The smooth-fit property in an exponential Lévy model

Damien Lamberton, Mohammed Mikou

Applied Probability Trust (September 13, 2011)

THE SMOOTH-FIT PROPERTY IN AN EXPONENTIAL LÉVY MODEL

DAMIEN LAMBERTON,* *Université Paris-Est*

MOHAMMED MIKOU,** *Université Paris-Est*

Abstract

We study the smooth-fit property of the American put price with finite maturity in an exponential Lévy model when the underlying stock pays dividends at a continuous rate. As in the perpetual case, a regularity property is sufficient for smooth-fit to occur. We also derive conditions on the Lévy measure under which smooth-fit fails.

Keywords: Optimal stopping; Lévy processes; smooth-fit property; American options.

2000 Mathematics Subject Classification: Primary 60G40; 60G51; 91G20.
Secondary 60J75

1. Introduction

The continuity of the derivative with respect to the underlying stock price of the American put price is a well known property in the Black-Scholes model, called the smooth-fit property. In the context of exponential Lévy models, this property may no longer be true. Figure 1 demonstrates that in the $CGMY$ model, one of the most used exponential Lévy models in practice (see [5]), the smooth-fit property holds when the parameter $Y = 1$ and it fails when $Y = 0.2$.

In the case of perpetual American options, a necessary and sufficient condition for smooth-fit was derived by Alili and Kyprianou [1] in an exponential Lévy model

* Postal address: Université Paris-Est, Laboratoire d'analyse et de mathématiques appliquées, UMR CNRS 8050, 5, Bld Descartes, F-77454 Marne-la-Vallée Cedex 2, France. E-mail: damien.lamberton@univ-mlv.fr

** Postal address: Ecole Internationale des Sciences du Traitement de l'Information, Laboratoire de mathématiques, avenue du Parc, 95011 Cergy-Pontoise Cedex. E-mail: mohammed.mikou@eisti.eu

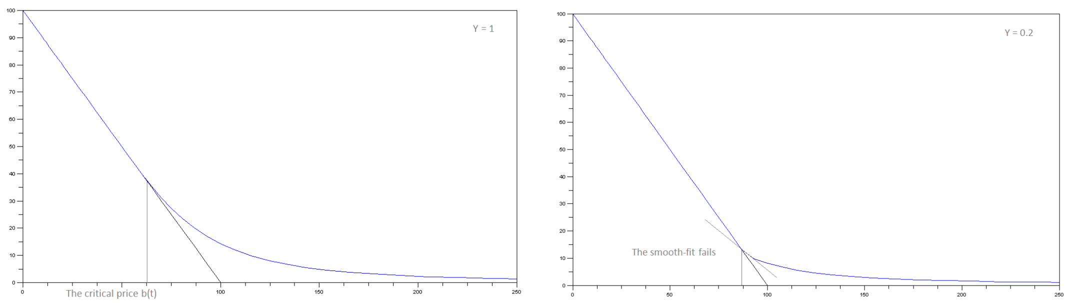

FIGURE 1: An example where the smooth-fit fails in the $CGMY$ model

without dividends. The picture is still unclear in the case of finite horizon, except for the jump-diffusion model (see [14]).

This paper deals with the smooth-fit property of the American put price in a general Lévy model when the underlying stock pays dividends at a continuous rate. We first show that the condition derived by Alili and Kyprianou is also sufficient for smooth-fit in the finite horizon case (see Theorem 4.1). This condition is typically not satisfied when the logarithmic stock price is a Lévy process with finite variation and positive drift. In this case, we prove that, for large maturities, the smooth-fit property is not satisfied (see Theorem 4.3). Under a slightly stronger condition, we derive a lower bound for the jump of the derivative (see Theorem 4.2 and Remark 4.1). In the perpetual case, we also propose a proof of a slightly weaker version of Alili and Kyprianou's result, based on the variational inequality.

The paper is organized as follows : in Section 2, we describe the exponential Lévy model with dividends and the basic properties of the perpetual and finite horizon American put price in this model. The third section is devoted to the properties of the free boundary in the infinite and finite horizon cases. The fourth section studies the smooth-fit property in the finite horizon case. The fifth section deals with the smooth-fit property in the perpetual case.

Acknowledgement

The authors are grateful to Goran Peskir for fruitful discussions and, especially, for providing the proof of Theorem 4.1.

2. The American put price in an exponential Lévy model

2.1. Lévy processes

A real Lévy process X is a real valued stochastic process, starting from 0, with stationary and independent increments. Without loss of generality, we may and shall assume that the sample paths of X are right continuous with left limits. The random process X can be interpreted as the independent superposition of a Brownian motion with drift and an infinite superposition of independent (compensated) Poisson processes. More precisely, the Lévy-Itô decomposition (see [13]) gives the following representation of X

$$X_t = \gamma t + \sigma B_t + Y_t, \tag{1}$$

$$Y_t = \tilde{X}_t + \lim_{\varepsilon \to 0} \tilde{X}_t^{\varepsilon}, \tag{2}$$

$$\tilde{X}_t = \int_0^t \int_{|x| \geq 1} x J_X(ds \times dx),\ \tilde{X}_t^{\varepsilon} = \int_0^t \int_{\varepsilon \leq |x| < 1} x \tilde{J}_X(ds \times dx),$$

