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An elementary proof of Catalan-Mihailescu theorem

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Abstract

(MSC=11D04) More than one century after its formulation by the Belgian mathematician Eugene Catalan, Preda Mihailescu has solved the open problem. But, is it all? Mihailescu's solution utilizes computation on machines, we propose here not really a proof as it is intended classically, but a resolution of an equation like the resolution of the polynomial equations of third and fourth degrees. This solution is totally algebraic and does not utilize, of course, computers or any kind of calculation. Then, we generalize the proof to Pillai conjecture.

(Keywords : Diophantine equations ; Catalan equation ; Pillai conjecture ; Algebraic resolution)

Introduction

Catalan theorem has been proved in 2002 by Preda Mihailescu. In 2004, it became officially Catalan-Mihailescu theorem. This theorem stipulates that there are not consecutive pure powers. There do not exist integers strictly greater than 1, $X > 1$ and $Y > 1$, for which with exponents strictly greater than 1, $p > 1$ and $q > 1$,

$$Y^p = X^q + 1$$

but for $(X, Y, p, q) = (2, 3, 2, 3)$. We can verify that

$$3^2 = 2^3 + 1$$

Euler has proved that the equation $X^3 + 1 = Y^2$ has this only solution. We propose in this study a general solution. The particular cases already solved concern $p = 2$, solved by Ko Chao in 1965, and $q = 3$ which has been solved in 2002. The case $q = 2$ has been solved by Lebesgue in 1850. We solve here the equation for the general case. We generalize the proof to Pillai conjecture

$$Y^p = X^q + a$$

And prove that it has always a finite number of solutions for a fixed a .

The approach

Let

$$c = \frac{X^p - 1}{Y^{\frac{p}{2}}}, \quad c' = \frac{7 - X^p}{Y^{\frac{p}{2}}}$$

And

$$d = X^p - Y^{\frac{p}{2}}, \quad d' = X^p + Y^{\frac{p}{2}}$$

We have

$$Y^{\frac{p}{2}} = X^p - d = d' - X^p = \frac{d' - d}{2}$$

And

$$X^p = \frac{d' + d}{2}$$

But

$$(c + c')Y^{\frac{p}{2}} = X^p - 1 + 7 - X^p = 6 \Rightarrow Y^{\frac{p}{2}} = \frac{6}{c + c'} = \frac{d' - d}{2}$$

And

$$X^p = cY^{\frac{p}{2}} + 1 = \frac{6c}{c + c'} + 1 = \frac{7c + c'}{c + c'} = \frac{d' + d}{2}$$

We deduce

$$d = \frac{7c + c' - 6}{c + c'}, \quad d' = \frac{7c + c' + 6}{c + c'}$$

And

$$X^q = Y^p - 1 = \frac{36 - (c + c')^2}{(c + c')^2}$$

We have

$$(c + c')Y^{\frac{p}{2}} = 6 > 0$$

Thus $c + c' > 0$. Also

$$cY^{\frac{p}{2}} = X^p - 1 > 0$$

Thus $c > 0$. But $X^p \geq 4$, hence

$$cY^{\frac{p}{2}} = X^p - 1 \geq 7 - X^p = c'Y^{\frac{p}{2}}$$

hence $c \geq c'$

$$7 - X^p = 7 - \frac{7c + c'}{c + c'} = \frac{6c'}{c + c'}$$

If

$$c' > 0$$

Thus $X^p < 7$ and $X^p = 4$, it means that $c' < 0$ and

$$(c + c')Y^{\frac{p}{2}} = 6 > 0$$

$$(c + c')X^p = 7c + c' > 0$$

thus $c + c' > 0$ and

$$c' + 1 > 0$$

And

$$Y^{\frac{p}{2}} = \frac{6}{c + c'} \geq 3 \Rightarrow c + c' \leq 2$$

main results What we must retain is

1) $c' < 0$

2) $0 < c + c' < 2$

3) $0 < 7c + c'$

And we will discuss two cases

I) $c^2 < 1$

II) $c^2 \geq 1$

case $c^2 < 1$ $c < 1$ means that $(c - 1)Y^{\frac{p}{2}} = d - 1 < 0$, we deduce

$$(c' + 1)Y^{\frac{p}{2}} = 7 - d > 0$$

Thus $c' > -1$. But

$$3X^p(c + c') - 4Y^{\frac{p}{2}}(c + c') = 3(7c + c') - 24 = 21(c - 1) + 3(c' - 1) < 0$$

