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An elementary proof of Catalan-Mihailescu theorem

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Abstract

(MSC=11D04) We begin with Fermat-Catalan equation $Z^c = X^a + Y^b > 2Y^b$ and see if it has solutions for $q + 1 = 2p$ or $q + 2 = 2p$ or $q = 2p$.

(Keywords : Diophantine equations, Fermat-Catalan equation ; Approach)

An attempt of resolution of Fermat-Catalan and Pillai equations

Let Fermat-Catalan equation

$$Y^p = X^q + Z^a > 2Z^a$$

The truth is Fermat-Catalan consists in three equations : the one higher and $Y^p = Z^a - X^q$ and $Y^p = X^q - Z^a$, we will treat only the first as the others have the same solutions by the same reasoning. Then let $Z^a = 1_a 1_a$. And Pillai one

$$Y^p = X^q + a = X^q + 1_a 1_a$$

Thus

$$Y^p = X^q + 1_a 1_a$$

We suppose firstly that

$$c = \frac{X^p - 1_a}{Y^{\frac{p}{2}}} > 0, \quad c' = \frac{7_a - X^p}{Y^{\frac{p}{2}}} < 0$$

Hence

$$Y^{\frac{p}{2}} = \frac{6_a}{c + c'}$$

$$X^p = cY^{\frac{p}{2}} + 1_a = \frac{6_a c}{c + c'} + 1_a = \frac{7_a c + 1_a c'}{c + c'}$$

And

$$X^q = Y^p - 1_a 1_a = \frac{36_a 1_a - 1_a 1_a (c + c')^2}{(c + c')^2}$$

As $X^p > 7_a$ then $c' < 0$ and $c > c'$ but $Y^{\frac{p}{2}} \geq 3$ then $0 < c + c' \leq 2_a$ Two cases $c^2 > 1$ et $0 < c^2 \leq 1$

$c^2 > 1$ We have

$$X^q - X^{2p} = \frac{36_a 1_a - 1_a 1_a (c + c')^2 - (7_a c + 1_a c')^2}{(c + c')^2}$$

$$< \frac{36_a 1_a - 36_a 1_a c^2}{(c + c')^2} < 0$$

Or

$$q + 1 \leq 2p$$

But

$$c' Y^{\frac{p}{2}} = 7_a - X^p < -Y^{\frac{p}{2}}$$

Then $c' < -1$. But, let u, v verifying

$$\frac{7uc + vc'}{c + c'} = 4$$

Or

$$(7u - 4)c = (4 - v)c' > 0$$

(The reasoning is the same if it is negative). As $c > 0$ and $c' < 0$ it means $7u - 4 > 0$ and $4 - v < 0$. Then

$$(7u - 4)(X^p - 1_a) = (4 - v)(7_a - X^p)$$

Or

$$(7u - v)X^p = 24_a + 7_a(u - v)$$

Hence

$$X^p = \frac{7_a(u - v) + 24_a}{7u - v}$$

But

$$c + c' = \frac{4 - v}{7u - 4}c' + c' = \frac{7u - v}{7u - 4}c'$$

Then

$$Y^{\frac{p}{2}} = \frac{6_a}{c + c'} = \frac{6_a(7u - 4)}{(7u - v)c'}$$

And

$$\frac{X^p}{Y^{\frac{p}{2}}} = \frac{(7_a(u - v) + 24_a)c'}{6_a(7u - 4)} = \frac{(7(u - v) + 24)c'}{6(7u - 4)}$$

But as $X^p > 28_a$ (else X, p can be calculated and Pillai equation has a finite number of solutions!) Thus

$$(9c' + 7c)Y^{\frac{p}{2}} = 9(7_a - X^p) + 7(X^p - 1_a) = 56_a - 2X^p < 0 \Rightarrow 9c' + 7c < 0$$

And

$$(4 - v)(9c' + 7c) > 0$$

Thus

$$9(7u - 4)c = 9(4 - v)c' > 7(v - 4)c$$

It means

$$7(7u - 4) > 7v - 14u - 20 > 7(4 - v)c' + 2(4 - 7u)$$

But if $X^p < X^{\frac{q+1}{2}} + 7 < X^{\frac{q+2}{2}}$ then $X^{q+1} \leq X^{2p} < X^{q+2}$ else

$$\begin{aligned} X^q(X - 49) > X^q > 49a1_a &\Rightarrow (X^p - 7_a)^2 - 49a1_a \geq X^{q+1} - 49a1_a > 49X^q \\ \Rightarrow (X^p - 7_a)^2 &\geq 49(X^q + 1_a1_a) = 49Y^p \Rightarrow X^p - 7_a \geq 7Y^{\frac{p}{2}} \end{aligned}$$

