An elementary proof of Catalan-Mihailescu theorem

Jamel Ghannouchi

An elementary proof of Catalan-Mihailescu theorem

Jamel Ghanouchi

Ecole Supérieure des Sciences et Techniques de Tunis

jamel.ghanouchi@live.com

Abstract

(MSC=11D04) More than one century after its formulation by the Belgian mathematician Eugene Catalan, Preda Mihailescu has solved the open problem. But, is it all ? Mihailescu's solution utilizes computation on machines, we propose here not really a proof as it is entended classically, but a resolution of an equation like the resolution of the polynomial equations of third and fourth degrees. This solution is totally algebraic and does not utilize, of course, computers or any kind of calculation. We generalize our approach to Pillai and Fermat-Catalan equations and discuss the solutions.

(Keywords : Diophantine equations, Catalan, Fermat-Catalan, Pillai, Conjectures, Proofs, Algebraic resolution)

Introduction

Catalan theorem has been proved in 2002 by Preda Mihailescu. In 2004, it became officially Catalan-Mihailescu theorem. This theorem stipulates that there are not consecutive pure powers. There do not exist integers stricly greater than 1, $X > 1$ and $Y > 1$, for which with exponants strictly greater than 1, $p > 1$ and $q > 1$,

$$Y^p = X^q + 1$$

but for $(X, Y, p, q) = (2, 3, 2, 3)$. We can verify that

$$3^2 = 2^3 + 1$$

Euler has proved that the equation $X^3+1 = Y^2$ has this only solution. We propose in this study a general solution. The particular cases already solved concern $p = 2$, solved by Ko Chao in 1965, and $q = 3$ which has been solved in 2002. The case $q = 2$ has been solved by Lebesgue in 1850. We solve here the equation and prove that Pillai and Fermat-Catalan equations are related to this problem.

The resolution

Let Catalan equation

$$Y^p = X^q + 1$$

We pose

$$c = \frac{X^p - 1}{Y^{\frac{p}{2}}}; \quad c' = \frac{7 - X^p}{Y^{\frac{p}{2}}}$$

Thus

$$Y^{\frac{p}{2}} = \frac{6}{c + c'}$$

$$X^p = cY^{\frac{p}{2}} + 1 = \frac{6c}{c + c'} + 1 = \frac{7c + c'}{c + c'}$$

And

$$X^q = Y^p - 1 = \frac{36 - (c + c')^2}{(c + c')^2}$$

We have $X^p > 7$ hence $c' < 0$ and $c > c'$ and $Y^{\frac{p}{2}} \geq 3$ thus $0 < c + c' \leq 2$. We will prove now that $q + 1 = 2p$. We have to discuss two cases $c^2 > 1$ and $c^2 \leq 1$

$c^2 > 1$ We have

$$X^q - X^{2p} = \frac{36 - (c + c')^2 - (7c + c')^2}{(c + c')^2}$$

$$< \frac{36 - 36c^2}{(c + c')^2} < 0$$

We deduce

$$q + 1 \leq 2p$$

Also

$$2c^2 X^q - X^{2p} = \frac{72c^2 - 2c^2(c + c')^2 - (7c + c')^2}{(c + c')^2}$$

$$> \frac{64c^2 - (7c + c')^2}{(c + c')^2} = \frac{(c - c')(15c + c')}{(c + c')^2} > 0$$

And

$$c^2 X^q - X^{2p} = \frac{36c^2 - (c + c')^2 - (7c + c')^2}{(c + c')^2}$$

$$< \frac{36c^2 - (7c + c')^2}{(c + c')^2} = \frac{(c' - c)(13c + c')}{(c + c')^2} < 0$$

