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Overdetermined systems of lacunary equations

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Abstract

We show that the zeros of an overdetermined system of lacunary equations in n variables can be described in terms of systems with at most n equations.

Mathematics Subject Classification: 11Y16, 11G35

1 Introduction

Let $F, G \in \mathbb{Z}[x]$ be polynomials of degree $\leq d$ and logarithmic height $\leq h$, having at most N non zero coefficients. Assuming that at least one of F and G does not vanish on roots of unity, Filaseta, Granville and Schinzel (see [3]) proved that there exists an algorithm which computes the greatest common divisor of F and G in time $O_{h,N}(\log d)$.

This result relies heavily on a work of Bombieri and Zannier on the intersection of a subvariety of \mathbb{G}_m^n (as customary, we denote by \mathbb{G}_m the multiplicative group) of codimension ≥ 2 with subgroups of dimension 1, appeared for the first time as an appendix of [6] by Zannier and later in [1].

Recently, in his P.h.D Thesis [4] the second Author relaxes the condition above on roots of unity. He shows that there exists an algorithm which computes in time $O_{h,N}(\log d)$ a polynomial P which divides $\gcd(F, G)$ and such that $P/\gcd(F, G)$ vanishes only at roots of unity. Hence the equation $P = 0$ has the same set of solutions as the system of equations $F = G = 0$. In this respect, it may not be out of place to recall that D. A. Plaisted [5] has shown that computing the GCD of two sparse polynomials is a NP-hard problem.

In this article we are interested in multivariate generalizations of these problems. An arbitrary algebraic variety of codimension l cannot be written as an intersection of l hypersurfaces. Nevertheless it can be described as a union of complete intersections on non-empty open sets. Such a representation can be

regarded as a generalization of the computation of the previous polynomial P in the univariate case.

Assume now that V is defined by polynomials with integers coefficients each of degree $\leq d$, logarithmic height $\leq h$ and supported by at most N monomials (this means that there exist N monomials such that each of these polynomials is a linear combination of such monomials). We look for an algorithm which computes such a representation in time quasi-linear in $\log d$. It turns out that a such type of result relies on a generalization of the already quoted theorem of Bombieri and Zannier. This generalization is still an open conjecture of Zilber (see section 2.4 for details).

A first result in this direction was proved again in his P.h.D Thesis [4] by the second Author. Leroux' algorithm covered the two dimensional case, working with bivariate polynomials. Here we extend, in a non obvious way, Leroux' algorithm to arbitrary multivariate polynomials.

We state below our main result in a simplified form. For a more precise statement, see Theorem 3.4 in section 3.

Theorem 1.1 *We assume Zilber's conjecture. Let $V \subset \mathbb{G}_m^n$ be a subvariety defined over a number field K by at most N equations of degree $\leq d$, height $\leq h$ and supported by at most N monomials.*

Then we can find a finite collection Γ whose elements are lists

$$(P_1, \dots, P_L, Q) \quad \text{with} \quad L \leq n$$

of Laurent polynomials and such that

$$V = \bigcup_{\Gamma} \overline{Z(P_1, \dots, P_L) \setminus Z(Q)}. \quad (1.1)$$

The polynomials in the lists of Γ have degree $O_{h,N}(d)$, height $O_{h,N}(1)$ and they are defined over a cyclotomic extension of K of degree $O_{h,N}(1)$. The collection Γ consists of $O_{h,N}(1)$ elements and its computation take at most $O_{h,N}(\log d)$ time.

Moreover, let $X \subseteq V$ be an irreducible component of maximal dimension. Then there exists a list (P_1, \dots, P_L, Q) in Γ such that

$$X \subseteq \overline{Z(P_1, \dots, P_L) \setminus Z(Q)}$$

and in addition $\overline{Z(P_1, \dots, P_L) \setminus Z(Q)}$ has codimension L .

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2 Auxiliary results

2.1 Notations

We shall use the following notations.

For a point $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{G}_m^n$ and an integral vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$, we set $\alpha^{\mathbf{a}} := \alpha_1^{a_1} \cdots \alpha_n^{a_n}$. Let $\varphi: \mathbb{G}_m^n \rightarrow \mathbb{G}_m^N$ be a morphism. Thus

$$\varphi(\alpha) = (\alpha^{\mathbf{a}_1}, \dots, \alpha^{\mathbf{a}_N})$$

for some $\mathbf{a}_1, \dots, \mathbf{a}_N \in \mathbb{Z}^n$. We define the size of φ as the maximum of the absolute values of these vectors.

Following Bombieri, Masser and Zannier we call *torsion coset* a translate of a subtorus by a torsion point.

Let X be a subvariety of \mathbb{G}_m^n . Since the intersection of torsion cosets is again a torsion coset, we can talk about the minimal torsion coset containing X . We define the multiplicative rank of X as the dimension of this minimal torsion coset and we denote it by $\text{rank}(X)$.

Remark 2.1 *Let $\varphi: \mathbb{G}_m^n \rightarrow \mathbb{G}_m^N$ be a morphism and let $X \subseteq \mathbb{G}_m^n$ be an irreducible subvariety. Then*

$$\text{rank}(\varphi(X)) - \dim(\varphi(X)) = \text{rank}(X) - \dim(X).$$

A point α has rank 0 if it is a torsion point; it has rank n if its coordinates $\alpha_1, \dots, \alpha_n$ are multiplicatively independent. In the terminology of [1], p.5, let us denote by \mathcal{H}_r the union of all algebraic subgroups of \mathbb{G}_m^n with dimension r . Then α has rank r if and only if $\alpha \in \mathcal{H}_r \setminus \mathcal{H}_{r-1}$.

2.2 Zilber's conjecture

As we have mentioned in the introduction, the result of Filaseta, Granville and Schinzel ([3]) relies heavily on a theorem of Bombieri and Zannier. Let us recall this statement, in one ([1], Theorem 4.1) of its equivalent forms:

Theorem 2.2 *Let W be a subvariety of \mathbb{G}_m^N of codimension ≥ 2 . Then there exists a finite collection \mathcal{U}_W of torsion cosets such that $W \cap \mathcal{H}_1$ is contained in the union of the $(W \cap T) \cap \mathcal{H}_1$ for all T in \mathcal{U}_W .*

A natural generalisation of this theorem is the following conjecture ([2], p.309)

Conjecture 2.3 *Let W be a subvariety of \mathbb{G}_m^N of codimension $\geq s$. Then there exists a finite collection \mathcal{U}_W of torsion cosets such that $W \cap \mathcal{H}_s$ is the union of the $(W \cap T) \cap \mathcal{H}_{s-1}$ for all T in \mathcal{U}_W .*

It turns out that this last statement is equivalent to a conjecture of Zilber:

Conjecture 2.4 *Let W be a subvariety of \mathbb{G}_m^N . Then there exists a finite collection \mathcal{U}_W of torsion cosets of \mathbb{G}_m^N of codimension 1 with the following property. Let $T_0 \subset \mathbb{G}_m^N$ be a torsion coset and let Y be an irreducible component of $W \cap T_0$ of dimension*

$$\dim Y > \dim W - \text{codim } T_0.$$

(such a component Y is called atypical in [2]). Then there exists $T \in \mathcal{U}_W$ such that $Y \subseteq W \cap T$.

