Junction between a plate and a rod of comparable thickness in nonlinear elasticity. Part II

Dominique Blanchard, Georges Griso

Junction between a plate and a rod of comparable thickness in nonlinear elasticity.

D. Blanchard[1], G. Griso[2]

[1] Université de Rouen, UMR 6085, 76801 Saint Etienne du Rouvray Cedex, France,
E-mail: dominique.blanchard@univ-rouen.fr

[2] Laboratoire J.-L. Lions–CNRS, Boîte courrier 187, Université Pierre et Marie Curie,
4 place Jussieu, 75005 Paris, France, Email: griso@ann.jussieu.fr

Abstract

We analyze the asymptotic behavior of a junction problem between a plate and a perpendicular rod made of a nonlinear elastic material. The two parts of this multi-structure have small thicknesses of the same order δ. We use the decomposition techniques obtained for the large deformations and the displacements in order to derive the limit energy as δ tends to 0.

KEY WORDS: nonlinear elasticity, junctions, straight rod, plate.
Mathematics Subject Classification (2000): 74B20, 74K10, 74K30.

1 Introduction

In a former paper [12] we derive the limit energy of the junction problem between a plate and a rod under an assumption that couples their respective thicknesses δ and ε to the order of the Lamé's coefficients of the materials in the plate and in the rod. This assumption precludes the case where the thicknesses have the same order and the structure is made of the same material (see equation 1.1 in the introduction of [12]). The aim of the present paper is to analyze this specific case for a total energy of order δ^5. As in [12], the structure is clamped on a part of the lateral boundary of the plate and it is free on the rest of its boundary.

The main difference here is the behavior in the rod in which, for this level of energy (which is higher than the maximum allowed in [12]), the stretching-compression is of order δ while the bending is of order $\delta^{1/2}$. The most important consequence is that in the limit model for the rod the stretching-compression is actually given by the bending in the rod (through a nonlinear relation) and by the bending in the plate at the junction point (see (6.15) and (6.16)). The bending and torsion models in the rod are the standard

linear ones. In the plate the limit model is the Von Kármán system in which the action of the rod is modelized by a punctual force at the junction.

Let us emphasize that in order to obtain sharp estimates on the deformations in the junction area, see Lemma 4.2, we use the decomposition techniques in thin domains (see [12],[22], [8], [7]). In order to scale the applied forces which induce a total energy energy of order δ^5, from Lemma 4.2 and [7], we derive a nonlinear Korn's inequality for the rod (as far as the plate is concerned this type of inequality is already established in [12]). The nonlinear character of these Korn's inequalities prompt us to adopt smallness assumptions on some components of the forces. Then, we are in a position to study the asymptotic behavior of the Green-St Venant's strain tensors in the two parts of the structure. At last this allows us to characterize the limit of the rescaled infimum of the 3d energy as the minimum of a functional over a set of limit admissible displacements which includes the nonlinear relation between the stretching-compression and the bending in the rod.

In Section 2 we introduce a few general notations. Section 3 gives a few recalls on the decomposition technique of the deformations in thin structures. In Section 4, we derive first estimates on the terms of the decomposition of a deformation in the rod and sharp estimates in the junction area. In the same section we also obtain Korn's inequality in the rod. In Section 5 we introduce the elastic energy and the assumptions on the applied forces in order to obtain a total elastic energy of order δ^5. In Section 6 we analyze the asymptotic behavior of the Green-St-Venant's strain tensors in the plate and in the rod. In Section 7 we prove the main result of the paper namely the characterization of the limit of the rescaled infimum of the 3d energy.

As general references on the theory of elasticity we refer to [2] and [14]. The reader is referred to [1], [29], [20] for an introduction of rods models and to [17], [16], [13], [18] for plate models. As far as junction problems in multi-structures we refer to [15], [16], [26], [27], [28], [3], [24], [25], [21], [19], [4], [5], [6], [23], [10], [11]. For the decomposition method in thin structures we refer to [20], [21], [22], [23], [7], [8], [9], [11].

2 Notations.

Let us introduce a few notations and definitions concerning the geometry of the plate and the rod. Let ω be a bounded domain in $\mathbb{R}^2$ with lipschitzian boundary included in the plane $(O;\mathbf{e}_1,\mathbf{e}_2)$ and such that $O \in \omega$. The plate is the domain

$$\Omega_\delta = \omega \times]-\delta,\delta[.$$

Let γ_0 be an open subset of $\partial\omega$ which is made of a finite number of connected components (whose closure are disjoint). The corresponding lateral part of the boundary of Ω_δ is

$$\Gamma_{0,\delta} = \gamma_0 \times]-\delta,\delta[.$$

The rod is defined by

$$B_\delta = D_\delta \times]-\delta, L[, \qquad D_\delta = D(O,\delta), \qquad D = D(O,1)$$

where $\delta > 0$ and where $D_r = D(O,r)$ is the disc of radius r and center the origin O. We assume that $D \subset\subset \omega$. The whole structure is denoted

$$\mathcal{S}_\delta = \Omega_\delta \cup B_\delta$$

while the junction is

$$C_\delta = \Omega_\delta \cap B_\delta = D_\delta \times]-\delta, \delta[.$$

We denote I_d the identity map of $\mathbb{R}^3$. The set of admissible deformations of the structure is

$$\mathbb{D}_\delta = \Big\{ v \in H^1(\mathcal{S}_\delta;\mathbb{R}^3) \ \mid \ v = I_d \ \text{ on } \ \Gamma_{0,\delta} \Big\}.$$

3 Some recalls.

To any vector $F \in \mathbb{R}^3$ we associate the antisymmetric matrix $\mathbf{A}_F$ defined by

$$\forall \mathbf{x} \in \mathbb{R}^3, \qquad \mathbf{A}_F\, \mathbf{x} = F \wedge \mathbf{x}. \tag{3.1}$$

From now on, in order to simplify the notations, for any open set $\mathcal{O} \subset \mathbb{R}^3$ and any field $u \in H^1(\mathcal{O};\mathbb{R}^3)$, we set

$$\mathbf{G}_s(u,\mathcal{O}) = ||\nabla u + (\nabla u)^T||_{L^2(\mathcal{O};\mathbb{R}^{3\times 3})}$$

and

$$\mathbf{d}(u,\mathcal{O}) = ||dist(\nabla u, SO(3))||_{L^2(\mathcal{O})}.$$

3.1 Recalls on the decompositions of the plate-displacement.

We know (see [21] or [22]) that any displacement $u \in H^1(\Omega_\delta;\mathbb{R}^3)$ of the plate is decomposed as

$$u(x) = \mathcal{U}(x_1,x_2) + x_3 \mathcal{R}(x_1,x_2) \wedge \mathbf{e}_3 + \overline{u}(x), \qquad x \in \Omega_\delta \tag{3.2}$$

where $\mathcal{U}$ and $\mathcal{R}$ belong to $H^1(\omega;\mathbb{R}^3)$ and $\overline{u}$ belongs to $H^1(\Omega_\delta;\mathbb{R}^3)$. The sum of the two first terms $U_e(x) = \mathcal{U}(x_1,x_2) + x_3\mathcal{R}(x_1,x_2) \wedge \mathbf{e}_3$ is called the elementary displacement associated to u.

The following Theorem is proved in [21] for the displacements in $H^1(\Omega_\delta;\mathbb{R}^3)$ and in [22] for the displacements in $W^{1,p}(\Omega_\delta;\mathbb{R}^3)$ $(1 < p < +\infty)$.

Theorem 3.1. *Let $u \in H^1(\Omega_\delta;\mathbb{R}^3)$, there exists an elementary displacement $U_e(x) = \mathcal{U}(x_1,x_2) + x_3\mathcal{R}(x_1,x_2) \wedge \mathbf{e}_3$ and a warping $\overline{u}$ satisfying* (3.2) *such that*

$$\begin{aligned}
&||\overline{u}||_{L^2(\Omega_\delta;\mathbb{R}^3)} \le C\delta \mathbf{G}_s(u,\Omega_\delta), \quad ||\nabla \overline{u}||_{L^2(\Omega_\delta;\mathbb{R}^3)} \le C\mathbf{G}_s(u,\Omega_\delta),\\
&\Big\| \frac{\partial \mathcal{R}}{\partial x_\alpha} \Big\|_{L^2(\omega;\mathbb{R}^3)} \le \frac{C}{\delta^{3/2}} \mathbf{G}_s(u,\Omega_\delta),\\
&\Big\| \frac{\partial \mathcal{U}}{\partial x_\alpha} - \mathcal{R} \wedge \mathbf{e}_\alpha \Big\|_{L^2(\omega;\mathbb{R}^3)} \le \frac{C}{\delta^{1/2}} \mathbf{G}_s(u,\Omega_\delta),\\
&||\nabla u - \mathbf{A}_{\mathcal{R}}||_{L^2(\omega;\mathbb{R}^9)} \le C\mathbf{G}_s(u,\Omega_\delta),
\end{aligned} \tag{3.3}$$

where the constant C does not depend on δ.

The warping $\overline{u}$ satisfies the following relations

$$\int_{-\delta}^{\delta} \overline{u}(x_1,x_2,x_3)dx_3 = 0, \qquad \int_{-\delta}^{\delta} x_3\overline{u}_\alpha(x_1,x_2,x_3)dx_3 = 0 \quad \text{for a.e. } (x_1,x_2)\in\omega. \tag{3.4}$$

If a deformation v belongs to $\mathbb{D}_\delta$ then the displacement $u = v - I_d$ is equal to 0 on $\Gamma_{0,\delta}$. In this case the the fields $\mathcal{U}$, $\mathcal{R}$ and the warping $\overline{u}$ satisfy

$$\mathcal{U} = \mathcal{R} = 0 \qquad \text{on} \quad \gamma_0, \qquad \overline{u} = 0 \qquad \text{on} \quad \Gamma_{0,\delta}. \tag{3.5}$$

Then, from (3.3), for any deformation $v \in \mathbb{D}_\delta$ the corresponding displacement $u = v - I_d$ verifies the following estimates (see also [21]):

$$\begin{aligned} ||\mathcal{R}||_{H^1(\omega;\mathbb{R}^3)} + ||\mathcal{U}_3||_{H^1(\omega)} &\le \frac{C}{\delta^{3/2}}\mathbf{G}_s(u,\Omega_\delta), \\ ||\mathcal{R}_3||_{L^2(\omega)} + ||\mathcal{U}_\alpha||_{H^1(\omega)} &\le \frac{C}{\delta^{1/2}}\mathbf{G}_s(u,\Omega_\delta). \end{aligned} \tag{3.6}$$

The constants depend only on ω. From the above estimates we deduce the following Korn's type inequalities for the displacement u

$$\begin{aligned} &||u_\alpha||_{L^2(\Omega_\delta)} \le C_0\mathbf{G}_s(u,\Omega_\delta), \qquad ||u_3||_{L^2(\Omega_\delta)} \le \frac{C_0}{\delta}\mathbf{G}_s(u,\Omega_\delta), \\ &||u-\mathcal{U}||_{L^2(\Omega_\delta;\mathbb{R}^3)} \le \frac{C}{\delta}\mathbf{G}_s(u,\Omega_\delta), \quad ||\nabla u||_{L^2(\Omega_\delta;\mathbb{R}^9)} \le \frac{C}{\delta}\mathbf{G}_s(u,\Omega_\delta). \end{aligned} \tag{3.7}$$

Due to Theorem 3.3 established in [8], the displacement $u = v - I_d$ is also decomposed as

$$u(x) = \mathcal{U}(x_1,x_2) + x_3(\mathbf{R}(x_1,x_2) - \mathbf{I}_3)\mathbf{e}_3 + \overline{v}(x), \qquad x\in\Omega_\delta \tag{3.8}$$

where $\mathbf{R}\in H^1(\omega;\mathbb{R}^{3\times3})$, $\overline{v}\in H^1(\Omega_\delta;\mathbb{R}^3)$ and we have the following estimates

$$\begin{aligned} &||\overline{v}||_{L^2(\Omega_\delta;\mathbb{R}^3)} \le C\delta\mathbf{d}(v,\Omega_\delta) \qquad ||\nabla\overline{v}||_{L^2(\Omega_\delta;\mathbb{R}^9)} \le C\mathbf{d}(v,\Omega_\delta) \\ &\Big\|\frac{\partial\mathbf{R}}{\partial x_\alpha}\Big\|_{L^2(\omega;\mathbb{R}^9)} \le \frac{C}{\delta^{3/2}}\mathbf{d}(v,\Omega_\delta) \\ &\Big\|\frac{\partial\mathcal{U}}{\partial x_\alpha} - (\mathbf{R}-\mathbf{I}_3)\mathbf{e}_\alpha\Big\|_{L^2(\omega;\mathbb{R}^3)} \le \frac{C}{\delta^{1/2}}\mathbf{d}(v,\Omega_\delta) \\ &\big\|\nabla v - \mathbf{R}\big\|_{L^2(\Omega_\delta;\mathbb{R}^9)} \le C\mathbf{d}(v,\Omega_\delta) \end{aligned} \tag{3.9}$$

where the constant C does not depend on δ. The following boundary conditions are satisfied

$$\mathcal{U} = 0, \quad \mathbf{R} = \mathbf{I}_3 \qquad \text{on} \quad \gamma_0, \qquad \overline{v} = 0 \qquad \text{on} \quad \Gamma_{0,\delta}. \tag{3.10}$$

3.2 Recall on the decomposition of the rod-deformation.

Now, we consider a deformation $v \in H^1(B_\delta;\mathbb{R}^3)$ of the rod B_δ. This deformation can be decomposed as (see Theorem 2.2.2 of [7])

$$v(x) = \mathcal{V}(x_3) + \mathbf{Q}(x_3)\big(x_1\mathbf{e}_1 + x_2\mathbf{e}_2\big) + \overline{w}(x), \qquad x \in B_\delta, \tag{3.11}$$

where $\mathcal{V}(x_3) = \dfrac{1}{|D_\delta|}\displaystyle\int_{D_\delta} v(x)dx_1dx_2$ belongs to $H^1(-\delta, L;\mathbb{R}^3)$, where $\mathbf{Q}$ belongs to $H^1(-\delta, L; SO(3))$ and $\overline{w}$ belongs to $H^1(B_\delta;\mathbb{R}^3)$. Let us give a few comments on the above decomposition. The term $\mathcal{V}$ gives the deformation of the center line of the rod. The second term $\mathbf{Q}(x_3)\big(x_1\mathbf{e}_1 + x_2\mathbf{e}_2\big)$ describes the rotation of the cross section (of the rod) which contains the point $(0, 0, x_3)$. The sum of the terms $\mathcal{V}(x_3)+\mathbf{Q}(x_3)\big(x_1\mathbf{e}_1+x_2\mathbf{e}_2\big)$ is called an elementary deformation of the rod.

The following theorem (see Theorem 2.2.2 of [7]) gives a decomposition (3.11) of a deformation and estimates on the terms of this decomposition.

Theorem 3.2. *Let $v \in H^1(B_\delta;\mathbb{R}^3)$, there exists an elementary deformation $\mathcal{V}(x_3) + \mathbf{Q}(x_3)\big(x_1\mathbf{e}_1 + x_2\mathbf{e}_2\big)$ and a warping $\overline{w}$ satisfying* (3.11) *and such that*

$$\begin{aligned}
&||\overline{w}||_{L^2(B_\delta;\mathbb{R}^3)} \le C\delta\mathbf{d}(v, B_\delta),\\
&||\nabla\overline{w}||_{L^2(B_\delta;\mathbb{R}^{3\times3})} \le C\mathbf{d}(v, B_\delta),\\
&\Big\|\frac{d\mathbf{Q}}{dx_3}\Big\|_{L^2(-\delta,L;\mathbb{R}^{3\times3})} \le \frac{C}{\delta^2}\mathbf{d}(v, B_\delta),\\
&\Big\|\frac{d\mathcal{V}}{dx_3} - \mathbf{Q}\mathbf{e}_3\Big\|_{L^2(-\delta,L;\mathbb{R}^3)} \le \frac{C}{\delta}\mathbf{d}(v, B_\delta),\\
&\big\|\nabla v - \mathbf{Q}\big\|_{L^2(B_\delta;\mathbb{R}^{3\times3})} \le C\mathbf{d}(v, B_\delta),
\end{aligned} \tag{3.12}$$

where the constant C does not depend on δ and L.

