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Canonical pure spinor (Fermionic) T-duality

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Abstract

We establish that the recently discovered fermionic T-duality can be viewed as a canonical transformation in phase space. This requires a careful treatment of constrained Hamiltonian systems. Additionally, we show how the canonical transformation approach for bosonic T-duality can be extended to include Ramond–Ramond backgrounds in the pure spinor formalism.

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1 Introduction

An important recent development in the study of $\mathcal{N} = 4$ supersymmetric gauge theories has been the discovery of a connection between planar scattering amplitudes and Wilson loops and the related discovery of a dual superconformal symmetry. From the dual AdS perspective this result is understood (at least at strong coupling) as a consequence of T-duality [1, 2, 3, 4]. In [1] it was established that under a series of T-dualities the $AdS_5 \times S^5$ metric is self-dual and moreover, that a configuration corresponding to a scattering amplitude is dualised to one describing a light-like Wilson loop.

However, the dualities used in [1] do not leave the full AdS background strictly invariant; instead they result in a shifted dilaton and different Ramond–Ramond (RR) fields. To rectify this and produce an exact selfduality of the background Berkovits and Maldacena [2] introduced a novel “Fermionic T-duality” which leaves the metric and Kalb–Ramond fields invariant but transforms the dilaton and RR fields. Whilst it is only valid at tree level in string perturbation theory and not a full symmetry of string theory, Fermionic T-duality is clearly important and certainly has applications as a solution generating symmetry of supergravity [5].

The derivation of Fermionic T-duality in [2] essentially follows the Buscher procedure [6, 7] carried out along the direction of a fermionic isometry in superspace. It has long been known that an alternative way to think about T-duality, albeit classical in nature, is as a canonical transformation of the phase space variables. This was first shown in the context of the chiral $O(4)$ bosonic model (dualized using its non-abelian symmetry) in [8] and for abelian T-duality of the bosonic string in [9]. It was later extended to the RNS formalism of the superstring [10] and also to the more general notions of non-abelian and Poisson Lie T-duality in [11] and [12], respectively.

The canonical approach to T-duality has a certain elegance, in that it requires no extra-aneous structure, and provides insight into the nature of the phenomenon of duality. In the case of bosonic T-duality the canonical approach has been successfully used to understand, for instance, the structure of conservation laws in dual theories and the deeper reason why σ -models related merely by an interchange of equations of motion and Bianchi identities can be actually inequivalent at the quantum level (for the prototype examples see [13, 14]). The canonical approach also helped in elucidating the non-local nature of transformations induced by T-duality and the relation of T-duality to world-sheet and spacetime supersymmetry [15, 10, 16].

In this letter we will show how this canonical approach may be extended to Fermionic T-duality. Because we are dealing with fermions we will see that proving the canonical equivalence requires a careful application of the Dirac procedure in order to treat second class constraints. As a by-product of this study we shall also show that the canonical transformation approach for bosonic T-duality can be readily extended to the pure spinor form of the superstring thus incorporating the transformations of both NS, RR and fermionic background fields.

We believe that our work will be useful in extending the notion of T-duality in superstring theory in the presence of non-trivial RR background fields when non-abelian isometry structures are involved (we hope to report on this shortly [17]).

2 Bosonic T-duality as a canonical transformation

For later reference we begin by reviewing the canonical transformation approach to plain bosonic T-duality. We shall see that certain structures are the same even when many more fields are present, as is the case with the discussion in section 4 below.

We start with the σ -model Lagrangian density

$$\mathcal{L} = \frac{1}{2}Q_{IJ}\partial_+X^I\partial_-X^J, \quad Q_{IJ} = G_{IJ} + B_{IJ}, \quad (2.1)$$

where G and B are the metric and the antisymmetric tensor in the NS sector, respectively. We demand that the background fields are independent of some coordinate X_0 and denote the rest of them by X^i . With the definitions¹

$$J_+ = \frac{1}{2}Q_{i0}\partial_+X^i, \quad J_- = \frac{1}{2}Q_{0i}\partial_-X^i, \quad V = -\frac{1}{2}Q_{ij}\partial_+X^i\partial_-X^j, \quad (2.2)$$

the momenta conjugate to X_0 is given by

$$P_0 = \frac{\delta\mathcal{L}}{\delta\dot{X}_0} = G_{00}\dot{X}_0 + J_+ + J_-, \quad (2.3)$$

and the Hamiltonian density obtained by performing the Legendre transform only on the active field X_0 is then

$$\mathcal{H} = \frac{1}{2G_{00}}P_0^2 + \frac{G_{00}}{2}X_0'^2 - \frac{1}{G_{00}}P_0(J_+ + J_-) + (J_+ - J_-)X_0'$$

¹In our conventions $\sigma^\pm = \frac{1}{2}(\tau \pm \sigma)$.