For details on the equivalence between these two conjectures (and for more conjectures), see the appendix of [2].

2.3 Locally complete intersection stratification

Let W be a subvariety of \mathbb{G}_m^N of codimension l . It is well known that W is (as a set) a complete intersection on a non-empty open set. More precisely, we can find polynomials F_1, \dots, F_l and G such that

$$W \setminus Z(G) = Z(F_1, \dots, F_l) \setminus Z(G)$$

and such that $Z(G)$ does not contain any component of W of codimension l . This implies that $Z(F_1, \dots, F_l) \setminus Z(G)$ contains all the irreducible components of W of codimension l and that $W \cap Z(G)$ has codimension $l+1$. Proceeding in this way we compute inductively a decomposition of W as a *union of complete intersections on non-empty open sets*:

Definition 2.5 *Let W be a subvariety of \mathbb{G}_m^N defined over a field K . A locally complete intersection stratification of W is a family*

$$\{(F_{j,1}, \dots, F_{j,l_j}, G_j)\}_{j=1, \dots, t}$$

where $F_{j,i}, G_j \in K[x_1^{\pm 1}, \dots, x_N^{\pm 1}]$ are such that:

- For $j = 1, \dots, t$ let l_j be the codimension of $W \cap Z(G_1, \dots, G_{j-1})$. Then $\text{codim}(W) = l_1 < \dots < l_t$ and $G_t = 1$;
- $W \cap Z(G_1, \dots, G_{j-1}) \setminus Z(G_j) = Z(F_{j,1}, \dots, F_{j,l_j}) \setminus Z(G_j)$, for $j = 1, \dots, t$.

We remark that for a *complete intersection* W we obviously have $t = 1$.

In the rest of the paper, for any subvariety W of \mathbb{G}_m^N we fix once and for all a stratification $\{F_{j,1}^{(W)}, \dots, F_{j,l_j}^{(W)}, G_j^{(W)}\}_{j=1, \dots, t}$ of this kind.

3 Proof of the main result

3.1 Main algorithm

The following algorithm is the heart of the proof of theorem 3.4.

Lemma 3.1 *Let K be a number field and let n, N, k be fixed non-negative integers with $k < N$. There exists an algorithm which takes as input a triple $(W, \boldsymbol{\eta}, \varphi)$, where*

- $\boldsymbol{\eta} \in \mathbb{G}_m^k$ is a torsion point;
- $W \subset \mathbb{G}_m^{N-k}$ is a subvariety defined over $K(\boldsymbol{\eta})$ given together with a locally complete intersection stratification (definition 2.5)

$$\{F_{j,1}^{(W)}, \dots, F_{j,l_j}^{(W)}, G_j^{(W)}\}_{j=1, \dots, t};$$

- $\varphi: \mathbb{G}_m^n \rightarrow \mathbb{G}_m^N$ a morphism.

and gives as its output a set (eventually empty) of triples $(W_1, \boldsymbol{\eta}_1, \varphi_1)$, where

- $\boldsymbol{\eta}_1 \in \mathbb{G}_m^{k+1}$ is a torsion point;

- $W_1 \subset \mathbb{G}_m^{N-k-1}$ is a subvariety defined over $K(\boldsymbol{\eta}_1)$;
- $\varphi_1: \mathbb{G}_m^n \rightarrow \mathbb{G}_m^N$ is a morphism.

In this algorithm, the cardinality of the output set, the torsion points $\boldsymbol{\eta}_1$ and the subvarieties W_1 depend only on W and $\boldsymbol{\eta}$ and not on φ . Moreover, the size of φ_1 is $O_{W,\boldsymbol{\eta}}(d)$ and the computation time is $O_{W,\boldsymbol{\eta}}(\log d)$.

The output of the algorithm satisfies the following properties:

1) Let $(W_1, \boldsymbol{\eta}_1, \varphi_1)$ be a triple in the output set. Then

$$\varphi_1^{-1}(W_1 \times \{\boldsymbol{\eta}_1\}) \subseteq \varphi^{-1}(W \times \{\boldsymbol{\eta}\}) .$$

2) Let j with $1 \leq j \leq t$ and¹ $l_{j-1} \leq n$. Let $W' := W \cap Z(G_1^{(W)}, \dots, G_{j-1}^{(W)})$ and let $X \subseteq \varphi^{-1}(W' \times \{\boldsymbol{\eta}\})$ be an irreducible variety such that

$$\text{rank}(X) - \dim(X) < l_j = \text{codim}(W') .$$

Let B the minimal torsion coset containing X . Then there exists a triple $(W_1, \boldsymbol{\eta}_1, \varphi_1)$ in the output set such that

$$\varphi^{-1}(W' \times \{\boldsymbol{\eta}\}) \cap B \subseteq \varphi_1^{-1}(W_1 \times \{\boldsymbol{\eta}_1\}) .$$

In particular, $X \subseteq \varphi_1^{-1}(W_1 \times \{\boldsymbol{\eta}_1\})$.

Proof. The algorithm consists of two loops. The first one is indexed by $j \in [1, t]$ with $l_{j-1} \leq n$. For a such j we set

$$W' := W \cap Z(G_1^{(W)}, \dots, G_{j-1}^{(W)}) ,$$

The second loop is indexed by T in the collection $\mathcal{U}_{W'}$ of codimension one torsion cosets whose existence is ensured by Zilber's conjecture 2.4. For a such $T \in \mathcal{U}_{W'}$ of equation, say, $\mathbf{x}^\lambda = \xi$ we choose an automorphism τ of \mathbb{G}_m^{N-k} of size $O_W(1)$ such that $\tau(T) = \{x_N = \xi\}$. Thus

$$\tau(W' \cap T) = W'' \times \{\xi\} .$$

where $W'' \subset \mathbb{G}_m^{N-k-1}$ is a subvariety defined² over $K(\boldsymbol{\eta}, \xi)$.