where γ and σ are real numbers, $(B_t)_{t \geq 0}$ is a Brownian motion, J_X is a Poisson measure on $\mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})$ and $\tilde{J}_X$ is the compensated Poisson measure $\tilde{J}_X(dt, dx) = J(dt, dx) - dt\nu(dx)$. The measure ν is a positive Radon measure on $\mathbb{R}\backslash\{0\}$, called the Lévy measure of X, and it satisfies

$$\int_{\mathbb{R}} 1 \wedge x^2 \nu(dx) < \infty. \tag{3}$$

Notice that the terms in the right hand side of (1) are independent and the convergence of the last term is almost surely uniform with respect to t on $[0, T]$. The Lévy-Itô decomposition entails that the distribution of X is uniquely determined by (σ^2, γ, ν), called the characteristic triplet of the process X. The characteristic function of X_t, for $t \geq 0$, has the following Lévy-Khinchin representation (see [13])

$$\Phi_{X_t}(z) = \mathbb{E}\left(e^{iz.X_t}\right) = \exp[t\varphi(z)], \quad z \in \mathbb{C}, \tag{4}$$

with

$$\varphi(z) = -\frac{1}{2}\sigma^2 z^2 + i\gamma.z + \int (e^{izx} - 1 - izx\mathbf{1}_{|x| \leq 1})\nu(dx).$$

The Lévy process X is a Markov process and its infinitesimal generator is given by

$$\begin{aligned} lf(x) &= \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}(x) + \gamma\frac{\partial f}{\partial x}(x) \\ &\quad + \int \left(f(x+y) - f(x) - y\frac{\partial f}{\partial x}(x)\mathbf{1}_{|y|\leq 1}\right)\nu(dy), \end{aligned} \tag{5}$$

for every $f \in \mathcal{C}_b^2(\mathbb{R})$, where $\mathcal{C}_b^2(\mathbb{R})$ denotes the set of all bounded twice continuously differentiable functions with bounded derivatives. We complete this subsection by recalling two classical results about Lévy processes with finite variation (see [13]).

Proposition 2.1. *A Lévy process is of finite variation if and only if its characteristic triplet (σ^2, γ, ν) satisfies*

$$\sigma = 0 \qquad \textit{and} \qquad \int_{|x|\leq 1} |x|\nu(dx) < \infty. \tag{6}$$

Remark 1. It follows from Proposition 2.1 that, for a finite variation Lévy process with characteristic triplet (σ, ν, γ), we have

$$\lim_{a\to 0} a\nu((-\infty, -a]) = 0.$$

Theorem 2.1. *Let X be a finite variation Lévy process with characteristic triplet $(0, \gamma, \nu)$. We have*

$$\lim_{t\to 0^+} \frac{X_t}{t} = \gamma - \int_{|x|\leq 1} x\nu(dx) \quad a.s.$$

2.2. The exponential Lévy model

Let $(S_t)_{t\in[0,T]}$ be the price of a financial asset modeled as a stochastic process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}_0)$. We suppose that there exists an equivalent (risk neutral) probability $\mathbb{P}$ under which the discounted underlying process is a martingale. In the exponential Lévy model, the risk neutral dynamics of S_t under $\mathbb{P}$ is given by

$$S_t = S_0 e^{(r-\delta)t + X_t}, \tag{7}$$

where the interest rate r, the dividend rate δ are nonnegative constants and $(X_t)_{t\in[0,T]}$ is a real Lévy process with characteristic triplet (σ^2, γ, ν). We include r and δ in (7) for ease of notation.

Under $\mathbb{P}$, the discounted dividend adjusted stock price $(e^{-(r-\delta)t}S_t)_{t\in[0,T]}$ is a martingale, which is equivalent to the following two conditions on the characteristic triplet (see [6], Proposition 3.17)

$$\int_{|x|\geq 1} e^x \nu(dx) < \infty, \tag{8}$$

$$\text{and} \quad \frac{\sigma^2}{2} + \gamma + \int (e^x - 1 - x\mathbf{1}_{|x|\leq 1})\nu(dx) = 0. \tag{9}$$

We suppose that the conditions (8) and (9) are satisfied in the sequel. We deduce from (8) that the infinitesimal generator defined in (5) can be written as

$$lf(x) = \frac{\sigma^2}{2}\left(\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial x}\right)(x) + \int \left(f(x+y) - f(x) - (e^y - 1)\frac{\partial f}{\partial x}(x)\right)\nu(dy). \tag{10}$$

The stock price $(S_t)_{t\in[0,T]}$ is also a Markov process, we denote by L its infinitesimal generator. From (10), we deduce that

$$Lf(x) \quad = \quad \frac{x^2\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}(x) + x(r-\delta)\frac{\partial f}{\partial x}(x) + \mathcal{B}f(x), \tag{11}$$

where

$$\mathcal{B}f(x) = \int \nu(dy)\left(f(xe^y) - f(x) - x(e^y - 1)\frac{\partial f}{\partial x}(x)\right).$$

2.3. The American put price

In this model, the value at time t of an American put with maturity T and strike price K is given by

$$P_t = ess \sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}(e^{-r\tau}\psi(S_\tau) \mid \mathcal{F}_t),$$

where $\psi(x) = (K-x)_+$ and $\mathcal{T}_{t,T}$ denotes the set of stopping times satisfying $t \leq \tau \leq T$. Due to the Markov property (see [7] and [10]), we have

$$P_t = P(t, S_t),$$

where

$$P(t,x) = \sup_{\tau\in\mathcal{T}_{0,T-t}} \mathbb{E}(e^{-r\tau}\psi(S_\tau^x)), \tag{12}$$

with $S_t^x = xe^{(r-\delta)t+X_t}$. The following proposition follows easily from (12).