$$+ \frac{1}{2G_{00}}(J_+ + J_-)^2 + V. \quad (2.4)$$

The Poisson brackets between the conjugate phase space variables are

$$\{X_0(\sigma), P_0(\sigma')\} = \delta(\sigma - \sigma'), \quad \{X_0(\sigma), X_0(\sigma')\} = \{P_0(\sigma), P_0(\sigma')\} = 0, \quad (2.5)$$

where here and subsequently we suppress the τ -dependence since we deal with equal time brackets. Under the following transformation to a new set of phase space variables (which preserves the above symplectic structure)

$$P_0 = \tilde{X}'_0, \quad X'_0 = \tilde{P}_0, \quad (2.6)$$

the Hamiltonian density becomes

$$\begin{aligned} \mathcal{H}_{\text{CT}} &= \frac{1}{2G_{00}}\tilde{X}'_0{}^2 + \frac{G_{00}}{2}\tilde{P}_0^2 - \frac{1}{G_{00}}\tilde{X}'_0(J_+ + J_-) + (J_+ - J_-)\tilde{P}_0 \\ &+ \frac{1}{2G_{00}}(J_+ + J_-)^2 + V. \end{aligned} \quad (2.7)$$

The T-dual model has a Hamiltonian density $\tilde{\mathcal{H}}$ of the same form as that in (2.4) with X_0 , P_0 , G_{00} , J_{\pm} and V replaced with the corresponding tilded quantities. What is remarkable, and the crux of the issue, is that the dual Hamiltonian can be brought into exactly the same form as the original Hamiltonian after a redefinition of the background fields. That is by demanding

$$H_{\text{CT}} = \int d\sigma \mathcal{H}_{\text{CT}} = \int d\sigma \tilde{\mathcal{H}} = \tilde{H}, \quad (2.8)$$

we obtain

$$\tilde{J}_{\pm} = \mp \frac{J_{\pm}}{G_{00}}, \quad \tilde{V} = V + 2\frac{J_+ J_-}{G_{00}}, \quad (2.9)$$

from which we easily recover the Buscher T-duality rules

$$\begin{aligned} \tilde{G}_{00} &= \frac{1}{G_{00}}, & \tilde{Q}_{i0} &= -\frac{Q_{i0}}{G_{00}}, & \tilde{Q}_{0i} &= \frac{Q_{0i}}{G_{00}}, \\ \tilde{Q}_{ij} &= Q_{ij} - \frac{Q_{i0}Q_{0j}}{G_{00}}. \end{aligned} \quad (2.10)$$

We mention that the transformation of worldsheet derivatives under the canonical

transformation can be computed as

$$\partial_+ \tilde{X}_0 = G_{00} \partial_+ X_0 + Q_{i0} \partial_+ X^i, \quad \partial_- \tilde{X}_0 = -G_{00} \partial_- X_0 - Q_{0i} \partial_- X^i. \quad (2.11)$$

Hence, the transformation of the differential involves the Hodge dual on the worldsheet, i.e. $d\tilde{X}_0 = G_{00} \star dX_0 + \dots$

Finally, we note that the energy momentum tensor can be written with either set of background fields and worldsheet derivatives, i.e.

$$T_{\pm\pm} = \frac{1}{2} G_{IJ} \partial_\pm X^I \partial_\pm X^J = \frac{1}{2} \tilde{G}_{IJ} \partial_\pm \tilde{X}^I \partial_\pm \tilde{X}^J = \tilde{T}_{\pm\pm}. \quad (2.12)$$

The proof relies on the transformation of the background fields and worldsheet derivatives, (2.10) and (2.11). For completeness we also note that there is an obvious generating function (of the first kind) for the canonical transformation

$$\mathcal{F} = - \int d\sigma X'_0 \tilde{X}_0, \quad \Pi = \frac{\delta \mathcal{F}}{\delta X_0}, \quad \tilde{\Pi} = - \frac{\delta \mathcal{F}}{\delta \tilde{X}_0}. \quad (2.13)$$

3 Fermionic T-duality as a canonical transformation

We now consider the fermionic T-duality proposed by Berkovits and Maldacena. We begin by considering the Lagrangian density

$$\mathcal{L} = \frac{1}{2} L_{MN} \partial_+ Z^M \partial_- Z^N, \quad L_{MN} = G_{MN} + B_{MN}, \quad (3.1)$$

where $Z^M = (X^I, \theta^\alpha)$ are coordinates on a superspace so that the θ variables are anti-commuting fermions and the superfields G and B obey graded symmetrisation rules

$$G_{MN} = (-)^{MN} G_{NM}, \quad B_{MN} = -(-)^{MN} B_{NM}, \quad (3.2)$$

where $(-)^{MN}$ is equal to $+1$ unless both M and N are spinorial indices in which case it is equal to -1 . The lowest components of these superfields, when the indices run over bosonic coordinates, are the target space metric and B-field.

