{x_2}(f\widetilde{u})}{f\widetilde{u}} \right) \quad \text{and} \quad b = \partial_{x_1} a.$$

Assumption 3.4 $\mathbb{A}$, B_1, B_2 *and* a, b *are bounded.*

We can apply the Carleman estimate (3.4) for z and we obtain:

$$I(z) \leq C \left[s\lambda \int_0^T \int_{\mathbb{R}_{x_1}} \int_{\Gamma_1^+} e^{-2s\varphi} \varphi |\partial_\nu z|^2 \, \partial_\nu \psi \, d\sigma \, dt \right. \tag{3.6}$$

$$\left. \int_Q e^{-2s\varphi} \left(|\nabla z|^2 + |z|^2 \right) dx \, dt + \int_Q e^{-2s\varphi} \left(|\nabla w|^2 + |w|^2 \right) dx \, dt \right].$$

The second integral of the right hand side of the previous estimate is "absorbed" by the left hand side, for s sufficiently large. For the last integral, we need the following lemma which is an adaptation of a lemma proved in [K] and [KT]:

Lemma 3.5 *Let F be a function in $L^2(Q)\cap\mathcal{C}(\overline{Q})$. Then we have the following estimate:*

$$\int_Q \left| \int_\alpha^{x_1} F(t,\xi,x_2)d\xi \right|^2 e^{-2s\varphi}dx_1\, dx_2\, dt \leq \frac{C}{s^2}\int_Q |F(t,x)|^2 e^{-2s\varphi}dx\, dt.$$

Proof
We recall here the proof of this lemma.

$$\begin{aligned} I &:= \int_Q \left| \int_\alpha^{x_1} F(t,\xi,x_2)d\xi \right|^2 e^{-2s\varphi}dx_1\, dx_2\, dt \\ &= \int_0^T \int_{\mathcal{D}} \int_{-\infty}^{\alpha} \left| \int_\alpha^{x_1} F(t,\xi,x_2)d\xi \right|^2 e^{-2s\varphi}dx_1\, dx_2\, dt \\ &\quad + \int_0^T \int_{\mathcal{D}} \int_\alpha^{+\infty} \left| \int_\alpha^{x_1} F(t,\xi,x_2)d\xi \right|^2 e^{-2s\varphi}dx_1\, dx_2\, dt \\ &:= I_1 + I_2. \end{aligned}$$

Note that

$$e^{-2s\varphi} = \frac{2s\partial_{x_1}\varphi}{2s\partial_{x_1}\varphi}e^{-2s\varphi} = \frac{1}{2s\partial_{x_1}\varphi}\partial_{x_1}(-e^{2s\varphi})$$

and according to Assumption 3.1 there exists a positive constant κ such that

$$\partial_{x_1}\varphi \geq \kappa > 0.$$

So, we have

$$e^{-2s\varphi} = \frac{1}{2s\partial_{x_1}\varphi}\partial_{x_1}(-e^{-2s\varphi}) \leq \frac{1}{2s\kappa}\partial_{x_1}(-e^{-2s\varphi}).$$

We first give an estimate for I_1:

$$\begin{aligned} I_1 &= \int_0^T \int_{\mathcal{D}} \int_{-\infty}^{\alpha} \left| \int_\alpha^{x_1} F(t,\xi,x_2)d\xi \right|^2 e^{-2s\varphi}dx_1\, dx_2\, dt \\ &\leq \frac{1}{2s\kappa} \int_0^T \int_{\mathcal{D}} \int_{-\infty}^{\alpha} \left| \int_\alpha^{x_1} F(t,\xi,x_2)d\xi \right|^2 \partial_{x_1}(-e^{-2s\varphi})dx_1\, dx_2\, dt. \end{aligned}$$

Using integration by parts and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} I_1 &= \frac{1}{s\kappa}\int_Q e^{-s\varphi}F(t,x)\left(\int_\alpha^{x_1} F(t,\xi,x_2)d\xi\right)e^{-s\varphi}dx_1\, dx_2\, dt \\ &\leq \frac{C}{2s\kappa}\left(\int_Q \left| \int_\alpha^{x_1} F(t,\xi,x_2)d\xi \right|^2 e^{-2s\varphi}dx_1\, dx_2\, dt\right)^{1/2}\left(\int_Q |F(t,x)|^2 e^{-2s\varphi}dx\, dt\right)^{1/2}. \end{aligned}$$

That is

$$I_1 \leq \frac{C}{s\kappa}I^{1/2}\left(\int_Q |F(t,x)|^2 e^{-2s\varphi}dx\, dt\right)^{1/2}.$$

In the same way, we obtain

$$I_2 \le \frac{C}{s\kappa} I^{1/2} \left(\int_Q |F(t,x)|^2 \, e^{-2s\varphi} dx \; dt \right)^{1/2}$$

and therefore

$$I \le \frac{2C}{s\kappa} I^{1/2} \left(\int_Q |F(t,x)|^2 \, e^{-2s\varphi} dx \; dt \right)^{1/2}.$$

This complete the proof.