Proposition 2.2. *For $t \in [0,T]$, the function $x \mapsto P(t,x)$ is nonincreasing and convex on $[0,+\infty)$.*

For $x \in [0,+\infty)$, the function $t \mapsto P(t,x)$ is continuous and nonincreasing on $[0,T]$.

We recall the following proposition about the variational inequality related to the American put in the exponential Lévy model (see [9] Theorem 3.1).

Theorem 2.2. *The distribution $(\partial_t + L - r)P$ is a nonpositive measure on $(0,T)\times\mathbb{R}$, and, on the open set $\{(t,x)\in(0,T)\times\mathbb{R} \mid P(t,x) > \psi(x)\}$, we have $(\partial_t + L - r)P = 0$.*

2.4. The perpetual American put price

The perpetual American put price is an American put price with maturity T equal to infinity. So, as previously, the value at time t of a perpetual American put with strike price K is given by

$$P_t^* = ess \sup_{\tau\in\mathcal{T}_{t,\infty}} \mathbb{E}(e^{-r\tau}\psi(S_\tau) \mid \mathcal{F}_t),$$

where $\psi(x) = (K-x)_+$ and $\mathcal{T}_{t,\infty}$ denotes the set of stopping times satisfying $t \le \tau$. Due to the fact that the process X has stationary and independent increments, it can be proved that

$$P_t^* = P^*(S_t),$$

where

$$P^*(x) = \sup_{\tau\in\mathcal{T}_{0,\infty}} \mathbb{E}(e^{-r\tau}\psi(S_\tau^x)), \tag{13}$$

with $S_t^x = xe^{(r-\delta)t+X_t}$. The following proposition follows easily from (13).

Proposition 2.3. *The function $x \mapsto P^*(x)$ is nonincreasing and convex on $[0,+\infty)$.*

As in the finite horizon case, the perpetual American put in the exponential Lévy model satisfies the following variational inequality (see [9], Theorem 3.1).

Theorem 2.3. *The distribution $(L-r)P^*$ is a nonpositive measure on $\mathbb{R}^+$, and, on the open set $\{x\in\mathbb{R}^+ \mid P^*(x) > \psi(x)\}$, we have $(L-r)P^* = 0$.*

3. Properties of the free boundary

3.1. The finite horizon case

Throughout this subsection we will assume that at least one of the following conditions is satisfied:

$$\sigma \neq 0, \quad \nu((-\infty,0)) > 0 \quad \text{or} \quad \int_{(0,+\infty)} (x\wedge 1)\nu(dx) = +\infty. \tag{14}$$

Under this assumption, we have (as observed in [9])

$$\forall t \in [0,T), \quad \forall x \in [0,+\infty), \quad P(t,x) > 0.$$

We will also assume that $r > 0$. The *critical price* at time $t \in [0,T)$ is defined by

$$b(t) = \inf\{x \geq 0 \mid P(t,x) > \psi(x)\}.$$

Note that, since $t \mapsto P(t,x)$ is nonincreasing, the function $t \mapsto b(t)$ is nondecreasing. It follows from (14) that $b(t) \in [0,K)$. We obviously have $P(t,x) = \psi(x)$ for $x \in [0,b(t))$ and also for $x = b(t)$, due to the continuity of P and ψ. We also deduce from the convexity of $x \mapsto P(t,x)$ that

$$\forall t \in [0,T), \quad \forall x > b(t), \quad P(t,x) > \psi(x).$$

Then, the continuation region C can be written as

$$C = \{(t,x) \in [0,T) \times [0,+\infty) \mid x > b(t)\}.$$

The graph of b is called the *exercise boundary* or *free boundary.*

We recall the following properties of $t \mapsto b(t)$ (see [9] section 4).

Theorem 3.1. *The function $t \mapsto b(t)$ is continuous and $b(t) > 0$ on $[0,T)$.*

We also recall from [9] the following result, which characterizes the limit of the critical price $b(t)$ as t approaches T.

Theorem 3.2. *If $\int (e^x - 1)_+ \nu(dx) \leq r - \delta$, we have $\lim_{t\to T} b(t) = K$.*

If $\int (e^x - 1)_+ \nu(dx) > r - \delta$, we have $\lim_{t\to T} b(t) = \xi$, where ξ is the unique real number in the interval $(0,K)$ such that

$$\varphi(\xi) = rK,$$

where φ is the function defined by

$$\varphi(x) = \overline{\varphi}(x) + \delta x, \quad \text{and} \quad \overline{\varphi}(x) = \int (xe^y - K)_+ \nu(dy), \quad x \in (0,K).$$

3.2. The perpetual case

Assume that $P^* > 0$. The *critical price* in this case is defined by

$$b^* = \inf\{x \geq 0 \mid P^*(x) > \psi(x)\}.$$

Note that, since $x \mapsto P^*(x)$ is nonincreasing convex function, and $P^* > 0$, we have $b^* \in (0, K)$ (one can prove that $b^* < K$ by the same argument as in [9] page 574). In this case, the continuation region C^* can be written as

$$C^* = \{x \in [0, +\infty) \mid x > b^*\} = (b^*, +\infty).$$

4. The smooth-fit principle in an exponential Lévy model

4.1. The finite horizon put

Throughout this subsection we will assume that $r > 0$.

