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# An elementary proof of Catalan-Mihailescu theorem

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## Abstract

(MSC=11D04) We begin with Catalan equation  $Y^p = X^q + 1$  and solve it.

(Keywords : Diophantine equations, Catalan equation ; Approach)

## Resolution of Catalan equation

Let Catalan equation :

$$Y^p = X^q + 1$$

We have

$$X^{q-3}Y^2 - Y^{p-2}X^3 = A$$

If  $A = 0$  then  $X^{q-6} = Y^{p-4}$  leads, as  $GCD(X, Y) = 1$ , to  $p = 4$  and  $q = 6$ . This case has been studied by Lebesgue in the XIX century, it has no solution. Thus  $A \neq 0$ .

And if  $A = \pm 1$  then it means that both

$$X^{q-4}Y^2 = \pm \frac{1}{X} + X^2Y^{p-2} \text{ and}$$

$$Y^{p-3}X^3 = \mp \frac{1}{X} + X^{q-3}Y \text{ are rationals}$$

it means that  $q = 3$  and  $p = 2$ .

We have

$$\frac{X^{q-3}}{A}Y^2 - \frac{Y^{p-2}}{A}X^3 = 1 = Y^{p-2}Y^2 - X^{q-3}X^3$$

And we have simultaneously

$$(Y^{p-2} - \frac{X^{q-3}}{A})Y^2 = (X^{q-3} - \frac{Y^{p-2}}{A})X^3$$

And

$$(Y^2 + \frac{X^3}{A})Y^{p-2} = (X^3 + \frac{Y^2}{A})X^{q-3}$$

We have four cases with  $u$  and  $v$  integers

$$Y^2 = u(X^{q-3} - \frac{Y^{p-2}}{A}); \quad X^3 = u(-\frac{X^{q-3}}{A} + Y^{p-2})$$

$$Y^{p-2} = v(X^3 + \frac{Y^2}{A}); \quad X^{q-3} = v(\frac{X^3}{A} + Y^2)$$

Or

$$\begin{aligned} uY^2 &= X^{q-3} - \frac{Y^{p-2}}{A}; & uX^3 &= -\frac{X^{q-3}}{A} + Y^{p-2} \\ vY^{p-2} &= X^3 + \frac{Y^2}{A}; & vX^{q-3} &= \frac{X^3}{A} + Y^2 \end{aligned}$$

Or

$$\begin{aligned} Y^2 &= u(X^{q-3} - \frac{Y^{p-2}}{A}); & X^3 &= u(-\frac{X^{q-3}}{A} + Y^{p-2}) \\ vY^{p-2} &= X^3 + \frac{Y^2}{A}; & vX^{q-3} &= \frac{X^3}{A} + Y^2 \end{aligned}$$

Or

$$\begin{aligned} uY^2 &= X^{q-3} - \frac{Y^{p-2}}{A}; & uX^3 &= -\frac{X^{q-3}}{A} + Y^{p-2} \\ Y^{p-2} &= v(X^3 + \frac{Y^2}{A}); & X^{q-3} &= v(\frac{X^3}{A} + Y^2) \end{aligned}$$

First case

$$\begin{aligned} Y^p &= uv(X^q - \frac{Y^p}{A^2} + \frac{1}{A}(Y^2X^{q-3} - Y^{p-2}X^3)) \\ &= uv(X^q - \frac{Y^p}{A^2} + \frac{1}{A}(A)) = uv(X^q + 1 - \frac{Y^p}{A^2}) = uv(Y^p - \frac{Y^p}{A^2}) \end{aligned}$$

Thus

$$uv = \frac{A^2}{A^2 - 1}$$

As  $uv$  is integer, it means that it is impossible thus  $u = 0$  and  $A^2 = 1$  or  $A = \pm 1$   
( $A$  is an integer and can not equal to  $\sqrt{2}$ )  
it means that  $q = 3$  and  $p = 2$ .

Second case

$$\begin{aligned} uvY^p &= X^q - \frac{Y^p}{A^2} + \frac{1}{A}(Y^2X^{q-3} - Y^{p-2}X^3) \\ &= X^q - \frac{Y^p}{A^2} + \frac{1}{A}(A) = X^q + 1 - \frac{Y^p}{A^2} = Y^p - \frac{Y^p}{A^2} \end{aligned}$$

Thus

$$uv = \frac{A^2 - 1}{A^2}$$

As  $uv$  is integer, it means that it is impossible thus  $u = 0$

Third case :

We have here

$$\begin{aligned} Y^2 &= u(X^{q-3} - \frac{Y^{p-2}}{A}); & X^3 &= u(-\frac{X^{q-3}}{A} + Y^{p-2}) \\ vY^{p-2} &= X^3 + \frac{Y^2}{A}; & vX^{q-3} &= \frac{X^3}{A} + Y^2 \end{aligned}$$

And

$$vAY^{p-2} = AX^3 + Y^2; \quad AvX^{q-3} = X^3 + AY^2$$

Or

$$Y^2 = vAY^{p-2} - AX^3 = v(Y^2X^{q-3} - X^3Y^{p-2})Y^{p-2} + (X^3Y^{p-2} - Y^2X^{q-3})X^3$$

If  $q \geq 4$  then  $\frac{Y^2}{X}$  is surely an interger and it is not possible as  $GCD(X, Y) = 1$  And

$$X^3 = AvX^{q-3} - AY^2 = v(Y^2X^{q-3} - X^3Y^{p-2})X^{q-3} + (X^3Y^{p-2} - Y^2X^{q-3})Y^2$$

And if  $p \geq 3$  then  $\frac{X^3}{Y}$  is surely an integer and it is impossible as  $GCD(X, Y) = 1$  !

The only solution is  $(p, q) = (2, 3)$ .

Fourth case :

$$uY^2 = X^{q-3} - \frac{Y^{p-2}}{A}; \quad uX^3 = -\frac{X^{q-3}}{A} + Y^{p-2}$$

$$Y^{p-2} = v(X^3 + \frac{Y^2}{A}); \quad X^{q-3} = v(\frac{X^3}{A} + Y^2)$$

We have here

$$AuY^2 = AX^{q-3} - Y^{p-2}; \quad AuX^3 - AY^{p-2} = -X^{q-3}$$

Hence

$$Y^{p-2} = AX^{q-3} - AuY^2 = (Y^2X^{q-3} - X^3Y^{p-2})(X^{q-3} - uY^2)$$

If  $q \geq 4$  then  $\frac{Y^{p-2}}{X}$  is surely an integer and it is impossible as  $GCD(X, Y) = 1$  !  
And

$$X^{q-3} = AY^{p-2} - AuX^3 = (Y^2X^{q-3} - X^3Y^{p-2})(Y^{p-2} - uX^3)$$

If  $p \geq 3$  then  $\frac{X^{q-3}}{Y}$  is surely an integer and it is impossible as  $GCD(X, Y) = 1$  !  
The only solution is  $(p, q) = (2, 3)$

The only solution, in all cases, in  $p = 2$  and  $q = 3$ .

And  $Y^2 = X^3 + 1$  whose solution is  $(X, Y) = (2, \pm 3)$ .

### Conclusion

Catalan equation  $Y^p = X^q + 1$  has solutions only for  $q = 3$  and  $p = 2$ . We have shown a way to solve it.

## Références

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- [2] Robert Tijdeman, On the equation of Catalan *Acta Arith* , (1976).