We assume that the action is invariant under a shift symmetry in one of the fermionic directions θ^1 (which henceforth we will denote simply by θ) and that the background superfield is independent of this coordinate (this is much the same as working in adapted coordinates for regular bosonic T-duality). We define Z^μ as running over all

bosonic and fermionic directions except θ . It is also helpful to define

$$\mathcal{J}_+ = \frac{1}{2}L_{\mu 1}\partial_+ Z^\mu, \quad \mathcal{J}_- = -\frac{1}{2}(-1)^\mu L_{1\mu}\partial_- Z^\mu, \quad \mathcal{V} = -\frac{1}{2}L_{\mu\nu}\partial_+ Z^\mu\partial_- Z^\nu. \quad (3.3)$$

Note that \mathcal{J}_\pm are fermionic. Then the σ -model Lagrangian can be written as

$$\mathcal{L} = -B_{11}\dot{\theta}\theta' + (\dot{\theta} + \theta')\mathcal{J}_- + \mathcal{J}_+(\dot{\theta} - \theta') - \mathcal{V}. \quad (3.4)$$

The equation of motions from varying θ is

$$\delta\theta: \quad \dot{B}_{11}\theta' - B'_{11}\dot{\theta} + (\dot{\mathcal{J}}_+ - \dot{\mathcal{J}}_-) - (\mathcal{J}_+ + \mathcal{J}_-)' = 0. \quad (3.5)$$

The canonical momenta conjugate to θ is given by

$$\Pi = \frac{\delta\mathcal{L}}{\delta\dot{\theta}} = -B_{11}\theta' - \mathcal{J}_+ + \mathcal{J}_-, \quad (3.6)$$

and the corresponding equal time Poisson Brackets are

$$\{\theta(\sigma), \Pi(\sigma')\} = -\delta(\sigma - \sigma'), \quad \{\theta(\sigma), \theta(\sigma')\} = \{\Pi(\sigma), \Pi(\sigma')\} = 0. \quad (3.7)$$

The sign convention in the first bracket is a consequences of the fermionic nature of θ and the fact that derivatives act from the left.

3.1 The constrained system

Since the Lagrangian is first order in time derivatives the velocities can not be solved in terms of momenta, instead we have an anticommuting constraint

$$f = \Pi + B_{11}\theta' + \mathcal{J}_+ - \mathcal{J}_- \approx 0. \quad (3.8)$$

The naive Hamiltonian density is given as

$$\mathcal{H} = \dot{\theta}\Pi - \mathcal{L} = -\theta'(\mathcal{J}_+ + \mathcal{J}_-) + \mathcal{V}, \quad (3.9)$$

however, this should be amended to take account of the constraint. We follow the Dirac procedure² by first modifying the Hamiltonian with an, as yet unknown, local

²See [18] for a detailed treatment of constrained dynamics. Also, the treatment of a constraint was necessary in the discussion of T-duality in heterotic string theory [19].

function $\lambda(\tau, \sigma)$ which resembles an anticommuting Lagrange multiplier

$$H_{\text{tot}} = \int d\sigma (\mathcal{H} + \lambda f) . \quad (3.10)$$

We now need to check whether any secondary constraints are produced by considering the time evolution of the constraint and demanding that

$$\dot{f}(\sigma) = \{f(\sigma), H_{\text{tot}}\} \approx 0 . \quad (3.11)$$

In order to calculate this time evolution it is necessary to know

$$\begin{aligned} \{f(\sigma), f(\sigma')\} &= \{\Pi(\sigma) + B_{11}\theta'(\sigma), \Pi(\sigma') + B_{11}(\sigma')\theta'(\sigma')\} \\ &= B_{11}(\sigma')\{\Pi(\sigma), \theta'(\sigma')\} + B_{11}(\sigma)\{\theta'(\sigma), \Pi(\sigma')\} \\ &= (B_{11}(\sigma') - B_{11}(\sigma))\frac{\partial}{\partial\sigma}\delta(\sigma - \sigma') \\ &= B'_{11}(\sigma')\delta(\sigma - \sigma') , \end{aligned} \quad (3.12)$$

where we have made use of the identifications $x\delta(x) = 0$ and $x\delta'(x) = -\delta(x)$ (to be understood in a distributional sense). Note that since the Poisson bracket of these constraints is non-zero they are second class constraints; to consider the quantization of the theory one should upgrade Poisson brackets to Dirac brackets.

