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# An elementary proof of Catalan-Mihailescu theorem

## Jamel Ghanouchi

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### Abstract

(MSC=11D04) More than one century after its formulation by the Belgian mathematician Eugene Catalan, Preda Mihailescu has solved the open problem. But, is it all? Mihailescu's solution utilizes computation on machines, we propose here not really a proof as it is entended classically, but a resolution of an equation like the resolution of the polynomial equations of third and fourth degrees. This solution is totally algebraic and does not utilize, of course, computers or any kind of calculation.

(Keywords : Diophantine equations, Catalan equation ; Algebraic resolution)

### Introduction

Catalan theorem has been proved in 2002 by Preda Mihailescu. In 2004, it became officially Catalan-Mihailescu theorem. This theorem stipulates that there are not consecutive pure powers. There do not exist integers stricly greater than 1,  $X > 1$  and  $Y > 1$ , for which with exponents strictly greater than 1,  $p > 1$  and  $q > 1$ ,

$$Y^p = X^q + 1$$

but for  $(X, Y, p, q) = (2, 3, 2, 3)$ . We can verify that

$$3^2 = 2^3 + 1$$

Euler has proved that the equation  $X^3 + 1 = Y^2$  has this only solution. We propose in this study a general solution. The particular cases already solved concern  $p = 2$ , solved by Ko Chao in 1965, and  $q = 3$  which has been solved in 2002. The case  $q = 2$  has been solved by Lebesgue in 1850. We solve here the equation for the general case.

### The resolution

Let Catalan equation

$$Y^p = X^q + 1$$

We pose

$$c = \frac{X^p - 1}{Y^{\frac{p}{2}}}; \quad c' = \frac{7 - X^p}{Y^{\frac{p}{2}}}$$

Thus

$$Y^{\frac{p}{2}} = \frac{6}{c+c'}$$

$$X^p = cY^{\frac{p}{2}} + 1 = \frac{6c}{c+c'} + 1 = \frac{7c+c'}{c+c'}$$

And

$$X^q = Y^p - 1 = \frac{36 - (c+c')^2}{(c+c')^2}$$

We have  $X^p > 7$  hence  $c' < 0$  and  $c > c'$  and  $Y^{\frac{p}{2}} \geq 3$  thus  $0 < c+c' \leq 3$  Two cases  $c^2 > 1$  and  $c^2 \leq 1$

$c^2 > 1$  We have

$$X^q - X^{2p} = \frac{36 - (c+c')^2 - (7c+c')^2}{(c+c')^2}$$

$$< \frac{36 - 36c^2}{(c+c')^2} < 0$$

We deduce

$$q+1 \leq 2p$$

But let

$$Z = \frac{c+7c'}{c+c'}$$

We have

$$Z + X^p = 8$$

Thus

$$8X^q - (Z + X^p)^2 = 8X^q - 64 \geq 0$$

And

$$8X^{q-2p} \geq (ZX^{-p} + 1)^2$$

Let

$$n = \frac{-6X^p}{6X^p - 48} < 0 \Rightarrow n+1 = \frac{-48}{6X^p - 48} < 0$$

$$\Rightarrow \frac{n+1}{n} = \frac{8}{X^p} > 0$$

$$m = \frac{6Z}{-6Z + 48} < 0 \Rightarrow m+1 = \frac{48}{-6Z + 48} > 0$$

$$\Rightarrow \frac{m+1}{m} = \frac{8}{Z} < 0$$

But, we have

$$(n+1)X^p = 8n$$

$$(m+1)Z = 8m$$

And

$$(n+1)X^p - (m+1)Z = 8(n-m) = 64nX^{-p} - 64mZ^{-1}$$

Also

$$(n+1) - (m+1)ZX^{-p} = 8(n-m)X^{-p} = 64nX^{-2p} - 64mZ^{-1}X^{-p}$$

And

$$(n+1)X^pZ - (m+1)Z^2 = 8(n-m)Z = 64nZX^{-p} - 64m$$

We deduce

$$\begin{aligned} ZX^{-p} &= \frac{n+1 + 8(m-n)X^{-p}}{m+1} = \frac{(n+1)m}{(m+1)n} = \frac{8(n-m)Z + 64m}{64n} \\ &= \frac{8(n-m)(X^{-p} + Z) + 64m - n - 1}{64n - m - 1} \end{aligned}$$