And

$$\begin{aligned} 2X^q - X^{2p} &= \frac{72 - 2(c + c')^2 - (7c + c')^2}{(c + c')^2} \\ &= \frac{72 - 51c^2 - 3c'^2 - 18cc'}{(c + c')^2} \\ &= \frac{51(1 - c^2) - 3c'(c + c') - 15cc' + 21}{(c + c')^2} > 0 \end{aligned}$$

And

$$\begin{aligned} c^2 X^q - X^{2p} &= \frac{36c^2 - c^2(c + c')^2 - (7c + c')^2}{(c + c')^2} \\ &= \frac{36c^2 - c^4 - c^2c'^2 - 2c^3c' - 49c^2 - c'^2 - 14cc'}{(c + c')^2} \\ &= \frac{-13c(c + c') - c'c(1 + (c' + 2c)c) - c^4 - c'^2}{(c + c')^2} \\ &= \frac{c(-13c - 14c' - 2c^2c' - cc'^2 - c^3) - c'^2}{(c + c')^2} \\ &= \frac{c(-13c - 13c' - c'c(c + c') - c^2(c' + c)) - c'(c + c')}{(c + c')^2} \\ &< \frac{c(-12c - 12c') - (c + c')^2 - c(c + c')^2}{(c + c')^2} < 0 \end{aligned}$$

Also

$$\begin{aligned} 100c^2 X^q - 99X^{2p} &= \frac{3600c^2 - 100c^2(c + c')^2 - 99(7c + c')^2}{(c + c')^2} \\ &= \frac{(60c - \sqrt{99}(7c + c'))(60c + \sqrt{99}(7c + c')) - 100c^2(c + c')^2}{(c + c')^2} \\ &= \frac{(-9.64c - 9.94c')(129.64c + 9.94c') - 100c^2(c + c')^2}{(c + c')^2} \end{aligned}$$

$$\begin{aligned}
&< \frac{-1249.73c^2 - 98.8c'^2 - 1384.24cc'}{(c + c')^2} \\
&= \frac{-67.26c'(c + 1.1c') - 1249.73c(c + c') - 24.81c'(c' + 2.44c)}{(c + c')^2} \\
&= \frac{c'(-127.91c - 98.8c') - 9.77c(127.91c + 98.8c') - 29.11cc'}{(c + c')^2} \\
&= \frac{-(127.91c + 98.8c')(9.77c + c') - 29.11cc'}{(c + c')^2} < 0
\end{aligned}$$

because

$$\begin{aligned}
127.91c + 98.8 &> -29.11c' > 0 \\
9.77c + c' &> c > 0
\end{aligned}$$

We deduce

$$\frac{c^2}{2}X^{2p} < c^2X^q < X^{2p}$$

And

$$c^2X^q < \frac{99}{100}X^{2p}$$

But

$$X^{2p-q-1} < 2X^{-1} \leq 1$$

Thus

$$2p \leq q + 1$$

We will give two proofs : the first : as $c^2 - 1 < 0$ and $Y^p \geq 9$, let

$$9(c^2 - 1) = 9X^p(X^p - 2) - 9c^2X^q \geq (c^2 - 1)Y^p = X^p(X^p - 2) - X^q$$

Thus

$$8X^p(X^p - 2) \geq (9c^2 - 1)X^q$$

First case $9c^2 - 1 \geq 2$ then

$$8X^p(X^p - 2) \geq 2X^q$$

And

$$4X^{2p-q} \geq 1 + 8X^{p-q} > 1$$

Thus

$$4X^{2p-q+1} > X \geq 4$$

Consequently

$$q + 1 \geq 2p \geq q$$

As p and q do not have the same parity, we deduce that $q + 1 = 2p$. Then

$$4X^q = 4X^{2p-1} \geq 4X^p(X^p - 2) \geq X^q = X^{2p-1}$$

$$1 \geq X - 2X^{1-p}$$

$$X - 1 \leq 2X^{1-p} \leq 1$$

And $X = 2$, but

$$Y^p - 1 = (Y^{\frac{p}{2}} - 1)(Y^{\frac{p}{2}} + 1) = 2^q$$

If $Y^{\frac{p}{2}} = 2v + 1$

$$4v(v + 1) = 2^q \geq 2^3$$

It is possible if $v = 1$, then $Y = 3$ and $p = 2$ and $q = 3$ or $X = 2$. Second case $3c^2 < 1$

$$0 > c^2 - 1 = X^{2p} - 2X^p - c^2X^q > X^{2p} - 2X^p - \frac{99}{100}X^{2p} = \frac{1}{100}X^p(X^p - 200) > 0$$

