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An elementary proof of Catalan-Mihailescu theorem

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Abstract

(MSC=11D04) More than one century after its formulation by the Belgian mathematician Eugene Catalan, Preda Mihailescu has solved the open problem. But, is it all? Mihailescu's solution utilizes computation on machines, we propose here not really a proof as it is intended classically, but a resolution of an equation like the resolution of the polynomial equations of third and fourth degrees. This solution is totally algebraic and does not utilize, of course, computers or any kind of calculation. We generalize our approach to Pillai and Fermat-Catalan equations and discuss the solutions.

(Keywords : Diophantine equations, Catalan, Fermat-Catalan, Pillai, Conjectures, Proofs, Algebraic resolution)

Introduction

Catalan theorem has been proved in 2002 by Preda Mihailescu. In 2004, it became officially Catalan-Mihailescu theorem. This theorem stipulates that there are not consecutive pure powers. There do not exist integers strictly greater than 1, $X > 1$ and $Y > 1$, for which with exponents strictly greater than 1, $p > 1$ and $q > 1$,

$$Y^p = X^q + 1$$

but for $(X, Y, p, q) = (2, 3, 2, 3)$. We can verify that

$$3^2 = 2^3 + 1$$

Euler has proved that the equation $X^3 + 1 = Y^2$ has this only solution. We propose in this study a general solution. The particular cases already solved concern $p = 2$, solved by Ko Chao in 1965, and $q = 3$ which has been solved in 2002. The case $q = 2$ has been solved by Lebesgue in 1850. We solve here the equation and prove that Pillai and Fermat-Catalan equations are related to this problem.

The resolution

Let Catalan equation

$$Y^p = X^q + 1$$

We pose

$$c = \frac{X^p - 1}{Y^{\frac{p}{2}}}; \quad c' = \frac{7 - X^p}{Y^{\frac{p}{2}}}$$

Thus

$$Y^{\frac{p}{2}} = \frac{6}{c + c'}$$

$$X^p = cY^{\frac{p}{2}} + 1 = \frac{6c}{c + c'} + 1 = \frac{7c + c'}{c + c'}$$

And

$$X^q = Y^p - 1 = \frac{36 - (c + c')^2}{(c + c')^2}$$

We have $X^p > 7$ hence $c' < 0$ and $c > c'$ and $Y^{\frac{p}{2}} \geq 3$ thus $0 < c + c' \leq 2$. We will prove now that $q + 1 = 2p$. We have to discuss two cases $c^2 > 1$ and $0 < c^2 \leq 1$

$0 < c^2 \leq 1$

$$2X^q - X^{2p} = \frac{72 - 2(c + c')^2 - (7c + c')^2}{(c + c')^2}$$

$$> \frac{64 - 49c^2}{(c + c')^2} > 0$$

Thus

$$X^{q+1} \geq 2X^q \geq X^{2p-1}$$

Or

$$q + 2 \geq 2p$$

Also

$$2c^2 X^q - X^{2p} = \frac{72c^2 - 2c^2(c + c')^2 - (7c + c')^2}{(c + c')^2}$$

$$> \frac{64c^2 - (7c + c')^2}{(c + c')^2} = \frac{(c - c')(15c + c')}{(c + c')^2} > 0$$

And

$$c^2 X^q - X^{2p} = \frac{36c^2 - (c + c')^2 - (7c + c')^2}{(c + c')^2}$$

$$< \frac{36c^2 - (7c + c')^2}{(c + c')^2} = \frac{(c' - c)(13c + c')}{(c + c')^2} < 0$$

We have

$$c'Y^{\frac{p}{2}} = 7 - X^p > -Y^{\frac{p}{2}}$$

Thus $c' > -1$. But, let u, v verifying

$$\frac{7uc + vc'}{c + c'} = 4$$

Or

$$(7u - 4)c = (4 - v)c' > 0$$

Then $7u - 4 > 0$ and $4 - v < 0$ and

$$(7u - 4)(X^p - 1) = (4 - v)(7 - X^p)$$

Or

$$(7u - v)X^p = 24 + 7(u - v)$$

Hence

$$X^p = \frac{7(u - v) + 24}{7u - v}$$

But

$$c + c' = \frac{4 - v}{7u - 4}c' + c' = \frac{7u - v}{7u - 4}c'$$

Then

$$Y^{\frac{p}{2}} = \frac{6}{c + c'} = \frac{6(7u - 4)}{(7u - v)c'}$$

And

$$\frac{X^p}{Y^{\frac{p}{2}}} = \frac{(7(u - v) + 24)c'}{6(7u - 4)}$$

But

$$(3c' + 7c)Y^{\frac{p}{2}} = 3(7 - X^p) + 7(X^p - 1) = 4X^p + 14 > 0 \Rightarrow 3c' + 7c > 0$$

