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An elementary proof of Catalan-Mihailescu theorem and generalization to Fermat-Catalan conjecture

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Abstract

(MSC=11D04) We begin with an equation, for example : $Y^p = X^q \pm Z^c$ and solve it.

(Keywords : Diophantine equations, Fermat-Catalan equation; Approach)

Resolution of Catalan equation

Let Catalan equation :

$$Y^p = X^q + 1$$

We have

$$X^{q-3}Y^2 - Y^{p-2}X^3 = A$$

And

$$Y^{p-2}Y^2 - X^{q-3}X^3 = Y^p - X^q = 1$$

If $A = 0$ then $X^{q-6} = Y^{p-4}$ leads, as $GCD(X, Y) = 1$, to $p = 4$ and $q = 6$. This case has been studied by Lebesgue in the XIX century, it has no solution. Thus $A \neq 0$.

And if $A = \pm 1$ then it means that both

$$X^{q-4}Y^2 = \pm \frac{1}{X} + X^2Y^{p-2} \text{ and}$$

$$Y^{p-3}X^3 = \mp \frac{1}{X} + X^{q-3}Y \text{ are rationals}$$

it means, as it is impossible, that $q = 3$ and $p = 2$.

We have

$$\frac{X^{q-3}}{A}Y^2 - \frac{Y^{p-2}}{A}X^3 = 1 = Y^{p-2}Y^2 - X^{q-3}X^3$$

And we have simultaneously

$$(Y^{p-2} - \frac{X^{q-3}}{A})Y^2 = (X^{q-3} - \frac{Y^{p-2}}{A})X^3$$

Or

$$(AY^{p-2} - X^{q-3})Y^2 = (AX^{q-3} - Y^{p-2})X^3$$

And

$$(Y^2 + \frac{X^3}{A})Y^{p-2} = (X^3 + \frac{Y^2}{A})X^{q-3}$$

Or

$$(AY^2 + X^3)Y^{p-2} = (AX^3 + Y^2)X^{q-3}$$

But X and Y are coprimes. It means that we have four cases with u and v integers

$$Y^2 = u(AX^{q-3} - Y^{p-2}); \quad X^3 = u(-X^{q-3} + AY^{p-2})$$

$$Y^{p-2} = v(AX^3 + Y^2); \quad X^{q-3} = v(X^3 + AY^2)$$

Or

$$uY^2 = AX^{q-3} - Y^{p-2}; \quad uX^3 = -X^{q-3} + AY^{p-2}$$

$$vY^{p-2} = AX^3 + Y^2; \quad vX^{q-3} = X^3 + AY^2$$

Or

$$Y^2 = u(AX^{q-3} - Y^{p-2}); \quad X^3 = u(-X^{q-3} + AY^{p-2})$$

$$vY^{p-2} = AX^3 + Y^2; \quad vX^{q-3} = X^3 + AY^2$$

Or

$$uY^2 = AX^{q-3} - Y^{p-2}; \quad uX^3 = -X^{q-3} + AY^{p-2}$$

$$Y^{p-2} = v(AX^3 + Y^2); \quad X^{q-3} = v(X^3 + AY^2)$$

First case

$$\begin{aligned} Y^p &= uv(AX^q - Y^p + A(Y^2X^{q-3} - Y^{p-2}X^3)) \\ &= uv(A^2X^q - Y^p + A(A)) = uv(A^2X^q + A^2 - Y^p) = uv(A^2Y^p - Y^p) \end{aligned}$$

Thus

$$uv = \frac{1}{A^2 - 1}$$

As uv is integer, it means that it is impossible thus $u = 0$ and $A^2 = 1$ or $A = \pm 1$ it means that $q = 3$ and $p = 2$.