We also need that

$$\{f(\sigma), \mathcal{H}(\sigma')\} = -\{\Pi, \theta'(\mathcal{J}_+ + \mathcal{J}_-)\} = (\mathcal{J}_+ + \mathcal{J}_-)(\sigma')\frac{\partial}{\partial\sigma'}\delta(\sigma - \sigma') . \quad (3.13)$$

Then the time evolution is given by

$$\begin{aligned} \dot{f}(\sigma) &= \{f(\sigma), H_{\text{tot}}\} = \int d\sigma' (\mathcal{J}_+ + \mathcal{J}_-)(\sigma')\frac{\partial}{\partial\sigma'}\delta(\sigma - \sigma') - \lambda(\sigma')B'_{11}(\sigma')\delta(\sigma - \sigma') \\ &= -(\mathcal{J}_+ + \mathcal{J}_-)' - \lambda(\sigma)B'_{11}(\sigma) , \end{aligned} \quad (3.14)$$

note that the minus sign in the second factor is due to the fact that $\lambda(\sigma)$ is anticommuting. Demanding that $\dot{f}(\sigma) \approx 0$ does not produce a new constraint but instead fixes the Lagrange multiplier function as

$$\lambda(\sigma) = -\frac{(\mathcal{J}_+ + \mathcal{J}_-)' }{B'_{11}(\sigma)} . \quad (3.15)$$

Thus, the total Hamiltonian density is given by the integrand in (3.10) upon substitut-

ing (3.15). We obtain

$$\mathcal{H}_{\text{tot}} = -\theta'(\mathcal{J}_+ + \mathcal{J}_-) + \mathcal{V} - \frac{(\mathcal{J}_+ + \mathcal{J}_-)' }{B'_{11}}(\Pi + B_{11}\theta' + \mathcal{J}_+ - \mathcal{J}_-). \quad (3.16)$$

Having established the appropriate Hamiltonian for our constrained system let's verify that the time evolution of θ indeed gives rise to the equations of motion (3.5). We easily compute that

$$\dot{\theta} = \{\theta, H_{\text{tot}}\} = -\frac{(\mathcal{J}_+ + \mathcal{J}_-)' }{B'_{11}} \quad (3.17)$$

and that

$$\dot{\Pi} = \{\Pi, H_{\text{tot}}\} = \left(\frac{B_{11}}{B'_{11}}(\mathcal{J}_+ + \mathcal{J}_-)' - (\mathcal{J}_+ + \mathcal{J}_-) \right)' = -(B_{11}\dot{\theta} + \mathcal{J}_+ + \mathcal{J}_-)', \quad (3.18)$$

where in the second equality we have used (3.17). Then from the definition of Π in (3.6) the last equation becomes identical to (3.5).

3.2 The canonical transformation

Consider now the transformation of phase-space variables

$$\theta' = \tilde{\Pi}, \quad \Pi = -\tilde{\theta}', \quad (3.19)$$

which leaves invariant the Poisson brackets (3.7). Under this transformation the Hamiltonian density in (3.16) becomes

$$\mathcal{H}_{\text{tot,CT}} = -\tilde{\Pi}(\mathcal{J}_+ + \mathcal{J}_-) + \mathcal{V} - \frac{(\mathcal{J}_+ + \mathcal{J}_-)' }{B'_{11}}(-\tilde{\theta}' + B_{11}\tilde{\Pi} + \mathcal{J}_+ - \mathcal{J}_-). \quad (3.20)$$

As in the bosonic case, we would like to identify the Hamiltonian corresponding to this density with that for the T-dual model. This means with a Hamiltonian whose density is as in (3.16) with the replacement of θ , Π , B_{11} , \mathcal{J}_{\pm} and \mathcal{V} with their tilded counterparts. Comparing the coefficients of terms with $\tilde{\Pi}$ we obtain the condition

$$\tilde{\Pi}: \quad -(\mathcal{J}_+ + \mathcal{J}_-) + \frac{B_{11}}{B'_{11}}(\mathcal{J}_+ + \mathcal{J}_-)' = \frac{(\tilde{\mathcal{J}}_+ + \tilde{\mathcal{J}}_-)' }{\tilde{B}'_{11}}, \quad (3.21)$$

from which we deduce, after some algebraic manipulations, that

$$\tilde{B}_{11} = -\frac{1}{B_{11}}, \quad \tilde{\mathcal{J}}_+ + \tilde{\mathcal{J}}_- = \frac{\mathcal{J}_+ + \mathcal{J}_-}{B_{11}}. \quad (3.22)$$

In addition, comparing the coefficients of terms with $\tilde{\theta}'$ we obtain

$$\tilde{\theta}' : \quad -\frac{(\mathcal{J}_+ + \mathcal{J}_-)'}{B'_{11}} = -(\tilde{\mathcal{J}}_+ + \tilde{\mathcal{J}}_-) + \frac{\tilde{B}_{11}}{\tilde{B}'_{11}}(\tilde{\mathcal{J}}_+ + \tilde{\mathcal{J}}_-)', \quad (3.23)$$

which is the tilded counterpart of (3.21) leading again to (3.22). Comparing the rest of the terms we obtain

$$\mathcal{V} - \frac{1}{B'_{11}}(\mathcal{J}_+ + \mathcal{J}_-)'(\mathcal{J}_+ - \mathcal{J}_-) = \tilde{\mathcal{V}} - \frac{1}{\tilde{B}'_{11}}(\tilde{\mathcal{J}}_+ + \tilde{\mathcal{J}}_-)'(\tilde{\mathcal{J}}_+ - \tilde{\mathcal{J}}_-), \quad (3.24)$$

leading to the conditions

$$\tilde{\mathcal{J}}_+ - \tilde{\mathcal{J}}_- = \frac{\mathcal{J}_+ - \mathcal{J}_-}{B_{11}}, \quad \tilde{\mathcal{V}} = \mathcal{V} + 2\frac{\mathcal{J}_+\mathcal{J}_-}{B_{11}}. \quad (3.25)$$