And

$$\begin{aligned} X^p - 7_a > 7Y^{\frac{p}{2}} > Y^{\frac{p}{2}} \left(\frac{7(v-4) + 2(4-7u)}{7u-4} \right) \\ > Y^{\frac{p}{2}} \left(\frac{7(4-v)c' + 2(4-7u)}{7u-4} \right) \end{aligned}$$

Then

$$-7_a + X^p + 2Y^{\frac{p}{2}} > \left(\frac{7(4-v)c'}{7u-4} \right) Y^{\frac{p}{2}}$$

Or

$$\begin{aligned} &\frac{-7_a + X^p + 2Y^{\frac{p}{2}}}{6c'} \\ &= \left(\frac{2}{6c'} - \frac{1}{6} \right) Y^{\frac{p}{2}} < \frac{7(4-v)}{6(7u-4)} Y^{\frac{p}{2}} \end{aligned}$$

Or

$$\left(\frac{2}{6c'} - \frac{1}{6} \right) c' > \frac{7(4-v)c'}{6(7u-4)}$$

And

$$\begin{aligned} &\frac{c'}{6} + \frac{7(4-v)c'}{6(7u-4)} \\ &= \frac{(7(u-v) + 24)c'}{6(7u-4)} = \frac{X^p}{Y^{\frac{p}{2}}} < \frac{2}{6} < 1 \end{aligned}$$

Thus

$$X^{q+1} \leq X^{2p} < Y^p = X^q + 1_a1_a < X^{q+1}$$

It is impossible! Thus $q + 1 = 2p$ or $q + 2 = 2p$

$$0 < c^2 \leq 1$$

$$\begin{aligned} 2X^q - X^{2p} &= \frac{72a1_a - 2a1_a(c+c')^2 - (7ac + 1ac')^2}{(c+c')^2} \\ &> 1_a1_a \frac{64 - 49c^2}{(c+c')^2} > 0 \end{aligned}$$

Or

$$X^q \geq X^{2p-1}$$

Or

$$q + 1 \geq 2p$$

We have

$$c'Y^{\frac{p}{2}} = 7a - X^p > -Y^{\frac{p}{2}}$$

Thus $c' > -1$. But, let u, v verifying

$$\frac{7uc + vc'}{c + c'} = 4$$

Or

$$(7u - 4)c = (4 - v)c' > 0$$

As $c > 0$ and $c' < 0$ it means $7u - 4 > 0$ and $4 - v < 0$. Then $7u - 4 > 0$ and $4 - v < 0$ and

$$(7u - 4)(X^p - 1_a) = (4 - v)(7a - X^p)$$

Or

$$(7u - v)X^p = 24a + 7a(u - v)$$

Hence

$$X^p = 1_a \frac{7(u - v) + 24}{7u - v}$$

But

$$c + c' = \frac{4 - v}{7u - 4}c' + c' = \frac{7u - v}{7u - 4}c'$$

Then

$$Y^{\frac{p}{2}} = \frac{6a}{c + c'} = \frac{6a(7u - 4)}{(7u - v)c'}$$

And

$$\frac{X^p}{Y^{\frac{p}{2}}} = \frac{(7(u - v) + 24)c'}{6(7u - 4)}$$

But

$$(3c' + 7c)Y^{\frac{p}{2}} = 3(7a - X^p) + 7(X^p - 1_a) = 4X^p + 14a > 0 \Rightarrow 3c' + 7c > 0$$

And

$$(4 - v)(3c' + 7c) < 0$$

Or

$$3(7u - 4)c = 3(4 - v)c' < 7(v - 4)c$$

It means

$$7u - 4 < 7v - 14u - 20 < 7(4 - v)c' + 2(4 - 7u)$$

But

$$X^q + 1_a 1_a = Y^p \geq X^{2p} + 1_a 1_a > (X^p - 7a)^2$$

Thus

$$\begin{aligned} X^p - 7a &< Y^{\frac{p}{2}} < Y^{\frac{p}{2}} \left(\frac{7(v - 4) + 2(4 - 7u)}{7u - 4} \right) \\ &< Y^{\frac{p}{2}} \left(\frac{7(4 - v)c' + 2(4 - 7u)}{7u - 4} \right) \end{aligned}$$

Then

$$-7a + X^p + 2Y^{\frac{p}{2}} < \left(\frac{7(4 - v)c'}{7u - 4} \right) Y^{\frac{p}{2}}$$

Or

$$\begin{aligned}\frac{-7a + X^p + 2Y^{\frac{p}{2}}}{6c'} &= \left(\frac{2}{6c'} - \frac{1}{6}\right)Y^{\frac{p}{2}} \\ &> \frac{7(4-v)}{6(7u-4)}Y^{\frac{p}{2}}\end{aligned}$$