Or

$$8(n-m)(X^{-p} + Z)(m+1)n + (64m - (n+1))n(m+1) = (64n - (m+1))m(n+1)$$

Hence

$$\begin{aligned} &8(n-m)(X^{-p} + Z)(m+1)n \\ &= 64mn^2 + 64mn - m^2n - mn - m^2 - m - 64m^2n - 64mn + mn^2 + mn + n^2 + n \\ &= 65mn(n-m) + (n-m)(n+m+1) = (n-m)(65mn + m + n + 1) \end{aligned}$$

Thus

$$(n-m)(8X^{-p}n(m+1) + 8Zn(m+1) - 65mn - m - n - 1) = 0$$

But

$$\begin{aligned} &(8X^{-p}n(m+1) + 8Zn(m+1) - 65mn - m - n - 1)(X^p - 8)(-Z + 8) \\ &= -64(X^p - 8)(-Z + 8) - 64ZX^p(X^p - 8)(-Z + 8) - (X^p - 8)(-Z + 8) + 65X^pZ - Z(X^p - 8) + X^p(-Z + 8) \\ &= 64ZX^p - 512Z + 64^2 - 512X^p + 64Z^2X^{2p} + 64^2ZX^p - 512Z^2X^p - 512ZX^{2p} + \\ &\quad + X^pZ - 8X^p - 8Z + 64 + 65X^pZ - 2ZX^p + 8Z + 8X^p \\ &= 64Z^2X^{2p} + (128 + 64^2 - 512 \cdot 8)ZX^p = 64ZX^p(ZX^p + 2) \neq 0 \end{aligned}$$

It means that  $n = m$  or

$$\frac{-6X^p}{6X^p - 48} = \frac{6Z}{-6Z + 48}$$

Or

$$36ZX^p - 48.6X^p = 36ZX^p - 48.6Z$$

Thus

$$X^p = Z \Rightarrow 8X^{q-2p} > (ZX^{-p} + 1)^2 = 4$$

Hence

$$4X^{q+1} \geq 8X^q > 4X^{2p} \geq 4X^{q+1}$$

Or

$$q + 1 = 2p$$

And  $X = 2$  and

$$\begin{aligned} 7c + c' = 7c' + c &\Rightarrow (c - c')Y^{\frac{p}{2}} = 0 = X^p - 1 - 7 + X^p \\ &\Rightarrow (X, p) = (2, 2) \Rightarrow (Y, q) = (\pm 3, 3) \end{aligned}$$

$$c^2 \leq 1$$

$$\begin{aligned} 2X^q - X^{2p} &= \frac{72 - 2(c + c')^2 - (7c + c')^2}{(c + c')^2} \\ &> \frac{64 - 49c^2}{(c + c')^2} > 0 \end{aligned}$$

Thus

$$X^q < X^{2p}$$

Or

$$q + 1 \leq 2p$$

But let

$$Z = \frac{7c' + c}{c + c'}$$

We have

$$\begin{aligned} Z + X^p &= 8 \\ 8X^q - (Z + X^p)^2 &= 8X^q - 64 \geq 0 \end{aligned}$$

And

$$8X^{q-2p} \geq (ZX^{-p} + 1)^2$$

Let

$$\begin{aligned} n &= \frac{-6X^p}{6X^p - 48} < 0 \Rightarrow n + 1 = \frac{-48}{6X^p - 48} < 0 \\ &\Rightarrow \frac{n + 1}{n} = \frac{8}{X^p} > 0 \\ m &= \frac{6Z}{-6Z + 48} < 0 \Rightarrow m + 1 = \frac{48}{-6Z + 48} > 0 \\ &\Rightarrow \frac{m + 1}{m} = \frac{8}{Z} < 0 \end{aligned}$$

