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Eigenvalue variance bounds for Wigner random matrices

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Abstract. *This work is concerned with finite range bounds on the variance of individual eigenvalues of random Wigner matrices, both in the bulk and at the edge. Relying on the GUE example, which needs to be investigated first, the main bounds are extended to Wigner Hermitian matrices by means of the Tao and Vu Four Moment Theorem and recent localization results by Erdős, Yau and Yin. The case of real Wigner matrices is obtained from interlacing formulas.*

Random matrices were first introduced by Wigner in the fifties to study eigenvalues of infinite-dimensional operators in statistical physics (see [12]) and then propagated to various fields of mathematics involved in the study of spectra of random matrices. Under suitable symmetry assumptions, the asymptotic properties of the eigenvalues of a random matrix were soon conjectured to be universal, in the sense they do not depend on the individual distribution of the matrix entries. This opened the way to numerous developments on the asymptotics of various statistics of the eigenvalues of random matrices, such as for example the global behavior of the spectrum, the spacings between the eigenvalues in the bulk of the spectrum or the behavior of the extreme eigenvalues. Two main different types of models are considered. In the invariant matrix models, the matrix law is unitary invariant and the eigenvalue joint distribution can be written explicitly in terms of a potential V . In the Wigner models, the matrix entries are independent (up to symmetry conditions). The case where the entries are Gaussian is the only model belonging to both types. The joint distribution of the eigenvalues is thus explicitly known and the previous statistics have been completely studied. One of the main directions in random matrix theory in the past decades was to prove that these asymptotic behaviors were the same for non-Gaussian matrices (see for instance [1], [3] and [14]).

However, in several fields such as computer science or statistics for example, asymptotic statements are often not enough, and more quantitative finite range results are required. Several recent developments have thus been concerned with non-asymptotic random matrix theory towards quantitative bounds (for instance on the probability for a certain event to occur) which are valid for all N , where N is the size of the given matrix. See for example [21] for an introduction to some problems considered in non-asymptotic random matrix theory.

In this paper, we investigate in this respect variance bounds on the eigenvalues of families of Wigner random matrices. Wigner matrices are Hermitian or real symmetric matrices M_N such that, if M_N is complex, for $i < j$, the real and imaginary parts of $(M_N)_{ij}$ are independent and identically distributed (iid) with mean 0 and variance $\frac{1}{2}$, $(M_N)_{ii}$ are iid, with mean 0 and variance 1. In the real case, $(M_N)_{ij}$ are iid, with mean 0 and variance 1 and $(M_N)_{ii}$ are iid, with mean 0 and variance 2. In both cases, set $W_N = \frac{1}{\sqrt{N}}M_N$. An important example of Wigner matrices is the case where the entries are Gaussian. If M_N is complex, then it belongs to the so-called Gaussian Unitary Ensemble (GUE). If it is real, it belongs to the Gaussian Orthogonal Ensemble (GOE).

The matrix W_N has N real eigenvalues $\lambda_1 \leq \dots \leq \lambda_N$. In the Gaussian case, the joint law of the eigenvalues is known, allowing for complete descriptions of their limiting behavior both in the global and local regimes (cf. for example [1], [3] and [14]).

Among universality results, at the global level, the classical Wigner's theorem states that the empirical distribution $\frac{1}{N} \sum_{j=1}^N \delta_{\lambda_j}$ on the eigenvalues of W_N converges weakly almost surely to the semi-circle law $d\rho_{sc}(x) = \frac{1}{2\pi} \sqrt{4 - x^2} 1_{[-2,2]}(x) dx$ (see for example [3] for a proof in the more general setting). This gives the global asymptotic behavior of the spectrum. An interesting question concerns the rate of convergence of this empirical distribution. Under the hypothesis of an exponential decay of the matrix entries, Götze and Tikhomirov recently showed that, with high probability,

$$\sup_{x \in \mathbb{R}} \left| \frac{1}{N} \mathcal{N}_x - G(x) \right| \leq \frac{(\log N \log \log N)^c}{N}$$

for some universal constant $c > 0$, where \mathcal{N}_x is the number of eigenvalues which are in $(-\infty, x]$ (i.e. \mathcal{N}_x is the eigenvalue counting function) and G is the distribution function of the semicircular law (see [8]).

Several information on individual eigenvalues can be deduced from Wigner's theorem. Indeed, according to the Glivenko-Cantelli theorem (see for example [5]), the normalized eigenvalue function $\frac{1}{N} \mathcal{N}_x$ converges uniformly on \mathbb{R} almost surely to the distribution function of the semicircular law G (with no more assumptions on the matrix entries). Define, for all $1 \leq j \leq N$, the theoretical location of the j^{th} eigenvalue γ_j by $\int_{-2}^{\gamma_j} d\rho_{sc}(x) = \frac{j}{N}$. Using the fact that $\frac{1}{N} \mathcal{N}_{\lambda_j} = \frac{j}{N} = G(\gamma_j)$ and crude bounds on the semicircular density function, it can be shown that almost surely $\lambda_j - \gamma_j \rightarrow 0$ uniformly for $\eta N \leq j \leq (1 - \eta)N$ (for a fixed $\eta > 0$). At the edge of the spectrum, Wigner's theorem is not enough to deduce a convergence for the smallest or largest eigenvalues. Bai and Yin proved in [2] with assumptions on higher moments that almost surely these eigenvalues converge to their theoretical locations, which means that $\lambda_N \rightarrow 2$ and $\lambda_1 \rightarrow -2$ almost surely (see also [3] for a proof of this theorem).