Or

$$\left(\frac{2}{6c'} - \frac{1}{6}\right)c' < \frac{7(4-v)c'}{6(7u-4)}$$

And

$$\begin{aligned}\frac{c'}{6} + \frac{7(4-v)c'}{6(7u-4)} \\ = \frac{(7(u-v) + 24)c'}{6(7u-4)} = \frac{X^p}{Y^{\frac{p}{2}}} > \frac{2}{6}\end{aligned}$$

Thus

$$3X^{2p+1} > 36X^{2p} > 4Y^p > 3X^q$$

It means that finally $q = 2p$ or $q + 1 = 2p$ or $q + 2 = 2p$. We suppose now that

$$c = \frac{X^p - 1_a}{Y^{\frac{p}{2}}} < 0, \quad c' = \frac{7a - X^p}{Y^{\frac{p}{2}}} > 0$$

Hence

$$Y^{\frac{p}{2}} = \frac{6a}{c + c'} > 3$$

Thus

$$0 < c + c' < 2a$$

And

$$X^p = cY^{\frac{p}{2}} + 1_a = \frac{6ac}{c + c'} + 1_a = \frac{7ac + 1_ac'}{c + c'}$$

And

$$X^q = Y^p - 1_a 1_a = \frac{36a 1_a - 1_a 1_a (c + c')^2}{(c + c')^2}$$

As $X^p < 1_a < 7a$ then $c < 0$ and $c' > -c > c$ but $Y^{\frac{p}{2}} \geq 3$ then $0 < c + c' \leq 2a$ Two cases $c'^2 > 1$ et $0 < c'^2 \leq 1$

$c'^2 > 1$ We have

$$c^2 Y^p = (X^p - 1_a)^2 > (X^p - 7a)^2 = c'^2 Y^p > Y^p = X^q + 1_a 1_a > (X^{\frac{q}{2}} - 1_a)^2$$

Or

$$q + 1 \leq 2p$$

But if $7a < X^p + Y^{\frac{p}{2}}$ then $7a < 3X^p < 21a$ and X, p can be calculated and Pillai equation has a finite number of solutions, else

$$c'Y^{\frac{p}{2}} = 7a - X^p > Y^{\frac{p}{2}}$$

Then $-c' < -1$. But, let u, v verifying

$$\frac{7uc + vc'}{c + c'} = 4$$

Or

$$(7u - 4)c = (4 - v)c' > 0$$

As $c < 0$ and $c' > 0$ it means $7u - 4 < 0$ and $4 - v > 0$. Then

$$(7u - 4)(X^p - 1_a) = (4 - v)(7_a - X^p)$$

Or

$$(7u - v)X^p = 24_a + 7_a(u - v)$$

Hence

$$X^p = \frac{7_a(u - v) + 24_a}{7u - v}$$

But

$$c + c' = \frac{4 - v}{7u - 4}c' + c' = \frac{7u - v}{7u - 4}c'$$

Then

$$Y^{\frac{p}{2}} = \frac{6_a}{c + c'} = \frac{6_a(7u - 4)}{(7u - v)c'}$$

And

$$\frac{X^p}{Y^{\frac{p}{2}}} = \frac{(7_a(u - v) + 24_a)c'}{6_a(7u - 4)} = \frac{(7(u - v) + 24)c'}{6(7u - 4)}$$

But

$$(9c' + 7c)Y^{\frac{p}{2}} = 9(7_a - X^p) + 7(X^p - 1_a) = 56_a - 2X^p > 0 \Rightarrow 9c' + 7c > 0$$

And

$$(4 - v)(9c' + 7c) > 0$$

Thus

$$9(7u - 4)c = 9(4 - v)c' > 7(v - 4)c$$

It means

$$7(7u - 4) > 7v - 14u - 20 > 7(4 - v)c' + 2(4 - 7u)$$

But if $X^p < X^{\frac{q+1}{2}} + 7 < X^{\frac{q+2}{2}}$ then $X^{q+1} \leq X^{2p} < X^{q+2}$ else

$$\begin{aligned} X^q(X - 49) > X^q > 49_a 1_a &\Rightarrow (X^p - 7_a)^2 - 49_a 1_a \geq X^{q+1} - 49_a 1_a > 49X^q \\ \Rightarrow (X^p - 7_a)^2 &\geq 49(X^q + 1_a 1_a) = 49Y^p \Rightarrow X^p - 7_a \geq 7Y^{\frac{p}{2}} \end{aligned}$$

And

$$\begin{aligned} X^p - 7_a &> 7Y^{\frac{p}{2}} > Y^{\frac{p}{2}} \left(\frac{7(v - 4) + 2(4 - 7u)}{7u - 4} \right) \\ &> Y^{\frac{p}{2}} \left(\frac{7(4 - v)c' + 2(4 - 7u)}{7u - 4} \right) \end{aligned}$$

Then

$$-7_a + X^p + 2Y^{\frac{p}{2}} > \left(\frac{7(4 - v)c'}{7u - 4} \right) Y^{\frac{p}{2}}$$