4 Preliminaries results

Let v be a deformation in $\mathbb{D}_\delta$. We set $u = v - I_d$. We decompose u as (3.2) and (3.8) in the plate and we decompose the deformation v as (3.11) in the rod.

4.1 A complement to the mid-surface bending.

Let us set

$$H^1_{\gamma_0}(\omega) = \{\varphi \in H^1(\omega)\ ;\ \varphi = 0 \ \text{ on } \gamma_0\}.$$

We define the function $\widetilde{\mathcal{U}}_3$ as the solution of the following variational problem :

$$\begin{cases} \widetilde{\mathcal{U}}_3 \in H^1_{\gamma_0}(\omega), \\ \displaystyle\int_\omega \nabla\widetilde{\mathcal{U}}_3 \nabla\varphi = \int_\omega (\mathbf{R}-\mathbf{I}_3)\mathbf{e}_\alpha\cdot\mathbf{e}_3 \frac{\partial\varphi}{\partial x_\alpha}, \\ \forall\varphi \in H^1_{\gamma_0}(\omega) \end{cases} \tag{4.1}$$

where $\mathbf{R}$ appears in the decomposition (3.8) of u. Due to (3.9), the function $\widetilde{\mathcal{U}}_3$ belongs to $H^1_{\gamma_0}(\omega)\cap H^2(D)$ and satisfies the estimates:

$$\begin{aligned} &||\widetilde{\mathcal{U}}_3||_{H^1(\omega)} \le \frac{C}{\delta^{3/2}}\mathbf{d}(v,\Omega_\delta), \quad ||\mathcal{U}_3-\widetilde{\mathcal{U}}_3||_{H^1(\omega)} \le \frac{C}{\delta^{1/2}}\mathbf{d}(v,\Omega_\delta), \\ &||\widetilde{\mathcal{U}}_3||_{H^2(D)} \le \frac{C}{\delta^{3/2}}\mathbf{d}(v,\Omega_\delta), \quad \Big\|\frac{\partial\widetilde{\mathcal{U}}_3}{\partial x_\alpha} - (\mathbf{R}-\mathbf{I}_3)\mathbf{e}_\alpha\cdot\mathbf{e}_3\Big\|_{H^1(D_\delta)} \le \frac{C}{\delta^{3/2}}\mathbf{d}(v,\Omega_\delta), \\ &|\widetilde{\mathcal{U}}_3(0,0)| \le \frac{C}{\delta^{3/2}}\mathbf{d}(v,\Omega_\delta). \end{aligned} \tag{4.2}$$

The constants do not depend on δ.

4.2 A complement to the rod center-line displacement.

Let $\mathcal{V}$ given by (3.11), we consider $\mathcal{W}(x_3)=\mathcal{V}(x_3)-x_3\mathbf{e}_3 = \dfrac{1}{|D_\delta|}\displaystyle\int_{D_\delta} u(x)dx_1dx_2$ the rod center-line displacement. From the above Theorem 3.2, the estimate below holds true

$$\Big\|\frac{d\mathcal{W}}{dx_3} - \big(\mathbf{Q}-\mathbf{I}_3\big)\mathbf{e}_3\Big\|_{L^2(-\delta,L;\mathbb{R}^3)} \le \frac{C}{\delta}\mathbf{d}(v,B_\delta). \tag{4.3}$$

As in [7] we split the center line displacement $\mathcal{W}$ into two parts. The first one $\mathcal{W}^{(m)}$ stands for the main displacement of the rod which describes the displacement coming from the bending and the second one for the stretching of the rod.

$$\begin{aligned} \forall x_3\in[0,L], \qquad \mathcal{W}^{(m)}(x_3) &= \mathcal{W}(0) + \int_0^{x_3}\big(\mathbf{Q}(t)-\mathbf{I}_3\big)\mathbf{e}_3 dt, \\ \mathcal{W}^{(s)}(x_3) &= \mathcal{W}(x_3) - \mathcal{W}^{(m)}(x_3). \end{aligned} \tag{4.4}$$

In the lemma below we give estimates on $\mathcal{W}^{(s)}$ and $\mathcal{W}^{(m)}$.

Lemma 4.1. *We have*

$$||\mathcal{W}^{(s)}||_{H^1(-\delta,L;\mathbb{R}^3)} \le \frac{C}{\delta}\mathbf{d}(v,B_\delta), \tag{4.5}$$

and

$$\big\|\mathcal{W}^{(m)}_\alpha - \mathcal{W}_\alpha(0)\big\|_{H^2(-\delta,L)} \le \frac{C}{\delta^2}\mathbf{d}(v,B_\delta) + C|||\mathbf{Q}(0)-\mathbf{I}_3|||, \tag{4.6}$$

$$\begin{aligned}\big\|\mathcal{W}_3^{(m)}-\mathcal{W}_3^{(m)}(0)\big\|_{H^1(-\delta,L)}&\leq\frac{C}{\delta^4}\big[\mathbf{d}(v,B_\delta)\big]^2+C|\big(\mathbf{Q}(0)-\mathbf{I}_3\big)\mathbf{e}_3\cdot\mathbf{e}_3|,\\ \Big\|\frac{d\mathcal{W}_3^{(m)}}{dx_3}\Big\|_{L^2(-\delta,\delta)}&\leq\frac{C}{\delta^{5/2}}\big[\mathbf{d}(v,B_\delta)\big]^2+C\delta^{1/2}|\big(\mathbf{Q}(0)-\mathbf{I}_3\big)\mathbf{e}_3\cdot\mathbf{e}_3|.\end{aligned}\tag{4.7}$$

The constants do not depend on δ.

Proof. Taking into account the fact that $\mathcal{W}^{(s)}(0)=0$, the estimate in (4.3) leads to (4.5). From the third estimate in (3.12) we obtain

$$||\mathbf{Q}-\mathbf{Q}(0)||_{L^2(-\delta,L;\mathbb{R}^{3\times3})}\leq\frac{C}{\delta^2}\mathbf{d}(v,B_\delta),\tag{4.8}$$

Due to the definition (4.4) of $\mathcal{W}^{(m)}$ and (4.3) we immediately get (4.6). A straightforward calculation gives

$$\frac{d\mathcal{W}_3^{(m)}}{dx_3}=\big(\mathbf{Q}-\mathbf{I}_3\big)\mathbf{e}_3\cdot\mathbf{e}_3=-\frac{1}{2}||(\mathbf{Q}-\mathbf{I}_3)\mathbf{e}_3||_2^2.\tag{4.9}$$

Besides we have

$$\frac{d}{dx_3}\big((\mathbf{Q}-\mathbf{I}_3)\mathbf{e}_3\big)=\frac{d\mathbf{Q}}{dx_3}\mathbf{e}_3.$$

Then, the third estimate in (3.12) and the above equality give

$$\begin{aligned}\big\|(\mathbf{Q}-\mathbf{I}_3)\mathbf{e}_3-(\mathbf{Q}(0)-\mathbf{I}_3)\mathbf{e}_3\big\|_{L^2(-\delta,L;\mathbb{R}^3)}&\leq\frac{C}{\delta^2}\mathbf{d}(v,B_\delta),\\ \big\|(\mathbf{Q}-\mathbf{I}_3)\mathbf{e}_3-(\mathbf{Q}(0)-\mathbf{I}_3)\mathbf{e}_3\big\|_{L^2(-\delta,\delta;\mathbb{R}^3)}&\leq\frac{C}{\delta}\mathbf{d}(v,B_\delta).\end{aligned}\tag{4.10}$$

These estimates and again the third one in (3.12) lead to

$$\begin{aligned}\big\|(\mathbf{Q}-\mathbf{I}_3)\mathbf{e}_3-(\mathbf{Q}(0)-\mathbf{I}_3)\mathbf{e}_3\big\|^2_{L^4(-\delta,L;\mathbb{R}^3)}&\leq\frac{C}{\delta^4}\big[\mathbf{d}(v,B_\delta)\big]^2,\\ \big\|(\mathbf{Q}-\mathbf{I}_3)\mathbf{e}_3-(\mathbf{Q}(0)-\mathbf{I}_3)\mathbf{e}_3\big\|^2_{L^4(-\delta,\delta;\mathbb{R}^3)}&\leq\frac{C}{\delta^{5/2}}\big[\mathbf{d}(v,B_\delta)\big]^2.\end{aligned}\tag{4.11}$$

Finally, from (4.9) and the above inequality we obtain (4.7). □

4.3 First estimates in the junction area.

Lemma 4.2. *We have the following estimate on* $\mathbf{Q}(0)-\mathbf{I}_3$*:*

$$\begin{aligned}\big|\big(\mathbf{Q}(0)-\mathbf{I}_3\big)\mathbf{e}_3\cdot\mathbf{e}_3\big|&\leq\frac{C}{\delta^{3/2}}\big(\mathbf{G}_s(u,\Omega_\delta)+\mathbf{d}(v,B_\delta)\big),\\ |||\mathbf{Q}(0)-\mathbf{I}_3|||&\leq\frac{C}{\delta^{7/4}}\mathbf{G}_s(u,\Omega_\delta)+\frac{C}{\delta^{3/2}}\mathbf{d}(v,B_\delta)\end{aligned}\tag{4.12}$$

and those about $\mathcal{W}(0)$

$$|\mathcal{W}_\alpha(0)| \leq \frac{C}{\delta^{3/4}}\mathbf{G}_s(u,\Omega_\delta) + \frac{C}{\delta^{1/2}}\mathbf{d}(v,B_\delta) \tag{4.13}$$

and

$$\begin{aligned} &|\mathcal{W}_3(0) - \widetilde{\mathcal{U}}_3(0,0)| \leq \frac{C}{\delta^2}\big[\mathbf{d}(v,B_\delta)\big]^2 + \frac{C}{\delta^{1/2}}\big(\mathbf{d}(v,B_\delta) + \mathbf{G}_s(u,\Omega_\delta)\big) + \frac{C}{\delta}\mathbf{d}(v,\Omega_\delta), \\ &|\mathcal{W}_3(0)| \leq \frac{C}{\delta^{3/2}}\mathbf{d}(v,\Omega_\delta) + \frac{C}{\delta^2}\big[\mathbf{d}(v,B_\delta)\big]^2 + \frac{C}{\delta^{1/2}}\big(\mathbf{d}(v,B_\delta) + \mathbf{G}_s(u,\Omega_\delta)\big). \end{aligned} \tag{4.14}$$

The constants are independent of δ.

Proof. Step 1. We prove the estimate on $\mathbf{Q}(0) - \mathbf{I}_3$. We consider the last inequalities in Theorems 3.1 and 3.2. They give

$$\big\|\mathbf{Q} - \mathbf{I}_3 - \mathbf{A}_\mathcal{R}\big\|_{L^2(C_\delta;\mathbb{R}^9)} \leq C\big(\mathbf{G}_s(u,\Omega_\delta) + \mathbf{d}(v,B_\delta)\big). \tag{4.15}$$

Now, from the third estimate in (3.12), we get

$$\big\|\mathbf{Q} - \mathbf{Q}(0)\big\|_{L^2(-\delta,\delta;\mathbb{R}^{3\times 3})} \leq C\delta\Big\|\frac{d\mathbf{Q}}{dx_3}\Big\|_{L^2(-\delta,\delta;\mathbb{R}^{3\times 3})} \leq \frac{C}{\delta}\mathbf{d}(v,B_\delta).$$

Hence

$$\big\|\mathbf{Q}(0) - \mathbf{I}_3 - \mathbf{A}_\mathcal{R}\big\|^2_{L^2(D_\delta;\mathbb{R}^{3\times 3})} \leq \frac{C}{\delta}\Big(\big[\mathbf{G}_s(u,\Omega_\delta)\big]^2 + \big[\mathbf{d}(v,B_\delta)\big]^2\Big). \tag{4.16}$$

We recall that the matrix $\mathbf{A}_\mathcal{R}$ is antisymmetric, then (4.16) leads to the first estimate in (4.12). Due to (3.6) we have

$$||\mathcal{R}||^4_{L^2(D_\delta;\mathbb{R}^3)} \leq C\delta^3||\mathcal{R}||^4_{L^8(D_\delta;\mathbb{R}^3)} \leq C\delta^3||\mathcal{R}||^4_{H^1(\omega;\mathbb{R}^3)} \leq \frac{C}{\delta^3}\big[\mathbf{G}_s(u,\Omega_\delta)\big]^4. \tag{4.17}$$

Then, using the above estimate and (4.16) we deduce the second estimate in (4.12).

Step 2. We prove the estimate (4.13) *on* $\mathcal{W}_\alpha(0)$.

The two decompositions of $u = v - I_d$ give, for a.e. $x \in C_\delta$

$$\begin{aligned} &\mathcal{U}(x_1,x_2) + x_3\mathcal{R}(x_1,x_2) \wedge \mathbf{e}_3 + \overline{u}(x) \\ =&\mathcal{W}(x_3) + (\mathbf{Q}(x_3) - \mathbf{I}_3)(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) + \overline{w}(x). \end{aligned} \tag{4.18}$$

Taking the averages on the cylinder C_δ of the terms in this equality (4.18) give

$$\mathcal{M}_{D_\delta}(\mathcal{U}) = \frac{1}{|D_\delta|}\int_{D_\delta}\mathcal{U}(x_1,x_2)dx_1dx_2 = \mathcal{M}_{I_\delta}(\mathcal{W}) = \frac{1}{2\delta}\int_{-\delta}^{\delta}\mathcal{W}(x_3)dx_3. \tag{4.19}$$

Besides, proceeding as for $\mathcal{R}$ in (4.17) and from (3.6) we have

$$||\mathcal{U}_\alpha||_{L^2(D_\delta)} \leq C\delta^{1/4}\mathbf{G}_s(u,\Omega_\delta).$$

From this estimate we get

$$|\mathcal{M}_{I_\delta}(\mathcal{W}_\alpha)| = |\mathcal{M}_{D_\delta}(\mathcal{U}_\alpha)| \leq \frac{C}{\delta^{3/4}}\mathbf{G}_s(u,\Omega_\delta). \tag{4.20}$$

We set $y_\alpha(x_3) = \mathcal{W}_\alpha(x_3) - x_3(\mathbf{Q}(0)-\mathbf{I}_3)\mathbf{e}_3\cdot\mathbf{e}_\alpha$. The estimates (4.3) and (4.10) lead to

$$\Big\|\frac{dy_\alpha}{dx_3}\Big\|_{L^2(-\delta,\delta)} \leq \frac{C}{\delta}\mathbf{d}(v,B_\delta)$$

which in turn implies

$$\big\|y_\alpha - y_\alpha(0)\big\|_{L^2(-\delta,\delta)} \leq C\mathbf{d}(v,B_\delta).$$

Taking the average, it yields

$$|\mathcal{M}_{I_\delta}(\mathcal{W}_\alpha) - \mathcal{W}_\alpha(0)| \leq \frac{C}{\delta^{1/2}}\mathbf{d}(v,B_\delta). \tag{4.21}$$

Finally, from (4.20) and (4.21) we obtain (4.13).

