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An elementary proof of Catalan-Mihailescu theorem

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Abstract

(MSC=11D04) More than one century after its formulation by the Belgian mathematician Eugene Catalan, Preda Mihailescu has solved the open problem. But, is it all? Mihailescu's solution utilizes computation on machines, we propose here not really a proof as it is intended classically, but a resolution of an equation like the resolution of the polynomial equations of third and fourth degrees. This solution is totally algebraic and does not utilize, of course, computers or any kind of calculation.

(Keywords : Diophantine equations, Catalan equation ; Algebraic resolution)

Introduction

Catalan theorem has been proved in 2002 by Preda Mihailescu. In 2004, it became officially Catalan-Mihailescu theorem. This theorem stipulates that there are not consecutive pure powers. There do not exist integers strictly greater than 1, $X > 1$ and $Y > 1$, for which with exponents strictly greater than 1, $p > 1$ and $q > 1$,

$$Y^p = X^q + 1$$

but for $(X, Y, p, q) = (2, 3, 2, 3)$. We can verify that

$$3^2 = 2^3 + 1$$

Euler has proved that the equation $X^3 + 1 = Y^2$ has this only solution. We propose in this study a general solution. The particular cases already solved concern $p = 2$, solved by Ko Chao in 1965, and $q = 3$ which has been solved in 2002. The case $q = 2$ has been solved by Lebesgue in 1850. We solve here the equation for the general case.

The approach

Let

$$c = \frac{X^p - 1}{Y^{\frac{p}{2}}}, \quad c' = \frac{7 - X^p}{Y^{\frac{p}{2}}}$$

And

$$d = X^p - Y^{\frac{p}{2}}, \quad d' = X^p + Y^{\frac{p}{2}}$$

We have

$$Y^{\frac{p}{2}} = X^p - d = d' - X^p = \frac{d' - d}{2}$$

And

$$X^p = \frac{d' + d}{2}$$

But

$$(c + c')Y^{\frac{p}{2}} = X^p - 1 + 7 - X^p = 6 \Rightarrow Y^{\frac{p}{2}} = \frac{6}{c + c'} = \frac{d' - d}{2}$$

And

$$X^p = cY^{\frac{p}{2}} + 1 = \frac{6c}{c + c'} + 1 = \frac{7c + c'}{c + c'} = \frac{d' + d}{2}$$

We deduce

$$d = \frac{7c + c' - 6}{c + c'}, \quad d' = \frac{7c + c' + 6}{c + c'}$$

And

$$X^q = Y^p - 1 = \frac{36 - (c + c')^2}{(c + c')^2}$$

Lemma 1 We have

$$c + c' > 0, \quad c' + 1 > 0, \quad c - 1 \geq 0$$

Proof of lemma 1 We have

$$(c + c')Y^{\frac{p}{2}} = 6 > 0$$

Thus $c + c' > 0$. But

$$2X^p = \frac{14c + 2c'}{c + c'} < \frac{14c + 2c' + 6}{c + c'} = d' + 7$$

Thus

$$d' - X^p = Y^{\frac{p}{2}} > X^p - 7 = -c'Y^{\frac{p}{2}}$$

It means that $c' + 1 > 0$, also

$$cY^{\frac{p}{2}} = X^p - 1 > 0$$

Thus $c > 0$. But $X^p \geq 4$, hence

$$cY^{\frac{p}{2}} = X^p - 1 \geq 7 - X^p = c'Y^{\frac{p}{2}}$$

hence $c \geq c'$. And $c + c' > c - 1$, if we suppose $c < 1$

$$7 - X^p = 7 - \frac{7c + c'}{c + c'} = \frac{6c'}{c + c'}$$

If

$$c' > 0$$

Thus $X^p < 7$ and $X^p = 4$, it means that $c' < 0$, now let $c < 1$, it means $d < 1$, but

$$(c + c')X^p = 7c + c' > 0$$

And

$$c = -ac' > -c' > 0 \Rightarrow 0 < a < 1$$

But

$$\begin{aligned} X^p &= \frac{-7a + 1}{1 - a} \\ X^p - \frac{28}{a} &= \frac{1 - 7a}{1 - a} + \frac{-28}{a} \\ &= \frac{-7a^2 + a + 28a - 28}{a(1 - a)} = \frac{-7a^2 + 28a + 7a - 28 - 6a}{a(1 - a)} \\ &= \frac{(28 - 7a)(a - 1) - 6a}{a(1 - a)} < 0 \end{aligned}$$