And

$$(4 - v)(3c' + 7c) < 0$$

Or

$$3(7u - 4)c = 3(4 - v)c' < 7(v - 4)c$$

It means

$$7u - 4 < 7v - 14u - 20 < 7(4 - v)c' + 2(4 - 7u)$$

But

$$X^q + 1 = Y^p \geq X^{2p} + 1 > (X^p - 7)^2$$

Thus

$$\begin{aligned} X^p - 7 &< Y^{\frac{p}{2}} < Y^{\frac{p}{2}} \left(\frac{7(v - 4) + 2(4 - 7u)}{7u - 4} \right) \\ &< Y^{\frac{p}{2}} \left(\frac{7(4 - v)c' + 2(4 - 7u)}{7u - 4} \right) \end{aligned}$$

Then

$$-7 + X^p + 2Y^{\frac{p}{2}} < \left(\frac{7(4 - v)c'}{7u - 4} \right) Y^{\frac{p}{2}}$$

Or

$$\begin{aligned} \frac{-7 + X^p + 2Y^{\frac{p}{2}}}{6c'} &= \left(\frac{2}{6c'} - \frac{1}{6} \right) Y^{\frac{p}{2}} \\ &> \frac{7(4 - v)}{6(7u - 4)} Y^{\frac{p}{2}} \end{aligned}$$

Or

$$\left(\frac{2}{6c'} - \frac{1}{6} \right) c' < \frac{7(4 - v)c'}{6(7u - 4)}$$

And

$$\begin{aligned} & \frac{c'}{6} + \frac{7(4-v)c'}{6(7u-4)} \\ &= \frac{(7(u-v)+24)c'}{6(7u-4)} = \frac{X^p}{Y^{\frac{p}{2}}} > \frac{2}{6} \end{aligned}$$

Thus

$$3X^{2p+1} > 36X^{2p} > 4Y^p > 3X^q$$

And

$$q+2 \geq 2p \geq q$$

As q is odd then $q+1 = 2p$ and

$$0 > (c^2 - 1)Y^p = X^{2p} - 2X^p - X^q = X^{2p} - 2X^p - X^{2p-1} > 0$$

Thus $c^2 = 1$

$$(c^2 - 1)Y^p = 0 = X^p(X^{p-1} - 2) \Rightarrow (X, p, q) = (2, 2, 3) \Rightarrow Y = \pm 3$$

The only solution of Catalan equation is then

$$(Y, X, p, q) = (\pm 3, 2, 2, 3)$$

$c^2 > 1$ We have

$$\begin{aligned} X^q - X^{2p} &= \frac{36 - (c+c')^2 - (7c+c')^2}{(c+c')^2} \\ &< \frac{36 - 36c^2}{(c+c')^2} < 0 \end{aligned}$$

We deduce

$$q+1 \leq 2p$$

Also

$$\begin{aligned} 2c^2 X^q - X^{2p} &= \frac{72c^2 - 2c^2(c+c')^2 - (7c+c')^2}{(c+c')^2} \\ &> \frac{64c^2 - (7c+c')^2}{(c+c')^2} = \frac{(c-c')(15c+c')}{(c+c')^2} > 0 \end{aligned}$$

And

$$\begin{aligned} c^2 X^q - X^{2p} &= \frac{36c^2 - (c+c')^2 - (7c+c')^2}{(c+c')^2} \\ &< \frac{36c^2 - (7c+c')^2}{(c+c')^2} = \frac{(c'-c)(13c+c')}{(c+c')^2} < 0 \end{aligned}$$