Second case

$$\begin{aligned} uv \frac{Y^p}{A^2} &= X^q - \frac{Y^p}{A^2} + \frac{Y^2X^{q-3} - Y^{p-2}X^3}{A} \\ &= X^q - \frac{Y^p}{A^2} + 1 = X^q + 1 - \frac{Y^p}{A^2} = \left(\frac{A^2 - 1}{A^2}\right)Y^p \end{aligned}$$

Thus

$$uv = A^2 - 1$$

And

$$uv(Y^2X^{q-3} - X^3Y^{p-2}) = uvA = u(X^{2q-6} - Y^{2p-4})A = v(-X^6 + Y^4)A$$

Thus

$$\begin{aligned} u &= -X^6 + Y^4; \quad v = X^{2q-6} - Y^{2p-4} \\ uv &= A^2 - 1 = (-X^6 + Y^4)(X^{2q-6} - Y^{2p-4}) \end{aligned}$$

But

$$Y^p = X^q + 1 \Rightarrow Y^{p-2}X^3 = X^q \frac{X^3}{Y^2} + \frac{X^3}{Y^2}$$

And

$$Y^2X^{q-3} = X^q \frac{X^{q-3}}{Y^{p-2}} + \frac{X^{q-3}}{Y^{p-2}}$$

We deduce

$$A = Y^2X^{q-3} - X^3Y^{p-2} = (X^q + 1)\left(\frac{X^{q-3}}{Y^{p-2}} - \frac{X^3}{Y^2}\right)$$

But

$$Y^p > X^q \Rightarrow A < 0 \Rightarrow \frac{X^{q-3}}{Y^{p-2}} < \frac{X^3}{Y^2}$$

And if

$$X^3 < Y^2 \Rightarrow X^{q-3} < Y^{p-2} \Rightarrow A^2 - 1 = (-X^6 + Y^4)(X^{2q-6} - Y^{2p-4}) < 0$$

But

$$0 > Y^2(X^{q-3} - Y^{p-2}) = Y^2X^{q-3} - Y^p = Y^2X^{q-3} - X^q - 1 = X^{q-3}(Y^2 - X^3) - 1 > -1$$

And it is possible if $q - 3 = p - 2 = 0$.

And if

$$X^{q-3} > Y^{p-2} \Rightarrow X^3 > Y^2 \Rightarrow A^2 - 1 = (-X^6 + Y^4)(X^{2q-6} - Y^{2p-4}) < 0$$

But

$$0 < Y^2(X^{q-3} - Y^{p-2}) = Y^2X^{q-3} - Y^p = Y^2X^{q-3} - X^q - 1 = X^{q-3}(Y^2 - X^3) - 1 < 0$$

And it is possible if $q - 3 = p - 2 = 0$

$$X^{q-3} > Y^{p-2} \Rightarrow X^{q-3}Y^2 > Y^p = X^q + 1 \Rightarrow Y^2 > X^3 + X^{3-q}$$

And

$$Y^2 - X^3 > X^{3-q} > 0$$

And $p - 2 = q - 3 = 0$

Third case :

We have here

$$Y^2 = u(AX^{q-3} - Y^{p-2}); \quad X^3 = u(-X^{q-3} + AY^{p-2})$$

$$vY^{p-2} = AX^3 + Y^2; \quad vX^{q-3} = X^3 + AY^2$$

And

$$vY^p = u(A^2X^q - Y^p + A^2) = u(A^2 - 1)Y^p$$

$$v = u(A^2 - 1)$$

$$v(Y^2X^{q-3} - X^3Y^{p-2}) = vA = uvA(X^{2q-6} - Y^{2p-4}) = A(-X^6 + Y^4)$$

Thus

$$1 = u(X^{2q-6} - Y^{2p-4})$$

With u and $A^2 - 1$ integers, it means $A^2 = 1$! Fourth case :

$$u\frac{Y^2}{A} = X^{q-3} - \frac{Y^{p-2}}{A}; \quad u\frac{X^3}{A} = -\frac{X^{q-3}}{A} + Y^{p-2}$$

$$\frac{Y^{p-2}}{A} = v(X^3 + \frac{Y^2}{A}); \quad \frac{X^{q-3}}{A} = v(\frac{X^3}{A} + Y^2)$$