Step 3. We prove the estimate on $\mathcal{W}_3(0)$. Using (4.2) we deduce that

$$\begin{aligned} ||\mathcal{U}_3 - \widetilde{\mathcal{U}}_3||_{L^2(D_\delta)} &\leq C\delta^{1/2}||\mathcal{U}_3 - \widetilde{\mathcal{U}}_3||_{L^4(\omega)} \\ &\leq C\delta^{1/2}||\mathcal{U}_3 - \widetilde{\mathcal{U}}_3||_{H^1(\omega)} \leq C\mathbf{d}(v,\Omega_\delta). \end{aligned} \tag{4.22}$$

Then we replace $\mathcal{U}_3$ with $\widetilde{\mathcal{U}}_3$ and $\mathcal{W}_3$ with $\mathcal{W}_3^{(m)}$ in (4.19). Taking into account (4.5) we obtain

$$|\mathcal{M}_{D_\delta}(\widetilde{\mathcal{U}}_3) - \mathcal{M}_{I_\delta}(\mathcal{W}_3^{(m)})| \leq \frac{C}{\delta}\mathbf{d}(v,\Omega_\delta) + \frac{C}{\delta^{1/2}}\mathbf{d}(v,B_\delta). \tag{4.23}$$

We carry on by comparing $\mathcal{M}_{D_\delta}(\widetilde{\mathcal{U}}_3)$ with $\widetilde{\mathcal{U}}_3(0,0)$. Let us set

$$\mathbf{r}_\alpha = \frac{1}{\pi\delta^2}\int_{D_\delta}\big(\mathbf{R}(x_1,x_2)-\mathbf{I}_3\big)\mathbf{e}_\alpha\cdot\mathbf{e}_3\,dx_1dx_2$$

and consider the function $\Psi(x_1,x_2) = \widetilde{\mathcal{U}}_3(x_1,x_2) - \mathcal{M}_{D_\delta}(\widetilde{\mathcal{U}}_3) - x_1\mathbf{r}_2 - x_2\mathbf{r}_1$. Due to (4.2) we first obtain

$$\Big\|\frac{\partial^2\Psi}{\partial x_\alpha\partial x_\beta}\Big\|_{L^2(D_\delta,\mathbb{R}^3)} \leq \frac{C}{\delta^{3/2}}\mathbf{d}(v,\Omega_\delta). \tag{4.24}$$

Then, applying twice the Poincaré-Wirtinger inequality in the disc D_δ and using (3.3) and the fourth estimate in (4.2) lead to

$$||\nabla\Psi||^2_{L^2(D_\delta,\mathbb{R}^6)} \leq \frac{C}{\delta}\big[\mathbf{d}(v,\Omega_\delta)\big]^2, \qquad ||\Psi||^2_{L^2(D_\delta,\mathbb{R}^3)} \leq C\delta\big[\mathbf{d}(v,\Omega_\delta)\big]^2. \tag{4.25}$$

From the above inequalities (4.24) and (4.25) we deduce that

$$||\Psi||_{L^\infty(D_\delta,\mathbb{R}^3)} \leq \frac{C}{\delta^{1/2}}\mathbf{d}(v,\Omega_\delta) \quad\Longrightarrow\quad |\Psi(0,0)| = |\widetilde{\mathcal{U}}_3(0,0) - \mathcal{M}_{D_\delta}(\widetilde{\mathcal{U}}_3)| \leq \frac{C}{\delta^{1/2}}\mathbf{d}(v,\Omega_\delta).$$

From this last estimate and (4.23) we obtain

$$|\widetilde{\mathcal{U}}_3(0,0) - \mathcal{M}_{I_\delta}\big(\mathcal{W}_3^{(m)}\big)| \leq \frac{C}{\delta}\mathbf{d}(v,\Omega_\delta) + \frac{C}{\delta^{1/2}}\mathbf{d}(v,B_\delta). \tag{4.26}$$

Then using the second estimate in (4.7) and (4.12) we have

$$\Big\|\frac{d\mathcal{W}_3^{(m)}}{dx_3}\Big\|_{L^2(-\delta,\delta)} \leq \frac{C}{\delta^{5/2}}\big[\mathbf{d}(v,B_\delta)\big]^2 + \frac{C}{\delta}\big(\mathbf{G}_s(u,\Omega_\delta) + \mathbf{d}(v,B_\delta)\big). \tag{4.27}$$

Finally, recalling that $\mathcal{W}_3(0) = \mathcal{W}_3^{(m)}(0)$, the above inequality leads to

$$|\mathcal{M}_{I_\delta}\big(\mathcal{W}_3^{(m)}\big) - \mathcal{W}_3(0)| \leq C\delta^{1/2}\Big\|\frac{d\mathcal{W}_3^{(m)}}{dx_3}\Big\|_{L^2(-\delta,\delta)} \leq \frac{C}{\delta^2}\big[\mathbf{d}(v,B_\delta)\big]^2 + \frac{C}{\delta^{1/2}}\big(\mathbf{G}_s(u,\Omega_\delta) + \mathbf{d}(v,B_\delta)\big)$$

which in turn with (4.26) and (4.2) lead to (4.14). □

4.4 Global estimates of u: Korn's type inequality.

Now, we give the last estimates of the displacement $u = v - I_d$ in the rod B_δ.

Lemma 4.3. *For any deformation v in $\mathbb{D}_\delta$ we have the following inequalities for the displacement $u = v - I_d$ in the rod B_δ:*

$$\begin{aligned}
&||u - \mathcal{W}||_{L^2(B_\delta;\mathbb{R}^3)} \leq C\big(\mathbf{d}(v,B_\delta) + \delta^{1/4}\mathbf{G}_s(u,\Omega_\delta)\big),\\
&\big\|\mathcal{W}_\alpha\big\|_{L^2(-\delta,L)} + \big\|\mathcal{W}_\alpha^{(m)}\big\|_{L^2(-\delta,L)} \leq C\Big(\frac{\mathbf{d}(v,B_\delta)}{\delta^2} + \frac{\mathbf{G}_s(u,\Omega_\delta)}{\delta^{7/4}}\Big),\\
&\big\|\mathcal{W}_3\big\|_{L^2(-\delta,L)} + \big\|\mathcal{W}_3^{(m)}\big\|_{L^2(-\delta,L)} \leq C\frac{\big[\mathbf{d}(v,B_\delta)\big]^2}{\delta^4} + \frac{C}{\delta^{3/2}}\big[\mathbf{G}_s(u,\Omega_\delta) + \mathbf{d}(v,\Omega_\delta) + \mathbf{d}(v,B_\delta)\big].
\end{aligned} \tag{4.28}$$

The constants do not depend on δ.

Proof. From (4.8) and (4.12) we get

$$\big\|\mathbf{Q} - \mathbf{I}_3\big\|_{L^2(-\delta,L;\mathbb{R}^{3\times 3})} \leq C\Big(\frac{\mathbf{d}(v,B_\delta)}{\delta^2} + \frac{\mathbf{G}_s(u,\Omega_\delta)}{\delta^{7/4}}\Big). \tag{4.29}$$

Then, from (3.12) and the above inequality we deduce that

$$||u - \mathcal{W}||_{L^2(B_\delta;\mathbb{R}^3)} \leq C\big(\mathbf{d}(v,B_\delta) + \delta^{1/4}\mathbf{G}_s(u,\Omega_\delta)\big). \tag{4.30}$$

From (4.6) again (4.12) and (4.13) we obtain

$$\big\|\mathcal{W}_\alpha^{(m)}\big\|_{H^1(-\delta,L)} \leq \frac{C}{\delta^2}\mathbf{d}(v,B_\delta) + \frac{C}{\delta^{7/4}}\mathbf{G}_s(u,\Omega_\delta). \tag{4.31}$$

Then since $\mathcal{W} = \mathcal{W}^{(m)} + \mathcal{W}^{(s)}$, (4.5) and (4.31) give the second estimate in (4.28).

From (4.7) and (4.12) we deduce that

$$\Big\|\frac{d\mathcal{W}_3^{(m)}}{dx_3}\Big\|_{L^2(-\delta,L)} \leq \frac{C}{\delta^4}\big[\mathbf{d}(v,B_\delta)\big]^2 + \frac{C}{\delta^{3/2}}\big[\mathbf{G}_s(u,\Omega_\delta) + \mathbf{d}(v,B_\delta)\big].$$

which in turn using (4.14) lead to

$$\big\|\mathcal{W}_3^{(m)}\big\|_{L^2(-\delta,L)} \leq \frac{C}{\delta^4}\big[\mathbf{d}(v,B_\delta)\big]^2 + \frac{C}{\delta^{3/2}}\big[\mathbf{G}_s(u,\Omega_\delta) + \mathbf{d}(v,\Omega_\delta) + \mathbf{d}(v,B_\delta)\big]$$

and then due to (4.5) we get the last estimate in (4.28). □

Corollary 4.4. *For any deformation v in $\mathbb{D}_\delta$ we have the following Korn's type inequality for the displacement $u = v - I_d$ in the rod B_δ:*

$$\begin{aligned}
&\big\|\nabla u\big\|_{L^2(B_\delta;\mathbb{R}^{3\times 3})} \leq \frac{C}{\delta}\mathbf{d}(v,B_\delta) + \frac{C}{\delta^{3/4}}\mathbf{G}_s(u,\Omega_\delta),\\
&||u_\alpha||_{L^2(B_\delta)} \leq \frac{C}{\delta}\mathbf{d}(v,B_\delta) + \frac{C}{\delta^{3/4}}\mathbf{G}_s(u,\Omega_\delta),\\
&||u_3||_{L^2(B_\delta)} \leq C\frac{\big[\mathbf{d}(v,B_\delta)\big]^2}{\delta^3} + \frac{C}{\delta^{1/2}}\big[\mathbf{G}_s(u,\Omega_\delta) + \mathbf{d}(v,\Omega_\delta) + \mathbf{d}(v,B_\delta)\big].
\end{aligned} \tag{4.32}$$

The constants do not depend on δ.

Proof. From (3.12) and (4.29) we obtain

$$\big\|\nabla u\big\|_{L^2(B_\delta;\mathbb{R}^{3\times 3})} \leq \frac{C}{\delta}\mathbf{d}(v,B_\delta) + \frac{C}{\delta^{3/4}}\mathbf{G}_s(u,\Omega_\delta). \tag{4.33}$$

The second and third inequalities are immediate consequences of Lemma 4.3. □

5 Elastic structure

5.1 Elastic energy.

In this section we assume that the structure $\mathcal{S}_\delta$ is made of an elastic material. The associated local energy $\widehat{W} : \mathbf{X}_3 \longrightarrow \mathbb{R}^+$ is the following St Venant-Kirchhoff's law (see [9])

$$\widehat{W}(F) = \begin{cases} Q(F^TF - \mathbf{I}_3) & \text{if} \quad \det(F) > 0 \\ +\infty & \text{if} \quad \det(F) \leq 0. \end{cases} \tag{5.1}$$

where $\mathbf{X}_3$ is the space of 3×3 symmetric matrices and where the quadratic form Q is given by

$$Q(E) = \frac{\lambda}{8}\big(tr(E)\big)^2 + \frac{\mu}{4}tr\big(E^2\big), \tag{5.2}$$

and where (λ,μ) are the Lamé's coefficients of the material. Let us recall (see e.g. [18] or [7]) that for any 3×3 matrix F such that $\det(F) > 0$ we have

$$[tr(F^TF - \mathbf{I}_3)]^2 = |||F^TF - \mathbf{I}_3|||^2 \geq \text{ dist } (F, SO(3))^2. \tag{5.3}$$

5.2 Assumptions on the forces and final estimates.

Now we assume that the structure $\mathcal{S}_\delta$ is submitted to applied body forces $f_\delta \in L^2(\mathcal{S}_\delta;\mathbb{R}^3)$ and we define the total energy $J_\delta(v)$[1] over $\mathbb{D}_\delta$ by

$$J_\delta(v) = \int_{\mathcal{S}_\delta} \widehat{W}_\delta(\nabla v)(x)dx - \int_{\mathcal{S}_\delta} f_\delta(x)\cdot(v(x)-I_d(x))dx. \tag{5.4}$$

Assumptions on the forces. To introduce the scaling on f_δ, let us consider f_r, g_1, g_2 in $L^2(0,L;\mathbb{R}^3)$ and $f_p \in L^2(\omega;\mathbb{R}^3)$. We assume that the force f_δ is given by

$$\begin{aligned} f_\delta(x) &= \delta^{5/2}\Big[f_{r,1}(x_3)\mathbf{e}_1 + f_{r,2}(x_3)\mathbf{e}_2 + \frac{1}{\delta^{1/2}}f_{r,3}(x_3)\mathbf{e}_3 + \frac{x_1}{\delta^2}g_1(x_3) + \frac{x_2}{\delta^2}g_2(x_3)\Big] \\ &\qquad x\in B_\delta, \quad x_3>\delta, \\ f_{\delta,\alpha}(x) &= \delta^2 f_{p,\alpha}(x_1,x_2), \qquad f_{\delta,3}(x) = \delta^3 f_{p,3}(x_1,x_2), \qquad x\in\Omega_\delta. \end{aligned} \tag{5.5}$$

We denote

$$F_{r,3}(x_3) = \int_{x_3}^{L} f_{r,3}(s)ds, \qquad \text{for a. e. } x_3\in]0,L[. \tag{5.6}$$

Theorem 5.1. *There exist two constants C_0 and C_1, which depend only on ω, L and μ, such that if*

$$||f_p||_{L^2(\Omega;\mathbb{R}^3)} \le C_0 \tag{5.7}$$

and if either

$$\begin{aligned} &\textit{Case 1: for a. e. } x_3\in]0,L[, \quad F_{r,3}(x_3)\ge 0, \\ &\textit{or} \\ &\textit{Case 2: } \ ||f_{r,3}||_{L^2(0,L)} \le C_1 \end{aligned} \tag{5.8}$$

then for δ small enough and for any $v\in\mathbb{D}_\delta$ satisfying $J_\delta(v)\le 0$ we have

$$\mathbf{d}(v,B_\delta) + \mathbf{d}(v,\Omega_\delta) \le C\delta^{5/2} \tag{5.9}$$

where the constant does not depend on δ.

Proof. From (3.7) and the assumptions (5.5) on the body forces, we obtain on the one hand for any $v\in\mathbb{D}_\delta$ and with $u = v - I_d$

$$\Big|\int_{\Omega_\delta} f_\delta(x)\cdot u(x)dx\Big| \le C\delta^{5/2}||f_p||_{L^2(\omega;\mathbb{R}^3)}\mathbf{G}_s(u,\Omega_\delta). \tag{5.10}$$

As far as the term involving the forces in the rod are concerned we first have

$$\begin{aligned} \int_{B_\delta} f_\delta(x)\cdot u(x)dx &= \pi\delta^{9/2}\int_\delta^L f_{r,\alpha}(x_3)\mathcal{W}_\alpha(x_3)dx_3 + \pi\delta^4\int_\delta^L f_{r,3}(x_3)\mathcal{W}_3(x_3)dx_3 \\ &\quad + \int_{B_\delta} f_\delta(x)\cdot(u(x)-\mathcal{W}(x_3))dx. \end{aligned}$$

[1]For later convenience, we have added the term $\int_{\mathcal{S}_\delta} f_\delta(x)\cdot I_d(x)dx$ to the usual standard energy, indeed this does not affect the minimizing problem for J_δ.

Then, using Lemma 4.3 and (5.5) we first get

$$\begin{aligned}
&\Big|\int_\delta^L f_{r,\alpha}(x_3)\mathcal{W}_\alpha(x_3)dx_3\Big| \le \frac{C}{\delta^2}\sum_{\alpha=1}^{2}||f_{r,\alpha}||_{L^2(0,L)}\big(\mathbf{d}(v,B_\delta)+\delta^{1/4}\mathbf{G}_s(u,\Omega_\delta)\big),\\
&\Big|\int_{B_\delta} f_\delta(x)\cdot\big(u(x)-\mathcal{W}(x_3)\big)dx\Big| \le C\delta^{5/2}\big(||g_1||_{L^2(0,L;\mathbb{R}^3)}+||g_2||_{L^2(0,L;\mathbb{R}^3)}\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\big(\mathbf{d}(v,B_\delta)+\delta^{1/4}\mathbf{G}_s(u,\Omega_\delta)\big)
\end{aligned}\tag{5.11}$$

Now we estimate $\displaystyle\int_\delta^L f_{r,3}\mathcal{W}_3(x_3)dx_3$. From (4.5) we first obtain

$$\Big|\int_\delta^L f_{r,3}(x_3)\mathcal{W}_3^{(s)}(x_3)dx_3\Big| \le \frac{C}{\delta}||f_{r,3}||_{L^2(0,L)}\mathbf{d}(v,B_\delta).\tag{5.12}$$

Then we have

$$\int_\delta^L f_{r,3}(x_3)\mathcal{W}_3^{(m)}(x_3)dx_3 = F_{r,3}(\delta)\mathcal{W}_3^{(m)}(\delta)+\int_\delta^L F_{r,3}(x_3)\frac{d\mathcal{W}_3^{(m)}}{dx_3}(x_3)dx_3.\tag{5.13}$$

Taking to account (4.27) and (4.14) we get

$$\begin{aligned}
|\mathcal{W}_3^{(m)}(\delta)| &\le |\mathcal{W}_3(0)|+\delta^{1/2}\Big\|\frac{d\mathcal{W}_3^{(m)}}{dx_3}\Big\|_{L^2(-\delta,\delta)}\\
&\le \frac{C}{\delta^{3/2}}\mathbf{d}(v,\Omega_\delta)+\frac{C}{\delta^2}\big[\mathbf{d}(v,B_\delta)\big]^2+\frac{C}{\delta^{1/2}}\big(\mathbf{G}_s(u,\Omega_\delta)+\mathbf{d}(v,B_\delta)\big).
\end{aligned}\tag{5.14}$$

Observe now that due to the expression (4.9) of $\dfrac{d\mathcal{W}_3^{(m)}}{dx_3}$, this derivative is nonpositive for a.e. $x_3\in]0,L[$ (see (4.9)).