Combining (3.22) with (3.25) we obtain the Fermionic T-duality rules

$$\begin{aligned} \tilde{B}_{11} &= -\frac{1}{B_{11}}, & \tilde{L}_{\mu 1} &= \frac{L_{\mu 1}}{B_{11}}, & \tilde{L}_{1\mu} &= \frac{L_{1\mu}}{B_{11}}, \\ \tilde{L}_{\mu\nu} &= L_{\mu\nu} - \frac{L_{1\nu}L_{\mu 1}}{B_{11}}. \end{aligned} \quad (3.26)$$

Similarly to the bosonic case one may compute the transformation of the worldsheet derivatives of θ under the canonical transformation. We compute that

$$\partial_+ \tilde{\theta} = B_{11} \partial_+ \theta + L_{\mu 1} \partial_+ Z^\mu, \quad \partial_- \tilde{\theta} = B_{11} \partial_- \theta - (-1)^\mu L_{1\mu} \partial_- Z^\mu. \quad (3.27)$$

Hence, unlike the bosonic case the transformation of the differential does not involve the Hodge dual on the worldsheet, i.e. $d\tilde{\theta} = B_{11}d\theta + \dots$

As in the bosonic case the energy momentum tensor can be written with either set of background fields and worldsheet derivatives

$$T_{\pm\pm} = \frac{1}{2}G_{MN}\partial_\pm Z^M\partial_\pm Z^N = \frac{1}{2}\tilde{G}_{MN}\partial_\pm \tilde{Z}^M\partial_\pm \tilde{Z}^N = \tilde{T}_{\pm\pm}, \quad (3.28)$$

where in the proof we have used (3.26) and (3.27). In addition, the analog of the bosonic generating function (2.13) in the fermionic case is

$$\mathcal{F} = \int d\sigma \theta' \tilde{\theta}, \quad \Pi = \frac{\delta \mathcal{F}}{\delta \theta}, \quad \tilde{\Pi} = -\frac{\delta \mathcal{F}}{\delta \tilde{\theta}}. \quad (3.29)$$

We note that we could have followed a similar path leading to the fermionic T-duality

transformation rules by using the Dirac brackets for our canonical variables which we include for completeness

$$\begin{aligned}
\{\theta(\sigma_1), \theta(\sigma_2)\}_D &= -\frac{\delta(\sigma_1 - \sigma_2)}{B'_{11}(\sigma_2)}, \\
\{\Pi(\sigma_1), \Pi(\sigma_2)\}_D &= -\partial_{\sigma_1} \partial_{\sigma_2} \left(\frac{B_{11}^2(\sigma_1)}{B'_{11}(\sigma_1)} \delta(\sigma_1 - \sigma_2) \right), \\
\{\theta(\sigma_1), \Pi(\sigma_2)\}_D &= -\delta(\sigma_1 - \sigma_2) - \frac{B_{11}(\sigma_1)}{B'_{11}(\sigma_1)} \delta'(\sigma_1 - \sigma_2).
\end{aligned} \tag{3.30}$$

This procedure would have allowed us to set the constraint (3.8) strongly to zero in various expressions. Then, in addition to equating the Hamiltonians, one should require that the constraint (3.8) is actually preserved by the transformation.

Finally we remark on the issue of the shift of the dilaton field. Being classical in its nature, the canonical transformation for both the bosonic and fermionic T-duality does not highlight the shift of the dilaton that is required at the quantum level. For the bosonic case the shift, computed first in [6, 7], is such that the target space measure $e^{-2\Phi} \sqrt{\det(G)}$ remains invariant, i.e. $\tilde{\Phi} = \Phi - \frac{1}{2} \ln G_{00}$. This dilaton shift is important to ensure the conformality of dual backgrounds. The opposite shift happens in the fermionic case and must be considered in direct applications, for instance in the amplitude / Wilson loop connection [2]. Within the canonical approach it has been suggested that the dilaton shift might arise from a careful definition of the functional measure on phase space [20]. However, in both the bosonic and fermionic (especially) cases this remains an interesting open question which we do not seek to address in this paper.

4 Canonical T-duality in the pure spinor formalism

Many important string backgrounds have non zero RR fluxes, the most notable being $AdS_5 \times S^5$. It is thus important to understand the action of T-duality on RR fields. This was first established from a supergravity perspective [21] and later by means of a Buscher procedure in the Green Schwarz form of the superstring [22, 23]. More recently the Buscher procedure was applied to the pure spinor form of the superstring [24, 25].