At the fluctuation level, eigenvalues inside the bulk and at the edge of the spectrum do not have the same behavior. Tracy and Widom showed in [19] that the largest eigenvalue fluctuates around 2 according to the so-called Tracy-Widom law F_2 . Namely,

$$N^{2/3}(\lambda_N - 2) \rightarrow F_2$$

in distribution as N goes to infinity. They proved this result for Gaussian matrices only, later extended to families of non-Gaussian Wigner matrices by Soshnikov (see [15]). Recent results by Tao and Vu ([18]) and by Erdős, Yau and Yin ([6]) provide alternate proofs of this fact for larger families of Wigner matrices. According to this asymptotic property, the variance of the largest eigenvalue λ_N is thus of the order of $N^{-4/3}$. In the bulk, Gustavsson proved in [10] that, for a fixed $\eta > 0$, for all $\eta N \leq j \leq (1 - \eta)N$,

$$\frac{\lambda_j - \gamma_j}{\sqrt{\frac{\log N}{8(4 - \gamma_j^2)N^2}}} \rightarrow \mathcal{N}(0, 1) \quad (1)$$

in distribution as N goes to infinity. This result was extended by Tao and Vu [17] to large families of non-Gaussian Wigner matrices. The variance of an eigenvalue λ_j in the bulk is thus of the order of $\frac{\log N}{N^2}$.

The previous results are however asymptotic. The purpose of this work is to provide quantitative bounds on the variance of the eigenvalues of the correct order, both in the bulk and at the edge. Let M_N be a complex (respectively real) Wigner matrix satisfying condition (C0). This condition, which will be detailed in Section 2.1, provides an exponential decay of the matrix entries. Assume furthermore that the entries have the same first four moments as the entries of a GUE matrix (respectively GOE). Set $W_N = \frac{1}{\sqrt{N}}M_N$.

Theorem 1 (in the bulk). *For all $\eta > 0$, there exists a constant $C(\eta) > 0$ such that, for all $N \geq 2$ and for all $\eta N \leq j \leq (1 - \eta)N$,*

$$\text{Var}(\lambda_j) \leq C(\eta) \frac{\log N}{N^2}. \quad (2)$$

Theorem 2 (at the edge). *There exists a constant $C > 0$ such that, for all $N \geq 1$,*

$$\text{Var}(\lambda_N) \leq CN^{-4/3}. \quad (3)$$

It should be mentioned that Theorem 1 does not seem to be known even for Gaussian matrices. The first step is thus to prove it for the GUE. This will be achieved via the analysis of eigenvalue counting function, which due to the particular determinantal structure in this case, has a binomial distribution. Sharp standard deviation inequalities are thus available in this case. These may then be transferred to the eigenvalues in the bulk together with Gustavsson's bounds on the variance of the eigenvalue counting function. As a result, we actually establish that

$$E[|\lambda_j - \gamma_j|^2] \leq C(\eta) \frac{\log N}{N^2}$$

leading thus to Theorem 1 in this case. On the other hand, Theorem 2 for the GUE and GOE has been known for some time (cf. [11]).

On the basis of these results for the GUE (and GOE), Theorems 1 and 2 are then extended to families of Wigner matrices by a suitable combination of Tao and Vu's Four Moment Theorem ([17], [18]) and Erdős, Yau and Yin's Localization Theorem [6]. The basic idea is that while the localization properties almost yield the correct

order, the Four Moment Theorem may be used to reach the optimal bounds by comparison with the Gaussian models. Theorems 1 and 2 are established first in the complex case. The real case is deduced by means of interlacing formulas.

The method developed here do not seem powerful enough to strengthen the variance bounds into exponential tail inequalities. In the recent contribution [16], Tao and Vu proved such exponential tail inequalities by a further refinement of the replacement method leading to the Four Moment Theorem. While much more powerful than variance bounds, they do not seem to yield at this moment the correct order of the variance bounds of Theorems 1 and 2.

Right-side intermediate eigenvalues consist in the λ_j with $\frac{N}{2} \leq j \leq N$ such that $\frac{j}{N} \rightarrow 1$ but $N - j \rightarrow \infty$ when N goes to infinity (the left side can be deduced by symmetry). Gustavsson proved a Central Limit Theorem for these eigenvalues (see [10]), from which their variance is guessed to be of the order of $\frac{\log(N-j)}{N^{4/3}(N-j)^{2/3}}$. However, up to now, it is not clear whether the method we develop here gives a sharp non-asymptotic bound in this case.

Turning now to the content of this paper, Section 1 describes the GUE case. Section 2 emphasizes to start with the Four Moment Theorem of Tao and Vu ([17] and [18]) and the localization results of Erdős, Yau and Yin [6]. On the basis of these results and the GUE case, the main Theorems 1 and 2 are then established for families of Wigner matrices. Section 3 is devoted to the corresponding statements for real matrices.

1 Deviation inequalities and variance bounds in the GUE case

The results and proofs developed in this section for the GUE model heavily rely on its determinantal structure which allows for a complete description of the joint law of the eigenvalues (see [1]). In particular the eigenvalue counting function is known to have a binomial distribution (cf. [4], [1]), whose mean and variance were computed by Gustavsson (cf. [10]). Deviation inequalities for individual eigenvalues can thus be established, leading to the announced bounds on the variance.

1.1 Eigenvalues in the bulk of the spectrum

The aim of this section is to establish the following result.