Or

$$\frac{-7a + X^p + 2Y^{\frac{p}{2}}}{6c'} \\ = \left(\frac{2}{6c'} - \frac{1}{6}\right)Y^{\frac{p}{2}} < \frac{7(4-v)}{6(7u-4)}Y^{\frac{p}{2}}$$

Or

$$\left(\frac{2}{6c'} - \frac{1}{6}\right)c' > \frac{7(4-v)c'}{6(7u-4)}$$

And

$$\frac{c'}{6} + \frac{7(4-v)c'}{6(7u-4)} \\ = \frac{(7(u-v) + 24)c'}{6(7u-4)} = \frac{X^p}{Y^{\frac{p}{2}}} < \frac{2}{6} < 1$$

Thus

$$X^{q+1} \leq X^{2p} < Y^p = X^q + 1_a 1_a < X^{q+1}$$

It is impossible! Thus $q+1 = 2p$ or $q+2 = 2p$

$$0 < c'^2 \leq 1$$

$$(X^p - 7a)^2 < Y^p = X^q + 1_a 1_a$$

Or

$$X^q \geq X^{2p-1}$$

Or

$$q+1 \geq 2p$$

We have

$$c'Y^{\frac{p}{2}} = 7a - X^p > -Y^{\frac{p}{2}}$$

Thus $c' > -1$. But, let u, v verifying

$$\frac{7uc + vc'}{c + c'} = 4$$

Or

$$(7u-4)c = (4-v)c' > 0$$

As $c < 0$ and $c' > 0$ it means $7u-4 < 0$ and $4-v > 0$. And

$$(7u-4)(X^p - 1_a) = (4-v)(7a - X^p)$$

Or

$$(7u-v)X^p = 24a + 7a(u-v)$$

Hence

$$X^p = 1_a \frac{7(u-v) + 24}{7u-v}$$

But

$$c + c' = \frac{4-v}{7u-4}c' + c' = \frac{7u-v}{7u-4}c'$$

Then

$$Y^{\frac{p}{2}} = \frac{6a}{c + c'} = \frac{6a(7u - 4)}{(7u - v)c'}$$

And

$$\frac{X^p}{Y^{\frac{p}{2}}} = \frac{(7(u - v) + 24)c'}{6(7u - 4)}$$

But

$$(3c' + 7c)Y^{\frac{p}{2}} = 3(7a - X^p) + 7(X^p - 1a) = 4X^p + 14a > 0 \Rightarrow 3c' + 7c > 0$$

And

$$(4 - v)(3c' + 7c) > 0$$

Or

$$3(7u - 4)c = 3(4 - v)c' > 7(v - 4)c$$

It means $3(7u - 4) < 7(v - 4)$ and

$$7u - 4 < 7v - 14u - 20 < 7(4 - v)c' + 2(4 - 7u)$$

But

$$X^q + 1a1a = Y^p \geq X^{2p} + 1a1a > (X^p - 7a)^2$$

Thus

$$\begin{aligned} X^p - 7a &< Y^{\frac{p}{2}} < Y^{\frac{p}{2}} \left(\frac{7(v - 4) + 2(4 - 7u)}{7u - 4} \right) \\ &< Y^{\frac{p}{2}} \left(\frac{7(4 - v)c' + 2(4 - 7u)}{7u - 4} \right) \end{aligned}$$

Then

$$-7a + X^p + 2Y^{\frac{p}{2}} < \left(\frac{7(4 - v)c'}{7u - 4} \right) Y^{\frac{p}{2}}$$

Or

$$\begin{aligned} \frac{-7a + X^p + 2Y^{\frac{p}{2}}}{6c'} &= \left(\frac{2}{6c'} - \frac{1}{6} \right) Y^{\frac{p}{2}} \\ &> \frac{7(4 - v)}{6(7u - 4)} Y^{\frac{p}{2}} \end{aligned}$$

Or

$$\left(\frac{2}{6c'} - \frac{1}{6} \right) c' < \frac{7(4 - v)c'}{6(7u - 4)}$$

And

$$\begin{aligned} \frac{c'}{6} + \frac{7(4 - v)c'}{6(7u - 4)} \\ = \frac{(7(u - v) + 24)c'}{6(7u - 4)} = \frac{X^p}{Y^{\frac{p}{2}}} > \frac{2}{6} \end{aligned}$$

Thus

$$3X^{2p+1} > 36X^{2p} > 4Y^p > 3X^q$$

It means that finally $q = 2p$ or $q + 1 = 2p$ or $q + 2 = 2p$. We suppose finally

$$c = \frac{X^p - 1a}{Y^{\frac{p}{2}}} > 0, \quad c' = \frac{7a - X^p}{Y^{\frac{p}{2}}} > 0$$