To a fixed level $x \in \mathbb{R}$ we associate the first strict passage time τ_x^- below x for the process $(\log \frac{S_t}{S_0})$ i.e.

$$\tau_x^- = \inf\{t \in (0, T) \mid (r - \delta)t + X_t < x\},$$

with the convention that $\inf \emptyset = T$. Recall that 0 is regular for $(-\infty, 0)$ if and only if $\mathbb{P}(\tau_0^- = 0) = 1$. Denote

$$d := r - \delta - \int (e^x - 1)\nu(dx).$$

Note that, if X has finite variation, we have, from Theorem 2.1,

$$\lim_{t \to 0} \frac{(r - \delta)t + X_t}{t} = d.$$

So that d appears as the drift of the logarithmic stock price. The following proposition is a summary of what is known from the literature (see [1], Proposition 7, or [8], Theorem 6.5).

Proposition 4.1. *The point* 0 *is regular for* $(-\infty, 0)$ *if and only if one of the following three conditions holds*

1. *X has finite variation and $d < 0$,*

2. X *has finite variation,* $d = 0$ *and*

$$\int_{-1}^{0^-} \frac{|x|\nu(dx)}{\int_0^{|x|} \nu(y,+\infty)dy} = +\infty,$$

3. X *has infinite variation.*

The second case was added to the class of processes exhibiting regularity of 0 for the lower half line in Bertoin [4] and, for the other cases, one refers to the discussion at the beginning of Bertoin [3], section VI.3.

The following theorem gives a sufficient condition for the smooth-fit property.

Theorem 4.1. *If* 0 *is regular for* $(-\infty, 0)$*, then the smooth-fit principle is satisfied.*

The proof of this result was given to us by G. Peskir. Note that the idea goes back to J. Bather [2] in the case of Brownian motion (see [12] Section 9.2). In fact, it was conjectured in [1] that regularity is a necessary and sufficient condition for smooth fit in the case of strong Markov processes. This conjecture was disproved for diffusions in [11].

Proof of Theorem 4.1. Suppose that 0 is regular for $(-\infty, 0)$ and fix $t \in [0, T)$. We want to show that $x \mapsto P(t, x)$ is differentiable at $b(t)$ and that $\partial_x P(t, b(t)) = \psi'(b(t))$ (smooth-fit), where $b(t) \in (0, K)$ is the critical price. To simplify the proof we consider $t = 0$. First note that, for $h > 0$,

$$\frac{P(0, b(0) + h) - P(0, b(0))}{h} \geq \frac{\psi(b(0) + h) - \psi(b(0))}{h},$$

since $P \geq \psi$ and $P(t, b(0)) = \psi(b(0))$. So, it follows that

$$\liminf_{h \to 0^+} \left(\frac{P(0, b(0) + h) - P(0, b(0))}{h} \right) \geq \psi'(b(0)). \tag{15}$$

Next we consider the optimal stopping time related to $P(0, b(0) + h)$

$$\begin{aligned}
\tau_h &= \inf\{t \in [0, T) \mid S_t^{b(0)+h} < b(t)\} \\
&= \inf\left\{t \in [0, T) \mid (r - \delta)t + X_t \leq \ln\left(\frac{b(t)}{b(0) + h}\right)\right\} \\
&\leq \inf\left\{t \in [0, T) \mid (r - \delta)t + X_t \leq \ln\left(\frac{b(0)}{b(0) + h}\right)\right\} \\
&=: \tau_h^*,
\end{aligned}$$

where the inequality follows from the fact that $(b(t))_{t\in[0,T)}$ is nondecreasing. Recall that $\mathbb{P}(\tau_0^- = 0) = 1$. On the set $\{\tau_0^- = 0\}$, given a fixed $t \in (0,T)$, there exists $s \in [0,t]$ such that $(r-\delta)s + X_s < 0$. For h small enough, we have $(r-\delta)s + X_s < \ln\left(\frac{b(0)}{b(0)+h}\right)$, so that $\tau_h^* \leq s$. Therefore, $\lim_{h\to 0}\tau_h^* \leq t$. Since t is arbitrary, we deduce that $\tau_h^* \to 0$ almost surely when h goes to 0. Hence

$$\lim_{h\to 0}\tau_h = 0,$$

almost surely. Moreover, since

$$P(0,b(0)) \geq \mathbb{E}\left(e^{-r\tau_h}\psi(b(0)e^{(r-\delta)\tau_h + X_{\tau_h}})\right),$$

we have

$$\frac{P(0,b(0)+h) - P(0,b(0))}{h} = \frac{\mathbb{E}\left(e^{-r\tau_h}\psi(b(0)e^{(r-\delta)\tau_h+X_{\tau_h}})\right) - P(0,b(0))}{h}$$
$$\leq \mathbb{E}\left(e^{-r\tau_h}\frac{\psi((b(0)+h)e^{(r-\delta)\tau_h+X_{\tau_h}}) - \psi(b(0)e^{(r-\delta)\tau_h+X_{\tau_h}})}{h}\right).$$

Since ψ is continuously differentiable in a neighborhood of $b(0)$, we have

$$\lim_{h\to 0}\frac{\psi((b(0)+h)e^{(r-\delta)\tau_h+X_{\tau_h}}) - \psi(b(0)e^{(r-\delta)\tau_h+X_{\tau_h}})}{h} = \psi(b(0)).$$

Then, using the Lipchitz continuity of ψ, by dominated convergence we get,

$$\limsup_{h\to 0}\left(\frac{P(0,b(0)+h) - P(0,b(0))}{h}\right) \leq \psi'(b(0)). \tag{16}$$

Combining (15) and (16), we deduce the theorem.