Theorem 3. *Let M_N be a GUE matrix. Set $W_N = \frac{1}{\sqrt{N}}M_N$. For all $\eta > 0$, there exists a constant $C(\eta) > 0$ such that for all $N \geq 2$ and for all $\eta N \leq j \leq (1 - \eta)N$,*

$$\mathbb{E}[|\lambda_j - \gamma_j|^2] \leq C(\eta) \frac{\log N}{N^2}. \quad (4)$$

In particular,

$$\text{Var}(\lambda_j) \leq C(\eta) \frac{\log N}{N^2}. \quad (5)$$

As announced, the proof is based on the connection between the distribution of eigenvalues and the eigenvalue counting function. For every $t \in \mathbb{R}$, let $\mathcal{N}_t = \sum_{i=1}^N 1_{\lambda_i \leq t}$ be the eigenvalue counting function. Due to the determinantal structure of the GUE, it is known (cf. [4]) that \mathcal{N}_t has a binomial distribution. Bernstein's inequality for example (although other, even sharper, inequalities may be used) may then be applied to get that for every $u \geq 0$,

$$\mathbb{P}(|\mathcal{N}_t - \mathbb{E}[\mathcal{N}_t]| \geq u) \leq 2 \exp\left(-\frac{u^2}{2\sigma_t^2 + u}\right), \quad (6)$$

where σ_t^2 is the variance of \mathcal{N}_t (see for example [20]). Note that the upper-bound is non-increasing in u while non-decreasing in the variance.

Set for simplicity $\rho_t = \rho_{sc}((-\infty, t])$, $t \in \mathbb{R}$. It has been shown in [9] that for some numerical constant $C_1 > 0$,

$$\sup_{t \in \mathbb{R}} |\mathbb{E}[\mathcal{N}_t] - N\rho_t| \leq C_1.$$

In particular thus, together with (6), for every $u \geq 0$,

$$\mathbb{P}(|\mathcal{N}_t - N\rho_t| \geq u + C_1) \leq 2 \exp\left(-\frac{u^2}{2\sigma_t^2 + u}\right). \quad (7)$$

As a main conclusion of the work by Gustavsson [10], for every $\kappa \in (0, 2)$, there exists $c_\kappa > 0$ such that

$$\sup_{t \in I_\kappa} \sigma_t^2 \leq c_\kappa \log N, \quad (8)$$

where $I_\kappa = [-2 + \kappa, 2 - \kappa]$.

On the basis of inequalities (7) and (8), it is then possible to derive a deviation inequality for eigenvalues λ_j in the bulk from their theoretical locations $\gamma_j \in [-2, 2]$, $1 \leq j \leq N$, defined by $\rho_{\gamma_j} = \frac{j}{N}$.

Proposition 4 (Deviation inequality for λ_j). *Let $\eta \in (0, 1)$ and $\eta N \leq j \leq (1 - \eta)N$. There exist $C_3 > 0$, $C_4 > 0$, $c > 0$ and $\kappa \in (0, 2)$ (all depending on η) such that, for all $C_4 \leq u \leq cN$,*

$$\mathbb{P}\left(|\lambda_j - \gamma_j| \geq \frac{u}{N}\right) \leq 4 \exp\left(-\frac{C_3^2 u^2}{2c_\kappa \log N + C_3 u}\right). \quad (9)$$

Proof. Let $\eta \in (0, 1)$. To start with, evaluate, for $\frac{N}{2} \leq j \leq (1 - \eta)N$ and $u \geq 0$, the probability $\mathbb{P}(|\lambda_j - \gamma_j| > \frac{u}{N})$. We have

$$\begin{aligned} \mathbb{P}\left(\lambda_j > \gamma_j + \frac{u}{N}\right) &= \mathbb{P}\left(\sum_{i=1}^N 1_{\lambda_i \leq \gamma_j + \frac{u}{N}} < j\right) \\ &= \mathbb{P}\left(\mathcal{N}_{\gamma_j + \frac{u}{N}} < j\right) \\ &= \mathbb{P}\left(N\rho_{\gamma_j + \frac{u}{N}} - \mathcal{N}_{\gamma_j + \frac{u}{N}} > N\rho_{\gamma_j + \frac{u}{N}} - j\right) \\ &= \mathbb{P}\left(N\rho_{\gamma_j + \frac{u}{N}} - \mathcal{N}_{\gamma_j + \frac{u}{N}} > N(\rho_{\gamma_j + \frac{u}{N}} - \rho_{\gamma_j})\right) \end{aligned}$$

where it has been used that $\rho_{\gamma_j} = \frac{j}{N}$. Then

$$\mathbb{P}\left(\lambda_j > \gamma_j + \frac{u}{N}\right) \leq \mathbb{P}\left(|\mathcal{N}_{\gamma_j + \frac{u}{N}} - N\rho_{\gamma_j + \frac{u}{N}}| > N(\rho_{\gamma_j + \frac{u}{N}} - \rho_{\gamma_j})\right).$$

But

$$\begin{aligned} \rho_{\gamma_j + \frac{u}{N}} - \rho_{\gamma_j} &= \int_{\gamma_j}^{\gamma_j + \frac{u}{N}} \frac{1}{2\pi} \sqrt{4 - x^2} dx \\ &\geq \frac{1}{\sqrt{2\pi}} \int_{\gamma_j}^{\gamma_j + \frac{u}{N}} \sqrt{2 - x} dx \\ &\geq \frac{1}{\sqrt{2\pi}} \left[-\frac{2}{3}(2 - x)^{3/2} \right]_{\gamma_j}^{\gamma_j + \frac{u}{N}} \\ &\geq \frac{\sqrt{2}}{3\pi} (2 - \gamma_j)^{3/2} \left(1 - \left(1 - \frac{\frac{u}{N}}{2 - \gamma_j} \right)^{3/2} \right) \\ &\geq \frac{\sqrt{2}}{3\pi} (2 - \gamma_j)^{1/2} \frac{u}{N} \\ &\geq C_2 \frac{u}{N}, \end{aligned}$$

where $C_2 = C_2(\eta) > 0$ (indeed, as $\frac{N}{2} \leq j \leq (1 - \eta)N$, $\sqrt{2 - \gamma_j}$ is bounded away from zero). Therefore