Hence

$$Y^{\frac{p}{2}} = \frac{6a}{c+c'}$$

$$X^p = cY^{\frac{p}{2}} + 1_a = \frac{6ac}{c+c'} + 1_a = \frac{7ac + 1_ac'}{c+c'}$$

And

$$X^q = Y^p - 1_a 1_a = \frac{36a1_a - 1_a 1_a (c+c')^2}{(c+c')^2}$$

As $1_a < X^p < 7a$ then X, p can be calculated and Pillai equation has a finite number of solutions!

main result and relation with Fermat and Catalan equations It means that finally $q = 2p$ or $q + 1 = 2p$ or $q + 2 = 2p$. We see that for Catalan equation (as q and $2p$ do not have the same parity) $q + 1 = 2p$ (for this equation, we have $c > 0$ and $c' < 0$) and for Fermat equation there are only two solutions $q + 2 = 2p = 2n = n + 2 = 4$ or $q + 1 = 2p = n + 1 = 2n = 2$. We will prove now that there is only one value of p for the solution, it is $p = 2$. If the exponent of X is even, we must do as it follows

$$Y^p = X^{2p-2u} + X + 1_a 1_a - X = (X+1)X(X^{2p-2u-2} - X^{2p-2u-3} + \dots - X+1) + 1_a 1_a - X = uX(X+1) + 1_a 1_a - X$$

We have also

$$Y^p = X^{2p-2u} - X + 1_a 1_a + X = (X-1)X(X^{2p-2u-2} + X^{2p-2u-3} + \dots + X+1) + 1_a 1_a + X = vX(X-1) + 1_a 1_a + X$$

Else if the exponent of X is odd and it will be the case that we will treat :

$$Y^p = X^{2p-1} + 1 + 1_a 1_a - 1 = (X+1)(X^{2p-2} - X^{2p-3} + \dots - X+1) + 1_a 1_a - 1 = u(X+1) + 1_a 1_a - 1$$

We have also

$$Y^p = X^{2p-2u} - 1 + 1_a 1_a + 1 = (X-1)(X^{2p-2} + X^{2p-3} + \dots + X+1) + 1_a 1_a + 1 = v(X-1) + 1_a 1_a + 1$$

Hence in all cases

$$2 = u(X+1) - v(X-1) = X+1 - (X-1)$$

And

$$2Y^p - 2_a 1_a = uX(X+1) + vX(X-1)$$

Thus

$$(u-1)(X+1) = (v-1)(X-1)$$

Thus

$$X = \frac{u+v-2}{v-u}$$

And

$$2Y^p - 2_a 1_a = (u+v)X + u - v = \frac{(u+v)(u+v-2)}{v-u} + u - v$$

$$= \frac{u^2 + v^2 + 2uv - 2u - 2v - u^2 - v^2 + 2uv}{v-u} = \frac{2u(v-1) + 2v(u-1)}{v-u}$$

Hence

$$Y^p = \frac{u(v-1) + v(u-1) + 1_a 1_a (v-u)}{v-u} = X^{2p-1} + 1_a 1_a$$

If we suppose p even

$$Y^p - Y = v'Y(Y-1) = X^{2p-1} + 1_a 1_a - Y$$

$$Y^p + Y = u'Y(Y+1) = X^{2p-1} + 1_a 1_a + Y$$

$$2Y = u'Y(Y+1) - v'Y(Y-1) = Y(Y+1) - Y(Y-1)$$

And

$$2X^{2p'-1} + 2_a 1_a = u'Y(Y+1) + v'Y(Y-1)$$

Or

$$(u'-1)(Y+1) = (v'-1)(Y-1)$$

Thus

$$Y = \frac{u' + v' - 2}{v' - u'}$$

And

$$2X^{2p'-1} + 2_a 1_a = Y((u' + v')Y + (u' - v')) = \frac{u' + v' - 2}{v' - u'} \left(\frac{2u'(v' - 1) + 2v'(u' - 1)}{v' - u'} \right)$$

Or

$$Y^p = X^{2p-1} + 1_a 1_a = \frac{u' + v' - 2}{v' - u'} \left(\frac{u'(v' - 1) + v'(u' - 1)}{v' - u'} \right)$$

We have for equations with four unknowns

$$\begin{aligned} Y^p &= \left(\frac{u' + v' - 2}{v' - u'} \right)^p = X^{2p-1} + 1_a 1_a = \left(\frac{u + v - 2}{v - u} \right)^{2p-1} + 1_a 1_a \\ &= \frac{u' + v' - 2}{v' - u'} \left(\frac{u'(v' - 1) + v'(u' - 1)}{v' - u'} \right) = \frac{u(v-1) + v(u-1) + 1_a 1_a (v-u)}{v-u} \end{aligned}$$