4.1 A brief introduction and generalities

The pure spinor approach to the superstring proposed by Berkovits combines the virtues of the RNS formalism with the those of the GS formalism. In particular, it allows one to describe the superstring in general curved backgrounds with non-trivial Ramond–Ramond sectors. We refer the reader to the original papers as well as the helpful reviews on this subject for more of the details of the formalism [26, 27, 28].

The Lagrangian density in a curved background is given by

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}L_{MN}(Z)\partial_+Z^M\partial_-Z^N + P^{\alpha\hat{\beta}}(Z)d_\alpha\hat{d}_{\hat{\beta}} + E_M^\alpha(Z)d_\alpha\partial_-Z^M \\ & + E_M^{\hat{\alpha}}(Z)\hat{d}_{\hat{\alpha}}\partial_+Z^M + \Omega_{M\alpha}{}^\beta(Z)\lambda^\alpha w_\beta\partial_-Z^M + \hat{\Omega}_{M\hat{\alpha}}{}^{\hat{\beta}}(Z)\hat{\lambda}^{\hat{\alpha}}\hat{w}_{\hat{\beta}}\partial_+Z^M \quad (4.1) \\ & + C_\alpha^{\beta\hat{\gamma}}(Z)\lambda^\alpha w_\beta\hat{d}_{\hat{\gamma}} + \hat{C}_{\hat{\alpha}}^{\hat{\beta}\gamma}(Z)\hat{\lambda}^{\hat{\alpha}}\hat{w}_{\hat{\beta}}d_\gamma + S_{\alpha\hat{\gamma}}^{\beta\hat{\delta}}(Z)\lambda^\alpha w_\beta\hat{\lambda}^{\hat{\gamma}}\hat{w}_{\hat{\delta}} + \mathcal{L}_\lambda + \hat{\mathcal{L}}_{\hat{\lambda}}. \end{aligned}$$

In this action the fields Z^M describe a mapping of the worldsheet into a superspace $\mathbb{R}^{10|32}$ and can be broken up into a bosonic part and fermionic parts $Z^M = (Z^m, \theta^\alpha, \hat{\theta}^{\hat{\alpha}})$. In the type-IIA theory θ and $\hat{\theta}$ have opposing chiralities whereas in the type-IIB theory they have the same chirality. The remaining fields ω_α and λ^α (and their hatted counterparts) are conjugate variables and are bosonic spinor ghosts with kinetic terms \mathcal{L}_λ . λ^α obey the pure spinor constraints $\lambda^\alpha\gamma_{\alpha\beta}^m\lambda^\beta = 0$ so that their contribution to the central charge cancels that coming from the Z^M .

The fermionic field d_α is vital in the pure spinor construction since it is used in forming the BRST operator

$$Q = \oint \lambda^\alpha d_\alpha, \quad \hat{Q} = \oint \hat{\lambda}^{\hat{\alpha}} \hat{d}_{\hat{\alpha}}. \quad (4.2)$$

In flat space the nilpotency of this BRST operator is shown using the OPE of the d_α and also the pure spinor constraint. For the curved space theory defined above demanding that Q is nilpotent and holomorphic constrains the background fields to obey the equations of motion of type-II supergravity.

Other than the inclusion of a Fradkin–Tseytlin term, the action (4.1) represents the most general σ -model coupled to background fields whose interpretation we now summarise. The superfield $L_{MN}(Z)$ is defined as in (3.1) and contains the metric and NS two-form. The field $P^{\alpha\hat{\beta}}$ contains the RR field strengths and has lowest component

$$P^{\alpha\hat{\beta}}|_{\theta, \hat{\theta}=0} = -\frac{i}{4}e^\phi F^{\alpha\hat{\beta}}$$

$$= -\frac{i}{4}e^\phi \left((\gamma^m)^{\alpha\hat{\beta}} F_m + \frac{1}{3!} (\gamma^{m_1 m_2 m_3})^{\alpha\hat{\beta}} F_{m_1 m_2 m_3} + \frac{1}{2} \frac{1}{5!} (\gamma^{m_1 \dots m_5})^{\alpha\hat{\beta}} F_{m_1 \dots m_5} \right) (4.3)$$

with a similar expression for the type-IIA theory involving even forms. The field E_M^α is part of the super-vielbein, $\hat{\Omega}_{\mu\hat{\alpha}}^{\hat{\beta}}$ contains the (torsionfull) spin connection and $S_{\alpha\hat{\gamma}}^{\beta\hat{\delta}}$ contains curvature terms.