It means that $0 \leq c^2 - 1 \leq 0$, or $c^2 = 1$ and

$$X^p - 2 = X^{q-p} \geq 2$$

$X^{p-1} - X^{q-p-1} = \frac{2}{X}$ is an integer, then $X = 2$. And

$$2^{p-1} = 1 + 2^{q-p-1}$$

In one side an even number, in the other an odd one, the solution is $q = p + 1$, then $2^{p-1} = 2$, or $p = 2$ and $Y = \pm 3$. Another proof: We have $1 + \frac{1}{99} < X^\epsilon = \frac{X^{2p-q}}{c^2} < \frac{X}{c^2}$

$$-\epsilon \log(X) > (2p - q - \epsilon) \log(X) = 2 \log(c)$$

$$\epsilon \log(X) = (2p - q) \log(X) - 2 \log(c) = \log\left(\frac{X^{2p-q}}{c^2}\right) > 1$$

Thus $X^\epsilon > 1$

$$2 > 2p - q + 1 = \frac{\log(c^2)}{\log(X)} + 1 > 0$$

because

$$\frac{\log(c^2)}{\log(X)} > -1$$

or

$$c^2 > X^{-1}$$

or

$$c^2 - 1 > X^{-1} - 1 > \frac{1 - X}{X}$$

or

$$(c^2 - 1)X > 1 - X$$

or

$$(c^2 - 1)X = X(X^{2p} - 2X^p - c^2X^q) > 1 - X$$

and

$$X(X^{2p} - 2X^p - c^2X^q) > 1 - X$$

and

$$X(X^p - 1)^2 - c^2X^{q+1} > 1$$

And with $X^\epsilon > 1$

$$X(X^p - 1)^2 - c^2X^{q+1} = X(X^p - 1)^2 - X^{2p-\epsilon+1} = X(X^{2p} - X^{2p-\epsilon} - 2X^p + 1) > 1$$

because

$$X^p \geq 2X^{p-\epsilon}$$

$$X^\epsilon > 1 + \frac{1}{99}$$

$$X^p > 200$$

And

$$\frac{99X^p}{100} = X^{p-\epsilon}$$

$$\frac{X^p}{100} > 2$$

Thus we always have

$$q \leq 2p \leq q + 1$$

and as p and q do not have the same parity, we have

$$q + 1 \leq 2p \leq q + 1$$

And $2p = q + 1$. Then

$$0 \geq (c^2 - 1)Y^p = X^{2p} - 2X^p - X^q = X^{2p} - 2X^p - X^{2p-1} = X^p(X^{p-1}(X-1) - 2) \geq 0$$

Thus $c^2 = 1$

$$X^p - X^{p-1} = 2$$

$$X^{p-1} - X^{p-2} = \frac{2}{X}$$

This expression is an integer, hence $X = 2$ and

$$2^{p-1} = 1 + 2^{p-2}$$

In one side an even number, in the other an odd one, it is possible if $p = 2$, thus

$$q = 2p - 1 = 3$$

Case $c \geq 1$ We must retain here that

$$1) c^2 > 1$$

$$2) c + c' < 2$$

$$3) c' < 0$$

We have

$$\begin{aligned} 2c^2 X^q - X^{2p} &= \frac{72c^2 - 2c^2(c + c')^2 - (7c + c')^2}{(c + c')^2} \\ &= \frac{72c^2 - 49c^2 - c'^2 - 14cc' - 2c^2(c^2 + c'^2 + 2cc')}{(c + c')^2} \\ &= \frac{c^2(23 - 2(c + c')^2) - c'(c + c') - 13cc'}{(c + c')^2} \\ &> \frac{15c^2 - c'(c + c') - 13cc'}{(c + c')^2} > 0 \end{aligned}$$

And

$$\begin{aligned}
c^2 X^q - X^{2p} &= \frac{36c^2 - c^2(c+c')^2 - (7c+c')^2}{(c+c')^2} \\
&= \frac{36c^2 - 50c^2 - 2c'^2 - 16cc' + (1-c^2)(c+c')^2}{(c+c')^2} \\
&= \frac{-2(7c+c')(c+c') + (c+c')^2(1-c^2)}{(c+c')^2} < 0
\end{aligned}$$

We deduce

$$\begin{aligned}
\frac{1}{2}X^{2p} < c^2 X^q < X^{2p} < c^2 X^{2p} \\
X^{2p-q} > c^2 > 1
\end{aligned}$$