The solutions of those equations are $u' = v' = 2$. Let us prove it : we have also

$$Y^p = Y^p - Y + Y = Y(Y-1)v' + Y = X^{2p-1} + 1_a 1_a = u(X+1) + 1_a 1_a - 1 \quad (1)$$

$$Y^p = Y^p - Y + Y = Y(Y-1)v' + Y = X^{2p-1} - 1 + 1_a 1_a + 1 = v(X-1) + 1_a 1_a + 1 \quad (2)$$

$$Y^p = Y^p + Y - Y = Y(Y+1)u' - Y = X^{2p-1} + 1 + 1_a 1_a - 1 = u(X+1) + 1_a 1_a - 1 \quad (3)$$

$$Y^p = Y^p + Y - Y = Y(Y+1)u' - Y = X^{2p-1} - 1 + 1_a 1_a + 1 = v(X-1) + 1_a 1_a + 1 \quad (4)$$

Hence, with

$$X = \frac{u + v - 2}{v - u}, \quad Y = \frac{u' + v' - 2}{v' - u'}$$

We have

$$(1) \quad v' \frac{u' + v' - 2}{v' - u'} \frac{2u' - 2}{v' - u'} + \frac{u' + v' - 2}{v' - u'} = u \frac{2v - 2}{v - u} + 1_a 1_a - 1$$

$$(2) \quad v' \frac{u' + v' - 2}{v' - u'} \frac{2u' - 2}{v' - u'} + \frac{u' + v' - 2}{v' - u'} = v \frac{2u - 2}{v - u} + 1_a 1_a + 1$$

$$(3) \quad u' \frac{u' + v' - 2}{v' - u'} \frac{2v' - 2}{v' - u'} - \frac{u' + v' - 2}{v' - u'} = u \frac{2v - 2}{v - u} + 1_a 1_a - 1$$

$$(4) \quad u' \frac{u' + v' - 2}{v' - u'} \frac{2v' - 2}{v' - u'} - \frac{u' + v' - 2}{v' - u'} = v \frac{2u - 2}{v - u} + 1_a 1_a + 1$$

Or

$$(1) \quad (v - u)(u' + v' - 2)(2v'(u' - 1) + v' - u') = (2u(v - 1) + (1_a 1_a - 1)(v - u))(v' - u')^2$$

$$(2) \quad (v - u)(u' + v' - 2)(2v'(u' - 1) + v' - u') = 2v(u - 1)(v' - u')^2 + (1_a 1_a + 1)(v - u)(v' - u')^2$$

$$(3) \quad (v - u)(u' + v' - 2)(2u'(v' - 1) - (v' - u')) = (2u(v - 1) + (1_a 1_a - 1)(v - u))(v' - u')^2$$

$$(4) \quad (v - u)(u' + v' - 2)(2u'(v' - 1) - (v' - u')) = 2v(u - 1)(v' - u')^2 + (1_a 1_a + 1)(v - u)(v' - u')^2$$

But

$$(1), (3) \quad (v - u)(u' + v' - 2)(2u'v' - u' - v') = (2uv - 2u)(v' - u')^2 + (1_a 1_a - 1)(v - u)(v' - u')^2$$

$$(2), (4) \quad (v - u)(u' + v' - 2)(2u'v' - u' - v') = (2vu - 2v)(v' - u')^2 + (1_a 1_a + 1)(v - u)(v' - u')^2$$

Hence if we suppose $u' - v' \neq 0$

$$\begin{aligned} & (v - u)((u' - 1)(Y - 1) + (v' - 1)(Y - 1))(u'(v' - 1)(Y - 1)) + v'(u' - 1)(Y - 1) \\ &= (v - u)((u' - 1)(Y - 1) + (u' - 1)(Y + 1))(u'(u' - 1)(Y + 1) + v'(u' - 1)(Y - 1)) \\ & \quad = 2(v - u)(u' - 1)^2 Y(u'(Y + 1) + v'(Y - 1)) \\ & \quad = 4(v - u)(u' - 1)^2 Y(X^{2p-1} + 1) \\ &= (2vu - 2u)(v' - u')^2(Y - 1)(Y - 1) + (1_a 1_a - 1)(v - u)(v' - u')^2(Y - 1)(Y + 1) \\ &= (2vu - 2u + (1_a 1_a - 1)(v - u))(v' - 1 - (u' - 1))(Y - 1)(v' - 1 - (u' - 1))(Y - 1) \\ &= (2vu - 2u + (1_a 1_a - 1)(v - u))((u' - 1)(Y + 1) - (u' - 1)(Y - 1))((u' - 1)(Y + 1) - (u' - 1)(Y - 1)) \\ & \quad = (8u(v - 1) + 4(1_a 1_a - 1)(v - u))(u' - 1)^2 \\ & \quad (u' - 1)^2(2u(v - 1) + (1_a 1_a - 1)(v - u) - (v - u)Y(X^{2p-1} + 1_a 1_a)) = 0 \end{aligned}$$