Let τ' be the automorphism of $\mathbb{G}_m^N = \mathbb{G}_m^{N-k} \times \mathbb{G}_m^k$ whose restriction to \mathbb{G}_m^{N-k} is τ and which is the identity on \mathbb{G}_m^k . The algorithm gives as its output a set of triples

$$(W_1, \boldsymbol{\eta}_1, \varphi_1) = (W'', \{\xi\} \times \{\boldsymbol{\eta}\}, \tau' \circ \varphi)$$

built in such a way, for

$$1 \leq j \leq t \text{ with } l_{j-1} \leq n \quad \text{and } T \in \mathcal{U}_{W'}$$

varying.

¹we agree that $l_0 = 1$.

²A set of equations of W'' can be easily deduced from the given equations of W' . Assume that $W' = Z(F_1, \dots, F_l)$. Let $\rho = \tau^{-1} \circ \iota$ where $\iota: \mathbb{G}_m^{N-k-1} \rightarrow \mathbb{G}_m^{N-k}$ is the inclusion $\iota(\mathbf{x}) = (\mathbf{x}, 1)$. Then the variety $W'' \subset \mathbb{G}_m^{N-k-1}$ is defined by the equations $F_i \circ \rho$, $i = 1, \dots, l$.

It is straightforward to justify the assertions concerning the cardinality of the output set, the size of its elements and the computation time. .

We now prove that the output of this algorithm satisfies the desired two properties. We prove 1). Let

$$(W_1, \boldsymbol{\eta}_1, \varphi_1) = (W'', \{\xi\} \times \{\boldsymbol{\eta}\}, \tau' \circ \varphi)$$

be a triple in the output set, builded choosing $j \leq t$ with $l_{j-1} \leq n$ and $T \in \mathcal{U}_{W'}$, as described above. Then

$$\begin{aligned} (\tau' \circ \varphi)^{-1}(W'' \times \{\xi\} \times \{\boldsymbol{\eta}\}) &= \varphi^{-1}(\tau^{-1}(W'' \times \{\xi\}) \times \{\boldsymbol{\eta}\}) \\ &= \varphi^{-1}(W' \cap T) \times \{\boldsymbol{\eta}\} \\ &\subseteq \varphi^{-1}(W \times \{\boldsymbol{\eta}\}) \end{aligned}$$

We now prove 2). By assumption, $\varphi(X) = X' \times \{\boldsymbol{\eta}\}$ with $X' \subseteq W'$. Let T_0 be the projection on the first $N - k$ coordinates of the minimal torsion coset through $\varphi(X)$. Thus $\dim T_0 \leq \text{rank}(\varphi(X))$ and $X' \subseteq W' \cap T_0$. Let Y be the irreducible component of $W' \cap T_0$ through X' . By remark 2.1 and by assumption,

$$\text{rank}(\varphi(X)) - \dim(\varphi(X)) = \text{rank}(X) - \dim(X) < l_j = \text{codim}(W') .$$

Thus

$$\begin{aligned} \dim Y &\geq \dim X' = \dim \varphi(X) > \text{rank}(\varphi(X)) - \text{codim}(W') \geq \dim T_0 - \text{codim} W' \\ &= \dim W' - \text{codim} T_0 . \end{aligned}$$

By Zilber's conjecture 2.4 there exists a torsion coset $T \in \mathcal{U}_{W'}$ of equation $\mathbf{x}^\lambda = \xi$ such that $X' \subseteq T$. Thus X is contained in the torsion coset $\varphi^{-1}(T \times \{\boldsymbol{\eta}\})$. By minimality of B , we have $B \subseteq \varphi^{-1}(T \times \{\boldsymbol{\eta}\})$.

Let τ be an automorphism of \mathbb{G}_m^{N-k} such that $\tau(T) = \{x_N = \xi\}$. Let $\tau(W' \cap T) = W'' \times \{\xi\}$ with $W'' \subseteq \mathbb{G}_m^{N-k-1}$. Define τ' to be the automorphism of $\mathbb{G}_m^N = \mathbb{G}_m^{N-k} \times \mathbb{G}_m^k$ whose restriction to \mathbb{G}_m^{N-k} is τ and which is the identity on \mathbb{G}_m^k . Then there exists a choice of τ such that the triple

$$(W'', \{\xi\} \times \{\boldsymbol{\eta}\}, \tau' \circ \varphi)$$

constructed in such a way is one of the output triples of the algorithm. Since $B \subseteq \varphi^{-1}(T \times \{\boldsymbol{\eta}\})$, we have

$$\begin{aligned} (\tau' \circ \varphi)(\varphi^{-1}(W' \times \{\boldsymbol{\eta}\}) \cap B) &\subseteq \tau'((W' \cap T) \times \{\boldsymbol{\eta}\}) \\ &= \tau(W' \cap T) \times \{\boldsymbol{\eta}\} = W'' \times \{\xi\} \times \{\boldsymbol{\eta}\} . \end{aligned}$$

Thus

$$\varphi^{-1}(W' \times \{\boldsymbol{\eta}\}) \cap B \subseteq (\tau' \circ \varphi)^{-1}(W'' \times \{\xi\} \times \{\boldsymbol{\eta}\}) .$$

as required. □

Remark 3.2

i) In some cases the algorithm of lemma 3.1 can be considerably simplified. Let $(W, \boldsymbol{\eta}, \varphi)$ be an input triple of this algorithm. Let $\pi_1: \mathbb{G}_m^N \rightarrow \mathbb{G}_m^{N-k}$ be the projections on the first $N - k$ coordinates and assume that there exists a codimension one torsion coset T of \mathbb{G}_m^{N-k} which does not depend on φ and such that

$$\text{Im}(\pi_1 \circ \varphi) \subseteq T.$$

In this case we can perform the second loop of the algorithm of lemma 3.1 *only* for this special T instead that for *all* $T \in \mathcal{U}_{W'}$. A straightforward inspection of the proof show that assertion 2) of lemma 3.1 still holds.

ii) Moreover, if in assertion 2) of lemma 3.1 we are only interested in subvarieties X of rank n , then we can born ourself to the search of codimension one torsion cosets T such that $\text{Im}(\pi_1 \circ \varphi) \subseteq T$. Indeed, let us assume that there exists a torsion coset $T \in \mathcal{U}_{W'}$ such that $(\pi_1 \circ \varphi)(X) \subseteq T$. Since $\text{rank}(X) = n$, we have $\text{Im}(\pi_1 \circ \varphi) \subseteq T$.