• If we are in Case 1 in (7.34), we have

$$\int_\delta^L f_{r,3}(x_3)\mathcal{W}_3(x_3)dx_3 \le C||f_{r,3}||_{L^2(0,L)}\Big[\frac{\mathbf{d}(v,\Omega_\delta)}{\delta^{3/2}}+\frac{\big[\mathbf{d}(v,B_\delta)\big]^2}{\delta^2}+\frac{\mathbf{G}_s(u,\Omega_\delta)+\mathbf{d}(v,B_\delta)}{\delta^{1/2}}\Big].$$

Hence, we obtain

$$\begin{aligned}
\int_{B_\delta} f_\delta(x)\cdot u(x)dx \le& C\delta^{5/2}\sum_{\alpha=1}^{2}\big(||f_{r,\alpha}||_{L^2(0,L)}+||g_\alpha||_{L^2(0,L;\mathbb{R}^3)}\big)\big(\mathbf{d}(v,B_\delta)+\delta^{1/4}\mathbf{G}_s(u,\Omega_\delta)\big)\\
&+C||f_{r,3}||_{L^2(0,L)}\Big[\delta^{5/2}\mathbf{d}(v,\Omega_\delta)+\delta^2[\mathbf{d}(v,B_\delta)\big]^2+\delta^{7/2}\big(\mathbf{G}_s(u,\Omega_\delta)+\mathbf{d}(v,B_\delta)\big)\Big].
\end{aligned}$$

We recall that (see [8])

$$\mathbf{G}_s(u,\Omega_\delta) \le C\mathbf{d}(v,\Omega_\delta)+\frac{C}{\delta^{5/2}}\big[\mathbf{d}(v,\Omega_\delta)\big]^2\tag{5.15}$$

where the constant does not depend on δ. Then due to (5.10) and the above inequalities we obtain that

$$\begin{aligned}\int_{\Omega_\delta} f_\delta(x)\cdot u(x)dx \leq & C||f_r||_{L^2(\omega;\mathbb{R}^3)}\delta^{5/2}\mathbf{d}(v,\Omega_\delta) + C^*||f_r||_{L^2(\omega;\mathbb{R}^3)}\big[\mathbf{d}(v,\Omega_\delta)\big]^2\\ \int_{B_\delta} f_\delta(x)\cdot u(x)dx \leq & C(f_r,g_1,g_2)\delta^{5/2}\big(\mathbf{d}(v,B_\delta)+\mathbf{d}(v,\Omega_\delta)\big)\\ &+ C(f_r,g_1,g_2)\delta^{1/4}\big[\mathbf{d}(v,\Omega_\delta)\big]^2 + C||f_{r,3}||_{L^2(0,L)}\delta^2\big[\mathbf{d}(v,B_\delta)\big]^2.\end{aligned} \tag{5.16}$$

Now, for any $v\in\mathbb{D}_\delta$ such that $J_\delta(v)\leq 0$, the assumptions (5.1), (5.2),(5.3) and the above estimates lead to

$$\begin{aligned}&\frac{\mu}{8}\big([\mathbf{d}(v,B_\delta)]^2+\big[\mathbf{d}(v,\Omega_\delta)\big]^2\big)\leq\int_{\mathcal{S}_\delta}\widehat{W}(\nabla v)(x)dx\leq\int_{\mathcal{S}_\delta}f_\delta(x)\cdot u(x)dx\\ &\leq C(f_r,g_1,g_2)\delta^{5/2}\big(\mathbf{d}(v,B_\delta)+\mathbf{d}(v,\Omega_\delta)\big)+C(f_r,g_1,g_2)\delta^{1/4}\big[\mathbf{d}(v,\Omega_\delta)\big]^2\\ &+C||f_{r,3}||_{L^2(0,L)}\delta^2\big[\mathbf{d}(v,B_\delta)\big]^2+C\delta^{5/2}||f_p||_{\omega;\mathbb{R}^3}\mathbf{d}(v,\Omega_\delta)+C^*||f_p||_{\omega;\mathbb{R}^3}\big[\mathbf{d}(v,\Omega_\delta)\big]^2.\end{aligned}$$

wich in turn gives

$$\begin{aligned}&\Big(\frac{\mu}{8}-C||f_{r,3}||_{L^2(0,L)}\delta^2\Big)\big[\mathbf{d}(v,B_\delta)\big]^2+\Big(\frac{\mu}{8}-C^*||f_p||_{\omega;\mathbb{R}^3}-C(f_r,g_1,g_2)\delta^{1/4}\Big)\big[\mathbf{d}(v,\Omega_\delta)\big]^2\\ &\leq C(f_r,g_1,g_2)\delta^{5/2}\big(\mathbf{d}(v,B_\delta)+\mathbf{d}(v,\Omega_\delta)\big)+C\delta^{5/2}||f_p||_{\omega;\mathbb{R}^3}\mathbf{d}(v,\Omega_\delta).\end{aligned}$$

Indeed the two quantities $C||f_{r,3}||_{L^2(0,L)}\delta^2$ and $C(f_r,g_1,g_2)\delta^{1/4}$ tend to 0 as δ tends to 0, then, under the condition $C^*||f_p||_{L^2(\omega;\mathbb{R}^3)}\leq\mu/32$ and for δ small enough we obtain

$$\mathbf{d}(v,B_\delta)+\mathbf{d}(v,\Omega_\delta)\leq C\delta^{5/2}.$$

The constant does not depend on δ.

• If we are in Case 2 in (7.34), from (4.28) we immediately have

$$\int_\delta^L f_{r,3}(x_3)\mathcal{W}_3(x_3)dx_3\leq C||f_{r,3}||_{L^2(0,L)}\Big[\frac{\big[\mathbf{d}(v,B_\delta)\big]^2}{\delta^4}+\frac{\mathbf{G}_s(u,\Omega_\delta)+\mathbf{d}(v,\Omega_\delta)+\mathbf{d}(v,B_\delta)}{\delta^{3/2}}\Big].$$

Then, proceeding as in Case 1 leads to

$$\begin{aligned}\int_{B_\delta} f_\delta(x)\cdot u(x)dx\leq & C\delta^{5/2}\sum_{\alpha=1}^2\big(||f_{r,\alpha}||_{L^2(0,L)}+||g_\alpha||_{L^2(0,L;\mathbb{R}^3)}\big)\big(\mathbf{d}(v,B_\delta)+\delta^{1/4}\mathbf{G}_s(u,\Omega_\delta)\big)\\ &+C||f_{r,3}||_{L^2(0,L)}\Big[\big[\mathbf{d}(v,B_\delta)\big]^2+\delta^{5/2}\big(\mathbf{G}_s(u,\Omega_\delta)+\mathbf{d}(v,\Omega_\delta)+\mathbf{d}(v,B_\delta)\big)\Big].\end{aligned} \tag{5.17}$$

Then for δ small enough, we get

$$\begin{aligned}&\frac{\mu}{8}\big([\mathbf{d}(v,B_\delta)]^2+\big[\mathbf{d}(v,\Omega_\delta)\big]^2\big)\leq\int_{\mathcal{S}_\delta}\widehat{W}(\nabla v)(x)dx\leq\int_{\mathcal{S}_\delta}f_\delta(x)\cdot u(x)dx\\ &\leq C\delta^{5/2}C(f,g)\big(\mathbf{d}(v,B_\delta)+\mathbf{d}(v,\Omega_\delta)\big)+C^{**}||f_{r,3}||_{L^2(0,L)}\Big[\big[\mathbf{d}(v,B_\delta)\big]^2+\big[\mathbf{d}(v,\Omega_\delta)\big]^2\Big]\\ &+C\delta^{5/2}||f_p||_{\omega;\mathbb{R}^3}\mathbf{d}(v,\Omega_\delta)+C^*||f_p||_{\omega;\mathbb{R}^3}\big[\mathbf{d}(v,\Omega_\delta)\big]^2.\end{aligned}$$

Hence, under the conditions $C^*||f_p||_{L^2(\omega;\mathbb{R}^3)} \leq \mu/32$ and $C^{**}||f_{r,3}||_{L^2(0,L)} \leq \mu/32$ we deduce that

$$\mathbf{d}(v, B_\delta) + \mathbf{d}(v, \Omega_\delta) \leq C\delta^{5/2}.$$

In the both cases, we finally obtain (5.9) □

As a consequence of Theorem 5.1 and estimates (5.16)-(5.17), we deduce that for δ small enough and for any $v \in \mathbb{D}_\delta$ satisfying $J_\delta(v) \leq 0$ we have ($u = v - I_d$)

$$\int_{\mathcal{S}_\delta} f_\delta \cdot u \leq C\delta^5, \qquad \int_{\mathcal{S}_\delta} \widehat{W}_\delta(\nabla v)(x)dx \leq C\delta^5. \tag{5.18}$$

From (5.18) we also obtain for any $v \in \mathbb{D}_\delta$ such that $J_\delta(v) \leq 0$

$$c\delta^5 \leq J_\delta(v) \tag{5.19}$$

where c is a nonpositive constant which does not depend on δ. We set

$$m_\delta = \inf_{v \in \mathbb{D}_\delta} J_\delta(v).$$

As a consequence of (5.19) we have

$$c \leq \frac{m_\delta}{\delta^5} \leq 0. \tag{5.20}$$

In general, a minimizer of J_δ does not exist on $\mathbb{D}_\delta$.

6 Asymptotic behavior of a sequence of deformations of the whole structure $\mathcal{S}_\delta$.

In this subsection and the following one, we consider a sequence of deformations (v_δ) belonging to $\mathbb{D}_\delta$ and satisfying

$$\mathbf{d}(v_\delta, B_\delta) + \mathbf{d}(v_\delta, \Omega_\delta) \leq C\delta^{5/2} \tag{6.1}$$

where the constant does not depend on δ. Setting $u_\delta = v_\delta - I_d$, then, due to (6.1) and (5.15) we obtain that

$$\mathbf{G}_s(u_\delta, \Omega_\delta) \leq C\delta^{5/2}. \tag{6.2}$$

For any open subset $\mathcal{O} \subset \mathbb{R}^2$ and for any field $\psi \in H^1(\mathcal{O}; \mathbb{R}^3)$, we denote

$$\gamma_{\alpha\beta}(\psi) = \frac{1}{2}\Big(\frac{\partial \psi_\alpha}{\partial x_\beta} + \frac{\partial \psi_\beta}{\partial x_\alpha}\Big), \qquad (\alpha, \beta) \in \{1, 2\}. \tag{6.3}$$

6.1 The rescaling operators

Before rescaling the domains, we introduce the reference domain Ω for the plate and the one B for the rod

$$\Omega = \omega\times]-1,1[, \qquad B = D\times]0,L[= D(O,1)\times]0,L[.$$

As usual when dealing with thin structures, we rescale Ω_δ and B_δ using -for the plate- the operator

$$\Pi_\delta(w)(x_1,x_2,X_3) = w(x_1,x_2,\delta X_3) \text{ for any } (x_1,x_2,X_3)\in\Omega$$

defined for e.g. $w\in L^2(\Omega_\delta)$ for which $\Pi_\delta(w)\in L^2(\Omega)$ and using -for the rod- the operator

$$P_\delta(w)(X_1,X_2,x_3) = w(\delta X_1,\delta X_2,x_3) \text{ for any } (X_1,X_2,x_3)\in B$$

defined for e.g. $w\in L^2(B_\delta)$ for which $P_\delta(w)\in L^2(B)$.

6.2 Asymptotic behavior in the plate.

Following Section 2 we decompose the restriction of $u_\delta = v_\delta - I_d$ to the plate. The Theorem 3.1 gives $\mathcal{U}_\delta$, $\mathcal{R}_\delta$ and $\overline{u}_\delta$, then estimates (3.6) lead to the following convergences for a subsequence still indexed by δ

$$\begin{aligned}
\frac{1}{\delta}\mathcal{U}_{\delta,3} &\longrightarrow \mathcal{U}_3 \quad \text{strongly in} \quad H^1(\omega),\\
\frac{1}{\delta^2}\mathcal{U}_{\delta,\alpha} &\rightharpoonup \mathcal{U}_\alpha \quad \text{weakly in} \quad H^1(\omega),\\
\frac{1}{\delta}\mathcal{R}_\delta &\rightharpoonup \mathcal{R} \quad \text{weakly in} \quad H^1(\omega;\mathbb{R}^3),\\
\frac{1}{\delta^3}\Pi_\delta(\overline{u}_\delta) &\rightharpoonup \overline{u} \quad \text{weakly in} \quad L^2(\omega;H^1(-1,1;\mathbb{R}^3),\\
\frac{1}{\delta^2}\Big(\frac{\partial\mathcal{U}_\delta}{\partial x_\alpha} - \mathcal{R}_\delta\wedge\mathbf{e}_\alpha\Big) &\rightharpoonup \mathcal{Z}_\alpha \quad \text{weakly in} \quad L^2(\omega;\mathbb{R}^3).
\end{aligned} \tag{6.4}$$

Denoting by $\mathbf{A}_\mathcal{R}$ the field of antisymmetric matrices associated to $\mathcal{R}$ as in Section 2, we also have

$$\begin{aligned}
\frac{1}{\delta^2}\Pi_\delta(u_\delta - \mathcal{U}_\delta) &\longrightarrow X_3\mathcal{R}\wedge\mathbf{e}_3 \quad \text{strongly in} \quad L^2(\Omega;\mathbb{R}^3),\\
\frac{1}{\delta}\Pi_\delta(\nabla u_\delta) &\longrightarrow \mathbf{A}_\mathcal{R} \quad \text{strongly in} \quad L^2(\Omega;\mathbb{R}^9).
\end{aligned} \tag{6.5}$$

The boundary conditions (3.5) give here

$$\mathcal{U}_3 = 0, \quad \mathcal{U}_\alpha = 0, \quad \mathcal{R} = 0 \qquad \text{on} \quad \gamma_0, \tag{6.6}$$

while (6.4) show that $\mathcal{U}_3\in H^2(\omega)$ with

$$\frac{\partial\mathcal{U}_3}{\partial x_1} = -\mathcal{R}_2, \qquad \frac{\partial\mathcal{U}_3}{\partial x_2} = \mathcal{R}_1. \tag{6.7}$$

In [8] (see Theorem 7.3) the limit of the Green-St Venant's strain tensor of the sequence v_δ is also derived. Let us set

$$\overline{u}_p = \overline{u} + \frac{X_3}{2}\big(\mathcal{Z}_1 \cdot \mathbf{e}_3\big)\mathbf{e}_1 + \frac{X_3}{2}\big(\mathcal{Z}_2 \cdot \mathbf{e}_3\big)\mathbf{e}_2 \tag{6.8}$$

and

$$\mathcal{Z}_{\alpha\beta} = \gamma_{\alpha\beta}(\mathcal{U}) + \frac{1}{2}\frac{\partial \mathcal{U}_3}{\partial x_\alpha}\frac{\partial \mathcal{U}_3}{\partial x_\beta}. \tag{6.9}$$

Then we have

$$\frac{1}{2\delta^2}\Pi_\delta\big((\nabla v_\delta)^T \nabla v_\delta - \mathbf{I}_3\big) \rightharpoonup \mathbf{E}_p \qquad \text{weakly in} \quad L^1(\Omega;\mathbb{R}^9),$$

where the symmetric matrix $\mathbf{E}_p$ is defined by

$$\mathbf{E}_p = \begin{pmatrix} -X_3\dfrac{\partial^2 \mathcal{U}_3}{\partial x_1^2} + \mathcal{Z}_{11} & -X_3\dfrac{\partial^2 \mathcal{U}_3}{\partial x_1 \partial x_2} + \mathcal{Z}_{12} & \dfrac{1}{2}\dfrac{\partial \overline{u}_{p,1}}{\partial X_3} \\ * & -X_3\dfrac{\partial^2 \mathcal{U}_3}{\partial x_2^2} + \mathcal{Z}_{22} & \dfrac{1}{2}\dfrac{\partial \overline{u}_{p,2}}{\partial X_3} \\ * & * & \dfrac{\partial \overline{u}_{p,3}}{\partial X_3} \end{pmatrix}. \tag{6.10}$$