It is well known that if X has infinite variation, 0 is regular (see Theorem 4.1), so that we have smooth-fit. We will now assume that X has finite variation. Denote $d^+ := r - \delta - \int(e^x-1)_+\nu(dx)$. Note that $d = d^+ + \int(e^y-1)_-\nu(dy)$. Recall that, if $d < 0$, 0 is regular for $(-\infty,0)$, so that the smooth-fit property is satisfied. We will prove below (see Theorem 4.3) that if $d > 0$, the smooth fit property cannot be satisfied, at least for large maturities. Under the stronger condition $d_+ \geq 0$, we have a more precise result.

Theorem 4.2. *If X has finite variation and $d^+ \geq 0$, we have*

$$\partial_x^+ P(t,b(t)) \neq \partial_x^- P(t,b(t)),$$

for every $t \in (0, T)$.

Proof. Let $t \in [0, T)$, $x \geq 0$ and suppose that X is a finite variation Lévy process such that $d^+ \geq 0$. In this case, the infinitesimal generator in (11) can be written as

$$Lf(x) = x\left[r - \delta - \int (e^y - 1)\nu(dy)\right]\frac{\partial f}{\partial x}(x) + \int \nu(dy)[f(xe^y) - f(x)], \quad (17)$$

for all $f \in \mathcal{C}_b^1(\mathbb{R})$, where $\mathcal{C}_b^1(\mathbb{R})$ denotes the set of all bounded $\mathcal{C}^1$ functions with bounded derivative. Recall that from Theorem 2.2, $(\partial_t + L - r)P = 0$ in the sense of distributions on the continuation region C. So $(L-r)P \geq 0$ since $t \mapsto P(t, x)$ is nonincreasing. Also, $x \mapsto P(t, x)$ is convex, so its right derivative $\partial_x^+ P$ is bounded and right continuous. Then, from (17) we deduce

$$\begin{aligned} b(t)\left[r - \delta - \int (e^y - 1)\nu(dy)\right]\partial_x^+ P(t, b(t)) + \int \nu(dy)[P(t, b(t)e^y) - P(t, b(t))] \\ \geq rP(t, b(t)). \quad (18)\end{aligned}$$

Note that $P(t, b(t)) = \psi(b(t)) = K - b(t)$, $P(t, b(t)e^y) = \psi(b(t)e^y) = K - b(t)e^y$ if $y < 0$ and $P(t, b(t)e^y) \leq P(t, b(t))$ if $y > 0$. So, from (18) we get

$$\begin{aligned} b(t)\left[r - \delta - \int (e^y - 1)\nu(dy)\right]\partial_x^+ P(t, b(t)) &\geq -\int_{y<0} \nu(dy)[(K - b(t)e^y) - (K - b(t))] \\ &\quad + r(K - b(t)) \\ &= b(t)\int_{y<0} (e^y - 1)\nu(dy) + r(K - b(t)) \\ &= -b(t)\int (e^y - 1)_- \nu(dy) + r(K - b(t)) \quad (19)\end{aligned}$$

Note also that $d = d^+ + \int (e^y - 1)_- \nu(dy)$. If we had $d = 0$, we would deduce that $\nu(-\infty, 0) = 0$ and (19) would become

$$r(K - b(t)) \leq b(t)\int (e^y - 1)_- \nu(dy) = 0,$$

which is in contradiction with the fact that $r > 0$ and $b(t) \in (0, K)$. Therefore, we must have $d > 0$, and (19) now gives

$$\begin{aligned} \partial_x^+ P(t, b(t)) &\geq \frac{-\int (e^y - 1)_- \nu(dy) + r(\frac{K}{b(t)} - 1)}{r - \delta - \int (e^y - 1)_+ \nu(dy) + \int (e^y - 1)_- \nu(dy)} \quad (20) \\ &> -1. \end{aligned}$$

We conclude the theorem since $\partial_x^- P(t, b(t)) = \psi'(b(t)) = -1$.

Remark 4.1. If $d^+ > 0$, we can see from (20) the following explicit lower bound for the jump of the derivative

$$\partial_x^+ P(t, b(t)) + 1 \quad \geq \quad \frac{d^+}{d} > 0,$$

for every $t \in [0, T]$.

We will now prove that if d is positive the smooth-fit property fails, at least for large values of the maturity.

Theorem 4.3. *If X is a finite variation Lévy process and $d > 0$, and if $T > \frac{K}{db^*}$, where b^* is the critical price of the perpetual put, there exists $t \in [0, T)$ such that*

$$\partial_x^+ P(t, b(t)) > -1.$$

We first show the following lemma.

Lemma 4.1. *Asume X is a finite variation Lévy process and $d > 0$. Fix $t \in [0, T)$ and assume $\partial_+ P(t, b(t)) = \partial_- P(t, b(t))$. Then, we have*

$$\limsup_{h \to 0} \frac{b(t+h) - b(t)}{h} \geq b^* d,$$

where b^* is the critical price of the perpetual put.