And

$$2p \geq q + 1$$

There are two cases again :

$$I) c'^2 < 1$$

$$II) c'^2 > 1$$

Case $c'^2 < 1$ We have with $u = q - p$

$$\frac{X^{2p} - 7X^p + 49}{X^q + X^p} < c'^2 = \frac{(X^p - 7)^2}{X^q + 1} < 1 = \frac{X^p}{X^{q-u}}$$

$u = q - p$, We have

$$X^{2p+q-u} - 14X^{p+q-u} + 49X^{q-u} \leq X^{p+q} + X^{2p}$$

$$X^{2p} - 14X^p + 49 \leq X^{p+u} + X^{2p-q+u}$$

$$X^{2p} - X^{p+u} - X^{2p-q+u} \leq 14X^p - 49$$

$$X^p - X^u - X^{p+u-q} \leq 14 - 49X^{-p}$$

$$X^p - X^u - 14 \leq X^{p+u-q} - 49X^{-p} = -48 + 49 - 49X^{-p} < -48 + 49 = 1$$

$$0 < X^p - X^{p-1} < X^p - X^{q-p} \leq 15$$

There are 6 possibilities $2k \in \{2, 4, 6, 8, 10, 12, 14\}$ and $X^{p-1} - X^{q-p-1} = \frac{2k}{X}$ is an integer thus $X = 2m \in \{2, 4, 6, 8, 10, 12, 14\}$

$$(2m)^{p-1} = 2v + 1 + (2m)^{q-p-1}$$

when $m \in 2, 6, 10, 12, 14$ in one side an even number, in the other an odd one, the solution is $q = p + 1$ which leads to

$$X^{p-1} = 2 \Rightarrow (X, p) = (2, 2)$$

And

$$(2m)^{p-2} = \frac{2v}{X} + (2m)^{q-p-2}$$

If $q \geq p + 2$, it means $X = 2v \in 2, 4$ and it is impossible! Thus $X = 2$ and $(p, q) = (2, 3)$ or $Y = \pm 3$

Case $c'^2 > 1$ In this case

$$X^q = \frac{36 - (c + c')^2}{(c + c')^2} > \frac{36 - 4}{(c + c')^2} = \frac{32}{36} \left(\frac{36}{(c + c')^2} \right) = \frac{8}{9} Y^p > \frac{8}{9} X^p$$

More generally, we have

$$9(c^2 - 1) = 9X^{2p} - 18X^p - 9c^2 X^q \leq (c^2 - 1)Y^p = X^{2p} - 2X^p - X^q$$

$$8X^{2p} - 16X^p = 8c^2 Y^p - 8 < (9c^2 - 1)X^q$$

We deduce

$$X^{q-2p} > \frac{8X^p(X^p - 2)}{(9c^2 - 1)X^{2p}}$$

But

$$18X^{2p} - 36X^p - 9c^2 X^{2p} = 18c^2 Y^p - 18 - 9c^2 Y^p + 9 - 18X^p$$

$$= 9c^2 Y^p - 9 - 18X^p = 9c^2 X^q + 9(c^2 - 1) - 18X^p > (c^2 - 1)(18X^p + 9) > 0 > X^{2p} - 2X^p - X^{2p}$$

In consequence

$$17X^p(X^p - 2) > (9c^2 - 1)X^{2p}$$

Or

$$X^{q-2p} > \frac{8X^p(X^p - 2)}{(9c^2 - 1)X^{2p}} > \frac{8}{17}$$

And

$$X^{q+1-2p} > \frac{8X}{17} \geq 1$$

Thus

$$q > 2p$$

Hence

$$0 < (c^2 - 1)Y^p = X^{2p} - 2X^p - X^q < X^{2p} - 2X^p - X^{2p} < 0$$

We have simultaneously $q + 1 = 2p$ and

$$(c^2 - 1)Y^p X^{-p-1} = 0 = X^{p-1} - X^{p-2} - \frac{2}{X} = 0$$

We deduce $(X, p) = (2, 2)$ or Catalan solution.

Conclusion

Catalan equation is solved, an original solution exists! We have generalized the approach to Pillai equation and proved that it always has a finite number of solutions. It is the proof of Pillai conjecture. It seems that many problems of number theory can be solved like this.

Références

- [1] P. MIHAILESCU A class number free criterion for catalan's conjecture, *Journal of Number theory*, **99** (2003).