∎

Now we come back to the inequality (3.6) in order to estimate the last integral of the right hand side.
A direct application of Lemma 3.5 leads to:

$$\int_Q e^{-2s\varphi} \left(|\nabla w|^2 + |w|^2 \right) \, dx \; dt \le \frac{C}{s^2} \int_Q e^{-2s\varphi} \left(|\nabla z|^2 + |z|^2 \right) \, dx \; dt.$$

So (3.6) becomes

$$I(z) \le C s\lambda \int_0^T \int_{\mathbb{R}_{x_1}} \int_{\Gamma_1^+} e^{-2s\varphi} \varphi |\partial_\nu z|^2 \; \partial_\nu \psi \; d\sigma \; dt + \frac{C}{s^2} \int_Q e^{-2s\varphi} \left(|\nabla z|^2 + |z|^2 \right) \, dx \; dt.$$

The last integral of the right hand side is "absorbed" by the left hand side, for s sufficiently large. We thus obtain a Carleman estimate for z solution of (3.5):

$$I(z) \le C s\lambda \int_0^T \int_{\mathbb{R}_{x_1}} \int_{\Gamma_1^+} e^{-2s\varphi} \varphi |\partial_\nu z|^2 \; \partial_\nu \psi \; d\sigma \; dt$$

3.3. Stability Estimate

In this subsection we consider u $\widetilde{u}$, v, w and z introduced in the previous subsection. We will exploit the fact that in the wave guide Ω derivations with respect to x_1 do not alter the Dirichlet condition. The main result of this subsection is the following stability estimate.

Theorem 3.6 *Let Assumptions 3.3 and 3.4 be fulfilled. Let $\alpha \in \mathbb{R}$ and $r > 0$ be such that $r \ge max\left(\|q\|_{L^2((0,T)\times\mathcal{D})}, \|\widetilde{q}\|_{L^2((0,T)\times\mathcal{D})} \right)$. Then, for any $0 < \varepsilon < \frac{T}{2}$, there exists a constant $C_\varepsilon > 0$ depending of ε, b, u_0, $k^\pm$ and r such that*

$$\|q - \widetilde{q}\|^2_{L^2((\varepsilon, T-\varepsilon)\times\mathcal{D})} \le C_\varepsilon \Big[\|\partial_\nu \partial_{x_1} \widetilde{u} - \partial_\nu \partial_{x_1} u\|^2_{L^2((0,T)\times\mathbb{R}_{x_1}\times\Gamma_1^+)} + \|\widetilde{u}(\cdot,\alpha,\cdot) - u(\cdot,\alpha,\cdot)\|^2_{H_t^1\left(0,T,\; H^2_{x_2}(\mathcal{D})\right)} \Big] \tag{3.7}$$

Proof

According to [LSU] and the maximum principle, Assumptions 3.3 implies that $u, \widetilde{u}, \partial_{x_2} u, \partial_{x_2} \widetilde{u} \in \mathcal{C}^1(Q)$ and $f\widetilde{u} \ge c_1 > 0$. Thus, w, $\partial_{x_2} w \in \mathcal{C}^1(Q)$ and one can write

$$w(t,x) = \int_\alpha^{x_1} z(t,x',x_2) \; dx' + w(t,\alpha,x_2), \quad (t,x_1,x_2) \in Q,$$

$$\partial_{x_2} w(t,x) = \int_\alpha^{x_1} \partial_{x_2} z(t,x',x_2)\ dx' + \partial_{x_2} w(t,\alpha,x_2), \quad (t,x_1,x_2) \in Q.$$

Let us consider the source term $B_2\partial_{x_2}w + bw$ of (3.5). Combining Lemma 3.5 with some arguments used in Theorem 2.6, we obtain

$$\int_Q e^{-2s\varphi}(B_2\partial_{x_2}w + bw)^2\ dx\ dt \le \frac{C}{s^2}\left(\int_Q e^{-2s\varphi}\left(|\partial_{x_2}(z)|^2 + |z|^2\right)\ dx\ dt \right. \tag{3.8}$$
$$\left. + \int_Q e^{-2s\varphi}\left(|w(t,\alpha,x_2)|^2 + |\partial_{x_2} w(t,\alpha,x_2)|^2\right)\ dx\ dt\right).$$

