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► **To cite this version:**

| Jamel Ghannouchi. An elementary proof of Catalan-Mihailescu theorem. 2011. hal-00677731v19

HAL Id: hal-00677731

<https://hal.science/hal-00677731v19>

Preprint submitted on 2 Jul 2012 (v19), last revised 25 Oct 2014 (v35)

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An elementary proof of Catalan-Mihailescu theorem

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Abstract

(MSC=11D04) More than one century after its formulation by the Belgian mathematician Eugene Catalan, Preda Mihailescu has solved the open problem. But, is it all? Mihailescu's solution utilizes computation on machines, we propose here not really a proof as it is entended classically, but a resolution of an equation like the resolution of the polynomial equations of third and fourth degrees. This solution is totally algebraic and does not utilize, of course, computers or any kind of calculation. Then, we generalize the proof to Pillai conjecture.

(Keywords : Diophantine equations ; Catalan equation ; Pillai conjecture ; Algebraic resolution)

Introduction

Catalan theorem has been proved in 2002 by Preda Mihailescu. In 2004, it became officially Catalan-Mihailescu theorem. This theorem stipulates that there are not consecutive pure powers. There do not exist integers stricly greater than 1, $X > 1$ and $Y > 1$, for which with exponants strictly greater than 1, $p > 1$ and $q > 1$,

$$Y^p = X^q + 1$$

but for $(X, Y, p, q) = (2, 3, 2, 3)$. We can verify that

$$3^2 = 2^3 + 1$$

Euler has proved that the equation $X^3 + 1 = Y^2$ has this only solution. We propose in this study a general solution. The particular cases already solved concern $p = 2$, solved by Ko Chao in 1965, and $q = 3$ which has been solved in 2002. The case $q = 2$ has been solved by Lebesgue in 1850. We solve here the equation for the general case. We generalize the proof to Pillai conjecture

$$Y^p = X^q + a$$

And prove that it has always a finite number of solutions for a fixed a .

The approach

Let

$$c = \frac{X^p - 1}{Y^{\frac{p}{2}}}, \quad c' = \frac{7 - X^p}{Y^{\frac{p}{2}}}$$

But

$$(c + c')Y^{\frac{p}{2}} = X^p - 1 + 7 - X^p = 6 \Rightarrow Y^{\frac{p}{2}} = \frac{6}{c + c'}$$

And

$$X^p = cY^{\frac{p}{2}} + 1 = \frac{6c}{c + c'} + 1 = \frac{7c + c'}{c + c'}$$

And

$$X^q = Y^p - 1 = \frac{36 - (c + c')^2}{(c + c')^2}$$

We have

$$(c + c')Y^{\frac{p}{2}} = 6 > 0$$

Thus $c + c' > 0$. Also $cY^{\frac{p}{2}} = X^p - 1 > 0$ Thus $c > 0$. But $X^p \geq 4$, hence

$$cY^{\frac{p}{2}} = X^p - 1 \geq 7 - X^p = c'Y^{\frac{p}{2}}$$

hence $c \geq c'$ and

$$7 - X^p = 7 - \frac{7c + c'}{c + c'} = \frac{6c'}{c + c'}$$

If $c' > 0$ Thus $X^p < 7$ and $X^p = 4$, it means that $c' < 0$ and

$$(c + c')Y^{\frac{p}{2}} = 6 > 0$$

$$(c + c')X^p = 7c + c' > 0$$

thus $c + c' > 0$ and

$$Y^{\frac{p}{2}} = \frac{6}{c + c'} \geq 3 \Rightarrow c + c' \leq 2$$

main results What we must retain is

1) $c' < 0$

2) $0 < c + c' < 2$

3) $0 < 7c + c'$

We will discuss two cases I) $c^2 > 1$

II) $c^2 \leq 1$

$c^2 > 1$ We have

$$X^q - X^{2p} = \frac{36 - (c + c')^2 - (7c + c')^2}{(c + c')^2} < \frac{(6 - 6c)(6 + 6c) - (c + c')^2}{(c + c')^2} < 0$$