Because $28 - 7a > 0$, $a - 1 < 0$, $-6a < 0$, thus

$$X^p - \frac{28}{a} = X^p + \frac{28c'}{c} < 0$$

Or

$$\frac{7c + c'}{c + c'} + \frac{28c'}{c} < 0$$

And

$$7c^2 + 29cc' + 28c'^2 < 0$$

It means

$$(7c^1 + 29cc' + 28c'^2)Y^p < 0$$

Or

$$7(X^p - 1)^2 + 28(X^p - 7)^2 - 29(X^p - 1)(X^p - 7) < 0$$

And

$$\begin{aligned} 6X^{2p} - 638X^p + 1176 &< 0 \\ \frac{638 - 616}{32} < X^p < \frac{638 + 616}{32} \end{aligned}$$

Or

$$\frac{11}{16} < X^p < \frac{627}{16}$$

And $X^p = 4$ is the only solution! Another proof : We have for u, v, a_1, a_2, b_1, b_2 positive numbers and

$$uv + vc' > 0$$

The following inequality

$$a_1c + b_1c' < \frac{b_1u - a_1v}{b_2u - a_2v}(a_2c + b_2c')$$

With the following conditions

$$a_1c + b_1c' > 0$$

$$(b_1u - a_1v)(b_2u - a_2v) > 0$$

Thus

$$a_2c + b_2c' > 0$$

And

$$(a_2c + b_2c')Y^{\frac{p}{2}} = a_2(X^p - 1) + b_2(7 - X^p) = (a_2 - b_2)X^p + 7b_2 - a_2 > 0$$

Or

$$7b_2 - a_2 > (b_2 - a_2)X^p > 0$$

Or

$$X^p < \frac{7b_2 - a_2}{b_2 - a_2}$$

And we can always calculate the pure power for which this inequality is satisfied, here is an example :

$$2c + 3c' < \frac{19}{6}(c + c')$$

If $2c + 3c' > 0$

$$Y^{\frac{p}{2}} = \frac{6}{c + c'} < \frac{19}{2c + 3c'}$$

And

$$X^p = \frac{7c + c'}{6}Y^{\frac{p}{2}} < \frac{19}{6}\left(\frac{7c + c'}{2c + 3c'}\right)$$

$$X^p + \frac{19}{6}a = \frac{19}{6}\left(\frac{(7 + 2a)c + (1 + 3a)c'}{2c + 3c'}\right) < 0$$

which means

$$(7 + 2a)c + (1 + 3a)c' < 0$$

$$7 + 2a = \frac{5c'}{2c + 3c'}$$

$$a = \frac{-14c - 16c'}{2(2c + 3c')}$$

thus

$$1 + 3a = \frac{4c + 6c' - 42c - 48c'}{2(2c + 3c')}$$

$$10cc' - (-38c - 42c')c' = (48c + 42c')c' < 0$$

$$48c + 42c' > -72c' + 42c' = -30c' > 0$$

Thus

$$X^p < \frac{19}{6}\left(\frac{14c + 16c'}{4c + 6c'}\right) < 36$$

Because

$$266c + 304c' < 36(24c + 36c')$$

Because

$$598c + 892c' > -897c' + 892c' = -5c' > 0$$

Thus

$$X^p < 36$$

$$X^p \neq 25$$

$$X^p \neq 16$$

And

$$X^p \neq 9$$

Thus

$$X^p = 4$$

And if $c \geq 1$

Resolution of Catalan equation We have

$$7c^2 + 7cc' - 7c - c' + 6 > (7c + c')(c - 1) + 6 = (c + c')X^p(c - 1) + 6 \geq 6$$

Thus

$$X^p - d = Y^{\frac{p}{2}} = \frac{6}{c + c'} < \frac{7c^2 + 7cc' - 7c - c' + 6}{c + c'} = 7c - \frac{7c + c' - 6}{c + c'} = 7c - d$$

And

$$X^p - 1 = cY^{\frac{p}{2}} < 7c - 1$$

Or

$$3 \leq Y^{\frac{p}{2}} < \frac{7c - 1}{c} < 8$$

Thus

$$9 \leq Y^p < 64$$

And

$$Y^p = X^q + 1 \neq 49$$

$$Y^p = X^q + 1 \neq 36$$

$$Y^p = X^q + 1 \neq 25$$

$$Y^p = X^q + 1 \neq 16$$

And

$$Y^p = X^q + 1 = 9$$

And

$$(X, Y, p, q) = (2, \pm 3, 2, 3)$$

Conclusion

Catalan equation implies a second one, and an original solution of Catalan equation exists. It seems that many problems of number theory can be solved like this. How? We showed an example.

Références

- [1] P. MIHAILESCU A class number free criterion for catalan's conjecture, *Journal of Number theory*, **99** (2003).