We have here

$$uY^2 = AX^{q-3} - Y^{p-2}; \quad uX^3 - AY^{p-2} = -X^{q-3}$$

And

$$Y^{p-2} = AX^{q-3} - uY^2 = (Y^2X^{q-3} - X^3Y^{p-2})X^{q-3} - uY^2$$

Hence

$$u\frac{Y^p}{A^2} = v(X^q - \frac{Y^p}{A^2} + 1) = v(1 - \frac{1}{A^2})Y^p$$

Thus

$$u = v(A^2 - 1)$$

$$u(Y^2X^{q-3} - X^3Y^{p-2}) = uA = A(X^{2q-6} - Y^{2p-4}) = uv(-X^6 + Y^4)A$$

Thus

$$1 = v(-X^6 + Y^4)$$

With v and $A^2 - 1$ integers, it means $A^2 - 1 = 0$!

The only solution, in all cases, in $p = 2$ and $q = 3$.

And $Y^2 = X^3 + 1$ whose solution is $(X, Y) = (2, \pm 3)$.

Resolution of Fermat equation

Let Fermat equation :

$$Y^n = X^n + Z^n$$

We have

$$X^{n-2}Y^2 - Y^{n-2}X^2 = A$$

And

$$Y^{n-2}Y^2 - X^{n-2}X^2 = Y^n - X^n = Z^n$$

If $A = 0$ then $X^{n-4} = Y^{n-4}$ leads, as $GCD(X, Y) = 1$, to $n = 4$. This case has been studied by Fermat, it has no solution. Thus $A \neq 0$.

And if $A = \pm Z^n$ then it means that both

$$X^{n-3}Y^2 = \pm \frac{Z^n}{X} + XY^{n-2} \text{ and}$$

$$Y^{n-3}X^2 = \mp \frac{Z^n}{Y} + X^{n-2}Y \text{ are rationals}$$

it means, as it is impossible, that n is not greater than 2 or $n = 2$.

We have

$$\frac{X^{n-2}Z^n}{A}Y^2 - \frac{Y^{n-2}Z^n}{A}X^2 = Z^n = Y^{n-2}Y^2 - X^{n-2}X^2$$

And we have simultaneously

$$(Y^{n-2} - \frac{X^{n-2}Z^n}{A})Y^2 = (X^{n-2} - \frac{Y^{n-2}Z^n}{A})X^2$$

Or

$$(AY^{n-2} - X^{n-2}Z^n)Y^2 = (AX^{n-2} - Y^{n-2}Z^n)X^2$$

And

$$(Y^2 + \frac{X^2Z^n}{A})Y^{n-2} = (X^2 + \frac{Y^2Z^n}{A})X^{n-2}$$

Or

$$(AY^2 + X^2Z^n)Y^{p-2} = (AX^2 + Y^2Z^n)X^{n-2}$$

We have four cases with u and v integers

$$\frac{Y^2}{A} = u(X^{n-2} - \frac{Y^{n-2}Z^n}{A}); \quad \frac{X^2}{A} = u(-\frac{X^{n-2}Z^n}{A} + Y^{p-2})$$

$$\frac{Y^{n-2}}{A} = v(X^2 + \frac{Y^2Z^n}{A}); \quad \frac{X^{n-2}}{A} = v(\frac{X^2Z^n}{A} + Y^2)$$

Or

$$u\frac{Y^2}{A} = X^{n-2} - \frac{Y^{n-2}Z^n}{A}; \quad u\frac{X^2}{A} = -\frac{X^{n-2}Z^n}{A} + Y^{n-2}$$

$$v\frac{Y^{n-2}}{A} = X^2 + \frac{Y^2Z^n}{A}; \quad v\frac{X^{n-2}}{A} = \frac{X^2Z^n}{A} + Y^2$$

Or

$$\frac{Y^2}{A} = u(X^{n-2} - \frac{Y^{n-2}Z^n}{A}); \quad \frac{X^2}{A} = u(-\frac{X^{n-2}Z^n}{A} + Y^{n-2})$$

$$v\frac{Y^{n-2}}{A} = X^2 + \frac{Y^2Z^n}{A}; \quad v\frac{X^{n-2}}{A} = \frac{X^2Z^n}{A} + Y^2$$