Lemma 3.3 *We assume Zilber's conjecture. Let $V \subset \mathbb{G}_m^n$ be a subvariety defined over a number field K by at most $N \geq 2$ equations of degree $\leq d$, height $\leq h$ and supported by at most N monomials. Then, for $k = 0, 1, \dots, N - 1$ there exists a finite (possibly empty) set Λ_k of cardinality $O_{h,N}(1)$ whose elements are triples $(W, \boldsymbol{\eta}, \varphi)$, where*

- $\boldsymbol{\eta} \in \mathbb{G}_m^k$ is a torsion point of order $O_{h,N}(1)$;
- $W \subset \mathbb{G}_m^{N-k}$ is a subvariety defined over $K(\boldsymbol{\eta})$ and given together with a locally complete intersection stratification $\{(F_{j,1}^{(W)}, \dots, F_{j,l_j}^{(W)}, G_j^{(W)})\}_{j=1, \dots, t}$ of Laurent polynomials of height and degree $O_{h,N}(1)$;
- $\varphi: \mathbb{G}_m^n \rightarrow \mathbb{G}_m^N$ is a morphism of size $O_{h,N}(d)$;

such that the following assertions hold.

- 1) For $k = 0, \dots, N - 1$ and for $(W, \boldsymbol{\eta}, \varphi) \in \Lambda_k$ we have $\varphi^{-1}(W \times \{\boldsymbol{\eta}\}) \subseteq V$.
- 2) Let $X \subseteq V$ be an irreducible subvariety. Then there exist an index k and $(W, \boldsymbol{\eta}, \varphi) \in \Lambda_k$ such that

$$X \subseteq \varphi^{-1} \left(\overline{Z(F_{j,1}^{(W)}, \dots, F_{j,l_j}^{(W)}) \setminus Z(G_j^{(W)})} \times \{\boldsymbol{\eta}\} \right)$$

for some $j \in [1, t]$ such that $l_j \leq \text{rank}(X) - \dim(X)$.

Moreover, the sets $\Lambda_0, \dots, \Lambda_N$ can be computed in time $O_{h,N}(\log d)$.

Proof. We construct Λ_k inductively. Let $F_i(\mathbf{x}) = \sum_{j=1}^N f_{ij} \mathbf{x}^{\lambda_j}$ be the equations defining V . We set $\Lambda_0 = \{(W_0, 1, \varphi_0)\}$ where W_0 is defined in \mathbb{G}_m^N by the linear equations $\sum_{j=1}^N f_{ij} y_j$ and where $\varphi_0(\mathbf{x}) = (\mathbf{x}^{\lambda_1}, \dots, \mathbf{x}^{\lambda_N})$.

Let $k \in \{0, \dots, N - 2\}$ and assume Λ_k already constructed. We choose $(W, \boldsymbol{\eta}, \varphi) \in \Lambda_k$. We apply the algorithm described in lemma 3.1 and we define

Λ_{k+1} as the set of the output of this algorithm.

The assertions concerning the cardinality of the sets Λ_k , the size of their elements and the computation time, directly follow from the corresponding assertions of lemma 3.1.

Similarly, statements 1) and 2) easily follow from statements 1) and 2) of lemma 3.1. Let us check 1). By construction of $\Lambda_0 = \{(W_0, 1, \varphi_0)\}$ we have $\varphi_0(V) \subseteq W_0$. Let k be an index with $0 \leq k \leq N - 1$ and let us assume that 1) is satisfied for all $(W, \boldsymbol{\eta}, \varphi) \in \Lambda_k$. Let $(W_1, \boldsymbol{\eta}_1, \varphi_1) \in \Lambda_{k+1}$. Assertion 1) of lemma 3.1 and the inductive hypothesis ensure that

$$\varphi_1^{-1}(W_1 \times \{\boldsymbol{\eta}_1\}) \subseteq \varphi^{-1}(W \times \{\boldsymbol{\eta}\}) \subseteq V .$$

Let now prove that assertion 2) holds. Let S be the set of integers $k \in [0, N - 1]$ such that there exists $(W, \boldsymbol{\eta}, \varphi) \in \Lambda_k$ satisfying

$$X \subseteq \varphi^{-1}(W \times \{\boldsymbol{\eta}\}) .$$

Observe that this set is not empty, because $0 \in S$. We define k as the largest element of S . Let us assume, by contradiction, that assertion 2) does not hold. Let us define $j^\dagger = 0$ if $l_1 > \text{rank}(X) - \dim(X)$ and j^\dagger as the largest integer $\in [1, t]$ such that $l_j \leq \text{rank}(X) - \dim(X)$ otherwise. By definition 2.5, we have

$$X \not\subseteq \overline{\varphi^{-1}(W \cap Z(G_1^{(W)}, \dots, G_{j-1}^{(W)}) \setminus Z(G_j^{(W)}))} \times \{\boldsymbol{\eta}\}$$

for $j = 1, \dots, j^\dagger$. Since $X \subseteq \varphi^{-1}(W \times \{\boldsymbol{\eta}\})$ and since X is irreducible, we see by induction that

$$X \subseteq \varphi^{-1}(W \cap Z(G_1^{(W)}, \dots, G_{j^\dagger}^{(W)})) \times \{\boldsymbol{\eta}\} .$$

This implies that $G_{j^\dagger-1}^{(W)} \neq 1$, thus $j^\dagger + 1 \leq t$. By definition of j^\dagger we have

$$l_{j^\dagger+1} > \text{rank}(X) - \dim(X) .$$

Then assertion 2) of lemma 3.1 (with j replaced by $j^\dagger + 1$) shows that there exists $(W_1, \boldsymbol{\eta}_1, \varphi_1) \in \Lambda_{k+1}$ such that

$$X \subseteq \varphi_1^{-1}(W_1 \times \{\boldsymbol{\eta}_1\}) .$$

This contradicts the choice of k .