6.3 Asymptotic behavior in the rod.

Now, we decompose the restriction of v_δ to the rod (see Section 2). The Theorem 3.2 gives $\mathcal{W}_\delta$, $\mathbf{Q}_\delta$, $\overline{w}_\delta$ and thanks to (4.4) we define $\mathcal{W}_\delta^{(m)}$ and $\mathcal{W}_\delta^{(s)}$. Then the estimates in Lemma 4.1 allow to claim that

$$\begin{aligned}
&||\overline{w}_\delta||_{L^2(B_\delta;\mathbb{R}^3)} \le C\delta^{7/2}, \quad ||\nabla \overline{w}_\delta||_{L^2(B_\delta;\mathbb{R}^3)} \le C\delta^{5/2}, \quad ||\mathcal{W}_\delta^{(s)}||_{H^1(-\delta,L;\mathbb{R}^3)} \le C\delta^{3/2}, \\
&||\mathcal{W}_{\delta,\alpha}^{(m)} - \mathcal{W}_{\delta,\alpha}(0)||_{H^2(-\delta,L)} \le C\delta^{1/2} + C|||\mathbf{Q}_\delta(0) - \mathbf{I}_3|||, \\
&\Big\|\frac{d\mathcal{W}_{\delta,3}^{(m)}}{dx_3}\Big\|_{L^2(-\delta,L)} \le C\delta + C|(\mathbf{Q}_\delta(0) - \mathbf{I}_3)\mathbf{e}_3 \cdot \mathbf{e}_3|, \\
&||\mathbf{Q}_\delta - \mathbf{Q}_\delta(0)||_{H^1(-\delta,L;\mathbb{R}^9)} \le C\delta^{1/2}.
\end{aligned} \tag{6.11}$$

Moreover from Lemma 4.2 we get

$$\begin{aligned}
&|||\mathbf{Q}_\delta(0) - \mathbf{I}_3||| \le C\delta^{3/4}, \qquad |(\mathbf{Q}_\delta(0) - \mathbf{I}_3)\mathbf{e}_3 \cdot \mathbf{e}_3| \le C\delta, \\
&|\mathcal{W}_{\delta,\alpha}^{(m)}(0)| \le C\delta^{7/4}, \qquad |\mathcal{W}_{\delta,3}(0) - \widetilde{\mathcal{U}}_{\delta,3}(0,0)| \le C\delta^{3/2}, \\
&|\mathcal{W}_{\delta,3}(0)| \le C\delta.
\end{aligned} \tag{6.12}$$

Finally we obtain the following estimates of the terms $\mathcal{W}_{\delta,\alpha}^{(m)}$, $\mathcal{W}_{\delta,3}^{(m)}$ and $\mathbf{Q}_\delta - \mathbf{I}_3$:

$$\begin{aligned}
&||\mathbf{Q}_\delta - \mathbf{I}_3||_{H^1(-\delta,L;\mathbb{R}^9)} \le C\delta^{1/2}, \quad ||\mathcal{W}_{\delta,\alpha}^{(m)}||_{H^2(-\delta,L)} \le C\delta^{1/2}, \\
&||\mathcal{W}_{\delta,3}^{(m)}||_{H^1(-\delta,L)} \le C\delta.
\end{aligned} \tag{6.13}$$

Now we are in a position to prove the following lemma:

Lemma 6.1. *There exists a subsequence still indexed by δ such that*

$$\begin{aligned}
&\frac{1}{\delta^{1/2}}\mathcal{W}_{\delta,\alpha}^{(m)} \rightharpoonup \mathcal{W}_\alpha \quad \textit{weakly in} \quad H^2(0,L),\\
&\frac{1}{\delta^{1/2}}\mathcal{W}_{\delta,\alpha},\ \frac{1}{\delta^{1/2}}\mathcal{W}_{\delta,\alpha}^{(m)} \longrightarrow \mathcal{W}_\alpha \quad \textit{strongly in} \quad H^1(0,L),\\
&\frac{1}{\delta}\mathcal{W}_{\delta,3},\ \frac{1}{\delta}\mathcal{W}_{\delta,3}^{(m)} \rightharpoonup \mathcal{W}_3 \quad \textit{weakly in} \quad H^1(0,L),\\
&\frac{1}{\delta^{3/2}}\mathcal{W}_{\delta}^{(s)} \rightharpoonup \mathcal{W}^{(s)} \quad \textit{weakly in} \quad H^1(0,L;\mathbb{R}^3),\\
&\frac{1}{\delta^{1/2}}\big(\mathbf{Q}_\delta - \mathbf{I}_3\big) \rightharpoonup \mathbf{A}_{\mathcal{Q}} \quad \textit{weakly in} \quad H^1(0,L;\mathbb{R}^9),\\
&\frac{1}{\delta^{5/2}}P_\delta(\overline{w_\delta}) \rightharpoonup \overline{w} \quad \textit{weakly in} \quad L^2(0,L;H^1(D;\mathbb{R}^3)).
\end{aligned} \tag{6.14}$$

We also have $\mathcal{W}_\alpha \in H^2(0,L)$ and for a.e. $x_3 \in]0,L[$ we have

$$\begin{aligned}
&\frac{d\mathcal{W}_1}{dx_3}(x_3) = \mathcal{Q}_2(x_3), \qquad \frac{d\mathcal{W}_2}{dx_3}(x_3) = -\mathcal{Q}_1(x_3),\\
&\frac{d\mathcal{W}_3}{dx_3}(x_3) + \frac{1}{2}\Big[\Big|\frac{d\mathcal{W}_1}{dx_3}(x_3)\Big|^2 + \Big|\frac{d\mathcal{W}_2}{dx_3}(x_3)\Big|^2\Big] = 0.
\end{aligned} \tag{6.15}$$

The junction conditions

$$\mathcal{W}_\alpha(0) = 0, \quad \mathcal{Q}(0) = 0, \qquad \mathcal{W}^{(s)}(0) = 0, \qquad \mathcal{W}_3(0) = \mathcal{U}_3(0,0) \tag{6.16}$$

hold true. We have

$$\frac{1}{2\delta^{3/2}}P_\delta\big((\nabla v_\delta)^T\nabla v_\delta - \mathbf{I}_3\big) \rightharpoonup \mathbf{E}_r \qquad \textit{weakly in} \quad L^1(B;\mathbb{R}^{3\times 3}), \tag{6.17}$$

where the symmetric matrix $\mathbf{E}_r$ is defined by

$$\mathbf{E}_r = \begin{pmatrix}
\gamma_{11}(\overline{w}_r) & \gamma_{12}(\overline{w}_r) & -\frac{1}{2}X_2\frac{d\mathcal{Q}_3}{dx_3} + \frac{1}{2}\frac{\partial\overline{w}_{r,3}}{\partial X_1} + \frac{1}{2}\frac{d\mathcal{W}_1^{(s)}}{dx_3}\\
* & \gamma_{22}(\overline{w}_r) & \frac{1}{2}X_1\frac{d\mathcal{Q}_3}{dx_3} + \frac{1}{2}\frac{\partial\overline{w}_{r,3}}{\partial X_2} + \frac{1}{2}\frac{d\mathcal{W}_2^{(s)}}{dx_3}\\
* & * & -X_1\frac{d^2\mathcal{W}_1}{dx_3^2} - X_2\frac{d^2\mathcal{W}_2}{dx_3^2} + \frac{d\mathcal{W}_3^{(s)}}{dx_3}
\end{pmatrix}. \tag{6.18}$$

Proof. First, taking into account (6.11), (6.13) and upon extracting a subsequence it follows that the convergences (6.14) hold true. First, due to the definition of $\mathcal{W}_\delta^{(m)}$ and the weak convergence in $H^1(0,L;\mathbb{R}^9)$ of the sequence $\frac{1}{\delta^{1/2}}\big(\mathbf{Q}_\delta - \mathbf{I}_3\big)$ towards the antisymmetric matrix $\mathbf{A}_{\mathcal{Q}}$ we deduce that $\frac{d\mathcal{W}}{dx_3} = \mathcal{Q}\wedge\mathbf{e}_3$ wich gives the two first equalities

in (6.15). Then, the strong convergence in $L^\infty(0,L;\mathbb{R}^3)$ of the sequence $\frac{1}{\delta^{1/2}}(\mathbf{Q}_\delta - \mathbf{I}_3)\mathbf{e}_3$ towards $\mathcal{Q}\wedge\mathbf{e}_3$, hence $\frac{1}{\delta}||(\mathbf{Q}_\delta - \mathbf{I}_3)\mathbf{e}_3||_2^2$ convergences towards $||\mathcal{Q}\wedge\mathbf{e}_3||_2^2 = \left|\frac{d\mathcal{W}_1}{dx_3}\right|^2 + \left|\frac{d\mathcal{W}_2}{dx_3}\right|^2$ strongly in $L^\infty(0,L)$. Finally using equality (4.9) we obtain the last equality in (6.15). The junction conditions on $\mathcal{Q}$ and $\mathcal{W}_\alpha$ are immediate consequences of (6.12) and the convergences (6.14).

In order to obtain the junction condition between the bending in the plate and the stretching in the rod, note first that the sequence $\frac{1}{\delta}\widetilde{\mathcal{U}}_{\delta,3}$ converges strongly in $H^1(\omega)$ to $\mathcal{U}_3$ because of (4.2) and the first convergence in (6.4). Besides this sequence is uniformly bounded in $H^2(D)$, hence it converges strongly to the same limit $\mathcal{U}_3$ in $C^0(D)$. Moreover the weak convergence of the sequence $\frac{1}{\delta}\mathcal{W}_{\delta,3}^{(m)}$ in $H^1(0,L)$, implies the convergence of $\frac{1}{\delta}\mathcal{W}_{\delta,3}^{(m)}(0) = \frac{1}{\delta}\mathcal{W}_{\delta,3}(0)$ to $\mathcal{W}_3(0)$. Using the third estimate in (6.12) gives the last condition in (6.16).

Once the convergences (6.14) are established, the limit of the rescaled Green-St Venant strain tensor of the sequence v_δ is analyzed in [7] (see Subsection 3.3) and it gives (6.18). □

7 Asymptotic behavior of the sequence $\frac{m_\delta}{\delta^5}$.

The goal of this section is to establish Theorem 7.2. Let us first introduce a few notations. We set

$$\begin{aligned}
\mathbb{D}_0 = \Big\{ &(\mathcal{U},\mathcal{W},\mathcal{Q}_3) \in H^1(\omega;\mathbb{R}^3)\times H^1(0,L;\mathbb{R}^3)\times H^1(0,L) \mid \\
&\mathcal{U}_3 \in H^2(\omega), \quad \mathcal{W}_\alpha \in H^2(0,L), \quad \mathcal{U}=0, \quad \frac{\partial \mathcal{U}_3}{\partial x_\alpha} = 0 \quad \text{on } \gamma_0, \\
&\frac{d\mathcal{W}_3}{dx_3} + \frac{1}{2}\Big[\Big|\frac{d\mathcal{W}_1}{dx_3}\Big|^2 + \Big|\frac{d\mathcal{W}_2}{dx_3}\Big|^2\Big] = 0 \quad \text{in }]0,L[, \\
&\mathcal{W}_3(0) = \mathcal{U}_3(0,0), \qquad \mathcal{W}_\alpha(0) = \frac{d\mathcal{W}_\alpha}{dx_3}(0) = \mathcal{Q}_3(0) = 0 \Big\}
\end{aligned} \tag{7.1}$$

Let us notice that $\mathbb{D}_0$ is a closed subset of $H^1(\omega;\mathbb{R}^3)\times H^1(0,L;\mathbb{R}^3)\times H^1(0,L)$.

We introduce below the "limit" elastic energies for the plate and the rod whose

expressions are well known for such structures

$$\mathcal{J}_p(\mathcal{U}) = \frac{E}{3(1-\nu^2)}\int_\omega \Big[(1-\nu)\sum_{\alpha,\beta=1}^{2}\Big|\frac{\partial^2 \mathcal{U}_3}{\partial x_\alpha \partial x_\beta}\Big|^2 + \nu\big(\Delta \mathcal{U}_3\big)^2\Big] + \frac{E}{(1-\nu^2)}\int_\omega \Big[(1-\nu)\sum_{\alpha,\beta=1}^{2}\big|\mathcal{Z}_{\alpha\beta}\big|^2 + \nu\big(\mathcal{Z}_{11}+\mathcal{Z}_{22}\big)^2\Big], \tag{7.2}$$

$$\mathcal{J}_r(\mathcal{W}_1,\mathcal{W}_2,\mathcal{Q}_3) = \frac{E\pi}{8}\int_0^L \Big[\Big|\frac{d^2\mathcal{W}_1}{dx_3^2}\Big|^2 + \Big|\frac{d^2\mathcal{W}_2}{dx_3^2}\Big|^2\Big] + \frac{\mu\pi}{8}\int_0^L \Big|\frac{d\mathcal{Q}_3}{dx_3}\Big|^2$$

where $\mathcal{Z}_{\alpha\beta}$ is given by

$$\mathcal{Z}_{\alpha\beta} = \gamma_{\alpha\beta}(\mathcal{U}) + \frac{1}{2}\frac{\partial \mathcal{U}_3}{\partial x_\alpha}\frac{\partial \mathcal{U}_3}{\partial x_\beta}.$$

The total energy of the plate-rod structure is given by the functional $\mathcal{J}$ defined over $\mathbb{D}_0$

$$\mathcal{J}(\mathcal{U},\mathcal{W},\mathcal{Q}_3) = \mathcal{J}_p(\mathcal{U}) + \mathcal{J}_r(\mathcal{W}_1,\mathcal{W}_2,\mathcal{Q}_3) - \mathcal{L}(\mathcal{U},\mathcal{W},\mathcal{Q}_3) \tag{7.3}$$

with

$$\mathcal{L}(\mathcal{U},\mathcal{W},\mathcal{Q}_3) = 2\int_\omega f_p\cdot\mathcal{U} + \pi\int_0^L f_r\cdot\mathcal{W}dx_3 + \frac{\pi}{2}\int_0^L g_\alpha\cdot\big(\mathcal{Q}\wedge\mathbf{e}_\alpha\big)dx_3 \tag{7.4}$$

where

$$\mathcal{Q} = -\frac{d\mathcal{W}_2}{dx_3}\mathbf{e}_1 + \frac{d\mathcal{W}_1}{dx_3}\mathbf{e}_2 + \mathcal{Q}_3\mathbf{e}_3.$$

Below we prove the existence of at least a minimizer of $\mathcal{J}$.

Lemma 7.1. *There exist two constants C_p^*, C_r^* such that, if $(f_{p,1}, f_{p,2})$ satisfies*

$$||f_{p,1}||^2_{L^2(\omega)} + ||f_{p,2}||^2_{L^2(\omega)} < C_p^* \tag{7.5}$$

and if $f_{r,3}$ satisfies

$$||f_{r,3}||_{L^2(0,L)} < C_r^* \tag{7.6}$$

then the minimization problem

$$\min_{(\mathcal{U},\mathcal{W},\mathcal{Q}_3)\in\mathbb{D}_0}\mathcal{J}(\mathcal{U},\mathcal{W},\mathcal{Q}_3) \tag{7.7}$$

admits at least a solution.