And But, we have

$$\begin{aligned} (n + 1)X^p &= 8n \\ (m + 1)Z &= 8m \end{aligned}$$

And

$$(n + 1)X^p - (m + 1)Z = 8(n - m) = 64nX^{-p} - 64mZ^{-1}$$

Also

$$(n + 1) - (m + 1)ZX^{-p} = 8(n - m)X^{-p} = 64nX^{-2p} - 64mZ^{-1}X^{-p}$$

And

$$(n + 1)X^pZ - (m + 1)Z^2 = 8(n - m)Z = 64nZX^{-p} - 64m$$

We deduce

$$\begin{aligned} ZX^{-p} &= \frac{n + 1 + 8(m - n)X^{-p}}{m + 1} = \frac{(n + 1)m}{(m + 1)n} = \frac{8(n - m)Z + 64m}{64n} \\ &= \frac{8(n - m)(X^{-p} + Z) + 64m - n - 1}{64n - m - 1} \end{aligned}$$

Or

$$8(n-m)(X^{-p}+Z)(m+1)n + (64m - (n+1))n(m+1) = (64n - (m+1))m(n+1)$$

Hence

$$\begin{aligned} & 8(n-m)(X^{-p}+Z)(m+1)n \\ = & 64mn^2 + 64mn - m^2n - mn - m^2 - m - 64m^2n - 64mn + mn^2 + mn + n^2 + n \\ = & 65mn(n-m) + (n-m)(n+m+1) = (n-m)(65mn + m + n + 1) \end{aligned}$$

Thus

$$(n-m)(8X^{-p}n(m+1) + 8Zn(m+1) - 65mn - m - n - 1) = 0$$

But

$$\begin{aligned} & (8X^{-p}n(m+1) + 8Zn(m+1) - 65mn - m - n - 1)(X^p - 8)(-Z + 8) \\ = & -64(X^p - 8)(-Z + 8) - 64ZX^p(X^p - 8)(-Z + 8) - (X^p - 8)(-Z + 8) + 65X^pZ - Z(X^p - 8) + X^p(-Z + 8) \\ = & 64ZX^p - 512Z + 64^2 - 512X^p + 64Z^2X^{2p} + 64^2ZX^p - 512Z^2X^p - 512ZX^{2p} + \\ & + X^pZ - 8X^p - 8Z + 64 + 65X^pZ - 2ZX^p + 8Z + 8X^p \\ = & 64Z^2X^{2p} + (128 + 64^2 - 512 \cdot 8)ZX^p = 64ZX^p(ZX^p + 2) \neq 0 \end{aligned}$$

It means that  $n = m$  or

$$\frac{-6X^p}{6X^p - 48} = \frac{6Z}{-6Z + 48}$$

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Thus

$$X^p = Z \Rightarrow 8X^{q-2p} > (ZX^{-p} + 1)^2 = 4$$

Hence

$$4X^{q+1} \geq 8X^q > 4X^{2p} \geq 4X^{q+1}$$

Or

$$q + 1 = 2p$$

And  $X = 2$  and

$$\begin{aligned} 7c + c' = 7c' + c & \Rightarrow (c - c')Y^{\frac{p}{2}} = 0 = X^p - 1 - 7 + X^p \\ & \Rightarrow (X, p) = (2, 2) \Rightarrow (Y, q) = (\pm 3, 3) \end{aligned}$$

The only solution of Catalan equation is

$$(Y, X, p, q) = (\pm 3, 2, 2, 3)$$

### Generalization to Pillai equation

The Pillai equation is  $Y^p = X^q + a$  let us pose  $a = 1_a 1_a$  thus

$$Y^p = X^q + 1_a 1_a$$

We pose

$$c = \frac{X^p - 1_a}{Y^{\frac{p}{2}}}, \quad c' = \frac{7_a - X^p}{Y^{\frac{p}{2}}}$$