Proof. To simplify the proof we consider the case $t = 0$. Let $h > 0$ and suppose that the smooth-fit property is satisfied at $t = 0$. Let τ_h be the optimal stopping time related to $P(0, b(0) + h)$,

$$\begin{aligned} \tau_h &= \inf\{t \in [0, T) \mid S_t^{b(0)+h} < b(t)\} \\ &= \inf\left\{t \in [0, T) \mid (r - \delta)t + X_t < \ln\left(\frac{b(t)}{b(0) + h}\right)\right\}, \end{aligned}$$

with the convention $\inf \emptyset = T$. Note that τ_h is nonnegative and nondecreasing with respect to h. We denote by τ_0 the limit of τ_h when h goes to 0. Note also that by the zero-one law, we have $\mathbb{P}(\tau_0 = 0) \in \{0, 1\}$. We now discuss both cases.

Case 1 : $\mathbb{P}(\tau_0 = 0) = 0$.

Note that $\tau_0 \leq \tau_h$ and

$$\begin{aligned} P(0, b(0) + h) &= \mathbb{E}\left(e^{-r\tau_h} \psi((b(0) + h) e^{(r-\delta)\tau_h + X_{\tau_h}})\right) \\ &\geq \mathbb{E}\left(e^{-r\tau_0} \psi((b(0) + h) e^{(r-\delta)\tau_0 + X_{\tau_0}})\right). \end{aligned}$$

So, by letting h go to 0, we have

$$P(0,b(0)) = \mathbb{E}\left(e^{-r\tau_0}\psi(b(0)e^{(r-\delta)\tau_0+X_{\tau_0}})\right). \tag{21}$$

Then, using the convexity of ψ, we get

$$\begin{aligned}
&\frac{P(0,b(0)+h)-P(0,b(0))}{h} \geq \\
&\quad \mathbb{E}\left(e^{-r\tau_0}\frac{\psi((b(0)+h)e^{(r-\delta)\tau_0+X_{\tau_0}})-\psi(b(0)e^{(r-\delta)\tau_0+X_{\tau_0}})}{h}\right) \\
&\geq \mathbb{E}\left(e^{-r\tau_0}\psi'_d(b(0)e^{(r-\delta)\tau_0+X_{\tau_0}})e^{(r-\delta)\tau_0+X_{\tau_0}}\right) \\
&= -\mathbb{E}\left(e^{-\delta\tau_0+X_{\tau_0}}\mathbf{1}_{\{(r-\delta)\tau_0+X_{\tau_0}<\ln(\frac{K}{b(0)})\}}\right).
\end{aligned} \tag{22}$$

Now, suppose $\delta > 0$. Since $\tau_0 > 0$ a.s. and e^X is a martingale, we obviously have

$$\begin{aligned}
\liminf_{h\to 0}\frac{P(0,b(0)+h)-P(0,b(0))}{h} &\geq -\mathbb{E}\left(e^{-\delta\tau_0+X_{\tau_0}}\right) \\
&= -\mathbb{E}\left(e^{-\delta\tau_0}\right) \\
&> -1.
\end{aligned} \tag{23}$$

On the other hand, if $\delta = 0$, (21) becomes

$$\begin{aligned}
P(0,b(0)) &= \mathbb{E}\left(e^{-r\tau_0}\psi(b(0)e^{r\tau_0+X_{\tau_0}})\right) \\
&= K\mathbb{E}\left(e^{-r\tau_0}\mathbf{1}_{\{r\tau_0+X_{\tau_0}\leq\ln(\frac{K}{b(0)})\}}\right) - b(0)\mathbb{E}\left(e^{X_{\tau_0}}\mathbf{1}_{\{r\tau_0+X_{\tau_0}\leq\ln(\frac{K}{b(0)})\}}\right).
\end{aligned}$$

Since $P(0,b(0)) = K - b(0)$, we derive

$$K\left[1-\mathbb{E}\left(e^{-r\tau_0}\mathbf{1}_{\{r\tau_0+X_{\tau_0}\leq\ln(\frac{K}{b(0)})\}}\right)\right] = b(0)\left[1-\mathbb{E}\left(e^{X_{\tau_0}}\mathbf{1}_{\{r\tau_0+X_{\tau_0}\leq\ln(\frac{K}{b(0)})\}}\right)\right].$$

Note that the left hand side is positive because $\tau_0 > 0$ a.s. and $r > 0$. Therefore $\mathbb{E}\left(e^{X_{\tau_0}}\mathbf{1}_{\{r\tau_0+X_{\tau_0}\leq\ln(\frac{K}{b(0)})\}}\right) < 1$ and (22) gives

$$\begin{aligned}
\liminf_{h\to 0}\frac{P(0,b(0)+h)-P(0,b(0))}{h} &\geq -\mathbb{E}\left(e^{X_{\tau_0}}\mathbf{1}_{\{r\tau_0+X_{\tau_0}<\ln(\frac{K}{b(0)})\}}\right) \\
&> -1.
\end{aligned} \tag{24}$$

We deduce from (23) and (24) that the smooth-fit fails for every $\delta \geq 0$.

Case 2 : $\mathbb{P}(\tau_0 = 0) = 1$.