$$\begin{aligned} \mathbb{P}\left(\lambda_j > \gamma_j + \frac{u}{N}\right) &\leq \mathbb{P}\left(|\mathcal{N}_{\gamma_j + \frac{u}{N}} - N\rho_{\gamma_j + \frac{u}{N}}| > C_2 u\right) \\ &\leq \mathbb{P}\left(|\mathcal{N}_{\gamma_j + \frac{u}{N}} - N\rho_{\gamma_j + \frac{u}{N}}| > C_3 u + C_1\right), \end{aligned}$$

with $C_3 = \frac{C_2}{2}$, when $u \geq \frac{2C_1}{C_2} = C_4$. Then, applying (7) leads to

$$\mathbb{P}\left(\lambda_j > \gamma_j + \frac{u}{N}\right) \leq 2 \exp\left(-\frac{C_3^2 u^2}{2\sigma_{\gamma_j + \frac{u}{N}}^2 + C_3 u}\right).$$

As γ_j is in the bulk, there exist κ and c such that $\gamma_j + \frac{u}{N} \in I_\kappa$, for all $\eta N \leq j \leq (1 - \eta)N$ and for all $C_4 \leq u \leq cN$, for all $N \geq 1$ (both κ and c depend on η). Then

$$\mathbb{P}\left(\lambda_j > \gamma_j + \frac{u}{N}\right) \leq 2 \exp\left(-\frac{C_3^2 u^2}{2c_\kappa \log N + C_3 u}\right).$$

Repeating the argument leads to the same bound on $\mathbb{P}\left(\lambda_j < \gamma_j - \frac{u}{N}\right)$. Therefore,

$$\mathbb{P}\left(|\lambda_j - \gamma_j| \geq \frac{u}{N}\right) \leq 4 \exp\left(-\frac{C_3^2 u^2}{2c_\kappa \log N + C_3 u}\right).$$

The proposition is thus established. □

Proof of Theorem 3. Note first that, for every j ,

$$\mathbb{E}[\lambda_j^4] \leq \sum_{i=1}^N \mathbb{E}[\lambda_i^4] = \mathbb{E}[\text{Tr}(W_N^4)].$$

The mean of this trace can be computed: $\mathbb{E}[\text{Tr}(W_N^4)] = 2N + \frac{1}{N}$. Consequently, for all $N \geq 1$,

$$\mathbb{E}[\lambda_j^4] \leq 3N. \quad (10)$$

Choose next $A = A(\eta) > 0$ large enough such that $\frac{C_3^2 A^2}{2c_\kappa + C_3 A} > 5$. Setting $Z = N|\lambda_j - \gamma_j|$,

$$\begin{aligned} \mathbb{E}[Z^2] &= \int_0^\infty \mathbb{P}(Z \geq v) 2v \, dv \\ &= \int_0^{C_4} \mathbb{P}(Z \geq v) 2v \, dv + \int_{C_4}^{A \log N} \mathbb{P}(Z \geq v) 2v \, dv + \int_{A \log N}^\infty \mathbb{P}(Z \geq v) 2v \, dv \\ &\leq C_4^2 + I_1 + I_2. \end{aligned}$$

The two latter integrals are handled in different ways. The first one I_1 will be bounded using (9) while I_2 will be controlled using the Cauchy-Schwarz inequality and (10).

Starting thus with I_2 ,

$$\begin{aligned} I_2 &= \int_{A \log N}^{+\infty} \mathbb{P}(Z \geq v) 2v \, dv \\ &\leq \mathbb{E}[Z^2 \mathbf{1}_{Z \geq A \log N}] \\ &\leq \sqrt{\mathbb{E}[Z^4]} \sqrt{\mathbb{P}(Z \geq A \log N)} \\ &\leq KN^{5/2} \sqrt{\mathbb{P}\left(|\lambda_j - \gamma_j| \geq \frac{A \log N}{N}\right)} \\ &\leq 2KN^{5/2} \exp\left(-\frac{1}{2} \frac{C_3^2 A^2}{2c_\kappa + C_3 A} \log N\right) \\ &\leq 2K \exp\left(\frac{1}{2} \left(5 - \frac{C_3^2 A^2}{2c_\kappa + C_3 A}\right) \log N\right), \end{aligned}$$

where $K > 0$ is a numerical constant. As $\exp\left(\frac{1}{2} \left(5 - \frac{C_3^2 A^2}{2c_\kappa + C_3 A}\right) \log N\right) \xrightarrow{N \rightarrow \infty} 0$, there exists a constant $C_6(\eta) > 0$ such that

$$I_2 \leq C_6(\eta).$$

Turning to I_1 , recall that Proposition 4 gives, for $C_4 \leq v \leq cN$,

$$\mathbb{P}(Z \geq v) = \mathbb{P}\left(|\lambda_j - \gamma_j| \geq \frac{v}{N}\right) \leq 4 \exp\left(-\frac{C_3^2 v^2}{2c_\kappa \log N + C_3 v}\right).$$

Hence in the range $v \leq A \log N$,

$$\mathbb{P}(Z \geq v) \leq 4 \exp\left(-\frac{B}{\log N} v^2\right),$$

where $B = B(\eta) = \frac{C_3^2}{2c_\kappa + C_3 A}$. As a consequence,

$$I_2 \leq 4 \int_{C_4}^{A \log N} \exp\left(-\frac{B}{\log N} v^2\right) 2v \, dv \leq \frac{4 \log N}{B} \left(e^{-\frac{BC_4^2}{\log N}} - e^{-BA^2 \log N}\right).$$