Hence

$$q \leq 2p - 1$$

And

$$c^2 X^q - X^{2p} = \frac{36c^2 - c^2(c + c')^2 - (7c + c')^2}{(c + c')^2} = \frac{-(c + c')(13c + c') - c^2(c + c')^2}{(c + c')^2} < 0$$

And

$$2c^2 X^q - X^{2p} = \frac{72c^2 - 2c^2(c+c')^2 - (7c+c')^2}{(c+c')^2}$$

$$> \frac{64c^2 - (7c+c')^2}{(c+c')^2} = \frac{(c-c')(15c+c')}{(c+c')^2} > 0$$

Thus

$$2c^2 > X^{2p-q} \geq X$$

We deduce

$$2c^2 > X$$

If we suppose $c^2 < X$ then

$$2X > 2c^2 > X^{2p-q} \Rightarrow 1 \geq \frac{2}{X} > X^{2p-q-2}$$

As q is odd $q+1 = 2p$ then $c^2 > X$ and let $c^2 > 2^{m-1}X$ and m verifying $c^2 < 2^m X$ and

$$2^{m+1}X > 2c^2 > X^{2p-q} > 2^m X > c^2$$

Or

$$2^{m+1} > X^{2p-q-1} > 2^m$$

As X even, it means $X = 2$ and $2p - q - 1 = m$ Thus

$$c^2 < 2^m X = 2^{2p-q} < 2c^2 < 2^{m+1} = 2^{2p-q}$$

And

$$2c^2 = 2^{2p-q}$$

Or

$$2^{2p-q-1} = c^2 = \frac{2^{2p} - 2^{p+1} + 1}{2^q + 1}$$

Then

$$2^{2p-q-1}(2^q + 1) = 2^{2p} - 2^{p+1} + 1$$

It is possible if $2p = q + 1$, thus $c^2 = 1$ and

$$c^2 - 1 = 2^{2p} - 2^{p+1} - 2^q = 0 \Rightarrow 2^{p-1} - 1 - 2^{q-p-1} = 0 \Rightarrow q = p + 1 = 3$$

Catalan equation becomes

$$Y^2 = 2^3 + 1 \Rightarrow Y = \pm 3$$

$c^2 \leq 1$ We have

$$2X^q - X^{2p} = \frac{72 - 2(c+c')^2 - (7c+c')^2}{(c+c')^2} > \frac{64 - 49c^2}{(c+c')^2} > 0$$

Thus

$$q \geq 2p - 1$$

And

$$c^2 X^q - X^{2p} = \frac{36c^2 - c^2(c+c')^2 - (7c+c')^2}{(c+c')^2} = \frac{-(c+c')(13c+c') - c^2(c+c')^2}{(c+c')^2} < 0$$

Or

$$\begin{aligned} 2c^2 X^q - X^{2p} &= \frac{72c^2 - 2c^2(c+c')^2 - (7c+c')^2}{(c+c')^2} \\ &> \frac{64c^2 - (7c+c')^2}{(c+c')^2} = \frac{(c-c')(15c+c')}{(c+c')^2} > 0 \end{aligned}$$

And

$$X^{-1} \leq X^{q-2p} < \frac{1}{c^2}$$

If

$$\begin{aligned} \frac{1}{2c^2} &< \frac{1}{X} \\ X^{q-2p} &< \frac{2}{X} \end{aligned}$$

As q is odd, it means $q+1 = 2p$ thus $\frac{1}{2c^2} > \frac{1}{X}$ let m verifying

$$\frac{1}{2c^2} < \frac{1}{2^m X}$$

Thus

$$\frac{1}{2c^2} < \frac{1}{2^m X} < X^{q+1-2p} < \frac{1}{c^2} < \frac{1}{2^{m-1} X}$$