But

$$2c^2 X^{2p-1} \geq 2c^2 X^q > X^{2p} \Rightarrow 2c^2 > X$$

Let

$$2c^2 = \frac{n + 2}{n + 1}, \quad X = \frac{m + 2}{m + 1}$$

We want to prove that $X > 2c^2$. It is evident that $n < 0$ and $m < 0$ ($2X - 1 - X - 1 = X - 2 = \frac{-m}{m+1} > 0$). We have for u, v verifying $v = -u$, $(X - 1)(2c^2 - 1) > u = \frac{2}{3(n+1)} > \frac{2}{3}$

$$\Rightarrow (2u - v)^2 n^2 + (12u^2 - 6uv - 16u - 4v)n + 9u^2 - 12u + 4$$

$$= 9u^2 n^2 + (18u^2 - 12u)n + 9u^2 - 12u + 4$$

$$= (9n^2 + 18n + 9)u^2 - (12n - 12)u + 4 = 9(n+1)^2u^2 - 12(n+1)u + 4 = 0$$

$$\Rightarrow ((2u+v)n + u - 2)^2 - 4u(2vn^2 + (2v - 2u + 2)n + 2 - 2u) = 0$$

$$\Rightarrow \frac{u}{(X-1)^2} + \frac{(2u+v)n + u - 2}{X-1} + 2vn^2 + (2v - 2u + 2)n + 2 - 2u > 0$$

$$\Rightarrow u(m+1)^2 + n(2vn + (2u+v)(m+1) - 2u - v + 3v + 2) + (u-2)(m+1) + 2 - 2u$$

$$= u(m+1)^2 + n(2vn + (2u+v)m + 3v + 2) + (u-2)m - u > 0$$

$$\Rightarrow (um + vn)(m-1) + (2n+2)(um+vn) + 2(u-1)m + 2(v+1)n$$

$$= u(m+1)^2 + n(2vn + (2u+v)m + 3v + 2) + (u-2)m - u > 0$$

$$\Rightarrow (um + vn)(m-1) + (2n+2)(um+vn) > -2(u-1)m - 2(v+1)n$$

$$\Rightarrow (um+vn)(m - 1 + 2n + 2) = (um+vn)(m + 2n + 1) > -2(u-1)m - 2(v+1)n$$

$$\Rightarrow (um+vn)(\frac{m}{2} + n + \frac{1}{2}) + (u-1)m + (v+1)n > 0$$

But

$$m + 1 + n = \frac{2-X}{X-1} + \frac{2-2c^2}{2c^2-1} + 1 = \frac{1}{2c^2-1} + \frac{2-X}{X-1}$$

$$= \frac{2X - 1 + 4c^2 - 2 - 2c^2X}{(X-1)(2c^2-1)} = \frac{X(2-c^2) + c^2(4-X) - 3}{(2c^2-1)(X-1)} < 0$$

$$\Rightarrow (2n+1)(m+1) < 0 < 2(n+1) \Rightarrow -1 - m < -1 + \frac{1}{-2n-1} = \frac{2n+2}{-2n-1}$$

$$\Rightarrow -m < \frac{1}{-2n-1} \Rightarrow m(2n+1) < 1$$

$$\Rightarrow m + 2mn - 1 < 0 \Rightarrow 2m + 2n + 2mn < m + 2n + 1$$

$$\Rightarrow (um+vn)(m + n + mn) + (u-1)m + (v+1)n > (um+vn)(\frac{m}{2} + n + \frac{1}{2}) + (u-1)m + (v+1)n > 0$$

$$\Rightarrow m(un + um + umn + u - 1) + n(vm + vn + vmn + v + 1) > 0$$

$$\Rightarrow um(m+1)(n+1) - m + vn(m+1)(n+1) + n > 0$$

$$\Rightarrow \frac{u(m+1)^2(n+1) + n + 1 + v(n+1)^2(m+1) - (m+1)}{(m+1)(n+1)} > u + v$$

$$\Rightarrow \frac{u(m+1)^2 + 1}{m+1} + \frac{v(n+1)^2 - 1}{n+1} > u + v$$

$$\Rightarrow um + u - u + \frac{1}{m+1} + vn + v - v - \frac{1}{n+1} = um + X - 1 + vn - 2c^2 + 1 > 0$$