There exists thus a constant $C_7(\eta) > 0$ such that

$$I_2 \leq C_7(\eta) \log N.$$

Summarizing the previous steps, $\mathbb{E}[Z^2] \leq C_8(\eta) \log N$. Therefore

$$\mathbb{E}[|\lambda_j - \gamma_j|^2] \leq C_8(\eta) \frac{\log N}{N^2},$$

which is the claim. \square

It may be shown similarly that, under the assumptions of Theorem 3,

$$\mathbb{E}[|\lambda_j - \gamma_j|^p] \leq C(p, \eta) \frac{(\log N)^{p/2}}{N^p}.$$

The proof relies on the formula $\mathbb{E}[Z^p] = \int_0^\infty \mathbb{P}(Z \geq v) p v^{p-1} dv$ and on the fact that $\mathbb{E}[\lambda_j^{2p}] \leq A(p)N$, where $A(p) > 0$ is a constant depending only on p . Indeed, $\mathbb{E}[\lambda_j^{2p}] \leq \mathbb{E}[\text{Tr}(W_N^{2p})]$ and it is known that $\frac{1}{N} \mathbb{E}[\text{Tr}(W_N^{2p})] \rightarrow c_p$ (when N goes to infinity) where c_p is the p^{th} Catalan number (see for example [1] or [3]).

1.2 Eigenvalues at the edge of the spectrum

In [11], Ledoux and Rider gave unified proofs of precise small deviation inequalities for the extreme eigenvalues of β -ensembles. The results hold in particular for GUE matrices ($\beta = 2$) and for GOE matrices ($\beta = 1$). The following theorem summarizes some of the relevant inequalities for the GUE.

Theorem 5. *Let M_N be a GUE matrix. Set $W_N = \frac{1}{\sqrt{N}}M_N$ and denote by λ_N the maximal eigenvalue of W_N . Then, for all $N \in \mathbb{N}$ and for all $0 < \varepsilon \leq 1$,*

$$\mathbb{P}(\lambda_N \geq 2(1 + \varepsilon)) \leq C \exp\left(-\frac{2N}{C}\varepsilon^{3/2}\right), \quad (11)$$

and

$$\mathbb{P}(\lambda_N \leq 2(1 - \varepsilon)) \leq C^2 \exp\left(-\frac{2N^2}{C}\varepsilon^3\right). \quad (12)$$

There exists also a right-tail large deviation inequality: for $\varepsilon = O(1)$,

$$\mathbb{P}(\lambda_N \geq 2(1 + \varepsilon)) \leq C \exp\left(-\frac{2N}{C}\varepsilon^2\right). \quad (13)$$

Similar inequalities hold for the smallest eigenvalue λ_1 .

As stated in [11], bounds on the variance straightly follow from these deviation inequalities.

Corollary 6. *Let M_N be a GUE matrix. Set $W_N = \frac{1}{\sqrt{N}}M_N$. Then there exists a constant $C > 0$ such that*

$$\forall N \in \mathbb{N}^*, \quad \text{Var}(\lambda_N) \leq \mathbb{E}[(\lambda_N - 2)^2] \leq CN^{-4/3}.$$

Similar results are probably true for the k^{th} smallest or largest eigenvalue (for $k \in \mathbb{N}$ fixed).

2 Variance bounds for Wigner Hermitian matrices

As announced, the goal of this section is to prove Theorem 1 and Theorem 2 for Wigner Hermitian matrices. The eigenvalues of a Wigner Hermitian matrix do not form a determinantal process. Therefore it does not seem easy to provide deviation inequalities for the counting function and for individual eigenvalues. However the sharp non-asymptotic bounds established in the Gaussian case can still be reached by a comparison procedure.

2.1 Tools

Two main recent theorems will be used in order to carry out this comparison procedure. First, Erdős, Yau and Yin proved in [6] a Localization Theorem which gives a high probability non-asymptotic bound on the distance between an eigenvalue λ_j and its theoretical value γ_j . Secondly, Tao and Vu's Four Moment Theorem ([17] and [18]) provides a very useful non-asymptotic bound on the error made by replacing a statistics of the eigenvalues of a Wigner matrix by the same statistics but with the eigenvalues of a GUE matrix.

Let M_N be a Wigner Hermitian matrix. Say that M_N satisfies condition (C0) if the real part ξ and the imaginary part $\tilde{\xi}$ of $(M_N)_{ij}$ are independent and have an exponential decay: there are two constants C_1 and C_2 such that

$$\mathbb{P}(|\xi| \geq t^{C_1}) \leq e^{-t} \quad \text{and} \quad \mathbb{P}(|\tilde{\xi}| \geq t^{C_1}) \leq e^{-t}$$

for all $t \geq C_2$.

Theorem 7 (Localization [6]). *Let M_N be a random Hermitian matrix whose entries satisfy condition (C0). There is a constant $c > 0$ such that, for any $j \in \{1, \dots, N\}$,*

$$\mathbb{P}\left(|\lambda_j - \gamma_j| \geq (\log N)^{c \log \log N} N^{-2/3} \min(j, N+1-j)^{-1/3}\right) \leq e^{-(\log N)^{c \log \log N}}. \quad (14)$$

This is a strong localization result and it almost yields the correct order on the bound on the variance. Indeed, by means of the Cauchy-Schwarz inequality and (10), it can be shown that, for $\eta N \leq j \leq (1-\eta)N$,

$$\text{Var}(\lambda_j) \leq \mathbb{E}[(\lambda_j - \gamma_j)^2] \leq \frac{C}{N^2} (\log N)^{2c \log \log N}.$$

In Tao and Vu's recent paper [16] on deviation inequalities, the authors proved a more precise localization result. They indeed established a bound similar to (14) but with $(\log N)^A$ instead of $(\log N)^{c \log \log N}$ (where $A > 0$ is fixed). However this more precise bound is of no help below, as the final bounds on the variances remain unchanged.