If the expression in the right of the parenthesis is equal to zero

$$\begin{aligned} (v - 1)(2u + 1_a 1_a - 1 - Y(X^{2p-1} + 1_a 1_a)) &= -(u - 1)Y(X^{2p-1} + 1_a 1_a - 1_a 1_a + 1) \\ &= \frac{(u - 1)(X + 1)}{X - 1}(2u + 1_a 1_a - 1 - Y(X^{2p-1} + 1_a 1_a)) \end{aligned}$$

Thus

$$\begin{aligned} (u - 1)((X + 1)(2u + 1_a 1_a - 1 - Y(X^{2p-1} + 1_a 1_a)) + (X - 1)Y(X^{2p-1} + 1_a 1_a - 1_a 1_a)) &= 0 \\ = (u - 1)(2Y^p + (1_a 1_a - 1)(X - 1) - 2Y(X^{2p-1} + 1_a 1_a) + (1 - 1_a 1_a)(X - 1)) &= 0 \\ = (u - 1)Y^p(2 - 2Y) &= 0 \end{aligned}$$

Consequently

$$u - 1 = 0 \Rightarrow (u - 1)(X + 1) = (v - 1)(X - 1) = 0 \Rightarrow v = 1$$

Impossible (because Y^p is not equal to $X + 1$)! Thus

$$v' = u'$$

$$(1), (3) \quad (v - u)(u' + v' - 2) = \frac{2u(v - 1)(v' - u')^2}{2v'u' - v' - u'}$$

$$(2), (4) \quad (v - u)(u' + v' - 2) = \frac{(2uv + 2v - 4u)(v' - u')^2}{u'v' - u' - v'} = \frac{(2uv - 2u)(v' - u')^2}{u'v' - u' - v'}$$

If $v - u = 0$ then

$$(v - u)X = u + v - 2 = 2u - 2 = 2v - 2 = 0$$

And it is impossible, because Y^p is different of $X + 1$. Thus $v' - u' = 0$, it means

$$(v' - u')Y = u' + v' - 2 = 2u' - 2 = 2v' - 2 = 0$$

Thus

$$v'Y(Y - 1) + Y = u(X + 1) = Y^2 = Y^p$$

And $p = 2$. And if p is odd

$$Y^p - 1 = v'(Y - 1)$$

$$Y^p + 1 = u'(Y + 1)$$

$$2 = u'(Y + 1) - v'(Y - 1) = (Y + 1) - (Y - 1)$$

$$2Y^p = 2X^{2p-1} + 2_a1_a = (u' + v')Y + u' - v'$$

And

$$(u' - 1)(Y + 1) = (v' - 1)(Y - 1)$$

Thus

$$Y = \frac{u' + v' - 2}{v' - u'}$$

$$X^{2p-1} + 1_a1_a = Y^p = \frac{u'(v' - 1) + v'(u' - 1)}{v' - u'}$$

We have four equations with four unknowns

$$\begin{aligned} Y^p &= \left(\frac{u' + v' - 2}{v' - u'}\right)^p = X^{2p-1} + 1_a1_a = \left(\frac{u + v - 2}{v - u}\right)^{2p-1} + 1_a1_a \\ &= \frac{u'(v' - 1) + v'(u' - 1)}{v' - u'} = \frac{u(v - 1) + v(u - 1) + 1_a1_a(v - u)}{v - u} \end{aligned}$$

The solutions are $u' = v' = 1$. Let us prove it : we have also

$$Y^p = Y^p - 1 + 1 = (Y - 1)v' + 1 = X^{2p-1} + 1_a1_a = u(X + 1) + 1_a1_a - 1 \quad (1)$$

$$Y^p = Y^p - 1 + 1 = (Y - 1)v' + 1 = X^{2p-1} - 1 + 1_a1_a + 1 = v(X - 1) + 1_a1_a + 1 \quad (2)$$

$$Y^p = Y^p + 1 - 1 = (Y + 1)u' - 1 = X^{2p-1} + 1 + 1_a1_a - 1 = u(X + 1) + 1_a1_a - 1 \quad (3)$$

$$Y^p = Y^p + 1 - 1 = (Y + 1)u' - 1 = X^{2p-1} - 1 + 1_a1_a + 1 + 2 = v(X - 1) + 1_a1_a + 1 \quad (4)$$

Hence, with

$$X = \frac{u + v - 2}{v - u}, \quad Y = \frac{u' + v' - 2}{v' - u'}$$

We have

$$(1) \quad v' \frac{2u' - 2}{v' - u'} + 1 = u \frac{2v - 2}{v - u} + 1_a 1_a - 1$$

$$(2) \quad v' \frac{2u' - 2}{v' - u'} + 1 = v \frac{2u - 2}{v - u} + 1_a 1_a + 1$$

$$(3) \quad u' \frac{2v' - 2}{v' - u'} - 1 = u \frac{2v - 2}{v - u} + 1_a 1_a - 1$$

$$(4) \quad u' \frac{2v' - 2}{v' - u'} - 1 = v \frac{2u - 2}{v - u} + 1_a 1_a + 1$$