Thus

$$Y^{\frac{p}{2}} = \frac{6_a}{c + c'}$$

$$X^p = cY^{\frac{p}{2}} + 1_a = \frac{6_a c}{c + c'} + 1_a = \frac{7_a c + 1_a c'}{c + c'}$$

And

$$X^q = Y^p - 1_a 1_a = \frac{36_a 1_a - 1_a 1_a (c + c')^2}{(c + c')^2}$$

We have  $X^p > 7_a$  hence  $c' < 0$  and  $c > c'$  and  $Y^{\frac{p}{2}} \geq 3_a$  thus  $0 < c + c' \leq 3$  Two cases  $c^2 > 1$  and  $c^2 \leq 1$

$c^2 > 1$  We have

$$\begin{aligned} X^q - X^{2p} &= \frac{36_a 1_a - 1_a 1_a (c + c')^2 - (7_a c + 1_a c')^2}{(c + c')^2} \\ &< \frac{36_a 1_a - 36_a 1_a c^2}{(c + c')^2} < 0 \end{aligned}$$

We deduce

$$q + 1 \leq 2p$$

But let

$$Z = \frac{1_a c + 7_a c'}{c + c'}$$

We have

$$Z + X^p = 8_a$$

Thus

$$8_a X^q - (Z + X^p)^2 = 8_a X^q - 64_a 1_a \geq 0$$

And

$$8_a X^{q-2p} \geq (ZX^{-p} + 1)^2$$

Let

$$\begin{aligned} n &= \frac{-6X^p}{6X^p - 48} < 0 \Rightarrow n + 1 = \frac{-48}{6X^p - 48} < 0 \\ &\Rightarrow \frac{n + 1}{n} = \frac{8}{X^p} > 0 \\ m &= \frac{6Z}{-6Z + 48} > 0 \Rightarrow m + 1 = \frac{48}{-6Z + 48} > 0 \\ &\Rightarrow \frac{m + 1}{m} = \frac{8}{Z} < 0 \end{aligned}$$

But, we have

$$(n + 1)X^p = 8n$$

$$(m + 1)Z = 8m$$

And

$$(n+1)X^p - (m+1)Z = 8(n-m) = 64nX^{-p} - 64mZ^{-1}$$

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We deduce

$$\begin{aligned} ZX^{-p} &= \frac{n+1 + 8(m-n)X^{-p}}{m+1} = \frac{(n+1)m}{(m+1)n} = \frac{8(n-m)Z + 64m}{64n} \\ &= \frac{8(n-m)(X^{-p} + Z) + 64m - n - 1}{64n - m - 1} \end{aligned}$$

Or

$$8(n-m)(X^{-p} + Z)(m+1)n + (64m - (n+1))n(m+1) = (64n - (m+1))m(n+1)$$

Hence Thus

$$(n-m)(8X^{-p}n(m+1) + 8Zn(m+1) - 65mn - m - n - 1) = 0$$

But

$$\begin{aligned} &(8X^{-p}n(m+1) + 8Zn(m+1) - 65mn - m - n - 1)(X^p - 8)(-Z + 8) \\ &= -64(X^p - 8)(-Z + 8) - 64ZX^p(X^p - 8)(-Z + 8) - (X^p - 8)(-Z + 8) + 65X^pZ - Z(X^p - 8) + X^p(-Z + 8) \\ &= 64ZX^p - 512Z + 64^2 - 512X^p + 64Z^2X^{2p} + 64^2ZX^p - 512Z^2X^p - 512ZX^{2p} + \\ &\quad + X^pZ - 8X^p - 8Z + 64 + 65X^pZ - 2ZX^p + 8Z + 8X^p \\ &= 64Z^2X^{2p} + (128 + 64^2 - 512 \cdot 8)ZX^p = 64ZX^p(ZX^p + 2) \neq 0 \end{aligned}$$

