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► **To cite this version:**

| Jamel Ghannouchi. An elementary proof of Catalan-Mihailescu theorem. 2011. hal-00677731v31

HAL Id: hal-00677731

<https://hal.science/hal-00677731v31>

Preprint submitted on 23 Mar 2014 (v31), last revised 25 Oct 2014 (v35)

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An elementary proof of Catalan-Mihailescu theorem

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Abstract

(MSC=11D04) We begin with Catalan equation $Y^p = X^q + 1$ and solve it.

(Keywords : Diophantine equations, Catalan equation ; Approach)

Resolution of Catalan equation

Let Catalan equation :

$$Y^p = X^q + 1$$

We have

$$Y^2 Y^{p-2} - X^3 X^{q-3} = 1 = v Y^{p-2} + u X^{q-3}$$

With $v = Y^2$ and $u = -X^3$. Hence

$$Y^{p-2} = \frac{1}{v} - \frac{u X^{q-3}}{v}$$

$$Y^{p-2} - 1 = \frac{1 - v - u X^{q-3}}{v}$$

$$X^{q-3} = \frac{1}{u} - \frac{v Y^{p-2}}{u}$$

$$X^{q-3} - 1 = \frac{1 - u - v Y^{p-2}}{u}$$

Thus

$$\begin{aligned} (X^{q-3} - 1)(Y^{p-2} - 1) &= \left(\frac{1 - v - u X^{q-3}}{u}\right) \left(\frac{1 - u - v Y^{p-2}}{v}\right) \\ &= \frac{1 - u - v + uv}{uv} + X^{q-3} Y^{p-2} + \frac{(u^2 - u) X^{q-3} + (v^2 - v) Y^{p-2}}{uv} \\ &= 1 + X^{q-3} Y^{p-2} - X^{q-3} - Y^{p-2} \end{aligned}$$

We deduce

$$1 - u - v + (u^2 - u + uv) X^{q-3} + (v^2 - v + uv) Y^{p-2} = 0$$

Thus

$$\frac{1}{X^3} ((1 - u - v) X^3 + 2(u^2 - u + uv) X^q) = \frac{1}{Y^2} ((u + v - 1) Y^2 - 2(v^2 - v + uv) Y^p)$$

Or

$$\begin{aligned} Y^2((1-u-v)X^3 + 2(u^2 - u + uv)X^q) &= X^3((u+v-1)Y^2 - 2(v^2 - v + uv)Y^p) \\ &= Y^2((1-u-v)X^3 - 2(u+v-1)X^3X^q) = X^3((u+v-1)Y^2 - 2(u+v-1)Y^pY^2) \end{aligned}$$

Thus

$$(u+v-1)(2X^3Y^2 - 2Y^pY^2 + 2X^3X^q) = 0$$

And

$$X^3Y^2 - Y^pY^2 + X^3X^q = (X^3 - Y^p)Y^2 + X^3X^q \neq 0$$

Because as X and Y are coprime, X^3 does not divide $X^3 - Y^p$ and does not divide Y^2 . Hence

$$u+v-1=0 = Y^2 - X^3 - 1$$

And $Y^2 = X^3 + 1$. The solution, Euler solved this equation, is $(X, Y) = (2, \pm 3)$: the equation becomes

$$3^p = 2^q + 1$$

It means $(p, q) = (2, 3)$.

Conclusion

Catalan equation $Y^p = X^q + 1$ has solutions only for $q = 3$ and $p = 2$. We have shown a way to solve it. By the same way, Fermat-Catalan equation can be solved.