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An elementary proof of Catalan-Mihailescu theorem

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Abstract

(MSC=11D04) We begin with Catalan equation $Y^p = X^q + 1$ and solve it.

(Keywords : Diophantine equations, Catalan equation ; Approach)

Resolution of Catalan equation

Let Catalan equation :

$$Y^p = X^q + 1$$

We have

$$X^{q-3}Y^2 - Y^{p-2}X^3 = A$$

If $A = 0$ then $X^{q-6} = Y^{p-4}$ leads, as $GCD(X, Y) = 1$, to $p = 4$ and $q = 6$. This case has been studied by Lebesgue in the XIX century, it has no solution. Thus $A \neq 0$.

And if $A = \pm 1$ then it means that both

$$X^{q-4}Y^2 = \pm \frac{1}{X} + X^2Y^{p-2} \text{ and}$$

$$Y^{p-3}X^3 = \mp \frac{1}{X} + X^{q-3}Y \text{ are rationals}$$

it means that $q = 3$ and $p = 2$.

We have

$$\frac{X^{q-3}}{A}Y^2 - \frac{Y^{p-2}}{A}X^3 = 1 = Y^{p-2}Y^2 - X^{q-3}X^3$$

And we have simultaneously

$$(Y^{p-2} - \frac{X^{q-3}}{A})Y^2 = (X^{q-3} - \frac{Y^{p-2}}{A})X^3$$

And

$$(Y^2 + \frac{X^3}{A})Y^{p-2} = (X^3 + \frac{Y^2}{A})X^{q-3}$$

We have four cases with u and v integers

$$Y^2 = u(X^{q-3} - \frac{Y^{p-2}}{A}); \quad X^3 = u(-\frac{X^{q-3}}{A} + Y^{p-2})$$

$$Y^{p-2} = v(X^3 + \frac{Y^2}{A}); \quad X^{q-3} = v(\frac{X^3}{A} + Y^2)$$

Or

$$uY^2 = X^{q-3} - \frac{Y^{p-2}}{A}; \quad uX^3 = -\frac{X^{q-3}}{A} + Y^{p-2}$$

$$vY^{p-2} = X^3 + \frac{Y^2}{A}; \quad vX^{q-3} = \frac{X^3}{A} + Y^2$$

Or

$$Y^2 = u(X^{q-3} - \frac{Y^{p-2}}{A}); \quad X^3 = u(-\frac{X^{q-3}}{A} + Y^{p-2})$$

$$vY^{p-2} = X^3 + \frac{Y^2}{A}; \quad vX^{q-3} = \frac{X^3}{A} + Y^2$$

Or

$$uY^2 = X^{q-3} - \frac{Y^{p-2}}{A}; \quad uX^3 = -\frac{X^{q-3}}{A} + Y^{p-2}$$

$$Y^{p-2} = v(X^3 + \frac{Y^2}{A}); \quad X^{q-3} = v(\frac{X^3}{A} + Y^2)$$

First case

$$Y^p = uv(X^q - \frac{Y^p}{A^2} + \frac{1}{A}(Y^2X^{q-3} - Y^{p-2}X^3))$$

$$= uv(X^q - \frac{Y^p}{A^2} + \frac{1}{A}(A)) = uv(X^q + 1 - \frac{Y^p}{A^2}) = uv(Y^p - \frac{Y^p}{A^2})$$

Thus

$$uv = \frac{A^2}{A^2 - 1}$$

As uv is integer, it means that it is impossible thus $u = 0$ and $A^2 = 1$ or $A = \pm 1$
(A is an integer and can not equal to $\sqrt{2}$)

it means that $q = 3$ and $p = 2$.

Second case

$$uvY^p = X^q - \frac{Y^p}{A^2} + \frac{1}{A}(Y^2X^{q-3} - Y^{p-2}X^3)$$

$$= X^q - \frac{Y^p}{A^2} + \frac{1}{A}(A) = X^q + 1 - \frac{Y^p}{A^2} = Y^p - \frac{Y^p}{A^2}$$

Thus

$$uv = \frac{A^2 - 1}{A^2}$$

As uv is integer, it means that it is impossible thus $u = 0$

Third case :

We have here

$$Y^2 = u(X^{q-3} - \frac{Y^{p-2}}{A}); \quad X^3 = u(-\frac{X^{q-3}}{A} + Y^{p-2})$$

$$vY^{p-2} = X^3 + \frac{Y^2}{A}; \quad vX^{q-3} = \frac{X^3}{A} + Y^2$$

And

$$vAY^{p-2} = AX^3 + Y^2; \quad AvX^{q-3} = X^3 + AY^2$$

Or

$$Y^2 = vAY^{p-2} - AX^3 = v(Y^2X^{q-3} - X^3Y^{p-2})Y^{p-2} + (X^3Y^{p-2} - Y^2X^{q-3})X^3$$

If $q \geq 4$ then $\frac{Y^2}{X}$ is surely an interger and it is not possible as $GCD(X, Y) = 1$ And

$$X^3 = AvX^{q-3} - AY^2 = v(Y^2X^{q-3} - X^3Y^{p-2})X^{q-3} + (X^3Y^{p-2} - Y^2X^{q-3})Y^2$$

And if $p \geq 3$ then $\frac{X^3}{Y}$ is surely an integer and it is impossible as $GCD(X, Y) = 1$!

The only solution is $(p, q) = (2, 3)$.

Fourth case :

$$uY^2 = X^{q-3} - \frac{Y^{p-2}}{A}; \quad uX^3 = -\frac{X^{q-3}}{A} + Y^{p-2}$$

$$Y^{p-2} = v(X^3 + \frac{Y^2}{A}); \quad X^{q-3} = v(\frac{X^3}{A} + Y^2)$$

We have here

$$AuY^2 = AX^{q-3} - Y^{p-2}; \quad AuX^3 - AY^{p-2} = -X^{q-3}$$

Hence

$$Y^{p-2} = AX^{q-3} - AuY^2 = (Y^2X^{q-3} - X^3Y^{p-2})(X^{q-3} - uY^2)$$

If $q \geq 4$ then $\frac{Y^{p-2}}{X}$ is surely an integer and it is impossible as $GCD(X, Y) = 1$!

And

$$X^{q-3} = AY^{p-2} - AuX^3 = (Y^2X^{q-3} - X^3Y^{p-2})(Y^{p-2} - uX^3)$$

If $p \geq 3$ then $\frac{X^{q-3}}{Y}$ is surely an integer and it is impossible as $GCD(X, Y) = 1$!

The only solution is $(p, q) = (2, 3)$

The only solution, in all cases, in $p = 2$ and $q = 3$.

And $Y^2 = X^3 + 1$ whose solution is $(X, Y) = (2, \pm 3)$.

Conclusion

Catalan equation $Y^p = X^q + 1$ has solutions only for $q = 3$ and $p = 2$. We have shown a way to solve it.

Références

- [1] Paolo Ribenboïm, The Catalan's conjecture *Academic press*, (1994).
- [2] Robert Tijdeman, On the equation of Catalan *Acta Arith*, (1976).