□

Theorem 3.4 *We assume Zilber's conjecture. Let $V \subset \mathbb{G}_m^n$ be a subvariety defined over a number field K by at most N equations of degree $\leq d$, height $\leq h$ and supported by at most N monomials. Then we can find a finite collection Γ whose elements are lists*

$$(P_1, \dots, P_L, Q) \quad \text{with} \quad L \leq n$$

of Laurent polynomials and such that

$$V = \bigcup_{\Gamma} \overline{Z(P_1, \dots, P_L) \setminus Z(Q)}. \quad (3.1)$$

The polynomials in the lists of Γ have degree $O_{h,N}(d)$, height $O_{h,N}(1)$ and they are defined over a cyclotomic extension of K of degree $O_{h,N}(1)$. The collection Γ consists of $O_{h,N}(1)$ elements and its computation take at most $O_{h,N}(\log d)$ time.

Moreover, let X be an irreducible component of V . Then there exists a list (P_1, \dots, P_L, Q) in Γ such that

$$X \subseteq \overline{Z(P_1, \dots, P_L) \setminus Z(Q)}$$

and

- 1) Among P_1, \dots, P_L there at least $n - \text{rank}(X)$ binomial equations of the shape $\mathbf{x}^{\lambda_h} = \xi_h$ ($\lambda_h \in \mathbb{Z}^n$, ξ_h root of unity).
- 2) $L \leq \text{codim}(X)$.
- 3) If X is of maximal dimension $\text{dim}(V)$, then $\overline{Z(P_1, \dots, P_L) \setminus Z(Q)}$ has codimension L .

Proof. Let us first explain how to construct the set Γ . We apply lemma 3.3. We choose an index k and we choose $(W, \boldsymbol{\eta}, \varphi) \in \Lambda_k$ with W of codimension $\leq n$. The variety W is given (cf. note ??) with a locally complete intersection stratification $\{F_{j,1}^{(W)}, \dots, F_{j,l_j}^{(W)}, G_j^{(W)}\}_{j=1, \dots, t}$.

Let $\pi_1: \mathbb{G}_m^N \rightarrow \mathbb{G}_m^{N-k}$ and $\pi_2: \mathbb{G}_m^N \rightarrow \mathbb{G}_m^k$ be the projections on the first $N-k$ coordinates and on the last k coordinates respectively. Let $B = (\pi_2 \circ \varphi)^{-1}(\boldsymbol{\eta}) \subseteq \mathbb{G}_m^n$. Thus B is a torsion coset of codimension, say, $s \leq n$. We compute s equations H_1, \dots, H_s of B of the desired shape $\mathbf{x}^\lambda = \xi$. For any index $j \in [1, t]$ such that $l_j \leq n - s$ we set $L := l_j + s$ and we add to Γ the list

$$(P_1, \dots, P_L, Q) := (F_{j,1}^{(W)} \circ \pi_1 \circ \varphi, \dots, F_{j,l_j}^{(W)} \circ \pi_1 \circ \varphi, H_1, \dots, H_s, G_j^{(W)} \circ \pi_1 \circ \varphi).$$

By construction and by proposition 2.4 1),

$$\begin{aligned} Z(P_1, \dots, P_L) \setminus Z(Q) &= \varphi^{-1}(Z(F_{j,1}^{(W)}, \dots, F_{j,l_j}^{(W)}) \setminus Z(G_j^{(W)}) \times \{\boldsymbol{\eta}\}) \\ &= \varphi^{-1}(W \cap Z(G_1^{(W)}, \dots, G_{j-1}^{(W)}) \setminus Z(G_j^{(W)}) \times \{\boldsymbol{\eta}\}) \\ &\subseteq \varphi^{-1}(W \times \{\boldsymbol{\eta}\}) \subseteq V. \end{aligned}$$

This shows that the union of the right hand side of (3.1) is contained in V .

We also remark that the elements of Γ have degree, height and field of definition as predicted by theorem 3.4. Moreover, Γ has cardinality $O_{h,N}(1)$ and its computation can be made in time

$$O_{h,N}(M(d) \log \log d).$$

This directly follow from lemma 3.3.

Let now X be an irreducible component of V . By 2) of lemma 3.3, there exists an index k and $(W, \boldsymbol{\eta}, \varphi) \in \Lambda_k$ such that

$$X \subseteq \varphi^{-1} \left(\overline{Z(F_{j,1}^{(W)}, \dots, F_{j,l_j}^{(W)}) \setminus Z(G_j^{(W)})} \times \{\boldsymbol{\eta}\} \right)$$

for some $j \in [1, t]$ such that $l_j \leq \text{rank}(X) - \dim(X)$. Let (P_1, \dots, P_L, Q) be the list in the output corresponding to $(W, \boldsymbol{\eta}, \varphi)$ and to j . By construction,

$$X \subseteq \overline{Z(P_1, \dots, P_L) \setminus Z(Q)} \subseteq V.$$

This shows that V is contained in the union of the right hand side of (3.1). We finally show the last three assertions of the theorem. We have $L = l_j + s$ with s the codimension of $B = (\pi_2 \circ \varphi)^{-1}(\boldsymbol{\eta})$. Since $X \subseteq B$, by definition of multiplicative rank we have $s \leq n - \text{rank}(X)$. This shows assertion 1). By remark 2.1,

$$L = l_j + s \leq \text{rank}(X) - \dim(X) + s = \text{rank}(X) - \dim(X) + s \leq \text{codim}(X)$$

which shows assertion 2). In particular, if X has dimension $\dim(V)$ then $\overline{Z(P_1, \dots, P_L) \setminus Z(Q)}$ has codimension L as claimed in assertion 3).

□

4 Appendix

In this appendix we highlight with two examples how the algorithms of lemma 3.1, lemma 3.3 and theorem 3.4 work.

4.1 Example 1

Let V be the subvarieties of \mathbb{G}_m^2 defined by the equations

$$f_1 = x^{3d+1}y^{3d} + x^2y + 5, \quad f_2 = x^{3d+2}y^{3d} + 5x + 25, \quad f_3 = x + x^2y + 25y.$$

supported by the 5 monomials $x^{3d+1}y^{3d}$, $x^{3d+2}y^{3d}$, x^2y , x , y . It is easy to see that V consists of two points $(5\zeta, 1/5)$ of rank 1 for ζ a third root of unity. We now describe how our algorithms give this result.

We first construct the sets $\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ following the method described in the proof of lemma 3.3. In order to simplify the exposition, we denote by $\iota = \iota_k$ the inclusion $\mathbb{G}_m^{4-k} \rightarrow \mathbb{G}_m^{5-k}$ given by $\iota(\mathbf{x}) = (\mathbf{x}, 1)$. Given an automorphism τ of \mathbb{G}_m^k we also let τ' be the automorphism of $\mathbb{G}_m^N = \mathbb{G}_m^{N-k} \times \mathbb{G}_m^k$ whose restriction to \mathbb{G}_m^{N-k} is τ and which is the identity on \mathbb{G}_m^k .