Or

$$u\frac{Y^2}{A} = X^{n-2} - \frac{Y^{n-2}Z^n}{A}; \quad u\frac{X^2}{A} = -\frac{X^{n-2}Z^n}{A} + Y^{n-2}$$

$$\frac{Y^{n-2}}{A} = v(X^2 + \frac{Y^2Z^n}{A}); \quad \frac{X^{n-2}}{A} = v(\frac{X^2Z^n}{A} + Y^2)$$

First case

$$\begin{aligned} Y^n &= uv(A^2X^n - Y^nZ^{2n} + AZ^n(Y^2X^{n-2} - Y^{n-2}X^2)) \\ &= uv(A^2X^n - Y^nZ^{2n} + A(AZ^n)) = uv(A^2X^n + A^2Z^n - Y^nZ^{2n}) = uv(A^2Y^n - Z^{2n}Y^n) \end{aligned}$$

Thus

$$uv = \frac{1}{A^2 - Z^{2n}}$$

As uv is integer, it means that it is impossible thus $u = 0$ and $A^2 = Z^{2n}$
it means that $p = 2$.

Second case

$$\begin{aligned} uv\frac{Y^n}{A^2} &= X^n - \frac{Y^nZ^{2n}}{A^2} + Z^n\frac{Y^2X^{n-2} - Y^{n-2}X^2}{A} \\ &= X^n - \frac{Y^nZ^{2n}}{A^2} + Z^n = X^n + Z^n - \frac{Y^nZ^{2n}}{A^2} = (\frac{A^2 - Z^{2n}}{A^2})Y^n \end{aligned}$$

Thus

$$uv = A^2 - Z^{2n}$$

And

$$uv(Y^2X^{n-2} - X^2Y^{n-2}) = uvA = u(X^{2n-4} - Y^{2n-4})A = v(-X^4 + Y^4)A$$

Thus

$$\begin{aligned} u &= -X^4 + Y^4; \quad v = X^{2n-4} - Y^{2n-4} \\ uv &= A^2 - Z^{2n} = (-X^4 + Y^4)(X^{2n-4} - Y^{2n-4}) \end{aligned}$$

But

$$Y^n = X^n + Z^n \Rightarrow Y^{n-2}X^2 = X^n \frac{X^2}{Y^2} + Z^n \frac{X^2}{Y^2}$$

And

$$Y^2 X^{n-2} = X^n \frac{X^{n-2}}{Y^{n-2}} + Z^n \frac{X^{n-2}}{Y^{n-2}}$$

We deduce

$$A = Y^2 X^{n-2} - X^2 Y^{n-2} = (X^n + Z^n) \left(\frac{X^{n-2}}{Y^{n-2}} - \frac{X^2}{Y^2} \right)$$

But

$$Y^n > X^n \Rightarrow A < 0 \Rightarrow \frac{X^{n-2}}{Y^{n-2}} < \frac{X^3}{Y^2}$$

And if

$$X^2 < Y^2 \Rightarrow X^{n-2} < Y^{n-2} \Rightarrow A^2 - 1 = (-X^4 + Y^4)(X^{2n-4} - Y^{2n-4}) < 0$$

But

$$X^{n-2} < Y^{n-2} \Rightarrow X^{n-2}Y^2 < Y^n = X^n + Z^n \Rightarrow Y^2 < X^2 + Z^n X^{2-n}$$

And

$$0 < Y^2 - X^2 < Z^n X^{2-n}$$

and if $X^n > z^n$ it means $n = 2$. And if

$$X^{n-2} > Y^{n-2} \Rightarrow X^3 > Y^2 \Rightarrow A^2 - 1 = (-X^4 + Y^4)(X^{2n-4} - Y^{2n-4}) > 0$$

But

$$X^{n-2} > Y^{n-2} \Rightarrow X^{n-2}Y^2 > Y^n = X^n + Z^n \Rightarrow Y^2 > X^2 + Z^n X^{2-n}$$

And

$$0 > Y^2 - X^2 > Z^n X^{2-n} > 0$$

And $n - 2 = 0$

Third case :