And

$$2^{-m} < X^{q+2-2p} < 2^{1-m}$$

Hence $X = 2$ and $2p - q - 1 = m$ or

$$2^{q+1-2p} < \frac{1}{c^2} < \frac{1}{2^{m-1} X} = \frac{1}{2^{2p-q-1}}$$

Then

$$\frac{1}{c^2} = 2^{q+1-2p} = \frac{2^q + 1}{2^{2p} - 2^{p+1} + 1}$$

It is possible if $q+1 = 2p$ then

$$2^{2p-1} - 1 = 2^{q-p-1} \Rightarrow q = p+1 = 3$$

The equation becomes

$$Y^2 = 2^3 + 1 \Rightarrow Y = \pm 3$$

Generalization to Pillai equation

Pillai equation is $Y^p = X^q + a$. We will define Pillai numbers : we note them : b_a they depend of a and have the same proprieties than b for $a = 1$. Thus, we write $1_a + 7_a = 8_a$ and $7_a - 1_a = 6_a$: our goal is to follow the approach as we have done

for Catalan equation. Of course, we have $1_a = a_1$ et $0_a = 0$. The equation becomes $Y^p = X^q + 1_a$. Let us pose for p, q, X, Y, a integers strictly greater to 1. Let

$$c = \frac{X^p - 1_a}{Y^{\frac{p}{2}}}, \quad c' = \frac{7_a - X^p}{Y^{\frac{p}{2}}}$$

And

$$(c + c')Y^{\frac{p}{2}} = X^p - 1_a + 7_a - X^p = 6_a \Rightarrow Y^{\frac{p}{2}} = \frac{6_a}{c + c'}$$

So

$$X^p = cY^{\frac{p}{2}} + 1_a = \frac{6_a c}{c + c'} + 1_a = \frac{7_a c + 1_a c'}{c + c'}$$

And

$$X^q = Y^p - 1_a = \frac{36_a 1_a - 1_a (c + c')^2}{(c + c')^2}$$

Also

$$(c + c')Y^{\frac{p}{2}} = 6_a > 0$$

Thus $c + c' > 0$. And

$$cY^{\frac{p}{2}} = X^p - 1_a > 0$$

So $c > 0$. But $X^p \geq 4_a$, then

$$cY^{\frac{p}{2}} = X^p - 1_a \geq 7_a - X^p = c'Y^{\frac{p}{2}}$$

And $c \geq c'$ and

$$7_a - X^p = 7_a - \frac{7_a c + 1_a c'}{c + c'} = \frac{6_a c'}{c + c'}$$

If $c' > 0$ Then $X^p < 7_a$ and we have a finite number of solutions, we have $c' < 0$, with

$$(c + c')Y^{\frac{p}{2}} = 6_a > 0$$

and $(c + c')X^p = 7_a c + 1_a c' > 0$ with $c + c' > 0$. And

$$Y^{\frac{p}{2}} = \frac{6_a}{c + c'} \geq 3_a \Rightarrow c + c' \leq 2 < 2_a$$

Main results We retain

1) $c' < 0$

2) $0 < c + c' < 2_a$

3) $0 < 7_a c + 1_a c'$

Two cases : I) $c^2 > 1_a$

II) $c^2 \leq 1_a$

$c^2 > 1_a$ We have

$$1_a X^q - X^{2p} = \frac{36_a 1_a 1_a - 1_a (c + c')^2 - (7_a c + 1_a c')^2}{(c + c')^2} < \frac{36_a 1_a (1_a - c^2) - 1_a (c + c')^2}{(c + c')^2} < 0$$

Hence $X^{q-2p} < \frac{1}{1_a} < 1$ and

$$q \leq 2p - 1$$

And

$$c^2 X^q - X^{2p} = \frac{36_a 1_a c^2 - c^2 (c + c')^2 - (7_a c + 1_a c')^2}{(c + c')^2} = \frac{-1_a 1_a (c + c')(13c + c') - c^2 (c + c')^2}{(c + c')^2} < 0$$

And

$$\begin{aligned} 2c^2 X^q - X^{2p} &= \frac{72_a 1_a c^2 - 2c^2 (c + c')^2 - (7_a c + 1_a c')^2}{(c + c')^2} \\ &> \frac{64_a 1_a c^2 - (7_a c + 1_a c')^2}{(c + c')^2} = \frac{1_a 1_a (c - c')(15c + c')}{(c + c')^2} > 0 \end{aligned}$$