We set $\Lambda_0 = \{(W_0, 1, \varphi_0)\}$ where W_0 is the complete intersection of \mathbb{G}_m^5 defined by the equations:

$$F_1 = x_1 + x_3 + 5, \quad F_2 = x_2 + 5x_4 + 25, \quad F_3 = x_4 + x_3 + 25x_5$$

and $\varphi_0: \mathbb{G}_m^2 \rightarrow \mathbb{G}_m^5$ is given by

$$\varphi_0(x, y) = (x^{3d+1}y^{3d}, x^{3d+2}y^{3d}, x^2y, x, y).$$

We now build on Λ_1 . We select the only element of Λ_0 and we apply the algorithm of lemma 3.1 with $k = 0$, $W = W_0$ and $\varphi = \varphi_0$. Since W is complete intersection, the locally complete intersection stratification of W is trivial and we have to choose $j = 1$ in the first loop of the algorithm. Thus $W' = W$. Since $\text{Im } \varphi \subseteq T = \{x_1x_2^{-1}x_4 = 1\}$ and since this torsion coset is independent from d , we can choose this T as the only torsion coset in the second loop of the algorithm (see remark 3.2, 1)). Let $\tau: \mathbb{G}_m^5 \rightarrow \mathbb{G}_m^5$ be the automorphism given by

$$\tau(\mathbf{x}) = (x_2, x_1x_2^{-1}, x_3, x_5, x_1x_2^{-1}x_4).$$

Thus $\tau(T) = \{x_5 = 1\}$ as required. Then

$$\tau(W' \cap T) = W'' \times \{1\}$$

where $W'' \subset \mathbb{G}_m^4$ is the complete intersection defined by the equations

$$F_1 \circ \rho = x_1x_2 + x_3 + 5, \quad F_2 \circ \rho = x_1 + 5x_2^{-1} + 25, \quad F_3 \circ \rho = x_2^{-1} + x_3 + 25x_4$$

(with $\rho(\mathbf{x}) = \tau^{-1} \circ \iota(\mathbf{x}) = (x_1x_2, x_1, x_3, x_2^{-1}, x_4)$). We set $\Lambda_1 = \{(W_1, \eta_1, \varphi_1)\}$ with $W_1 = W''$, $\eta_1 = \{1\}$ and

$$\varphi_1(x, y) = (\tau \circ \varphi)(x, y) = (x^{3d+2}y^{3d}, x^{-1}, x^2y, y, 1)$$

The construction of Λ_2 and Λ_3 is similar. Let $(W, \{1\}, \varphi)$ the only element of Λ_1 . Since W is again complete intersection, the locally complete intersection stratification of W is trivial and we have to choose $j = 1$ in the first loop of algorithm 3.1. Thus $W' = W$. Again, we can choose only one torsion coset $T = \{x_2^2x_3x_4^{-1} = 1\}$ in the second loop. We chose $\tau(\mathbf{x}) = (x_1, x_2^{-2}x_4, x_2, x_2^2x_3x_4^{-1})$ as one of the automorphisms of \mathbb{G}_m^4 such that $\tau(T) = \{x_4 = 1\}$. Then $\tau(W' \cap T) = W'' \times \{1\}$ where $W'' \subset \mathbb{G}_m^3$ is the complete intersection defined by the equations

$$F_1 \circ \rho = x_1x_3 + x_2 + 5, \quad F_2 \circ \rho = x_1 + 5x_3^{-1} + 25, \quad F_3 \circ \rho = x_3^{-1} + x_2 + 25x_2x_3^2$$

(with $\rho(\mathbf{x}) = \tau^{-1} \circ \iota(\mathbf{x}) = (x_1, x_3, x_2, x_2x_3^2)$). We set $\Lambda_2 = \{(W_1, \eta_1, \varphi_1)\}$ with $W_1 = W''$, $\eta_1 = \{1\}$ and

$$\varphi_1(x, y) = (\tau' \circ \varphi)(x, y) = (x^{3d+2}y^{3d}, x^2y, x^{-1}, 1, 1).$$

Let $(W, \{1\}, \varphi)$ this only element of Λ_2 . In the two loops of algorithm 3.1 we choose $j = 1$ (thus $W' = W$) and then $T = \{x_1x_3^2 = 1\}$. We chose $\tau(\mathbf{x}) = (x_1, x_3, x_1x_3^2)$ as one of the automorphisms of \mathbb{G}_m^3 such that $\tau(T) = \{x_3 = 1\}$. Then $\tau(W' \cap T) = W'' \times \{1\}$ where $W'' \subset \mathbb{G}_m^3$ is the complete intersection defined by the equations

$$F_1 \circ \rho = x_2^{-1} + x_1 + 5, \quad F_2 \circ \rho = x_2^{-2} + 5x_2^{-1} + 25$$

(with $\rho(\mathbf{x}) = (\tau^{-1} \circ \iota)(\mathbf{x}) = (x_2^{-2}, x_1, x_2)$). Remark indeed that the equation $F_3 \circ \rho = x_2^{-1} + x_1 + 25x_1x_2^2$ is in the ideal generated by the others. W'' is a zero-dimensional variety consisting of the two points $(5\zeta, \zeta/5)$ with ζ a third root of unity. We set $\Lambda_3 = \{(W_1, \eta_1, \varphi_1)\}$ with $W_1 = W''$, $\eta_1 = \{1\}$ and

$$\varphi_1(x, y) = (\tau' \circ \varphi)(x, y) = (x^2y, x^{-1}, (xy)^{3d}, 1, 1).$$

We finally build on Λ_4 . Let $(W, \{1\}, \varphi)$ the only element of Λ_3 . Since W is again complete intersection, the locally complete intersection stratification of W is trivial and we have to choose $j = 1$ and $W' = W$. We compute the collection $\mathcal{U}_{W'}$ of codimension one torsion cosets whose existence is ensured by Zilber conjecture 2.4. Since $\dim W' = 0$, we can choose for $\mathcal{U}_{W'}$ the set consisting of the two torsion cosets

$$T_\zeta = \{x_1x_2 = \zeta^2\},$$

torsion cosets through $(5\zeta, \zeta/5)$ with ζ a third root of unity. We fix one if this roots of unity. Let $\tau: \mathbb{G}_m^2 \rightarrow \mathbb{G}_m^2$ be the automorphism given by

$$\tau(x_1, x_2) = (x_1, x_1x_2)$$

which satisfies $\tau(T_\zeta) = \{x_2 = \zeta\}$ as required. Then

$$\tau(W' \cap T) = W'' \times \{1\}$$

where $W''_\zeta = \{5\zeta\} = Z(x_1 - 5\zeta) \subset \mathbb{G}_m$. We finally define Λ_4 as the set of two elements:

$$\Lambda_4 = \{(W''_\zeta, \{(\zeta^2, 1, 1, 1)\}, \tau' \circ \varphi)\}$$

where $(\tau' \circ \varphi)(x, y) = (x^2y, xy, (xy)^{3d}, 1, 1)$.