We turn now to Tao and Vu's Four Moment Theorem, in order to compare W_N with a GUE matrix W'_N . Say that two complex random variables ξ and ξ' match to order k if

$$\mathbb{E}[\Re(\xi)^m \Im(\xi)^l] = \mathbb{E}[\Re(\xi')^m \Im(\xi')^l]$$

for all $m, l \geq 0$ such that $m + l \leq k$.

Theorem 8 (Four Moment Theorem [17], [18]). *There exists a small positive constant c_0 such that the following holds. Let $M_N = (\xi_{ij})_{1 \leq i, j \leq N}$ and $M'_N = (\xi'_{ij})_{1 \leq i, j \leq N}$ be two random Wigner Hermitian matrices satisfying condition (C0). Assume that, for $1 \leq i < j \leq N$, ξ_{ij} and ξ'_{ij} match to order 4 and that, for $1 \leq i \leq N$, ξ_{ii} and ξ'_{ii} match to order 2. Set $A_N = \sqrt{N}M_N$ and $A'_N = \sqrt{N}M'_N$. Let $G : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function such that:*

$$\forall 0 \leq k \leq 5, \quad \forall x \in \mathbb{R}, \quad |G^{(k)}(x)| \leq N^{c_0}. \quad (15)$$

Then, for all $1 \leq i \leq N$ and for N large enough (depending on constants C_1 and C_2 in condition (C0)),

$$|\mathbb{E}[G(\lambda_i(A_N))] - \mathbb{E}[G(\lambda_i(A'_N))]| \leq N^{-c_0}. \quad (16)$$

Actually Tao and Vu proved this theorem in a more general form, involving a finite number of eigenvalues. In this work, it will only be used with one given eigenvalue. See [17] and [18] for more details.

It should be mentioned that Tao and Vu extended in [17] Gustavsson's result (see equation (1)) via this theorem. By means of a smooth bump function G , they compared the probability for λ_j to be in a given interval for a non-Gaussian matrix with almost the same probability but for a GUE matrix. Applying this technique to $\mathbb{P}(|\lambda_j - \gamma_j| > \frac{u}{N})$ in order to extend directly the deviation inequality leads to the following in the general Wigner case: for $C' \leq u \leq c'N$,

$$\mathbb{P}\left(|\lambda_j - \gamma_j| > \frac{u}{N}\right) \leq C \exp\left(-\frac{u^2}{c \log N + u}\right) + O(N^{-c_0}).$$

The bound is not exponential anymore and is not enough to conclude towards sharp bounds on the variance or higher moments.

2.2 Comparison with Gaussian matrices

Let M_N be a Hermitian Wigner matrix and M'_N be a GUE matrix such that they satisfy the hypotheses of Theorem 8. As the function $G : x \in \mathbb{R} \mapsto x^2$ does not satisfy (15), Theorem 8 will be applied to a truncation of G . Theorem 7 will provide a small area around the theoretical location γ_j where the eigenvalue λ_j is very likely to be in, so that the error due to the truncation will be well controlled. Note that this procedure is valid for eigenvalues in the bulk and at the edge of the spectrum.

Let $1 \leq j \leq N$. Set $R_N^{(j)} = (\log N)^{c \log \log N} N^{1/3} \min(j, N+1-j)^{-1/3}$ and $\varepsilon_N = e^{-(\log N)^{c \log \log N}}$. Then Theorem 7 leads to:

$$\mathbb{P}\left(|\lambda_j - \gamma_j| \geq \frac{R_N^{(j)}}{N}\right) \leq \varepsilon_N. \quad (17)$$

Let ψ be a smooth function with support $[-2, 2]$ and values in $[0, 1]$ such that $\psi(x) = \frac{1}{10}x^2$ for all $x \in [-1, 1]$. Set $G_j : x \in \mathbb{R} \mapsto \psi\left(\frac{x - N\gamma_j}{R_N^{(j)}}\right)$. We want to apply Tao and Vu's Four Moment Theorem 8 to G_j . As ψ is smooth and has compact

support, its first five derivatives are bounded by $M > 0$. Then, for all $0 \leq k \leq 5$, for all $x \in \mathbb{R}$,

$$|G_j^{(k)}(x)| \leq \frac{M}{(R_N^{(j)})^k} \leq N^{c_0},$$

where the last inequality holds for N large enough (depending only on M and c_0). Then, the Four Moment Theorem 8 yields:

$$|\mathbb{E}[G_j(\lambda_j(A_N))] - \mathbb{E}[G_j(\lambda_j(A'_N))]| \leq N^{-c_0} \quad (18)$$

for large enough N . But

$$\begin{aligned} \mathbb{E}[G_j(\lambda_j(A_N))] &= \frac{1}{10} \mathbb{E} \left[\left(\frac{\lambda_j(A_N) - N\gamma_j}{R_N^{(j)}} \right)^2 1_{\frac{|\lambda_j(A_N) - N\gamma_j|}{R_N^{(j)}} \leq 1} \right] + \mathbb{E} \left[G_j(\lambda_j(A_N)) 1_{\frac{|\lambda_j(A_N) - N\gamma_j|}{R_N^{(j)}} > 1} \right] \\ &= \frac{N^2}{10(R_N^{(j)})^2} \mathbb{E} \left[(\lambda_j - \gamma_j)^2 1_{|\lambda_j - \gamma_j| \leq \frac{R_N^{(j)}}{N}} \right] + \mathbb{E} \left[G_j(\lambda_j(A_N)) 1_{\frac{|\lambda_j(A_N) - N\gamma_j|}{R_N^{(j)}} > 1} \right]. \end{aligned}$$

On the one hand,

$$\begin{aligned} \mathbb{E} \left[G_j(\lambda_j(A_N)) 1_{\frac{|\lambda_j - N\gamma_j|}{R_N^{(j)}} > 1} \right] &\leq \mathbb{P}(|\lambda_j(A_N) - N\gamma_j| > R_N^{(j)}) \\ &\leq \mathbb{P} \left(|\lambda_j(W_N) - \gamma_j| > \frac{R_N^{(j)}}{N} \right) \\ &\leq \varepsilon_N. \end{aligned}$$