Thus

$$2c^2 > X^{2p-q} \geq X$$

We deduce

$$2c^2 > X$$

If $c^2 < X$

$$2X > c^2 > X^{2p-q} \Rightarrow 1 \geq \frac{2}{X} > X^{2p-q-2}$$

As q is odd $q + 1 = 2p$ thus $c^2 > X$ let $c^2 > 2^{m-1} X$ with m verifying $c^2 < 2^m X < X^{2p-q}$ and

$$2^{m+1} X > c^2 > X^{2p-q} > 2^m X > c^2$$

Or

$$2^{m+1} > X^{2p-q-1} > 2^m$$

If X is even, it means $X = 2$ and $2p - q - 1 = m$ then

$$c^2 < 2^m X = 2^{2p-q} < 2c^2 < 2^{m+1} = 2^{2p-q}$$

Thus

$$2c^2 = 2^{2p-q}$$

And

$$2^{2p-q-1} = c^2 = \frac{2^{2p} - 1_a 2^{p+1} + 1_a 1_a}{2^q + 1_a}$$

Or

$$2^{2p-q-1} (2^q + 1_a) = 2^{2p} - 1_a 2^{p+1} + 1_a 1_a$$

It is possible if $2p = q + 1$, and there is a finite number of solutions!

$c^2 \leq 1$ We have

$$2_a X^q - X^{2p} = \frac{72_a 1_a 1_a - 2_a (c + c')^2 - (7_a c + 1_a c')^2}{(c + c')^2} > \frac{64_a 1_a 1_a - 49_a 1_a c^2}{(c + c')^2} > 0$$

Thus

$$q \geq 2p - 1$$

And

$$c^2 X^q - X^{2p} = \frac{36_a 1_a c^2 - c^2 (c + c')^2 - (7_a c + 1_a c')^2}{(c + c')^2} = \frac{-1_a 1_a (c + c')(13c + c') - c^2 (c + c')^2}{(c + c')^2} < 0$$

And

$$\begin{aligned} 2c^2 X^q - X^{2p} &= \frac{72_a 1_a c^2 - 2c^2 (c + c')^2 - (7_a c + 1_a c')^2}{(c + c')^2} \\ &> \frac{64_a 1_a c^2 - (7_a c + 1_a c')^2}{(c + c')^2} = \frac{1_a 1_a (c - c')(15c + c')}{(c + c')^2} > 0 \end{aligned}$$

And

$$X^{-1} \leq X^{q-2p} < \frac{1}{c^2}$$

If

$$\begin{aligned} \frac{1}{2c^2} &< \frac{1}{X} \\ X^{q-2p} &< \frac{2}{X} \end{aligned}$$

As q is odd, it means that $q + 1 = 2p$ then $\frac{1}{2c^2} > \frac{1}{X}$ let m with

$$\frac{1}{2c^2} < \frac{1}{2^m X} < X^{q+1-2p}$$

Then

$$\frac{1}{2c^2} < \frac{1}{2^m X} < X^{q+1-2p} < \frac{1}{c^2} < \frac{1}{2^{m-1} X}$$

And

$$2^{-m} < X^{q+2-2p} < 2^{1-m}$$

Thus $X = 2$ and $2p - q - 1 = m$ or

$$2^{q+1-2p} < \frac{1}{c^2} < \frac{1}{2^{m-1} X} = \frac{1}{2^{2p-q-1}}$$

Hence

$$\frac{1}{c^2} = 2^{q+1-2p} = \frac{2^q + 1_a}{2^{2p} - 1_a 2^{p+1} + 1_a 1_a}$$

It is possible if $q + 1 = 2p$ and there is a finite number of solutions! In all cases, Pillai equation has a finite number of solutions, it is the formulation of the conjecture. Thus, we have proved it.

Conclusion

Catalan equation is solved, an original solution exists! We have generalized the approach to Pillai equation and proved that it always has a finite number of solutions. It is the proof of Pillai conjecture. It seems that many problems of number theory can be solved like this.

Références

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