But

$$c'Y^{\frac{p}{2}} = 7 - X^p < -Y^{\frac{p}{2}}$$

Then $c' < -1$. But, let u, v verifying

$$\frac{7uc + vc'}{c+c'} = 4$$

Or

$$(7u - 4)c = (4 - v)c' > 0$$

Then

$$(7u - 4)(X^p - 1) = (4 - v)(7 - X^p)$$

Or

$$(7u - v)X^p = 24 + 7(u - v)$$

Hence

$$X^p = \frac{7(u - v) + 24}{7u - v}$$

But

$$c + c' = \frac{4 - v}{7u - 4}c' + c' = \frac{7u - v}{7u - 4}c'$$

Then

$$Y^{\frac{p}{2}} = \frac{6}{c + c'} = \frac{6(7u - 4)c'}{7u - v}$$

And

$$\frac{X^p}{Y^{\frac{p}{2}}} = \frac{7(u - v) + 24}{6(7u - 4)c'}$$

But

$$(9c' + 7c)Y^{\frac{p}{2}} = 9(7 - X^p) + 7(X^p - 1) = 58 - 2X^p < 0 \Rightarrow 9c' + 7c < 0$$

And

$$(4 - v)(9c' + 7c) > 0$$

Thus

$$9(7u - 4)c = 9(4 - v)c' > 7(v - 4)c$$

It means

$$7(7u - 4) > 7v - 14u - 20 > 7(4 - v)c' + 2(4 - 7u)$$

But

$$X^q(X - 49) > X^q > 7 \rightarrow X^{2p} - 7 \geq X^{q+1} - 7 > 49X^q$$

And

$$\begin{aligned} X^p - 7 &> 7Y^{\frac{p}{2}} > Y^{\frac{p}{2}} \left(\frac{7(v - 4) + 2(4 - 7u)}{7u - 4} \right) \\ &> Y^{\frac{p}{2}} \left(\frac{7(4 - v)c' + 2(4 - 7u)}{7u - 4} \right) \end{aligned}$$

Then

$$-7 + X^p + 2Y^{\frac{p}{2}} > \left(\frac{7(4 - v)c'}{7u - 4} \right) Y^{\frac{p}{2}}$$

Or

$$\begin{aligned} &\frac{-7 + X^p + 2Y^{\frac{p}{2}}}{6c'} \\ &= \left(\frac{2}{6c'} - \frac{1}{6} \right) Y^{\frac{p}{2}} < \frac{7(4 - v)}{6(7u - 4)} Y^{\frac{p}{2}} \end{aligned}$$

Or

$$\left(\frac{2}{6c'} - \frac{1}{6}\right)c' > \frac{7(4-v)c'}{6(7u-4)}$$

And

$$\begin{aligned} & \frac{c'}{6} + \frac{7(4-v)c'}{6(7u-4)} \\ &= \frac{(7(u-v) + 24)c'}{6(7u-4)} = \frac{X^p}{Y^{\frac{p}{2}}} < \frac{2}{6} < 1 \end{aligned}$$

Thus

$$\begin{aligned} X^{2p} < Y^p = X^q + 1 < X^{q+1} \\ q + 1 \geq 2p \geq q + 1 \end{aligned}$$

Or $q + 1 = 2p$. But

$$\begin{aligned} c > -c' &\Rightarrow 6(4-v)c' + 3096c = 6(7u-4)c + 3096c = (42u + 3072)c > -3096c' + 6(4-v)c' = (-3072 - 6v)c' \\ &\Rightarrow (c + c')X^p(7u - v - 6) = (7c + c')(7u - v - 6) = 49uc - 7vc - 42c + 7uc' - vc' - 6c' \\ &> 7uc - 7vc - 3072c + 7uc' - 7vc' - 3072c' = (7u - 7v - 3072)(c + c') \\ &\Rightarrow (7u - 7 + 512)(X^p - 1) + (v - 1 + 512)(7 - X^p) > 0 \\ &\Rightarrow (7u - 7 + 512)c + (v - 1 + 512)c' = (7uc + vc') - 7c - c' + 512(c + c') = (c + c')(4 - X^p + 512) > 0 \\ &\Rightarrow X^p < 516 \end{aligned}$$

And $q = 2p - 1$, then, after calculating all the X, p verifying $X^p < 516$, we have $(X, p, q) = (2, 2, 3)$ and $Y = \pm 3$.

Conclusion

Catalan equation implies a second one, and an original solution of Catalan equation exists. We have proved that there exists only a couple of consecutive pure powers (8, 9).

Références

- [1] P. MIHAILESCU A class number free criterion for catalan's conjecture, *Journal of Number theory*, **99** (2003).