On the other hand,

$$\mathbb{E} \left[(\lambda_j - \gamma_j)^2 1_{|\lambda_j - \gamma_j| \leq \frac{R_N^{(j)}}{N}} \right] = \mathbb{E}[(\lambda_j - \gamma_j)^2] - \mathbb{E} \left[(\lambda_j - \gamma_j)^2 1_{|\lambda_j - \gamma_j| > \frac{R_N^{(j)}}{N}} \right].$$

From the Cauchy-Schwarz inequality and (10),

$$\begin{aligned} \mathbb{E} \left[(\lambda_j - \gamma_j)^2 1_{|\lambda_j - \gamma_j| > \frac{R_N^{(j)}}{N}} \right] &\leq \sqrt{\mathbb{E}[(\lambda_j - \gamma_j)^4] \mathbb{P} \left(|\lambda_j - \gamma_j| > \frac{R_N^{(j)}}{N} \right)} \\ &\leq K \sqrt{N \varepsilon_N}, \end{aligned}$$

where $K > 0$ is a numerical constant. Then

$$\begin{aligned} \mathbb{E}[G_j(\lambda_j(A_N))] &= \frac{N^2}{10(R_N^{(j)})^2} \left(\mathbb{E}[(\lambda_j - \gamma_j)^2] + O(N^{1/2} \varepsilon_N^{1/2}) \right) + O(\varepsilon_N) \\ &= \frac{N^2}{10(R_N^{(j)})^2} \mathbb{E}[(\lambda_j - \gamma_j)^2] + O(N^{5/2} \varepsilon_N^{1/2} (R_N^{(j)})^{-2}) + O(\varepsilon_N). \end{aligned}$$

Repeating the same computations gives similarly

$$\mathbb{E}[G_j(\lambda_j(A'_N))] = \frac{N^2}{10(R_N^{(j)})^2} \mathbb{E}[(\lambda'_j - \gamma_j)^2] + O(N^{5/2} \varepsilon_N^{1/2} (R_N^{(j)})^{-2}) + O(\varepsilon_N).$$

Then (18) leads to

$$\mathbb{E}[(\lambda_j - \gamma_j)^2] = \mathbb{E}[(\lambda'_j - \gamma_j)^2] + O(N^{1/2}\varepsilon_N^{1/2}) + O(N^{-2}(R_N^{(j)})^2\varepsilon_N) + O((R_N^{(j)})^2N^{-c_0-2}).$$

As the first two error terms are smaller than the third one, the preceding equation becomes

$$\mathbb{E}[(\lambda_j - \gamma_j)^2] = \mathbb{E}[(\lambda'_j - \gamma_j)^2] + O((R_N^{(j)})^2N^{-c_0-2}). \quad (19)$$

2.3 Combining the results

We distinguish between the bulk and the edge. Note that the constants $C'(\eta)$ and C' depend on the constants C_1 and C_2 in condition (C0).

2.3.1 Eigenvalues in the bulk of the spectrum

Let $\eta > 0$ and $\eta N \leq j \leq (1 - \eta)N$. From Theorem 3, $\mathbb{E}[(\lambda'_j - \gamma_j)^2] \leq C(\eta)\frac{\log N}{N^2}$. Thus, from (19), it remains to show that the error term is smaller than $\frac{\log N}{N^2}$. But

$$R_N^{(j)} = (\log N)^{c \log \log N} N^{1/3} \min(j, N + 1 - j)^{-1/3} \leq \eta^{-1/3} (\log N)^{c \log \log N}.$$

Then $(R_N^{(j)})^2 N^{-c_0-2} = o_\eta\left(\frac{\log N}{N^2}\right)$. As a consequence,

$$\mathbb{E}[(\lambda_j - \gamma_j)^2] = \mathbb{E}[(\lambda'_j - \gamma_j)^2] + o_\eta\left(\frac{\log N}{N^2}\right)$$

and we get the desired result:

$$\mathbb{E}[(\lambda_j - \gamma_j)^2] \leq C'(\eta)\frac{\log N}{N^2}.$$

2.3.2 Eigenvalues at the edge of the spectrum

From Corollary 6, $\mathbb{E}[(\lambda'_N - \gamma_N)^2] = \mathbb{E}[(\lambda'_N - 2)^2] \leq CN^{-4/3}$. By means of (19), it remains to prove that the error term is smaller than $N^{-4/3}$. We have

$$R_N^{(N)} = (\log N)^{c \log \log N} N^{1/3}.$$

Consequently $(R_N^{(N)})^2 N^{-c_0-2} = o(N^{-4/3})$. Then

$$\mathbb{E}[(\lambda_N - 2)^2] = \mathbb{E}[(\lambda'_N - 2)^2] + o(N^{-4/3})$$

and

$$\mathbb{E}[(\lambda_N - 2)^2] \leq C'N^{-4/3}.$$

As for Gaussian matrices, the same result is available for the smallest eigenvalue λ_1 .

3 Real matrices

The goal of this section is to prove Theorem 1 and Theorem 2 for real Wigner matrices. Tao and Vu's Four Moment Theorem (Theorem 8) as well as Erdős, Yau and Yin's Localization Theorem (Theorem 7) still hold for real Wigner matrices. Section 2 is therefore valid for real matrices. The point is then to establish the results in the GOE case.