We have here

$$\begin{aligned} Y^2 &= u(AX^{n-2} - Y^{n-2}Z^n); & X^2 &= u(-X^{n-2}Z^n + AY^{n-2}) \\ vY^{n-2} &= AX^2 + Y^2Z^n; & vX^{n-2} &= X^2Z^n + AY^2 \end{aligned}$$

And

$$\begin{aligned} vY^n &= u(A^2X^n - Y^nZ^{2n} + A^2Z^n) = u(A^2 - Z^{2n})Y^n \\ v &= u(A^2 - Z^{2n}) \\ v(Y^2X^{n-2} - X^2Y^{n-2}) &= vA = uvA(X^{2n-4} - Y^{2n-4}) = A(-X^4 + Y^4) \end{aligned}$$

Thus

$$u(X^{2n-4} - Y^{2n-4}) = 1$$

With u and $A^2 - Z^{2n}$ integers, it means $A^2 = Z^{2n}$! Fourth case :

$$\begin{aligned} u \frac{Y^2}{A} &= X^{n-2} - \frac{Y^{n-2}Z^n}{A}; & u \frac{X^2}{A} &= -\frac{X^{n-2}Z^n}{A} + Y^{n-2} \\ \frac{Y^{n-2}}{A} &= v(X^2 + \frac{Y^2Z^n}{A}); & \frac{X^{n-2}}{A} &= v(\frac{X^2Z^n}{A} + Y^2) \end{aligned}$$

Hence

$$u \frac{Y^n}{A^2} = v(X^n - \frac{Y^nZ^{2n}}{A^2} + Z^n) = v(1 - \frac{Z^{2n}}{A^2})Y^n$$

Thus

$$\begin{aligned} u &= v(A^2 - Z^{2n}) \\ u(Y^2X^{n-2} - X^2Y^{n-2}) &= uA = A(X^{2n-4} - Y^{2n-4}) = uv(-X^4 + Y^4)A \\ v(-X^4 + Y^4) &= 1 \end{aligned}$$

With v and $A^2 - Z^{2n}$ integers, it means $A^2 - Z^{2n} = 0$!

The only solution, in all cases, in $n = 2$.

Resolution of Fermat- Catalan equation

Let Fermat-Catalan equation :

$$Y^p = X^q + aZ^c$$

$$a = \pm 1$$

We have

$$X^{q-w}Y^2 - Y^{p-2}X^w = aA$$

And

$$Y^{p-2}Y^2 - X^{q-w}X^w = Y^p - X^q = aZ^c$$

If $A = 0$ then $X^{q-2w} = Y^{p-4}$ leads, as $GCD(X, Y) = 1$, to $p = 4$ and $q = 2w$. Thus $A \neq 0$.

And if $A = \pm Z^c$ then it means that both

$$X^{q-w-1}Y^2 = \pm \frac{aZ^c}{X} + X^{w-1}Y^{p-2} \text{ and}$$

$$Y^{p-3}X^w = \mp \frac{aZ^c}{Y} + X^{q-w}Y \text{ are rationals}$$

it means, as it is impossible, that $q = w$ and $p = 2$.

We have

$$\frac{X^{q-w}Z^c}{A}Y^2 - \frac{Y^{p-2}Z^c}{A}X^w = aZ^c = Y^{p-2}Y^2 - X^{q-w}X^w$$

And we have simultaneously

$$(Y^{p-2} - \frac{X^{q-w}Z^c}{A})Y^2 = (X^{q-w} - \frac{Y^{p-2}Z^c}{A})X^w$$

Or

$$(AY^{p-2} - X^{q-w}Z^c)Y^2 = (AX^{q-w} - Y^{p-2}Z^c)X^w$$

And

$$(Y^2 + \frac{X^wZ^c}{A})Y^{p-2} = (X^w + \frac{Y^2Z^c}{A})X^{q-w}$$

Or

$$(AY^2 + X^wZ^c)Y^{p-2} = (AX^w + Y^2Z^c)X^{q-w}$$

We have four cases with u and v integers

$$\frac{Y^2}{A} = u(X^{q-w} - \frac{Y^{p-2}Z^c}{A}) \quad \frac{X^w}{A} = u(-\frac{X^{q-w}Z^c}{A} + Y^{p-2})$$

$$\frac{Y^{p-2}}{A} = v(X^w + \frac{Y^2Z^c}{A}) \quad \frac{X^{q-w}}{A} = v(\frac{X^wZ^c}{A} + Y^2)$$