4.2 Bosonic T-duality

We demand that the background fields entering (4.1) are independent of some bosonic coordinate X_0 and denote the rest of them by Z^μ . The action (4.1) corresponds to a Hamiltonian with density of the form (2.4) but with J_\pm and V defined by

$$\begin{aligned} J_+ &= \frac{1}{2} L_{\mu 0} \partial_+ Z^\mu + E_0^\alpha d_\alpha + \Omega_{0\alpha}{}^\beta \lambda^\alpha \omega_\beta, \\ J_- &= \frac{1}{2} L_{0\mu} \partial_- Z^\mu + E_0^{\hat{\alpha}} \hat{d}_{\hat{\alpha}} + \hat{\Omega}_{0\hat{\alpha}}{}^{\hat{\beta}} \hat{\lambda}^{\hat{\alpha}} \hat{\omega}_{\hat{\beta}} \end{aligned} \quad (4.4)$$

and

$$\begin{aligned} V &= -\frac{1}{2} L_{\mu\nu} \partial_+ Z^\mu \partial_- Z^\nu - P^{\alpha\hat{\beta}} d_\alpha \hat{d}_{\hat{\beta}} - E_\mu^\alpha d_\alpha \partial_- Z^\mu - E_\mu^{\hat{\alpha}} \hat{d}_{\hat{\alpha}} \partial_+ Z^\mu \\ &\quad - \Omega_{\mu\alpha}{}^\beta \lambda^\alpha \omega_\beta \partial_- Z^\mu - \hat{\Omega}_{\mu\hat{\alpha}}{}^{\hat{\beta}} \hat{\lambda}^{\hat{\alpha}} \hat{\omega}_{\hat{\beta}} \partial_+ Z^\mu \\ &\quad - C_\alpha^{\beta\hat{\gamma}} \lambda^\alpha \omega_\beta \hat{d}_{\hat{\gamma}} - \hat{C}_{\hat{\alpha}}^{\beta\hat{\gamma}} \hat{\lambda}^{\hat{\alpha}} \hat{\omega}_{\hat{\beta}} d_{\hat{\gamma}} - S_{\alpha\hat{\gamma}}^{\beta\hat{\delta}} \lambda^\alpha \omega_\beta \hat{\lambda}^{\hat{\gamma}} \hat{\omega}_{\hat{\delta}} - \mathcal{L}_\lambda - \hat{\mathcal{L}}_{\hat{\lambda}}. \end{aligned} \quad (4.5)$$

In this case the transformation (2.6) should be accompanied with a transformations that changes the chirality of the spinors. We may choose to change the chirality of either component corresponding to the hatted or unhatted symbols. We choose to do so for the hatted ones, which implies that all hatted fermions transform as

$$\tilde{\hat{\psi}} = \Gamma \hat{\psi}, \quad \hat{\psi} = (\hat{\theta}, \hat{d}, \hat{\lambda}, \hat{\omega}), \quad (4.6)$$

where $\gamma_1 \equiv \Gamma$ is the gamma-matrix in the direction of the isometry. This transformation clearly leaves invariant the Poisson brackets between the spinorial fields due to the fact that $\Gamma^2 = \mathbb{I}$.

Using the first of (2.9) and (4.6) we obtain that

$$\tilde{G}_{00} = \frac{1}{G_{00}}, \quad \tilde{L}_{\mu 0} = -\frac{L_{\mu 0}}{G_{00}}, \quad \tilde{L}_{0\mu} = \frac{L_{0\mu}}{G_{00}},$$

$$\begin{aligned}\tilde{E}_0^\alpha &= -G_{00}^{-1}E_0^\alpha, & \tilde{E}_0^{\hat{\alpha}} &= G_{00}^{-1}E_0^{\hat{\beta}}\Gamma_{\hat{\beta}}^{\hat{\alpha}}, \\ \tilde{\Omega}_{0\alpha}^\beta &= -G_{00}^{-1}\Omega_{0\alpha}^\beta, & \tilde{\Omega}_{0\hat{\alpha}}^{\hat{\beta}} &= G_{00}^{-1}\hat{\Omega}_{0\hat{\gamma}}^{\hat{\delta}}\Gamma_{\hat{\alpha}}^{\hat{\gamma}}\Gamma_{\hat{\delta}}^{\hat{\beta}}.\end{aligned}\quad (4.7)$$

Using the second of (2.9) and (4.6) as well as introducing the matrices

$$A_M^N = \begin{pmatrix} -G_{00}^{-1} & 0 \\ -G_{00}^{-1}L_{0\mu} & \delta_{\mu}^\nu \end{pmatrix}, \quad \hat{A}_M^N = \begin{pmatrix} G_{00}^{-1} & 0 \\ -G_{00}^{-1}L_{\mu 0} & \delta_{\mu}^\nu \end{pmatrix}, \quad (4.8)$$

we obtain that

$$\begin{aligned}\tilde{L}_{\mu\nu} &= L_{\mu\nu} - \frac{L_{0\nu}L_{\mu 0}}{G_{00}}, \\ \tilde{E}_M^\alpha &= A_M^N E_N^\alpha, & \tilde{E}_M^{\hat{\alpha}} &= \hat{A}_M^N E_N^{\hat{\beta}}\Gamma_{\hat{\beta}}^{\hat{\alpha}}, \\ \tilde{\Omega}_{M\alpha}^\beta &= A_M^N \Omega_{N\alpha}^\beta, & \tilde{\Omega}_{M\hat{\alpha}}^{\hat{\beta}} &= \hat{A}_M^N \hat{\Omega}_{N\hat{\gamma}}^{\hat{\delta}}\Gamma_{\hat{\alpha}}^{\hat{\gamma}}\Gamma_{\hat{\delta}}^{\hat{\beta}}\end{aligned}\quad (4.9)$$