$$-um - vn = u(n - m) = u(\frac{X - 2c^2}{(X-1)(2c^2-1)})$$

$$(X - 2c^2)(1 - \frac{u}{(X-1)(2c^2-1)}) > 0$$

$$\Rightarrow X - 2c^2 > 0 \Rightarrow X > 2c^2 > X$$

$$\Rightarrow X^{q+1} > 2c^2X^q > X^{2p} \geq X^{q+1} \Rightarrow q + 1 = 2p$$

Thus

$$2c^2 > X > 2c^2 \Rightarrow X = 2c^2$$
$$\Rightarrow 2c^2 Y^p = 2(X^p - 1)^2 = XY^p = X^{q+1} + X$$
$$\Rightarrow \frac{2}{X} = X^q + 1 - 2X^{2p-1} + 4X^{p-1} \in \mathbb{N}$$
$$\Rightarrow X = 2 \Rightarrow -2^{2p-1} + 2^{p+1} = 0 \Rightarrow q = 2p - 1 = 3$$
$$\Rightarrow Y = \pm 3$$

The equation becomes

$$Y^p = 2^{2p-1} + 1$$

And also

$$(c^2 - 1)Y^p X^{-p-1} = 0 = X^{p-1} - 1 - X^{p-2} = 2^{p-1} - 1 - 2^{p-2} \Rightarrow 2^{p-2} = 1$$
$$\Rightarrow p = 2 \Rightarrow q = 2p - 1 = 3 \Rightarrow Y = \pm 3$$

$c^2 \leq 1$

$$2X^q - X^{2p} = \frac{72 - 2(c + c')^2 - (7c + c')^2}{(c + c')^2}$$
$$> \frac{64 - 49c^2}{(c + c')^2} > 0$$

Thus

$$X^q \geq X^{2p-1}$$

Or

$$q + 1 \geq 2p$$

Also

$$2c^2 X^q - X^{2p} = \frac{72c^2 - 2c^2(c + c')^2 - (7c + c')^2}{(c + c')^2}$$
$$> \frac{64c^2 - (7c + c')^2}{(c + c')^2} = \frac{(c - c')(15c + c')}{(c + c')^2} > 0$$

And

$$c^2 X^q - X^{2p} = \frac{36c^2 - (c + c')^2 - (7c + c')^2}{(c + c')^2}$$
$$< \frac{36c^2 - (7c + c')^2}{(c + c')^2} = \frac{(c' - c)(13c + c')}{(c + c')^2} < 0$$

Let

$$2c^{-2} = \frac{n + 2}{n + 1}, \quad X = \frac{m + 2}{m + 1}$$

It is evident that $n < 0$ and $m < 0$ ($2X - 1 - X - 1 = X - 2 = \frac{-m}{m+1} > 0$). And

$$2c^{-2} = 1 + \frac{1}{n + 1} > 3 \Rightarrow n < -\frac{1}{2}$$

We have for u, v verifying $v = -u$, $(X-1)(2c^{-2}-1) > u = \frac{2}{3(n+1)} > 0$

$$\Rightarrow (2u-v)^2n^2 + (12u^2 - 6uv - 16u - 4v)n + 9u^2 - 12u + 4$$

$$= 9u^2n^2 + (18u^2 - 12u)n + 9u^2 - 12u + 4$$

$$= (9n^2 + 18n + 9)u^2 - (12n - 12)u + 4 = 9(n+1)^2u^2 - 12(n+1)u + 4 = 0$$

$$\Rightarrow (2u-v)^2n^2 + (12u^2 - 6uv - 16u - 4v)n + 9u^2 - 12u + 4 = 0$$

$$\Rightarrow ((2u+v)n + u - 2)^2 - 4u(2vn^2 + (2v - 2u + 2)n + 2 - 2u) = 0$$

$$\Rightarrow \frac{u}{(X-1)^2} + \frac{(2u+v)n + u - 2}{X-1} + 2vn^2 + (2v - 2u + 2)n + 2 - 2u > 0$$

$$\Rightarrow u(m+1)^2 + n(2vn + (2u+v)(m+1) - 2u - v + 3v + 2) + (u-2)(m+1) + 2 - 2u$$