We then have $\lim_{h\to 0}\tau_h = 0$ a. s.. In particular $\tau_h < T$ for h close to 0 and from the definition of τ_h we have

$$\begin{aligned}(r-\delta)\tau_h + X_{\tau_h} &\leq \ln(b(\tau_h)) - \ln(b(0)+h)\\ &\leq \ln(b(\tau_h)) - \ln(b(0)).\end{aligned}$$

Therefore, using Theorem 2.1 and (9), we have

$$\begin{aligned}r-\delta+\lim_{h\to 0}\left(\frac{X_{\tau_h}}{\tau_h}\right) &= r-\delta+\gamma-\int_{\{|y|\leq 1\}} y\nu(dy)\\ &= d\\ &\leq \liminf_{h\to 0}\frac{\ln(b(\tau_h))-\ln(b(0))}{\tau_h}\\ &\leq \limsup_{t\to 0}\frac{\ln(b(t))-\ln(b(0))}{t}\\ &\leq \frac{1}{b(0)}\limsup_{t\to 0}\frac{b(t)-b(0)}{t}\\ &\leq \frac{1}{b^*}\limsup_{t\to 0}\frac{b(t)-b(0)}{t}.\end{aligned}$$

Hence

$$\limsup_{h\to 0}\frac{b(h)-b(0)}{h}\geq b^* d.$$

Proof of Theorem 4.3. Suppose that the smooth-fit is satisfied for every $t\in[0,T)$. Recall that $t\mapsto b(t)$ is a continuous nondecreasing function on $[0,T)$. So b is almost everywhere differentiable on $[0,T)$ and from Lemma 4.1 we have

$$b'(t)\geq b^* d>0 \quad \text{a.e. on} \quad [0,T).$$

Therefore, by integrating the last inequality, we get $K\geq b(t)-b(0)\geq db^*t$. Finally, we get a contradiction for $T>\frac{K}{db^*}$.

4.2. The perpetual put

The following Theorem can be proved by the same argument as in the finite horizon case.

Theorem 4.4. *If 0 is regular for $(-\infty,0)$, then the smooth-fit principle is satisfied.*

We also have the following result. This result was already proved by Alili and Kyprianou [1]. Our contribution is only to give a proof based on the variational inequality.

Theorem 4.5. *If X has finite variation and $d > 0$, the smooth-fit principle is not satisfied.*

Proof. Suppose that the smooth-fit principle is satisfied and $d > 0$. From Theorem 2.3 and (17) we have, for $x \geq b^*$,

$$x\left(r-\delta-\int(e^y-1)\nu(dy)\right)\partial_x P^*(x) + \int (P^*(xe^y)-P^*(x))\,\nu(dy) - rP^*(x) = 0. \tag{25}$$

In particular, for $x = b^*$, we deduce from $\partial_x P^*(b^*) = -1$ and $P^*(b^*) = K - b^*$

$$b^*\delta + \int (P^*(b^*e^y) - (K - b^*e^y))\,\nu(dy) = rK. \tag{26}$$

Note that (25) can be written as

$$x\delta + xd(\partial_x P^*(x) + 1)\int (P^*(xe^y) - P^*(x) + x(e^y-1))\,\nu(dy) = r(P^*(x)+x). \tag{27}$$

subtracting (26) from (27), we get

$$\begin{aligned}&(x-b^*)\delta + xd(\partial_x P^*(x)+1)\\ &+\int (P^*(xe^y)-P^*(x)+x(e^y-1)-(P^*(b^*e^y)-(K-b^*e^y)))\,\nu(dy)\\ &= r(P^*(x)-(K-x)).\end{aligned} \tag{28}$$

For $y \in \mathbb{R}$, let f_y the function defined by $f_y(x) = P^*(xe^y) - P^*(x) + x(e^y - 1)$. Then (28) becomes

$$(x-b^*)\delta + xd(\partial_x P^*(x)+1) + \int (f_y(x) - f_y(b^*))\,\nu(dy) = r(P^*(x)-(K-x)). \tag{29}$$

We see from (25) that $\partial_x P^*$ is continuous on (b^*, ∞), so $f_y \in \mathcal{C}^1(b^*, \infty)$ and

$$f_y'(x) = e^y(\partial_x P^*(xe^y)+1) - (\partial_x P^*(x)+1) \geq 0$$

if $y \geq 0$, because $x \mapsto P^*(x)$ is convex. So, for $x > b^*$ and $y > 0$,

$$f_y(x) - f_y(b^*) \geq 0.$$

Also, for $y \leq \ln(\frac{b^*}{x})$, by the mean value theorem we have

$$f_y(x) - f_y(b^*) = f_y'(\theta)(x-b^*), \tag{30}$$

for some $\theta \in (b^*, x)$, where

$$f'_y(\theta) = e^y(\partial_x P^*(\theta e^y) + 1) - (\partial_x P^*(\theta) + 1) \qquad (31)$$
$$\geq -(\partial_x P^*(\theta) + 1)$$
$$\geq -(\partial_x P^*(x) + 1).$$