We now construct the set Γ following the method described in the proof of theorem 3.4. We have to choose an index k and $(W, \boldsymbol{\eta}, \varphi) \in \Lambda_k$ with W of codimension ≤ 2 . By the construction above, we have only two possible choices, both corresponding to $k = 4$:

$$(W_\zeta, \{(\zeta^2, 1, 1, 1)\}, \varphi)$$

with ζ a third root of unity, where $W_\zeta = \{5\zeta\} \subset \mathbb{G}_m$ has equation

$$F = x_1 - 5\zeta = 0$$

and where $\varphi: \mathbb{G}_m^2 \rightarrow \mathbb{G}_m^5$ is defined by $\varphi(x, y) = (x^2y, xy, (xy)^{3d}, 1, 1)$. Let $\pi_1: \mathbb{G}_m^N \rightarrow \mathbb{G}_m^1$ and $\pi_2: \mathbb{G}_m^N \rightarrow \mathbb{G}_m^4$ be the projections on the first coordinate and on the last 4 coordinates respectively. Let

$$B = (\pi_2 \circ \varphi)^{-1}(\{\zeta^2, 1, 1, 1\}) = \{(t, \zeta^2t^{-1}), t \in \mathbb{G}_m\}.$$

Thus B is a torsion coset of codimension 1 of equation $H_1 = xy - \zeta^2$. We also compute $F \circ \pi_1 \circ \varphi = x^2y - 5\zeta$. The set Γ consists of two elements:

$$\Gamma = \{(x^2y - 5\zeta, xy - \zeta^2, 1)\}$$

with ζ a third root of unity.

4.2 Example 2

Let V be the subvarieties of \mathbb{G}_m^2 defined by the equations

$$\begin{aligned} f_1 &= x^d y^2 - 5x^d y - 2y + 10, \\ f_2 &= x^{d+1} y - 2x^d y - 2x + 4, \\ f_3 &= x^{d+1} y + x^d y^2 - 7x^d y - 2x - 2y + 14. \end{aligned}$$

These equations are supported by the 5 monomials $x^{d+1}y, x^d y^2, x^d y, y, x$. It is easy to see that V consists of the codimensional one component of equation $x^d y - 2$ and of the isolated point $(2, 5)$. We describe how our algorithms give this result.

We first construct the sets Λ_k , following the proof of lemma 3.3. We use the same conventions concerning ι and τ' adopted in the previous example. We set $\Lambda_0 = \{(W_0, 1, \varphi_0)\}$ where W_0 is the complete intersection of \mathbb{G}_m^5 defined by the equations:

$$F_1 = x_2 - 5x_3 - 2x_4 + 10, \quad F_2 = x_1 - 2x_3 - 2x_5 + 4, \quad F_3 = x_1 + x_2 - 7x_3 - 2x_4 - 2x_5 + 14$$

and $\varphi_0: \mathbb{G}_m^2 \rightarrow \mathbb{G}_m^5$ is given by

$$\varphi_0(x, y) = (x^{d+1}y, x^d y^2, x^d y, y, x).$$

We now build on Λ_1 . We select the only element of Λ_0 and we apply the algorithm of lemma 3.1 with $k = 0$, $W = W_0$ and $\varphi = \varphi_0$. Since W is complete intersection, the locally complete intersection stratification of W is trivial and we have to choose $j = 1$ in the first loop of algorithm 3.1. Thus $W' = W$. By remark 3.2, 1)), we can choose only one torsion coset $T = \{x_2 x_3^{-1} x_4^{-1} = 1\}$ in the second loop. We chose $\tau(\mathbf{x}) = (x_1, x_3, x_4, x_5, x_2 x_3^{-1} x_4^{-1})$ as one of the automorphisms of \mathbb{G}_m^5 such that $\tau(T) = \{x_5 = 1\}$. Thus

$$\tau(W' \cap T) = W'' \times \{1\}$$

where $W'' \subset \mathbb{G}_m^4$ is the complete intersection defined by the equations

$$\begin{aligned} F_1 \circ \rho &= x_2 x_3 - 5x_2 - 2x_3 + 10, \\ F_2 \circ \rho &= x_1 - 2x_2 - 2x_4 + 4 \end{aligned}$$

(with $\rho(\mathbf{x}) = (\tau^{-1} \circ \iota)(\mathbf{x}) = (x_1, x_2 x_3, x_2, x_3, x_4)$). Remark indeed that the equation $F_3 \circ \rho = x_1 + x_2 x_3 - 7x_2 - 2x_3 - 2x_4 + 14$ is in the ideal generated by the others. We set $\Lambda_1 = \{(W_1, \eta_1, \varphi_1)\}$ with $W_1 = W''$, $\eta_1 = \{1\}$ and

$$\varphi_1(x, y) = (\tau \circ \varphi)(x, y) = (x^{d+1}y, x^d y, y, x, 1).$$

The construction of Λ_2 is similar. In the two loops of algorithm 3.1 we choose $j = 1$ (thus $W' = W$) and then $T = \{x_1 x_2^{-1} x_4^{-1} = 1\}$. We chose $\tau(\mathbf{x}) = (x_2, x_3, x_4, x_1 x_2^{-1} x_4^{-1})$ as one of the automorphisms of \mathbb{G}_m^4 such that $\tau(T) = \{x_4 = 1\}$. Then $\tau(W' \cap T) = W'' \times \{1\}$ where $W'' \subset \mathbb{G}_m^3$ is defined by the equations

$$\begin{aligned} F_1 \circ \rho &= x_1 x_2 - 5x_1 - 2x_2 + 10, \\ F_2 \circ \rho &= x_1 x_3 - 2x_1 - 2x_3 + 4 \end{aligned}$$