Or

$$\frac{uY^2}{A} = X^{q-w} - \frac{Y^{p-2}Z^c}{A} \quad \frac{uX^w}{A} = -\frac{X^{q-w}Z^c}{A} + Y^{p-2}$$

$$\frac{vY^{p-2}}{A} = X^w + \frac{Y^2Z^c}{A} \quad \frac{vX^{q-w}}{A} = \frac{X^wZ^c}{A} + Y^2$$

Or

$$\frac{Y^2}{A} = u(X^{q-w} - \frac{Y^{p-2}Z^c}{A}) \quad \frac{X^w}{A} = u(-\frac{X^{q-w}Z^c}{A} + Y^{p-2})$$

$$\frac{Y^{p-2}}{A} = v(X^w + \frac{Y^2Z^c}{A}) \quad \frac{X^{q-w}}{A} = \frac{X^wZ^c}{A} + Y^2$$

Or

$$\frac{uY^2}{A} = X^{q-w} - \frac{Y^{p-2}Z^c}{A} \quad \frac{uX^w}{A} = -\frac{X^{q-w}Z^c}{A} + Y^{p-2}$$

$$\frac{Y^{p-2}}{A} = v(X^w + \frac{Y^2Z^c}{A}) \quad \frac{X^{q-w}}{A} = v(\frac{X^wZ^c}{A} + Y^2)$$

First case

$$Y^p = uv(A^2X^q - Y^pZ^{2c} + AZ^c(Y^2X^{q-w} - Y^{p-2}X^w))$$

$$= uv(A^2X^q - Y^pZ^{2c} + aA(A)Z^c) = uv(A^2X^q + A^2Z^c - Y^pZ^{2c}) = uv(A^2Y^p - Y^pZ^{2c})$$

Thus

$$uv = \frac{1}{A^2 - Z^{2c}}$$

As uv is integer, it means that it is impossible thus $u = 0$ and $A^2 = Z^{2c}$ or $A = \pm Z^c$ it means that $q = w$ and $p = 2$.

Second case

$$\begin{aligned} uv \frac{Y^p}{A^2} &= X^q - \frac{Y^p Z^{2c}}{A^2} + Z^c \frac{Y^2 X^{q-w} - Y^{p-2} X^w}{A} \\ &= X^q - \frac{Y^p Z^{2c}}{A^2} + aZ^c = X^q + aZ^c - \frac{Y^p Z^{2c}}{A^2} = \left(\frac{A^2 - Z^{2c}}{A^2}\right) Y^p \end{aligned}$$

Thus

$$uv = A^2 - Z^{2c}$$

And (with $a > 0$, the proof is similar for $a < 0$)

$$uv(Y^2 X^{q-w} - X^w Y^{p-2}) = uv a A = u(X^{2q-2w} - Y^{2p-4})A = v(-X^{2w} + Y^4)A$$

Thus

$$\begin{aligned} ua &= -X^{2w} + Y^4; \quad va = X^{2q-2w} - Y^{2p-4} \\ uv &= A^2 - Z^{2c} = (-X^{2w} + Y^4)(X^{2q-2w} - Y^{2p-4}) \end{aligned}$$

$$Y^b = X^a + Z^c \Rightarrow Y^{b-2} X^w = X^a \frac{X^w}{Y^2} + Z^n \frac{X^w}{Y^2}$$

And

$$Y^2 X^{a-w} = X^a \frac{X^{a-w}}{Y^{b-2}} + Z^c \frac{X^{a-w}}{Y^{b-2}}$$

We deduce

$$A = Y^2 X^{a-w} - X^w Y^{b-2} = (X^a + Z^c) \left(\frac{X^{a-w}}{Y^{b-2}} - \frac{X^w}{Y^2} \right)$$