Proof. Due to the boundary conditions on $\mathcal{U}_3$ in $\mathbb{D}_0$, we immediately have

$$||\mathcal{U}_3||^2_{H^2(\omega)} \le C\mathcal{J}_p(\mathcal{U}). \tag{7.8}$$

Then we get

$$\begin{aligned}\sum_{\alpha,\beta=1}^{2}||\gamma_{\alpha,\beta}(\mathcal{U})||^2_{L^2(\omega)} &\leq C\mathcal{J}_p(\mathcal{U})+C\big\|\nabla\mathcal{U}_3\big\|^4_{L^4(\omega;\mathbb{R}^2)}\\ &\leq C\mathcal{J}_p(\mathcal{U})+C[\mathcal{J}_p(\mathcal{U})]^2.\end{aligned}\tag{7.9}$$

Thanks to the 2D Korn's inequality we obtain

$$||\mathcal{U}_1||^2_{H^1(\omega)}+||\mathcal{U}_2||^2_{H^1(\omega)}\leq C\mathcal{J}_p(\mathcal{U})+C_p[\mathcal{J}_p(\mathcal{U})]^2.\tag{7.10}$$

Again, due to the boundary conditions on $\mathcal{W}_\alpha$ and $\mathcal{Q}_3$ in $\mathbb{D}_0$, we immediately have

$$||\mathcal{W}_1||^2_{H^2(0,L)}+||\mathcal{W}_2||^2_{H^2(0,L)}+||\mathcal{Q}_3||^2_{H^1(0,L)}\leq C\mathcal{J}_r(\mathcal{W}_1,\mathcal{W}_2,\mathcal{Q}_3).\tag{7.11}$$

Then, due to the definition of $\mathbb{D}_0$ and (7.11) we get

$$\Big\|\frac{d\mathcal{W}_3}{dx_3}\Big\|^2_{L^2(0,L)}\leq C\Big\{\Big\|\frac{d\mathcal{W}_1}{dx_3}\Big\|^4_{L^4(0,L)}+\Big\|\frac{d\mathcal{W}_2}{dx_3}\Big\|^4_{L^4(0,L)}\Big\}\leq C[\mathcal{J}_r(\mathcal{W}_1,\mathcal{W}_2,\mathcal{Q}_3)]^2.\tag{7.12}$$

From the above inequality and (7.8) we obtain

$$\begin{aligned}\big\|\mathcal{W}_3\big\|^2_{L^2(0,L)}&\leq C|\mathcal{W}_3(0)|^2+C\Big\|\frac{d\mathcal{W}_3}{dx_3}\Big\|^2_{L^2(0,L)}\\ &\leq C\mathcal{J}_p(\mathcal{U})+C_r[\mathcal{J}_r(\mathcal{W}_1,\mathcal{W}_2,\mathcal{Q}_3)]^2.\end{aligned}\tag{7.13}$$

Since $\mathcal{J}(0,0,0)=0$, let us consider a minimizing sequence $(\mathcal{U}^{(N)},\mathcal{W}^{(N)},\mathcal{Q}_3^{(N)})\in\mathbb{D}_0$ satisfying $\mathcal{J}(\mathcal{U}^{(N)},\mathcal{W}^{(N)},\mathcal{Q}_3^{(N)})\leq 0$

$$m=\inf_{(\mathcal{U},\mathcal{W},\mathcal{Q}_3)\in\mathbb{D}_0}\mathcal{J}(\mathcal{U},\mathcal{W},\mathcal{Q}_3)=\lim_{N\to+\infty}\mathcal{J}(\mathcal{U}^{(N)},\mathcal{W}^{(N)},\mathcal{Q}_3^{(N)})$$

where $m\in[-\infty,0]$.
With the help of (7.8)-(7.13) we get

$$\begin{aligned}&\mathcal{J}_p(\mathcal{U}^{(N)})+\mathcal{J}_r(\mathcal{W}_1^{(N)}\mathcal{W}_2^{(N)},\mathcal{Q}_3^{(N)})\leq C||f_{p,3}||\sqrt{\mathcal{J}_p(\mathcal{U}^{(N)})}\\ &\quad+2\big(||f_{p,1}||^2_{L^2(\omega)}+||f_{p,2}||^2_{L^2(\omega)}\big)^{1/2}\big(C\sqrt{\mathcal{J}_p(\mathcal{U}^{(N)})}+\sqrt{C_p}\mathcal{J}_p(\mathcal{U}^{(N)})\big)\\ &\quad+C\sum_{\alpha=1}^{2}\big(||f_{r,\alpha}||_{L^2(0,L)}+||g_\alpha||_{L^2(0,L;\mathbb{R}^3)}\big)\sqrt{\mathcal{J}_r(\mathcal{W}_1^{(N)}\mathcal{W}_2^{(N)},\mathcal{Q}_3^{(N)})}\\ &\quad+\pi||f_{r,3}||_{L^2(0,L)}\big(C\sqrt{\mathcal{J}_r(\mathcal{W}_1^{(N)}\mathcal{W}_2^{(N)},\mathcal{Q}_3^{(N)})}+\sqrt{C_r}\mathcal{J}_r(\mathcal{W}_1^{(N)}\mathcal{W}_2^{(N)},\mathcal{Q}_3^{(N)})\big)\end{aligned}\tag{7.14}$$

Choosing $C_p^*=\dfrac{1}{2C_p}$ and $C_r^*=\dfrac{1}{\pi\sqrt{C_r}}$, if the applied forces satisfy (7.5) and (7.6) then the following estimates hold true

$$\begin{aligned}&||\mathcal{U}_3^{(N)}||_{H^2(\omega)}+||\mathcal{U}_1^{(N)}||_{H^1(\omega)}+||\mathcal{U}_2^{(N)}||_{H^1(\omega)}+||\mathcal{W}_1^{(N)}||_{H^2(0,L)}\\ &+||\mathcal{W}_2^{(N)}||_{H^2(0,L)}+||\mathcal{Q}_3^{(N)}||_{H^1(0,L)}+||\mathcal{W}_3^{(N)}||_{H^1(0,L)}\leq C\end{aligned}\tag{7.15}$$

where the constant C does not depend on N.

As a consequence, there exists $(\mathcal{U}^{(*)}, \mathcal{W}^{(*)}, \mathcal{Q}_3^{(*)}) \in H^1(\omega;\mathbb{R}^3)\times H^1(0,L;\mathbb{R}^3)\times H^1(0,L)$ such that for a subsequence

$$\begin{aligned}
&\mathcal{U}_3^{(N)} \rightharpoonup \mathcal{U}_3^{(*)} \quad \text{weakly in } H^2(\omega) \text{ and strongly in } W^{1,4}(\omega),\\
&\mathcal{U}_\alpha^{(N)} \rightharpoonup \mathcal{U}_\alpha^{(*)} \quad \text{weakly in } H^1(\omega),\\
&\mathcal{W}_\alpha^{(N)} \rightharpoonup \mathcal{W}_\alpha^{(*)} \quad \text{weakly in } H^2(0,L) \text{ and strongly in } W^{1,4}(0,L),\\
&\mathcal{Q}_3^{(N)} \rightharpoonup \mathcal{Q}_3^{(*)} \quad \text{weakly in } H^1(0,L),\\
&\mathcal{W}_3^{(N)} \rightharpoonup \mathcal{W}_3^{(*)} \quad \text{weakly in } H^1(0,L).
\end{aligned}$$

Notice that we also get the following convergences:

$$\mathcal{Z}_{\alpha\beta}^{(N)} \rightharpoonup \mathcal{Z}_{\alpha\beta}^{(*)} = \gamma_{\alpha\beta}(\mathcal{U}^{(*)}) + \frac{1}{2}\frac{\partial \mathcal{U}_3^{(*)}}{\partial x_\alpha}\frac{\partial \mathcal{U}_3^{(*)}}{\partial x_\beta} \qquad \text{weakly in } L^2(\omega).$$

The above convergences show that $(\mathcal{U}^{(*)}, \mathcal{W}^{(*)}, \mathcal{Q}_3^{(*)}) \in \mathbb{D}_0$. Finally, since $\mathcal{J}$ is weakly sequentially continuous in

$$H^1(\omega;\mathbb{R}^2) \times H^2(\omega) \times L^2(\omega;\mathbb{R}^3) \times H^2(0,L;\mathbb{R}^2) \times H^1(0,L;\mathbb{R}^2)$$

with respect to

$$(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{Z}_{11}, \mathcal{Z}_{12}, \mathcal{Z}_{22}, \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{Q}_3)$$

The above weak and strong converges imply that

$$\mathcal{J}(\mathcal{U}^{(*)}, \mathcal{W}^{(*)}, \mathcal{Q}_3^{(*)}) = m = \min_{(\mathcal{U},\mathcal{W},\mathcal{Q}_3)\in\mathbb{D}_0} \mathcal{J}(\mathcal{U}, \mathcal{W}, \mathcal{Q}_3)$$

which ends the proof of the lemma. □

Theorem 7.2. *We have*

$$\lim_{\delta\to 0} \frac{m_\delta}{\delta^5} = \min_{(\mathcal{U},\mathcal{W},\mathcal{Q}_3)\in\mathbb{D}_0} \mathcal{J}(\mathcal{U}, \mathcal{W}, \mathcal{Q}_3), \tag{7.16}$$

where the functional $\mathcal{J}$ is defined by (7.3).

Proof. Step 1. In this step we show that

$$\min_{(\mathcal{U},\mathcal{W},\mathcal{Q}_3)\in\mathbb{D}_0} \mathcal{J}(\mathcal{U}, \mathcal{W}, \mathcal{Q}_3) \leq \liminf_{\delta\to 0} \frac{m_\delta}{\delta^5}. \tag{7.17}$$

Let $(v_\delta)_{\delta>0}$ be a sequence of deformations belonging to $\mathbb{D}_\delta$ and such that

$$\lim_{\delta\to 0} \frac{J_\delta(v_\delta)}{\delta^5} = \liminf_{\delta\to 0} \frac{m_\delta}{\delta^5}. \tag{7.18}$$

One can always assume that $J_\delta(v_\delta) \leq 0$ without loss of generality. From the analysis of the previous section and, in particular from estimate (5.9) the sequence v_δ satisfies

$$\mathbf{d}(v_\delta, \Omega_\delta) + \mathbf{d}(v_\delta, B_\delta) \leq C\delta^{5/2}. \tag{7.19}$$

From (5.1)-(5.2) and estimates (5.18), we obtain

$$\big\|\nabla v_\delta^T \nabla v_\delta - \mathbf{I}_3\big\|_{L^2(\Omega_\delta;\mathbb{R}^{3\times 3})} \leq C\delta^{5/2}, \quad \big\|\nabla v_\delta^T \nabla v_\delta - \mathbf{I}_3\big\|_{L^2(B_\delta;\mathbb{R}^{3\times 3})} \leq C\delta^{5/2}. \tag{7.20}$$

Firstly, for any fixed δ, the displacement $u_\delta = v_\delta - I_d$, restricted to Ω_δ, is decomposed as in Theorem 3.1. Due to estimate (7.19), we can apply the results of Subsection 6.2 to the sequence (v_δ). As a consequence there exist a subsequence (still indexed by δ) and $\mathcal{U}^{(0)},\ \mathcal{R}^{(0)} \in H^1(\omega;\mathbb{R}^3)$ and $\overline{u}_p^{(0)} \in L^2(\omega; H^1(-1,1;\mathbb{R}^3))$, such that the convergences (6.4) and (6.5) hold true. Due to (6.6) and (6.7) the field $\mathcal{U}_3^{(0)}$ belongs to $H^2(\omega)$, and we have the boundary conditions

$$\mathcal{U}^{(0)} = 0, \quad \nabla U_3^{(0)} = 0, \qquad \text{on} \quad \gamma_0. \tag{7.21}$$

Subsection 6.2 and estimates in (7.20) also show that

$$\frac{1}{2\delta^2}\Pi_\delta\big(\nabla v_\delta^T \nabla v_\delta - \mathbf{I}_3\big) \rightharpoonup \mathbf{E}_p^{(0)} \quad \text{weakly in} \quad L^2(\Omega;\mathbb{R}^9) \tag{7.22}$$

where $\mathbf{E}_p^{(0)}$ is defined

$$\mathbf{E}_p^{(0)} = \begin{pmatrix} -X_3\dfrac{\partial^2 \mathcal{U}_3^{(0)}}{\partial x_1^2} + \mathcal{Z}_{11}^{(0)} & -X_3\dfrac{\partial^2 \mathcal{U}_3^{(0)}}{\partial x_1 \partial x_2} + \mathcal{Z}_{12}^{(0)} & \dfrac{1}{2}\dfrac{\partial \overline{u}_{p,1}^{(0)}}{\partial X_3} \\ * & -X_3\dfrac{\partial^2 \mathcal{U}_3^{(0)}}{\partial x_2^2} + \mathcal{Z}_{22}^{(0)} & \dfrac{1}{2}\dfrac{\partial \overline{u}_{p,2}^{(0)}}{\partial X_3} \\ * & * & \dfrac{\partial \overline{u}_{p,3}^{(0)}}{\partial X_3} \end{pmatrix} \tag{7.23}$$

with

$$\mathcal{Z}_{\alpha\beta}^{(0)} = \gamma_{\alpha\beta}(\mathcal{U}^{(0)}) + \frac{1}{2}\frac{\partial \mathcal{U}_3^{(0)}}{\partial x_\alpha}\frac{\partial \mathcal{U}_3^{(0)}}{\partial x_\beta}. \tag{7.24}$$

Secondly, still for δ fixed, the displacement $u_\delta = v_\delta - I_d$, restricted to B_δ, is decomposed as in Theorem 3.2 and (4.4). Again due to the estimate in (7.19), we can apply the results of Subsection 6.3 to the sequence (v_δ). As a consequence there exist a subsequence (still indexed by δ) and $\mathcal{W}^{(0)},\ \mathcal{W}^{(s,0)},\ \mathcal{Q}^{(0)} \in H^1(0,L;\mathbb{R}^3)$ and $\overline{w}_r^{(0)} \in L^2(0,L;H^1(D;\mathbb{R}^3))$ such that the convergences (6.14) hold true. As a consequence of (6.15) the components $\mathcal{W}_\alpha^{(0)}$ belong to $H^2(0,L)$ and we have

$$\frac{d\mathcal{W}^{(0)}}{dx_3} = \mathcal{Q}^{(0)} \wedge \mathbf{e}_3 \quad \text{ and } \quad \frac{d\mathcal{W}_3^{(0)}}{dx_3}(x_3) + \frac{1}{2}\Big[\Big|\frac{d\mathcal{W}_1^{(0)}}{dx_3}(x_3)\Big|^2 + \Big|\frac{d\mathcal{W}_2^{(0)}}{dx_3}(x_3)\Big|^2\Big] = 0.$$

The junction conditions (6.16) give

$$\mathcal{Q}^{(0)}(0)=0,\quad \mathcal{W}_\alpha^{(0)}(0)=0,\quad \mathcal{W}^{(s,0)}(0)=0,\qquad \mathcal{W}_3^{(0)}(0)=\mathcal{U}_3^{(0)}(0,0). \tag{7.25}$$

As a first consequence, the triplet $(\mathcal{U}^{(0)},\mathcal{W}^{(0)},\mathcal{Q}_3^{(0)})$ belongs to $\mathbb{D}_0$.

Subsection 6.3 and the second estimate (7.20) also show that

$$\frac{1}{2\delta^{3/2}}P_\delta\big((\nabla v_\delta)^T\nabla v_\delta-\mathbf{I}_3\big)\rightharpoonup \mathbf{E}_r^{(0)}\qquad\text{weakly in}\quad L^2(B;\mathbb{R}^{3\times 3}), \tag{7.26}$$

where the symmetric matrices $\mathbf{E}_r^{(0)}$ is defined by

$$\mathbf{E}_r^{(0)}=\begin{pmatrix} \gamma_{11}(\overline{w}_r^{(0)}) & \gamma_{12}(\overline{w}_r^{(0)}) & -\frac{1}{2}X_2\frac{d\mathcal{Q}_3^{(0)}}{dx_3}+\frac{1}{2}\frac{\partial\overline{w}_{r,3}^{(0)}}{\partial X_1}+\frac{1}{2}\frac{d\mathcal{W}_1^{(s,0)}}{dx_3}\\ * & \gamma_{22}(\overline{w}_r^{(0)}) & \frac{1}{2}X_1\frac{d\mathcal{Q}_3^{(0)}}{dx_3}+\frac{1}{2}\frac{\partial\overline{w}_{r,3}^{(0)}}{\partial X_2}+\frac{1}{2}\frac{d\mathcal{W}_2^{(s,0)}}{dx_3}\\ * & * & -X_1\frac{d^2\mathcal{W}_1^{(0)}}{dx_3^2}-X_2\frac{d^2\mathcal{W}_2^{(0)}}{dx_3^2}+\frac{d\mathcal{W}_3^{(s,0)}}{dx_3}\end{pmatrix}. \tag{7.27}$$

In order to bound from below the quantity $\liminf_{\delta\to 0}\dfrac{J_\delta(v_\delta)}{\delta^5}$, using the assumptions on the forces (5.5) and the convergences (6.4) and (6.14) we first have

$$\lim_{\delta\to 0}\frac{1}{\delta^5}\int_{\mathcal{S}_\delta} f_\delta\cdot(v_\delta-I_d)=\mathcal{L}(\mathcal{U}^{(0)},\mathcal{W}^{(0)},\mathcal{Q}_3^{(0)}) \tag{7.28}$$

where $\mathcal{L}(\mathcal{U},\mathcal{W},\mathcal{Q}_3)$ is given by (7.4) for any triplet in $\mathbb{D}_0$.
As far as the elastic energy is concerned, we write

$$\begin{aligned}&\frac{1}{\delta^5}\int_{\mathcal{S}_\delta}\widehat{W}_\delta\big(\nabla v_\delta\big)=\frac{1}{\delta^5}\int_{\Omega_\delta}\widehat{W}_\delta\big(\nabla v_\delta\big)+\frac{1}{\delta^5}\int_{B_\delta\setminus C_\delta}\widehat{W}_\delta\big(\nabla v_\delta\big)\\ &=\int_\Omega Q\Big(\Pi_\delta\Big[\frac{1}{\delta^2}\big((\nabla v_\delta)^T\nabla v_\delta-\mathbf{I}_3\big)\Big]\Big)+\int_B Q\Big(\chi_{B\setminus D\times]0,\delta[}P_\delta\Big[\frac{1}{\delta^{3/2}}\big((\nabla v_\delta)^T\nabla v_\delta-\mathbf{I}_3\big)\Big]\Big)\end{aligned}$$

From the weak convergences of the Green-St Venant's tensors in (7.22), (7.26) and equality (7.28), we obtain

$$\liminf_{\delta\to 0}\frac{J_\delta(v_\delta)}{\delta^5}\geq\int_\Omega Q\big(\mathbf{E}_p^{(0)}\big)+\int_B Q\big(\mathbf{E}_r^{(0)}\big)-\mathcal{L}(\mathcal{U}^{(0)},\mathcal{W}^{(0)},\mathcal{Q}_3^{(0)}) \tag{7.29}$$

where $\mathbf{E}_p^{(0)}$ and $\mathbf{E}_r^{(0)}$ are given by (7.23) and (7.50).