Or

$$(1) \quad (v - u)(2v'(u' - 1) + v' - u') = 2u(v - 1)(v' - u') + (1_a 1_a - 1)(v - u)(v' - u')$$

$$(2) \quad (v - u)(2v'(u' - 1) + v' - u') = 2v(u - 1)(v' - u') + (1_a 1_a + 1)(v - u)(v' - u')$$

$$(3) \quad (v - u)(2u'(v' - 2) - (v' - u')) = 2u(v - 1)(v' - u') + (1_a 1_a - 1)(v - u)(v' - u')$$

$$(4) \quad (v - u)(2u'(v' - 1) - (v' - u')) = 2v(u - 1)(v' - u') + (1_a 1_a + 1)(v - u)(v' - u')$$

But

$$(1), (3) \quad (v - u)(2u'v' - u' - v') = 2u(v - 1)(v' - u') + (1_a 1_a - 1)(v - u)(v' - u')$$

$$(2), (4) \quad (v - u)(2u'v' - u' - v') = (2vu - 2v)(v' - u') + (1_a 1_a + 1)(v - u)(v' - u')$$

Hence if we suppose $u' - v' \neq 0$

$$\begin{aligned} & (v - u)(u'(v' - 1)(Y - 1) + v'(u' - 1)(Y - 1)) \\ &= (v - u)(u'(u' - 1)(Y + 1) + v'(u' - 1)(Y - 1)) \\ &= (v - u)(u' - 1)(2X^{2p-1} + 2_a 1_a - 2) \\ &= (2vu - 2u + (1_a 1_a - 1))(v' - u')(Y - 1) = (2u(v - 1) + (1_a 1_a - 1)(v - u))(v' - 1 - (u' - 1))(Y - 1) \\ &= (2u(v - 1) + (1_a 1_a - 1)(v - u))((u' - 1)(Y + 1) - (u' - 1)(Y - 1)) \\ &= (4u(v - 1) + 2(1_a 1_a - 1)(v - u))(u' - 1) \end{aligned}$$

Thus

$$(u' - 1)(2u(v - 1) + (1_a 1_a - 1)(v - u) - (v - u)(X^{2p-1} + 1_a 1_a)) = 0$$

If

$$\begin{aligned} (v - 1)(X^{2p'-1} + 1 - 2u) &= (u - 1)(X^{2p'-1} + 1) \\ &= \frac{(u - 1)(X + 1)}{X - 1}(X^{2p'-1} + 1 - 2u) \end{aligned}$$

And

$$\begin{aligned} (u - 1)((X + 1)(X^{2p'-1} + 1 - 2u) - (X - 1)(X^{2p'-1} + 1)) &= 0 \\ &= (u - 1)(2X^{2p'-1} + 3 - 2u) = 0 \end{aligned}$$

Consequently

$$u - 1 = 0 \Rightarrow (u - 1)(X + 1) = (v - 1)(X - 1) = 0 \Rightarrow v = 1$$

Impossible (because Y^p is not equal to $X + 1$)! Thus

$$v' = u'$$

And

$$u' + v' = 2 = 2u' = 2v'$$

Thus

$$v'(Y - 1) + 1 = u(X + 1) = Y = Y^p$$

It is impossible! Thus $p = 2$. We deduce that Fermat-Catalan equation implies six equations-solutions which are

$$(I) \quad Y^2 = X^3 + Z^a$$

$$(II) \quad Y^2 = X^4 + Z^a$$

$$(III) \quad Y^2 + X^3 = Z^a$$

$$(IV) \quad Y^2 + X^4 = Z^a$$

$$(V) \quad Y^2 + Z^a = X^3$$

$$(VI) \quad Y^2 + Z^a = X^4$$

Known solutions We have

$$1 + 2^3 = 3^2$$

$$7^2 + 3^5 = 3^4$$

$$13^2 + 7^3 = 2^9$$

$$71^2 = 17^3 + 2^7$$

$$3^5 + 11^4 = 122^2$$

$$1549304^2 + 33^8 = 15613^3$$

$$2213459^2 + 1413^3 = 65^7$$

$$15312283^2 = 113^7 - 9262^3$$

$$21062938^2 = 76271^3 + 17^7$$

$$30042907^2 = 96222^3 + 43^8$$

Conclusion

Fermat-Catalan equation $Z^c = X^a + Y^b$ does not have solutions for both $a > 2$, $b > 2$ and $c > 2$. In fact, at least one of the exponents must be equal to 2. We have shown a way to discuss the solutions.