(with $\rho(\mathbf{x}) = (\tau^{-1} \circ \iota)(\mathbf{x}) = (x_1 x_3, x_1, x_2, x_3)$). The subvariety W'' consists of two components, of codimension 1 and 2. Indeed, the two polynomials defining it have $x_1 - 2$ have a non trivial gcd, and it is easy to check that

$$W'' = Z(x_1 - 2) \cup Z(x_2 - 5, x_3 - 2).$$

Thus its locally complete intersection stratification has length two. We can choose

$$F_{1,1}^{(W'')} = x_1 - 2, \quad G_1^{(W'')} = x_3 - 2$$

and

$$F_{2,1}^{(W''')} = x_1x_2 - 5x_1 - 2x_2 + 10 = (x_1 - 2)(x_2 - 5), \quad F_{2,2}^{(W''')} = x_3 - 2, \quad G_2^{(W''')} = 1.$$

We set $\Lambda_2 = \{(W_1, \eta_1, \varphi_1)\}$ with $W_1 = W''$, $\eta_1 = \{1\}$ and with $\varphi_1(x, y) = (\tau' \circ \varphi)(x, y) = (x^d y, y, x, 1, 1)$.

Let $(W, \{(1, 1)\}, \varphi)$ the element of Λ_2 . The first loop of algorithm 3.1 consists of two steps: $j = 1$ and $j = 2$, which we describe now. For $j = 1$ we set $W' = W$ and it is easy to see that this variety does not have atypical subvarieties. Thus we can choose $\mathcal{U}_{W'} = \emptyset$ and $j = 1$ gives no contribution to the output. Let now consider the contribution of $j = 2$ to the output of the algorithm 3.1. We have

$$W' = W \cap Z(G_1^{(W)}) = Z((x_1 - 2)(x_2 - 5), x_3 - 2).$$

Let T be the codimension one torsion coset of equation $x_1x_3^{-1} = 1$. Then W' has only one maximal atypical subvariety, $W' \cap T = \{(2, t, 2) \mid t \in \mathbb{G}_m\}$ and we can choose $\mathcal{U}_{W'} = \{T\}$. Let $\tau: \mathbb{G}_m^3 \rightarrow \mathbb{G}_m^3$ be the automorphism given by $\tau(\mathbf{x}) = (x_1, x_2, x_1^{-1}x_4)$. Thus $\tau(T) = \{x_3 = 1\}$ as required and $\tau(W' \cap T) = W'' \times \{1\}$ where $W'' \subset \mathbb{G}_m^2$ is the hypersurface $Z(x_1 - 2)$. We set $\Lambda_3 = \{(W_1, \eta_1, \varphi_1)\}$ with $W_1 = W''$, $\eta_1 = \{(1, 1)\}$ and

$$\varphi_1(x, y) = (\tau' \circ \varphi)(x^d y, y, 1, 1, 1).$$

Since we can verify that the hypersurface $Z(x_1 - 2)$ does not have atypical subvarieties, $\Lambda_4 = \emptyset$ and the construction of lemma 3.3 end here.

We now construct the set Γ following the method described in the proof of theorem 3.4. We have to choose an index k and $(W, \boldsymbol{\eta}, \varphi) \in \Lambda_k$ with W of codimension ≤ 2 . Thus $k = 1, 2$ or 3 .

We first choose $k = 1$. Following the discussion above and the notations of the proof of theorem 3.4, we have $B := (\pi_2 \circ \varphi)^{-1}(1) = \mathbb{G}_m^2$ of codimension $s = 0$ and Λ_1 contribute with one element to Γ :

$$\begin{aligned} P_1 &= x^d y^2 - 5x^d y - 2y + 10 = (x^d y - 2)(y - 5), \\ P_2 &= x^{d+1} y - 2x^d y - 2x + 4 = (x^d y - 2)(x - 2), \\ Q &= 1. \end{aligned}$$

We now choose $k = 2$. We have again $B := (\pi_2 \circ \varphi)^{-1}(1, 1) = \mathbb{G}_m^2$ of codimension $s = 0$ and Λ_2 contribute with two elements to Γ :

$$P_1 = x^d y - 2, \quad P_2 = 0, \quad Q = x - 2$$

and

$$P_1 = (x^d y - 2)(y - 5), \quad P_2 = x - 2, \quad Q = 1.$$

We finally choose $k = 3$. We have $B := (\pi_2 \circ \varphi)^{-1}(1, 1, 1) = Z(x^{d-1}y - 1)$ of codimension $s = 1$ and Λ_3 contribute with one element to Γ :

$$P_1 = x^d y - 2, \quad P_2 = x^{d-1} y - 1, \quad Q = 1.$$

Thus theorem 3.4 represent $V = Z(x^d y - 2) \cup \{(2, 5)\}$ as a union of four pieces $V_1, V_{2,1}, V_{2,2}, V_3$:

$$V_1 = Z((x^d y - 2)(y - 5), (x^d y - 2)(x - 2)) = Z(x^d y - 2) \cup \{(2, 5)\}$$

$$V_{2,1} = Z(x^d y - 2) \setminus Z(x - 2) ,$$

$$V_{2,2} = Z((x^d y - 2)(y - 5), x - 2) = \{(2, 5)\} \cup \{(2, 2^{-d+1})\} ,$$

$$V_3 = Z(x^d y - 2, x^{d-1} y - 1) = \{(2, 2^{-d+1})\} .$$

References

- [1] E. Bombieri, D. Masser and U. Zannier, “Anomalous subvarieties-structure theorems and applications”. *Int. Math. Res. Not.* IMRN 2007, no. 19, 33p.
- [2] E. Bombieri, D. Masser and U. Zannier, “On unlikely intersections of complex varieties with tori”. *Acta Arith.* **133** (2008), no. 4, 309-323.
- [3] M. Filaseta, A. Granville and A. Schinzel, “Irreducibility and greatest common divisor algorithms for sparse polynomials”. In *Number theory and polynomials*, 155–176, London Math. Soc. Lecture Note Ser., **352**, Cambridge Univ. Press, Cambridge, 2008.
- [4] L. Leroux *Algorithmes pour les polynômes lacunaires*, Thèse, Université de Caen-Basse Normandie, 2011.
- [5] D. A. Plaisted, “Sparse complex polynomials and polynomial reducibility”, *J. Comput. System Sci.* **14** (1977), 210–221.
- [6] A. Schinzel, “Polynomials with special regard to reducibility”. With an appendix by Umberto Zannier. *Encyclopedia of Mathematics and its Applications*, **77**. Cambridge University Press, Cambridge, 2000.