From (30) and (31) we get

$$\int_{\{y \leq \ln(\frac{b^*}{x})\}} (f_y(x) - f_y(b^*))\, \nu(dy) \geq -(\partial_x P^*(x) + 1)(x - b^*)\nu\left((-\infty, \ln(\frac{b^*}{x}))\right)$$
$$= (\partial_x P^*(x) + 1) A_x^{b^*} B_x^{b^*}, \qquad (32)$$

where $A_x^{b^*} = \frac{1}{x}\frac{x-b^*}{\ln(x)-\ln(b^*)}$ and $B_x^{b^*} = \ln(\frac{x}{b^*})\nu\left((-\infty, \ln(\frac{b^*}{x}))\right)$. Note that $\lim_{x\to b^*} A_x^{b^*} = \frac{b^*}{x}$ and $\lim_{x\to b^*} B_x^{b^*} = 0$ (see Remark 1). Therefore, from (32) we can choose some $x_1 > b^*$ such that for $x \in (b^*, x_1)$

$$\int_{\{y \leq \ln(\frac{b^*}{x})\}} (f_y(x) - f_y(b^*))\, \nu(dy) \geq -\frac{xd}{4}(\partial_x P^*(x) + 1). \qquad (33)$$

Now, let $y \in (\ln(\frac{b^*}{x}), 0)$. Then, we have $f_y(b^*) = 0$ and

$$f_y(x) = P^*(xe^y) - P(x) + x(e^y - 1) \qquad (34)$$
$$= P^*(xe^y) - P(x) - x(e^y - 1)\partial_x P^*(x) + x(e^y - 1)(\partial_x P^*(x) + 1)$$
$$\geq x(e^y - 1)(\partial_x P^*(x) + 1),$$

since $x \mapsto P(t, x)$ is convex. We see from (6) that $y \mapsto e^y - 1$ is ν-integrable, so there exists some $x_2 > b^*$ such that for $x \in (b^*, x_2)$

$$\int_{\{\frac{b^*}{x}) < y < 0\}} (e^y - 1)\nu(dy) \geq -\frac{d}{4}. \qquad (35)$$

Therefore, from (34) and (35) we check that for $x \in (b^*, x_2)$

$$\int_{\{\ln(\frac{b^*}{x}) < y < 0\}} (f_y(x) - f_y(b^*))\nu(dy) \geq -\frac{xd}{4}(\partial_x P^*(x) + 1). \qquad (36)$$

On the other hand, the function f defined by $f(x) = P^*(x) - (K - x)$ is continuously differentiable and satisfies $f(b^*) = 0$ and $f'(x) = \partial_x P^*(x) + 1 \geq 0$. By the mean value theorem we have

$$f(x) - f(b^*) = (\partial_x P^*(\theta) + 1)(x - b^*),$$

for some $\theta \in (b^*, x)$. Therefore

$$f(x) - f(b^*) \leq x(\partial_x P^*(x) + 1)(1 - \frac{b^*}{x}),$$

since $x \mapsto P(t, x)$ is convex. So, there exists some $x_3 > b^*$ such that, for $x \in (b^*, x_3)$,

$$P^*(x) - (K - x) \leq \frac{xd}{4}(\partial_x P^*(x) + 1). \tag{37}$$

Denote $x_0 = x_1 \wedge x_2 \wedge x_3$. Recombining (29), (33), (36) and (37), we get for $x \in (b^*, x_0)$

$$(x - b^*)\delta + \frac{xd}{4}(\partial_x P^*(x) + 1) + \int_{\{y>0\}} (f_y(x) - f_y(b^*))\, \nu(dy) \geq 0.$$

This contradicts (30).

References

[1] Alili, L. and Kyprianou, A. E. (2005). Some Remarks on First Passage of Lévy Processes, the American Put and Pasting Principles. *Ann. Appl. Probab.* **15**, 2062-2080.

[2] Bather, J. (1970). Optimal Stopping Problems for Brownian Motion. *Adv. in Appl. Probab.* **2**, 259-286.

[3] Bertoin, J. (1996).*Lévy Processes.* Cambridge University Press, Reprint edition.

[4] Bertoin, J. (1997). Regularity of the Half-Line for Lévy Processes. *Bull. Sci. Math.* **121**,345-354.

[5] Carr, P., Geman, H., Mandan, D., B., Yor, M. (2002). The Fine Structure of Returns: An Empirical Investigation. *Journal of Business* **75**, 2, 305-332.

[6] Cont, R. and Tankov, P. (2004).*Financial Modelling with Jump Prosses.* Chapman and Hall/CRC.

[7] El Karoui, N., Lepeltier, J.P., Millet, A. (1992). A Probabilistic Approach to the Reduite in Optimal Stopping. *Probab. Math. Statist.* **13**, 1, 97-121.

[8] Kyprianou, A. (2006).*Introductory lectures on uctuations of Lévy processes with applications*. Springer-Verlag, Berlin.

[9] Lamberton, D. and Mikou, M. (2008). The Critical Price for the American Put in an Exponential Lévy Model, *Finance and Stochastics* **12**, 561-581.

[10] Mikou, M. (2009). *Options américaines dans le modèle exponentiel de Lévy.* Doctoral Thesis, Université Paris-Est.

[11] Peskir, G. (2007). Principle of smooth fit and diffusions with angles. *Stochastics* **79**, 293-302.

[12] Peskir, G. and Shiryaev, A. (2006). *Optimal Stopping and Free-Boundary Problems.* Bikhauser Verlag: Basel, Boston, Berlin.

[13] Sato, K. (1999). *Lévy Processes and Infinity Divisible Distributions.* Cambrigge University Press, Cambridge, UK.

[14] Zhang, X.L. (1997). Numerical Analysis of American Option Pricing in a Jump-Diffusion Model. *Math. Oper. Res.* **22**, 668-690.