The next step in the derivation of the limit energy consists in minimizing $\displaystyle\int_{-1}^1 Q\big(\mathbf{E}_p^{(0)}\big)dX_3$ (resp. $\displaystyle\int_D Q\big(\mathbf{E}_r^{(0)}\big)dX_1dX_2$) with respect to $\overline{u}_p^{(0)}$(resp. $\overline{w}_r^{(0)}$).

First the expressions of Q and of $\mathbf{E}_p^{(0)}$ under a few calculations show that

$$\begin{aligned}\int_{-1}^{1} Q\big(\mathbf{E}_p^{(0)}\big)dX_3 \geq & \frac{E}{3(1-\nu^2)}\Big[(1-\nu)\sum_{\alpha,\beta=1}^{2}\Big|\frac{\partial^2\mathcal{U}_3^{(0)}}{\partial x_\alpha\partial x_\beta}\Big|^2+\nu\big(\Delta\mathcal{U}_3^{(0)}\big)^2\Big]\\ & +\frac{E}{(1-\nu^2)}\Big[(1-\nu)\sum_{\alpha,\beta=1}^{2}\big|\mathcal{Z}_{\alpha\beta}^{(0)}\big|^2+\nu\big(\mathcal{Z}_{11}^{(0)}+\mathcal{Z}_{22}^{(0)}\big)^2\Big]\end{aligned} \tag{7.30}$$

the expression in the right hand side of (7.30) is obtained through replacing $\overline{u}_p^{(0)}$ by

$$\overline{\overline{u}}_p^{(0)}(\cdot,\cdot,X_3)=\frac{\nu}{1-\nu}\Big[\Big(\frac{X_3^2}{2}-\frac{1}{6}\Big)\Delta\mathcal{U}_3^{(0)}-X_3\big(\mathcal{Z}_{11}^{(0)}+\mathcal{Z}_{22}^{(0)}\big)\Big]\mathbf{e}_3. \tag{7.31}$$

Following [7] (see equation (4.56) and (4.57)), choosing $\mathcal{W}_3^{(s,0)}=0$ and $\overline{w}_r^{(0)}$ such that

$$\begin{aligned}\overline{\overline{w}}_{r,1}^{(0)}&=-\nu\Big[\frac{X_2^2-X_1^2}{2}\frac{d^2\mathcal{W}_1^{(0)}}{dx_3^2}-X_1X_2\frac{d^2\mathcal{W}_2^{(0)}}{dx_3^2}\Big],\\ \overline{\overline{w}}_{r,2}^{(0)}&=-\nu\Big[\frac{X_1^2-X_2^2}{2}\frac{d^2\mathcal{W}_2^{(0)}}{dx_3^2}-X_1X_2\frac{d^2\mathcal{W}_1^{(0)}}{dx_3^2}\Big],\\ \overline{\overline{w}}_{r,3}^{(0)}&=-X_1\frac{d\mathcal{W}_1^{(s,0)}}{dx_3}-X_2\frac{d\mathcal{W}_2^{(s,0)}}{dx_3},\end{aligned} \tag{7.32}$$

permit to obtain

$$\int_D Q\big(\mathbf{E}_r^{(0)}\big)dX_1dX_2\geq\frac{E\pi}{8}\Big[\Big|\frac{d^2\mathcal{W}_1^{(0)}}{dx_3^2}\Big|^2+\Big|\frac{d^2\mathcal{W}_2^{(0)}}{dx_3^2}\Big|^2\Big]+\frac{\mu\pi}{8}\Big|\frac{d\mathcal{Q}_3^{(0)}}{dx_3}\Big|^2. \tag{7.33}$$

In view of (7.29), (7.30) and (7.33), the proof of(7.17) is achieved.

Step 2. In this step we show that

$$\limsup_{\delta\to 0}\frac{m_\delta}{\delta^5}\leq\min_{(\mathcal{U},\mathcal{W},\mathcal{Q}_3)\in\mathbb{D}_0}\mathcal{J}(\mathcal{U},\mathcal{W},\mathcal{Q}_3).$$

Let $(\mathcal{U}^{(1)},\mathcal{W}^{(1)},\mathcal{Q}_3^{(1)})\in\mathbb{D}_0$ such that

$$\min_{(\mathcal{U},\mathcal{W},\mathcal{Q}_3)\in\mathbb{D}_0}\mathcal{J}(\mathcal{U},\mathcal{W},\mathcal{Q}_3)=\mathcal{J}(\mathcal{U}^{(1)},\mathcal{W}^{(1)},\mathcal{Q}_3^{(1)}).$$

We consider a sequence $\big(\mathcal{U}^{(n)},\mathcal{W}^{(n)},\mathcal{Q}_3^{(n)}\big)_{n\geq 2}$ of elements belonging to $\mathbb{D}_0$ such that

- $\mathcal{U}_\alpha^{(n)}\in W^{2,\infty}(\omega)\cap H^1_{\gamma_0}(\omega)$ and

$$\begin{aligned}&\nabla\mathcal{U}_\alpha^{(n)}=0\quad\text{in}\quad D_{1/n},\quad(\alpha,\beta)\in\{1,2\}^2,\\ &\mathcal{U}_\alpha^{(n)}\longrightarrow\mathcal{U}_\alpha^{(1)}\text{ strongly in }H^1(\omega),\end{aligned} \tag{7.34}$$

• $\mathcal{U}_3^{(n)} \in W^{3,\infty}(\omega) \cap H^2_{\gamma_0}(\omega)$ and

$$\begin{aligned} &\frac{\partial^2 \mathcal{U}_3^{(n)}}{\partial x_\alpha \partial x_\beta} = 0 \quad \text{in} \quad D_{1/n}, \quad (\alpha,\beta) \in \{1,2\}^2, \\ &\mathcal{U}_3^{(n)} \longrightarrow \mathcal{U}_3^{(1)} \text{ strongly in } H^2(\omega), \end{aligned} \tag{7.35}$$

• $\mathcal{W}_\alpha^{(n)} \in W^{3,\infty}(-1/n, L)$ with $\mathcal{W}_\alpha^{(n)} = 0$ in $[-1/n, 1/n]$ and

$$\mathcal{W}_\alpha^{(n)} \longrightarrow \mathcal{W}_\alpha^{(1)} \text{ strongly in } H^2(0,L), \tag{7.36}$$

• $\mathcal{Q}_3^{(n)} \in W^{1,\infty}(-1/n, L)$ with $\mathcal{Q}_3^{(n)} = 0$ in $[-1/n, 1/n]$ and

$$\mathcal{Q}_3^{(n)} \longrightarrow \mathcal{Q}_3^{(1)} \text{ strongly in } H^1(0,L), \tag{7.37}$$

We define $\mathcal{W}_3^{(n)} \in W^{2,\infty}(-1/n, L)$ by

$$\mathcal{W}_3^{(n)} = \mathcal{U}_3^{(n)}(0,0) \quad \text{and} \quad \frac{d\mathcal{W}_3^{(n)}}{dx_3} + \frac{1}{2}\Big[\Big|\frac{d\mathcal{W}_1^{(n)}}{dx_3}\Big|^2 + \Big|\frac{d\mathcal{W}_2^{(n)}}{dx_3}\Big|^2\Big] = 0 \qquad \text{in} \quad]-1/n, L[.$$

Obviously we have

$$\mathcal{W}_3^{(n)} \longrightarrow \mathcal{W}_3^{(1)} \text{ strongly in } H^1(0,L). \tag{7.38}$$

In order to define an admissible deformation of the whole structure, we introduce the two fields $\overline{\overline{u}}_p^{(1)} \in L^2(\omega; H^1(-1,1;\mathbb{R}^3))$ obtained through replacing $\mathcal{U}^{(0)}$ by $\mathcal{U}^{(1)}$ in (7.24)-(7.31) and $\overline{\overline{w}}_{r,\alpha}^{(1)} \in L^2(0,L;H^1(D))$ obtained through replacing $\mathcal{W}^{(0)}$ and $\mathcal{Q}_3^{(0)}$ by $\mathcal{U}^{(1)}$ and $\mathcal{Q}_3^{(0)}$ in (7.32) and taking $\overline{\overline{w}}_{r,3}^{(1)} = 0$.

Then, we consider two sequences of warpings $\overline{u}^{(n)}, \overline{w}^{(n)}$ such that

• $\overline{u}^{(n)} \in W^{1,\infty}(\Omega;\mathbb{R}^3)$ with $\overline{u}^{(n)} = 0$ on $\partial\omega\times]-1,1[$, $\overline{u}^{(n)} = 0$ in the cylinder $D(O,1/n)\times]-1,1[$ and

$$\overline{u}^{(n)} \longrightarrow \overline{\overline{u}}_p^{(1)} \text{ strongly in } L^2(\omega; H^1(-1,1;\mathbb{R}^3)),$$

• $\overline{w}^{(n)} \in W^{1,\infty}(]-1/n, L[\times D;\mathbb{R}^3)$ with $\overline{w}^{(n)} = 0$ in the cylinder $D\times]-1/n, 1/n[$ and

$$\overline{w}^{(n)} \longrightarrow \overline{\overline{w}}_r^{(1)} \text{ strongly in } L^2(0,L;H^1(D;\mathbb{R}^3)).$$

For n fixed, let us consider the sequence of deformations of the plate Ω_δ. We set

$$\begin{aligned} v_{\delta,1}^{(n)}(x) &= x_1 + \delta^2\Big[\mathcal{U}_1^{(n)}(x_1,x_2) - \frac{x_3}{\delta}\frac{\partial \mathcal{U}_3^{(n)}}{\partial x_1}(x_1,x_2) + \delta\overline{u}_1^{(n)}(x_1,x_2,\frac{x_3}{\delta})\Big], \\ v_{\delta,2}^{(n)}(x) &= x_2 + \delta^2\Big[\mathcal{U}_2^{(n)}(x_1,x_2) - \frac{x_3}{\delta}\frac{\partial \mathcal{U}_3^{(n)}}{\partial x_2}(x_1,x_2) + \delta\overline{u}_2^{(n)}(x_1,x_2,\frac{x_3}{\delta})\Big], \\ v_{\delta,3}^{(n)}(x) &= x_3 + \delta\Big[\mathcal{U}_3^{(n)}(x_1,x_2) + \delta^2\overline{u}_3^{(n)}(x_1,x_2,\frac{x_3}{\delta})\Big]. \end{aligned} \tag{7.39}$$

If δ is small enough (in order to have $\delta \leq 1/n$) the expression of $v_\delta^{(n)}$ in the cylinder C_δ is

$$\begin{aligned} v_{\delta,1}^{(n)}(x) &= x_1 + \delta^2\Big[\mathcal{U}_1^{(n)}(0,0) - \frac{x_3}{\delta}\frac{\partial \mathcal{U}_3^{(n)}}{\partial x_1}(0,0)\Big], \\ v_{\delta,2}^{(n)}(x) &= x_2 + \delta^2\Big[\mathcal{U}_2^{(n)}(0,0) - \frac{x_3}{\delta}\frac{\partial \mathcal{U}_3^{(n)}}{\partial x_2}(0,0)\Big], \\ v_{\delta,3}^{(n)}(x) &= x_3 + \delta\Big[\mathcal{U}_3^{(n)}(0,0) + x_1\frac{\partial \mathcal{U}_3^{(n)}}{\partial x_1}(0,0) + x_2\frac{\partial \mathcal{U}_3^{(n)}}{\partial x_2}(0,0)\Big]. \end{aligned} \tag{7.40}$$

We denote

$$\mathcal{Q}^{(n)} = -\frac{d\mathcal{W}_2^{(n)}}{dx_3}\mathbf{e}_1 + \frac{d\mathcal{W}_1^{(n)}}{dx_3}\mathbf{e}_2 + \mathcal{Q}_3^{(n)}\mathbf{e}_3, \qquad \mathcal{R}^{(n)} = \frac{\partial \mathcal{U}_3^{(n)}}{\partial x_2}(0,0)\mathbf{e}_1 - \frac{\partial \mathcal{U}_3^{(n)}}{\partial x_1}(0,0)\mathbf{e}_2.$$

The field $\mathcal{Q}^{(n)}$ belongs to $W^{1,\infty}(-1/n, L; \mathbb{R}^3)$. Let $\mathbf{R}_\delta^{(n)}$ be the matrix field defined by

$$\mathbf{R}_\delta^{(n)}(0) = \mathbf{I}_3, \qquad \frac{d\mathbf{R}_\delta^{(n)}}{dx_3} = \mathbf{A}_{F_\delta^{(n)}}\mathbf{R}_\delta^{(n)} \quad \text{in} \quad [-1/n, L] \tag{7.41}$$

where $F_\delta^{(n)} = \delta^{1/2}\dfrac{d\mathcal{Q}^{(n)}}{dx_3} + \delta\mathcal{R}^{(n)}$ (see (3.1)) and let $\mathcal{W}_\delta^{(n)}$ be defined in $[-1/n, L]$ by

$$\mathcal{W}_\delta^{(n)}(x_3) = \int_0^{x_3} \big(\mathbf{R}_\delta^{(n)}(t) - \mathbf{I}_3\big)\mathbf{e}_3 dt + \delta^2\mathcal{U}_1^{(n)}(0,0)\mathbf{e}_1 + \delta^2\mathcal{U}_2^{(n)}(0,0)\mathbf{e}_2 + \delta\mathcal{U}_3^{(n)}(0,0)\mathbf{e}_3. \tag{7.42}$$

We have $\mathbf{R}_\delta^{(n)} \in W^{1,\infty}(-1/n, L; SO(3))$, $\mathcal{W}_\delta^{(n)} \in W^{2,\infty}(-1/n, L; \mathbb{R}^3)$ and the following strong convergences (as δ tends towards 0)

$$\begin{aligned} \mathbf{R}_\delta^{(n)} &\longrightarrow \mathbf{I}_3 \text{ strongly in } W^{1,\infty}(-1/n, L; SO(3)), \\ \frac{1}{\delta^{1/2}}\frac{d\mathbf{R}_\delta^{(n)}}{dx_3} &\longrightarrow \mathbf{A}_{\frac{d\mathcal{Q}^{(n)}}{dx_3}} \text{ strongly in } L^\infty(-1/n, L; \mathbb{R}^9), \\ \frac{1}{\delta^{1/2}}\big(\mathbf{R}_\delta^{(n)} - \mathbf{I}_3\big) &\longrightarrow \mathbf{A}_{\mathcal{Q}^{(n)}} \text{ strongly in } W^{1,\infty}(-1/n, L; \mathbb{R}^9), \\ \frac{1}{\delta}\big(\mathbf{R}_\delta^{(n)} - \mathbf{I}_3\big)\mathbf{e}_3 \cdot \mathbf{e}_3 &\longrightarrow -\frac{1}{2}||\mathbf{A}_{\mathcal{Q}^{(n)}}\mathbf{e}_3||_2^2 \text{ strongly in } L^\infty(-1/n, L), \\ \frac{1}{\delta^{1/2}}\mathcal{W}_{\delta,\alpha}^{(n)} &\longrightarrow \mathcal{W}_\alpha^{(n)} \text{ strongly in } W^{1,\infty}(-1/n, L), \\ \frac{1}{\delta}\mathcal{W}_{\delta,3}^{(n)} &\longrightarrow \mathcal{W}_3^{(n)} \text{ strongly in } W^{1,\infty}(-1/n, L). \end{aligned} \tag{7.43}$$