$$= u(m+1)^2 + n(2vn + (2u+v)m + 3v + 2) + (u-2)m - u > 0$$

$$\Rightarrow (um+vn)(m-1) + (2n+2)(um+vn) + 2(u-1)m + 2(v+1)n$$

$$= u(m+1)^2 + n(2vn + (2u+v)m + 3v + 2) + (u-2)m - u > 0$$

$$\Rightarrow (um+vn)(m-1) + (2n+2)(um+vn) > -2(u-1)m - 2(v+1)n$$

$$\Rightarrow (um+vn)(m-1+2n+2) = (um+vn)(m+2n+1) > -2(u-1)m - 2(v+1)n$$

$$\Rightarrow (um+vn)(\frac{m}{2} + n + \frac{1}{2}) + (u-1)m + (v+1)n > 0$$

But

$$m + 1 + n = \frac{2-X}{X-1} + \frac{2-2c^{-2}}{2c^{-2}-1} + 1 = \frac{1}{2c^{-2}-1} + \frac{2-X}{X-1}$$

$$= \frac{2X - 1 + 4c^{-2} - 2 - 2c^{-2}X}{(X-1)(2c^{-2}-1)} = \frac{X(2-c^{-2}) + c^{-2}(4-X) - 3}{(2c^2-1)(X-1)} < 0$$

$$\Rightarrow (2n+1)(m+1) < 0 < 2(n+1) \Rightarrow -1 - m < -1 + \frac{1}{-2n-1} = \frac{2n+2}{-2n-1}$$

$$\Rightarrow -m < \frac{1}{-2n-1} \Rightarrow m(2n+1) < 1$$

$$\Rightarrow m + 2mn - 1 < 0 \Rightarrow 2m + 2n + 2mn < m + 2n + 1$$

$$\Rightarrow (um+vn)(m+n+mn) + (u-1)m + (v+1)n > (um+vn)(\frac{m}{2} + n + \frac{1}{2}) + (u-1)m + (v+1)n > 0$$

$$\Rightarrow m(un + um + umn + u - 1) + n(vm + vn + vmn + v + 1) > 0$$

$$\Rightarrow um(m+1)(n+1) - m + vn(m+1)(n+1) + n > 0$$

$$\Rightarrow \frac{u(m+1)^2(n+1) + n + 1 + v(n+1)^2(m+1) - (m+1)}{(m+1)(n+1)} > u + v = 0$$

$$\Rightarrow \frac{u(m+1)^2 + 1}{m+1} + \frac{v(n+1)^2 - 1}{n+1} > u + v = 0$$

$$\Rightarrow um + u + \frac{1}{m+1} + vn + v - \frac{1}{n+1} = um + u + X - 1 + vn + v - 2c^{-2} + 1 > 0$$

$$-um - vn = u(n-m) = u(\frac{X - 2c^{-2}}{(X-1)(2c^{-2}-1)})$$

$$(X - 2c^{-2})(1 - \frac{u}{(X-1)(2c^{-2}-1)}) > 0$$

$$\Rightarrow X - 2c^{-2} > 0 \Rightarrow X > 2c^{-2}$$

$$\Rightarrow X^{2p} > c^2 X^q > 2X^{q-1} \geq 2X^{2p-2} \Rightarrow q + 1 = 2p, q = 2p$$

But

$$0 > (c^2 - 1)Y^p = X^{2p} - 2X^p - X^q = X^{2p} - 2X^p - X^{2p-1} > X^{2p-1} - 2X^p > 0$$

$$\Rightarrow (c^2 - 1)Y^p = 0 = X^p(X^{p-1} - 2) \Rightarrow (X, p, q) = (2, 2, 3) \Rightarrow Y = \pm 3$$

The only solution of Catalan equation is then

$$(Y, X, p, q) = (\pm 3, 2, 2, 3)$$

Conclusion

Catalan equation implies a second one, and an original solution of Catalan equation exists. We have proved that there exists only a couple of consecutive pure powers (8, 9).

Références

[1] P. MIHAILESCU A class number free criterion for catalan's conjecture, *Journal of Number theory* , **99** (2003).