It means that  $n = m$  or

$$\frac{-6X^p}{6X^p - 48} = \frac{6Z}{-6Z + 48}$$

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Thus

$$X^p = Z \Rightarrow 8_a X^{q-2p} > (ZX^{-p} + 1)^2 = 4$$

Hence

$$4X^{q+1} \geq 8_a X^q > 4X^{2p} \geq 4X^{q+1}$$

Or

$$q + 1 = 2p$$

And  $X = 2_a$  and

$$7_a c + 1_a c' = 7_a c' + 1_a c \Rightarrow (c - c')Y^{\frac{p}{2}} = 0 = X^p - 1_a - 7_a + X^p$$

And

$$X^p = 4_a = 2X$$

$$\Rightarrow (X, p) = (2, 2) \Rightarrow (Y, q) = (\pm 3, 3)$$

Our approach is available only for Catalan equation and the other cases must be seen at the case by case!



$$c^2 \leq 1$$

$$\begin{aligned} 2X^q - X^{2p} &= \frac{72a1_a - 2a1_a(c + c')^2 - (7ac + 1ac')^2}{(c + c')^2} \\ &> 1_a1_a \frac{64 - 49c^2}{(c + c')^2} > 0 \end{aligned}$$

Thus

$$X^q < X^{2p}$$

Or

$$q + 1 \leq 2p$$

But let

$$Z = \frac{7c' + c}{c + c'}$$

We have

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And But, we have

$$\begin{aligned} (n + 1)X^p &= 8n \\ (m + 1)Z &= 8m \end{aligned}$$

And

$$(n + 1)X^p - (m + 1)Z = 8(n - m) = 64nX^{-p} - 64mZ^{-1}$$

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We deduce

$$\begin{aligned} ZX^{-p} &= \frac{n + 1 + 8(m - n)X^{-p}}{m + 1} = \frac{(n + 1)m}{(m + 1)n} = \frac{8(n - m)Z + 64m}{64n} \\ &= \frac{8(n - m)(X^{-p} + Z) + 64m - n - 1}{64n - m - 1} \end{aligned}$$

Or

$$8(n-m)(X^{-p}+Z)(m+1)n+(64m-(n+1))n(m+1)=(64n-(m+1))m(n+1)$$

Hence

$$\begin{aligned} & 8(n-m)(X^{-p}+Z)(m+1)n \\ = & 64mn^2+64mn-m^2n-mn-m^2-m-64m^2n-64mn+mn^2+mn+n^2+n \\ & = 65mn(n-m)+(n-m)(n+m+1)=(n-m)(65mn+m+n+1) \end{aligned}$$

Thus Thus

$$(n-m)(8X^{-p}n(m+1)+8Zn(m+1)-65mn-m-n-1)=0$$

But

$$\begin{aligned} & (8X^{-p}n(m+1)+8Zn(m+1)-65mn-m-n-1)(X^p-8)(-Z+8) \\ = & -64(X^p-8)(-Z+8)-64ZX^p(X^p-8)(-Z+8)-(X^p-8)(-Z+8)+65X^pZ-Z(X^p-8)+X^p(-Z+8) \\ = & 64ZX^p-512Z+64^2-512X^p+64Z^2X^{2p}+64^2ZX^p-512Z^2X^p-512ZX^{2p}+ \\ & +X^pZ-8X^p-8Z+64+65X^pZ-2ZX^p+8Z+8X^p \\ = & 64Z^2X^{2p}+(128+64^2-512.8)ZX^p=64ZX^p(ZX^p+2)\neq 0 \end{aligned}$$

It means that  $n = m$  or

$$\frac{-6X^p}{6X^p-48}=\frac{6Z}{-6Z+48}$$

Or

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Thus

$$X^p=Z\Rightarrow 8_aX^{q-2p}>(ZX^{-p}+1)^2=4$$

Hence

$$4X^{q+1}\geq 8_aX^q>4X^{2p}\geq 4X^{q+1}$$

Or

$$q+1=2p$$

and  $X = 2_a$ , also  $X^p = 4_a = 2X$ . We have Catalan equation. The Pillai equation must be seen at the case by case!

### Conclusion

Catalan equation implies a second one, and an original solution of Catalan equation exists. We have the generalized the approach to illai equation and proved that it has a finite number of solutions!

## Références

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