Since

$$\sup_{(t,x_2)\in(0,T)\times\mathcal{D}} \int_{\mathbb{R}} e^{-s\varphi(t,x_1,x_2)} dx_1 < +\infty$$

we deduce that

$$\left(\int_Q e^{-s\varphi}\left(|w(t,\alpha,x_2)|^2 + |\partial_{x_2} w(t,\alpha,x_2)|^2\right)\ dx\ dt\right)$$
$$\le C\left(\int_0^T\int_{\mathcal{D}}\left(|w(t,\alpha,x_2)|^2 + |\partial_{x_2} w(t,\alpha,x_2)|^2\right)\ dx_2\ dt\right).$$

Combining this estimate with (3.8), we obtain

$$\int_Q e^{-2s\varphi}(B_2\partial_{x_2}w + bw)^2\ dx\ dt \le \frac{C}{s^2}\left(\int_Q e^{-2s\varphi}\left(|\partial_{x_2}(z)|^2 + |z|^2\right)\ dx\ dt \right. \tag{3.9}$$
$$\left. + \int_0^T\int_{\mathcal{D}}\left(|w(t,\alpha,x_2)|^2 + |\partial_{x_2} w(t,\alpha,x_2)|^2\right)\ dx_2\ dt\right).$$

An application of the Carleman estimate (3.4) to z yields

$$s^3\lambda^4\int_Q e^{-2s\varphi}\varphi^3|z|^2\ dx\ dt + s\lambda\int_Q e^{-2s\varphi}\varphi|\nabla z|^2\ dx\ dt + \|M_1(e^{-s\varphi}z)\|^2_{L^2(Q)} + \|M_2(e^{-s\varphi}z)\|^2_{L^2(Q)}$$
$$\le C\left[s\lambda\int_0^T\int_{\mathbb{R}_{x_1}}\int_{\Gamma_1^+} e^{-2s\varphi}\varphi|\partial_\nu z|^2\ \partial_\nu\psi\ d\sigma\ dt + \int_Q e^{-2s\varphi}\ |B_2\partial_{x_2}w + bw|^2\ dx\ dt\right].$$

Combining this estimate with (3.9), we find

$$s^3\lambda^4\int_Q e^{-2s\varphi}\varphi^3|z|^2\ dx\ dt + s\lambda\int_Q e^{-2s\varphi}\varphi|\nabla z|^2\ dx\ dt + s^{-1}\int_Q e^{-2s\varphi}\varphi^{-1}\left(|\partial_t z|^2 + |\Delta z|^2\right)\ dx\ dt$$
$$\le Cs\lambda\int_0^T\int_{\mathbb{R}_{x_1}}\int_{\Gamma_1^+} e^{-2s\varphi}\varphi|\partial_\nu z|^2\ \partial_\nu\psi\ d\sigma\ dt + \frac{C}{s^2}\left(\int_Q e^{-2s\varphi}\left(|\partial_{x_2}(z)|^2 + |z|^2\right)\ dx\ dt\right)$$
$$+ C\left(\int_0^T\int_{\mathcal{D}}\left(|w(t,\alpha,x_2)|^2 + |\partial_{x_2} w(t,\alpha,x_2)|^2\right)\ dx_2\ dt\right).$$

Then, using Assumption 3.1, for s and λ sufficiently large we get

$$\int_{Q_\varepsilon}\left(|\partial_t z|^2 + |\Delta z|^2 + |\nabla z|^2 + |z|^2\right)\ dx\ dt \le C\left(\int_0^T\int_{\Gamma_1^+}\varphi|\partial_\nu z|^2\ \partial_\nu\psi\ d\sigma\ dt\right)$$

$$+C_\varepsilon \left(\int_0^T \int_{\mathcal{D}} \left(|w(t,\alpha,x_2)|^2 + |\partial_{x_2} w(t,\alpha,x_2)|^2 \right) \; dx_2 \; dt \right). \tag{3.10}$$

with $Q_\varepsilon = (\varepsilon, T-\varepsilon) \times (-R, R) \times \mathcal{D}$. Combining this estimate with some arguments used in Theorem 2.6 , we deduce easily (3.7) and the proof of Theorem 3.6 is complete.

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