But

$$Y^b > X^a \Rightarrow A < 0 \Rightarrow \frac{X^{a-w}}{Y^{b-2}} < \frac{X^w}{Y^2}$$

And if

$$X^w < Y^2 \Rightarrow X^{a-w} < Y^{b-2} \Rightarrow A^2 - 1 = (-X^{2w} + Y^4)(X^{2a-2w} - Y^{2b-4}) < 0$$

But

$$X^{a-w} < Y^{b-2} \Rightarrow X^{a-w} Y^2 < Y^n = X^a + Z^c \Rightarrow Y^2 < X^w + Z^c X^{w-a}$$

And

$$0 < Y^2 - X^w < Z^c X^{w-a}$$

and if $X^a > Z^c$ it means $b = 2$. And if

$$X^{a-w} > Y^{b-2} \Rightarrow X^w > Y^2 \Rightarrow A^2 - 1 = (-X^{2w} + Y^4)(X^{2a-2w} - Y^{2b-4}) > 0$$

But

$$X^{a-w} > Y^{b-2} \Rightarrow X^{a-w} Y^2 > Y^b = X^a + Z^c \Rightarrow Y^2 > X^w + Z^c X^{w-a}$$

And

$$0 > Y^2 - X^w > Z^c X^{w-a} > 0$$

And $b - 2 = 0$

Third case :

We have here

$$Y^2 = u(AX^{q-w} - Y^{p-2}Z^c); \quad X^w = u(-X^{q-w}Z^c + AY^{p-2})$$

$$vY^{p-2} = AX^w + Y^2Z^c; \quad vX^{q-w} = X^wZ^c + AY^2$$

And

$$vY^p = u(A^2X^q - Y^pZ^{2c} + aA^2Z^c) = u(A^2 - Z^{2c})Y^p$$

$$v = u(A^2 - Z^{2c})$$

$$v(Y^2X^{q-w} - X^wY^{p-2}) = vaA = uvA(X^{2q-2w} - Y^{2p-4}) = A(-X^{2w} + Y^4)$$

Thus $au(X^{2q-2w} - Y^{2p-4}) = 1$: it is possible only if $p = 2$ and $q = w$. Fourth case :

$$\begin{aligned} u \frac{Y^2}{A} &= X^{q-w} - \frac{Y^{p-2}Z^c}{A}; & u \frac{X^w}{A} &= -\frac{X^{q-w}Z^c}{A} + Y^{p-2} \\ \frac{Y^{p-2}}{A} &= v(X^w + \frac{Y^2Z^c}{A}); & \frac{X^{q-w}}{A} &= v(\frac{X^wZ^c}{A} + Y^2) \end{aligned}$$

We have here

$$uY^2 = AX^{q-w} - Y^{p-2}Z^c; \quad uX^w - AY^{p-2} = -X^{q-w}Z^c$$

And

$$Y^{p-2} = AX^{q-w} - uY^2Z^c = (Y^2X^{q-w} - X^wY^{p-2})X^{q-w} - uY^2Z^c$$

Hence

$$u \frac{Y^p}{A^2} = v(X^q - \frac{Y^pZ^{2c}}{A^2} + aZ^c) = v(1 - \frac{Z^{2c}}{A^2})Y^p$$

Thus

$$\begin{aligned} u &= v(A^2 - Z^{2c}) \\ u(Y^2X^{q-w} - X^wY^{p-2}) &= uaA = A(X^{2q-2w} - Y^{2p-4}) = uv(-X^{2w} + Y^4)A \end{aligned}$$

Thus $-av(X^{2w} - Y^4) = 1$: it is possible only if $u = 0$ and $p - 2 = q - w = 0$

The only solution, in all cases, is $p = 2$.

And $Y^2 = X^q + aZ^c$.

Conclusion

Fermat-Catalan equation $Y^p = X^q \pm Z^c$ has solutions only for $p = 2$. We have shown a way to solve it.

References

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