By definition, $\mathcal{Q}^{(n)}$ is equal to 0 in $[-1/n, 1/n]$, hence we have

$$\forall x_3 \in [-1/n, 1/n], \qquad \mathbf{R}_\delta^{(n)}(x_3) = \exp\big(\delta\mathbf{A}_{\mathcal{R}^{(n)}}x_3\big).$$

Now, we consider the fields $\overline{\mathbf{R}}_\delta^{(n)} \in W^{1,\infty}(-\delta, L; SO(3))$ and $\overline{\mathcal{W}}_\delta^{(n)} \in W^{2,\infty}(-\delta, L; \mathbb{R}^3)$ defined by

$$\begin{aligned}
&\overline{\mathbf{R}}_\delta^{(n)}(x_3) = \exp\big(\delta \mathbf{A}_{\mathcal{R}^{(n)}} x_3\big) \quad \text{in} \quad [-\delta, L],\\
&\overline{\mathcal{W}}_\delta^{(n)}(x_3) = \int_0^{x_3} \big(\overline{\mathbf{R}}_\delta^{(n)}(t) - \mathbf{I}_3\big)\mathbf{e}_3 dt + \mathcal{W}_\delta^{(n)}(0) \quad \text{in} \quad [-\delta, L].
\end{aligned} \tag{7.44}$$

We introduce a last displacement $\widetilde{v}_\delta^{(n)}$ belonging to $W^{1,\infty}(B_\delta; \mathbb{R}^3)$ (for $\delta \leq 1/n$)

$$\begin{aligned}
\widetilde{v}_\delta^{(n)}(x) = &\begin{pmatrix}
\delta^2\big(\mathcal{U}_1^{(n)}(0,0) - \dfrac{x_3}{\delta}\dfrac{\partial \mathcal{U}_3^{(n)}}{\partial x_1}(0,0)\big)\\
\delta^2\big(\mathcal{U}_2^{(n)}(0,0) - \dfrac{x_3}{\delta}\dfrac{\partial \mathcal{U}_3^{(n)}}{\partial x_2}(0,0)\big)\\
\delta\big(\mathcal{U}_3^{(n)}(0,0) + x_1\dfrac{\partial \mathcal{U}_3^{(n)}}{\partial x_1}(0,0) + x_2\dfrac{\partial \mathcal{U}_3^{(n)}}{\partial x_2}(0,0)\big)
\end{pmatrix}\\
&- \overline{\mathcal{W}}_\delta^{(n)}(x_3) - \big(\overline{\mathbf{R}}_\delta^{(n)}(x_3) - \mathbf{I}_3\big)(x_1\mathbf{e}_1 + x_2\mathbf{e}_2).
\end{aligned}$$

We have

$$||\nabla \widetilde{v}_\delta^{(n)}||_{L^\infty(B_\delta;\mathbb{R}^9)} \leq C^{(n)}\delta^2.$$

Now, we are in a position to define the deformation $v_\delta^{(n)}$ in the rod B_δ. We set

$$v_\delta^{(n)}(x) = x + \mathcal{W}_\delta^{(n)}(x_3) + \big(\mathbf{R}_\delta^{(n)}(x_3) - \mathbf{I}_3\big)(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) + \delta^{5/2}\overline{w}^{(n)}\big(\frac{x_1}{\delta}, \frac{x_2}{\delta}, x_3\big) + \widetilde{v}_\delta^{(n)}(x). \tag{7.45}$$

In the cylinder C_δ, the above expression of $v_\delta^{(n)}$ matches the one given by (7.40) if δ is small enough ($\delta \leq 1/n$).

By construction the deformation $v_\delta^{(n)}$ belongs to $\mathbb{D}_\delta$. Then we have

$$m_\delta \leq J_\delta(v_\delta^{(n)}). \tag{7.46}$$

The expression (7.39) of the displacement $v_\delta^{(n)} - I_d$ in the plate is similar to the decomposition (3.2) given in Section 2. Hence the results of Subsection 6.2 and the regularity of the terms $\mathcal{U}^{(n)}$ and $\overline{u}^{(n)}$ lead to

$$\frac{1}{2\delta^2}\Pi_\delta\big((\nabla v_\delta^{(n)})^T \nabla v_\delta^{(n)} - \mathbf{I}_3\big) \longrightarrow \mathbf{E}_p^{(n)} \qquad \text{strongly in} \quad L^\infty(\Omega; \mathbb{R}^9), \tag{7.47}$$

where the symmetric matrix $\mathbf{E}_p^{(n)}$ is defined by

$$\mathbf{E}_p^{(n)} = \begin{pmatrix}
-X_3\dfrac{\partial^2 \mathcal{U}_3^{(n)}}{\partial x_1^2} + \mathcal{Z}_{11}^{(n)} & -X_3\dfrac{\partial^2 \mathcal{U}_3^{(n)}}{\partial x_1 \partial x_2} + \mathcal{Z}_{12}^{(n)} & \dfrac{1}{2}\dfrac{\partial \overline{u}_1^{(n)}}{\partial X_3}\\
* & -X_3\dfrac{\partial^2 \mathcal{U}_3^{(n)}}{\partial x_2^2} + \mathcal{Z}_{22}^{(n)} & \dfrac{1}{2}\dfrac{\partial \overline{u}_2^{(n)}}{\partial X_3}\\
* & * & \dfrac{\partial \overline{u}_3^{(n)}}{\partial X_3}
\end{pmatrix}$$

Remark that here $\overline{u}_p^{(n)} = \overline{u}^{(n)}$ (see (6.8) and where the $\mathcal{Z}_{\alpha\beta}^{(n)}$'s are obtained through replacing $\mathcal{U}$ by $\mathcal{U}^{(n)}$ in (6.9).
Now, in the rod B_δ we have the following strong convergence in $L^\infty(B;\mathbb{R}^9)$ (as δ tends towards 0):

$$\frac{1}{\delta^{3/2}}P_\delta\big(\nabla v_\delta^{(n)} - \mathbf{R}_\delta^{(n)}\big) \longrightarrow \frac{d\mathbf{R}^{(n)}}{dx_3}(X_1\mathbf{e}_1 + X_2\mathbf{e}_2) + \begin{pmatrix} \gamma_{11}(\overline{w}^{(n)}) & \gamma_{12}(\overline{w}^{(n)}) & \frac{1}{2}\frac{\partial \overline{w}_3^{(n)}}{\partial X_1} \\ * & \gamma_{22}(\overline{w}^{(n)}) & \frac{1}{2}\frac{\partial \overline{w}_3^{(n)}}{\partial X_2} \\ * & * & 0 \end{pmatrix}. \tag{7.48}$$

Then we obtain

$$\frac{1}{2\delta^{3/2}}P_\delta\big((\nabla v_\delta^{(n)})^T\nabla v_\delta^{(n)} - \mathbf{I}_3\big) \longrightarrow \mathbf{E}_r^{(n)} \qquad \text{strongly in} \quad L^\infty(0;\mathbb{R}^9), \tag{7.49}$$

where the symmetric matrix $\mathbf{E}_r^{(n)}$ is defined by

$$\mathbf{E}_r^{(n)} = \begin{pmatrix} \gamma_{11}(\overline{w}^{(n)}) & \gamma_{12}(\overline{w}^{(n)}) & -\frac{1}{2}X_2\frac{d\mathcal{Q}_3^{(n)}}{dx_3} + \frac{1}{2}\frac{\partial \overline{w}_3^{(n)}}{\partial X_1} \\ * & \gamma_{22}(\overline{w}^{(n)}) & \frac{1}{2}X_1\frac{d\mathcal{Q}_3^{(n)}}{dx_3} + \frac{1}{2}\frac{\partial \overline{w}_3^{(n)}}{\partial X_2} \\ * & * & -X_1\frac{d^2\mathcal{W}_1^{(n)}}{dx_3^2} - X_2\frac{d^2\mathcal{W}_2^{(n)}}{dx_3^2} \end{pmatrix}. \tag{7.50}$$

Before passing to the limit, notice that

$$\Big|\frac{1}{\delta^5}\int_{\mathcal{S}_\delta}\widehat{W}_\delta(\nabla v_\delta^{(n)})(x)dx - \int_\Omega Q\Big(\Pi_\delta\big((\nabla v_\delta^{(n)})^T\nabla v_\delta^{(n)} - \mathbf{I}_3\big)\Big) - \int_B Q\Big(P_\delta\big((\nabla v_\delta^{(n)})^T\nabla v_\delta^{(n)} - \mathbf{I}_3\big)\Big)\Big| \leq \frac{C}{\delta^5}\int_{C_\delta}|||\big((\nabla v_\delta^{(n)})^T\nabla v_\delta^{(n)} - \mathbf{I}_3\big)|||^2,$$

then from the expression (7.40) of $v_\delta^{(n)}$ in C_δ and the strong convergences (7.47)-(7.49) we get

$$\lim_{\delta\to 0}\frac{1}{\delta^5}\int_{\mathcal{S}_\delta}\widehat{W}_\delta(\nabla v_\delta^{(n)})(x)dx = \int_\Omega Q(\mathbf{E}_p^{(n)}) + \int_B Q(\mathbf{E}_r^{(n)}).$$

From the expressions of $v_\delta^{(n)}$ in the plate and in the rod, from the convergences (7.43) and taking to account the expressions of the applied forces (5.5) we get

$$\lim_{\delta\to 0}\frac{1}{\delta^5}\int_{\mathcal{S}_\delta} f_\delta\cdot(v_\delta^{(n)} - I_d) = \mathcal{L}(\mathcal{U}^{(n)},\mathcal{W}^{(n)},\mathcal{Q}_3^{(n)}).$$

Then, from the above limits and (7.46) we finally get

$$\limsup_{\delta\to 0}\frac{m_\delta}{\delta^5}\leq\lim_{\delta\to 0}\frac{J_\delta(v_\delta^{(n)})}{\delta^5}=\int_\Omega Q\big(\mathbf{E}_p^{(n)}\big)+\int_B Q\big(\mathbf{E}_r^{(n)}\big)-\mathcal{L}(\mathcal{U}^{(n)},\mathcal{W}^{(n)},\mathcal{Q}_3^{(n)}).\quad(7.51)$$

Now, n goes to infinity, due to the definitions of the warpings $\overline{\overline{u}}_p^{(1)}$ and $\overline{\overline{w}}_r^{(1)}$ and the strong convergences (7.34)-(7.35)-(7.36)-(7.37)-(7.38) that give

$$\limsup_{\delta\to 0}\frac{m_\delta}{\delta^5}\leq\int_\Omega Q\big(\mathbf{E}_p^{(1)}\big)+\int_B Q\big(\mathbf{E}_r^{(1)}\big)-\mathcal{L}(\mathcal{U}^{(1)},\mathcal{W}^{(1)},\mathcal{Q}_3^{(1)})=\mathcal{J}(\mathcal{U}^{(1)},\mathcal{W}^{(1)},\mathcal{Q}_3^{(1)}).$$

This conclude the proof of the theorem. □

Remark 7.3. *Let us point out that Theorem 7.2 shows that for any minimizing sequence* $(v_\delta)_{\delta>0}$ *as in Step 1, the convergence of the rescaled Green-St Venant's strain tensor in* (7.22) *is a strong convergence in* $L^2(\Omega;\mathbb{R}^{3\times 3})$ *and the convergence* (7.26) *is a strong convergence in* $L^2(B;\mathbb{R}^{3\times 3})$.

References

[1] S. S. Antman. The theory of rods, in Handbuch der Physik, Band VIa/2, S. Flügge & C. Truesdell eds., Springer-Verlag, Berlin, (1972), 641–703.

[2] S. S. Antman. Nonlinear Problems of Elasticity, Applied Mathematical Sciences 107, Springer-Verlag, New York (1995).

[3] M. Aufranc. Junctions between three-dimensional and two-dimensional nonlinearly elastic structures. Asymptotic Anal. 4 (1991), 4, 319–338.

[4] D. Blanchard, A. Gaudiello, G. Griso. Junction of a periodic family of elastic rods with a 3*d* plate. I. J. Math. Pures Appl. (9) 88 (2007), no 1, 149-190.

[5] D. Blanchard, A. Gaudiello, G. Griso. Junction of a periodic family of elastic rods with a thin plate. II. J. Math. Pures Appl. (9) 88 (2007), no 2, 1-33.

[6] D. Blanchard, G. Griso. Microscopic effects in the homogenization of the junction of rods and a thin plate. Asympt. Anal. 56 (2008), no 1, 1-36.

[7] D. Blanchard, G. Griso. Decomposition of deformations of thin rods. Application to nonlinear elasticity, Ana. Appl. 7 (1) (2009) 21-71.

[8] D. Blanchard, G. Griso. Decomposition of the deformations of a thin shell. Asymptotic behavior of the Green-St Venant's strain tensor. J. of Elasticity, 101 (2), 179-205 (2010).

[9] D. Blanchard, G. Griso. A simplified model for elastic thin shells (to appear in Asympt. Anal.).

[10] D. Blanchard, G. Griso. Modeling of rod-structures in nonlinear elasticity. C. R. Math. Acad. Sci. Paris 348 (2010), 19-20, 1137–1141.

[11] D. Blanchard, G. Griso. Asymptotic Behavior of Structures Made of Straight Rods (to appear in J. of Elasticity).

[12] D. Blanchard, G. Griso. Asymptotic behavior of a structure made by a plate and a straight rod (hal-00611655, arXiv : 1107.5283).

[13] P.G. Ciarlet and P. Rabier. Les équations de Von Kármán. Lectures Notes in Mathematics. (Springer Verlag 1980).

[14] P.G. Ciarlet. Mathematical Elasticity, Vol. I, North-Holland, Amsterdam (1988).

[15] P.G. Ciarlet, H. Le Dret and R. Nzengwa. Junctions between three dimensional and two dimensional linearly elastic structures. J. Math. Pures Appl. 68 (1989), 261–295.

[16] P.G. Ciarlet. Plates and Junctions in Elastic Multi-Structures: An Asymptotic Analysis (Masson, Paris, 1990).

[17] P.G. Ciarlet. Mathematical Elasticity, Vol. II: Theory of plates, North-Holland, Amsterdam (1997).

[18] G. Friesecke, R. D. James and S. Müller. A hierarchy of plates models derived from nonlinear elasticity by Γ-convergence, Arch. Rat. Mech. Anal. 180 (2006) 183-236.

[19] A. Gaudiello, R. Monneau, J. Mossino, F. Murat, A. Sili. Junction of elastic plates and beams. ESAIM Control Optim. Calc. Var. 13 (2007), no. 3, 419–457.

[20] G. Griso. Asymptotic behavior of rods by the unfolding method. Math. Meth. Appl. Sci. 2004; 27: 2081-2110.

[21] G. Griso. Asymptotic behavior of structures made of plates. Analysis and Applications 3 (2005), 4, 325-356.

[22] G. Griso. Decomposition of displacements of thin structures. J. Math. Pures Appl. 89 (2008), 199-233.

[23] G. Griso. Asymptotic behavior of structures made of curved rods. Analysis and Applications, 6 (2008), 1, 11-22.

[24] I. Gruais. Modélisation de la jonction entre une plaque et une poutre en élasticité linéarisée. RAIRO: Modèl. Math. Anal. Numér. 27 (1993) 77-105.

[25] I. Gruais. Modeling of the junction between a plate and a rod in nonlinear elasticity. Asymptotic Anal. 7 (1993) 179-194.

[26] H. Le Dret. Modeling of the junction between two rods, J. Math. Pures Appl. 68 (1989), 365–397.

[27] H. Le Dret. Modeling of a folded plate, Comput. Mech., 5 (1990), 401–416.

[28] H. Le Dret. Problèmes Variationnels dans les Multi-domaines: Modélisation des Jonctions et Applications. Masson, Paris (1991).

[29] L. Trabucho and J.M. Viano. Mathematical Modelling of Rods, Handbook of Numerical Analysis 4. North-Holland, Amsterdam (1996).