As announced in Section 1.2, the variance of eigenvalues at the edge of the spectrum is known to be bounded by $N^{-4/3}$ for GOE matrices (see [11]). The conclusion for the smallest and largest eigenvalues is then established for large families of real symmetric Wigner matrices.

$$\mathrm{Var}(\lambda_N) \leq \frac{\tilde{C}}{N^{4/3}} \quad \text{and} \quad \mathrm{Var}(\lambda_1) \leq \frac{\tilde{C}}{N^{4/3}}.$$

For eigenvalues in the bulk of the spectrum, O'Rourke proved in [13] a Central Limit Theorem which is very similar to the one established by Gustavsson in [10]. In particular, the normalisation is still of the order of $\frac{\sqrt{\log N}}{N}$ and differs from the complex case only by a constant. It is therefore natural to expect the same bound on the variance for GOE matrices. But GOE matrices do not have the same determinantal properties as GUE matrices, and it is therefore not clear that a deviation inequality (similar to (7)) holds for the eigenvalue counting function. However, as explained by O'Rourke in [13], GOE and GUE matrices are linked by interlacing formulas established by Forrester and Rains (see [7]). These formulas lead to the following relation between the eigenvalue counting functions in the complex and in the real cases: for all $t \in \mathbb{R}$,

$$\mathcal{N}_t(W_N^{\mathbb{C}}) = \frac{1}{2}(\mathcal{N}_t(W_N^{\mathbb{R}}) + \mathcal{N}_t(\tilde{W}_N^{\mathbb{R}})) + \zeta_N(t), \quad (20)$$

where

- $W_N^{\mathbb{C}} = \frac{1}{\sqrt{N}}M_N^{\mathbb{C}}$ and $M_N^{\mathbb{C}}$ is from the GUE,
- $W_N^{\mathbb{R}} = \frac{1}{\sqrt{N}}M_N^{\mathbb{R}}$, $\tilde{W}_N^{\mathbb{R}} = \frac{1}{\sqrt{N}}\tilde{M}_N^{\mathbb{R}}$ and $M_N^{\mathbb{R}}$ and $\tilde{M}_N^{\mathbb{R}}$ are independent matrices from the GOE,
- $\zeta_N(t)$ takes values in $\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$.

See [13] for more details.

The aim is now to establish a deviation inequality for the eigenvalue counting function similar to (7). From (7), we know that for all $u \geq 0$,

$$\mathbb{P}\left(|\mathcal{N}_t(W_N^{\mathbb{C}}) - N\rho_t| \geq u + C_1\right) \leq 2 \exp\left(-\frac{u^2}{2\sigma_t^2 + u}\right).$$

Set $C'_1 = C_1 + 1$ and let $u \geq 0$. We can then write

$$\begin{aligned}
& \mathbb{P}(\mathcal{N}_t(W_N^{\mathbb{R}}) - N\rho_t \geq u + C'_1)^2 \\
&= \mathbb{P}\left(\mathcal{N}_t(W_N^{\mathbb{R}}) - N\rho_t \geq u + C'_1, \mathcal{N}_t(\tilde{W}_N^{\mathbb{R}}) - N\rho_t \geq u + C'_1\right) \\
&\leq \mathbb{P}\left(\frac{1}{2}(\mathcal{N}_t(W_N^{\mathbb{R}}) + \mathcal{N}_t(\tilde{W}_N^{\mathbb{R}})) - N\rho_t \geq u + C'_1\right) \\
&\leq \mathbb{P}(\mathcal{N}_t(W_N^{\mathbb{C}}) - N\rho_t - \zeta_N(t) \geq u + C'_1) \\
&\leq \mathbb{P}(\mathcal{N}_t(W_N^{\mathbb{C}}) - N\rho_t \geq u + C'_1 - 1) \\
&\leq \mathbb{P}(|\mathcal{N}_t(W_N^{\mathbb{C}}) - N\rho_t| \geq u + C_1) \\
&\leq 2 \exp\left(-\frac{u^2}{2\sigma_t^2 + u}\right).
\end{aligned}$$

Then, for all $u \geq 0$,

$$\mathbb{P}(\mathcal{N}_t(W_N^{\mathbb{R}}) - N\rho_t \geq u + C'_1) \leq \sqrt{2} \exp\left(-\frac{u^2}{4\sigma_t^2 + 2u}\right).$$

Repeating the computations for $\mathbb{P}(\mathcal{N}_t(W_N^{\mathbb{R}}) - N\rho_t \leq -u - C'_1)$ leads to

$$\mathbb{P}(\mathcal{N}_t(W_N^{\mathbb{R}}) - N\rho_t \leq -u - C'_1) \leq \sqrt{2} \exp\left(-\frac{u^2}{4\sigma_t^2 + 2u}\right).$$

Then

$$\mathbb{P}(|\mathcal{N}_t(W_N^{\mathbb{R}}) - N\rho_t| \geq u + C'_1) \leq 2\sqrt{2} \exp\left(-\frac{u^2}{4\sigma_t^2 + 2u}\right). \quad (21)$$

Note that σ_t^2 is still the variance of $\mathcal{N}_t(W_N^{\mathbb{C}})$ in the preceding formula.

What remains then to be proved is very similar to the complex case. From (21) and Gustavsson's bound on the variance σ_t^2 (equation (8)), a deviation inequality for individual eigenvalues in the bulk can be deduced. It is then straightforward to derive the announced bound on the variance for GOE matrices. The argument developed in Section 2 in order to extend the GUE result to large families of complex Wigner matrices can be reproduced to reach the desired bound on the variance of eigenvalues in the bulk for families of real Wigner matrices. Then there exists a constant $\tilde{C}(\eta) > 0$ depending only on η such that for all $\eta N \leq j \leq (1 - \eta)N$,

$$\text{Var}(\lambda_j) \leq \mathbb{E}[(\lambda_j - \gamma_j)^2] \leq \tilde{C}(\eta) \frac{\log N}{N^2}.$$

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