and

$$\begin{aligned}\tilde{P}^{\alpha\hat{\beta}} &= \left(P^{\alpha\hat{\gamma}} + \frac{2}{G_{00}} E_0^\alpha E_0^{\hat{\gamma}} \right) \Gamma_{\hat{\gamma}}^{\hat{\beta}}, \\ \tilde{C}_\beta^{\alpha\hat{\gamma}} &= \left(C_\beta^{\alpha\hat{\gamma}} - \frac{2}{G_{00}} \Omega_{0\beta}^\alpha E_0^{\hat{\delta}} \right) \Gamma_{\hat{\delta}}^{\hat{\gamma}}, & \tilde{C}_{\hat{\beta}}^{\hat{\alpha}\hat{\gamma}} &= \left(C_{\hat{\gamma}}^{\hat{\alpha}\hat{\delta}} - \frac{2}{G_{00}} E_0^{\hat{\gamma}} \hat{\Omega}_{0\hat{\delta}}^{\hat{\delta}} \right) \Gamma_{\hat{\delta}}^{\hat{\alpha}} \Gamma_{\hat{\beta}}^{\hat{\delta}}, \\ \tilde{S}_{\hat{\beta}\hat{\delta}}^{\alpha\hat{\gamma}} &= \left(S_{\hat{\beta}\hat{\zeta}}^{\alpha\hat{\epsilon}} - \frac{2}{G_{00}} \Omega_{0\beta}^\alpha \hat{\Omega}_{0\hat{\zeta}}^{\hat{\epsilon}} \right) \Gamma_{\hat{\epsilon}}^{\hat{\gamma}} \Gamma_{\hat{\delta}}^{\hat{\zeta}}.\end{aligned}\quad (4.10)$$

In practice, the expression (4.3) can be used to read off the transformation rules of the various forms in the theory. The fact that we have changed the chirality of the hatted fermions according to (4.6) has the consequence that the bosonic T-duality takes one from the type-IIA to the type-IIB and vice versa.

4.3 Fermionic T-duality

The treatment of fermionic T-duality in the pure spinor superstring follows exactly the steps described in section 3. The Lagrangian is of the same form as (3.4) but with \mathcal{J}_\pm in (3.3) replaced by

$$\begin{aligned}\mathcal{J}_+ &= \frac{1}{2}L_{\mu 1}\partial_+ Z^\mu + E_1^\alpha d_\alpha + \Omega_{1\alpha}^\beta \lambda^\alpha \omega_\beta, \\ \mathcal{J}_- &= -(-1)^\mu \frac{1}{2}L_{1\mu}\partial_- Z^\mu - E_1^{\hat{\alpha}} \hat{d}_{\hat{\alpha}} - \hat{\Omega}_{1\hat{\alpha}}^{\hat{\beta}} \hat{\lambda}^{\hat{\alpha}} \hat{\omega}_{\hat{\beta}}\end{aligned}\quad (4.11)$$

and the potential \mathcal{V} given by the lengthy expression in (4.5) but with the understanding that Z^μ runs over all coordinates except θ , the fermionic direction along which we are dualising.

We have seen, that under the canonical transformation (3.19), the transformed *total* Hamiltonian can be viewed as the Hamiltonian of a dual σ -model with the identifications

$$\tilde{B}_{11} = -\frac{1}{B_{11}}, \quad \tilde{\mathcal{J}}_{\pm} = \frac{\mathcal{J}_{\pm}}{B_{11}}, \quad \tilde{\mathcal{V}} = \mathcal{V} + 2\frac{\mathcal{J}_+\mathcal{J}_-}{B_{11}}. \quad (4.12)$$

Inserting the form of the currents (4.11) and potential (4.5) into these relations yields the fermionic T-duality rules³

$$\begin{aligned} \tilde{B}_{11} &= -\frac{1}{B_{11}}, & \tilde{L}_{\mu 1} &= \frac{L_{\mu 1}}{B_{11}}, & \tilde{L}_{1\mu} &= \frac{L_{1\mu}}{B_{11}}, \\ \tilde{L}_{\mu\nu} &= L_{\mu\nu} - \frac{L_{1\nu}L_{\mu 1}}{B_{11}}, \\ \tilde{p}^{\alpha\hat{\beta}} &= p^{\alpha\hat{\beta}} + 2\frac{E_1^\alpha E_1^{\hat{\beta}}}{B_{11}}, \end{aligned} \quad (4.13)$$

$$\begin{aligned} \tilde{E}_M^\alpha &= B_M^N E_N^\alpha, & \tilde{E}_M^{\hat{\alpha}} &= \hat{B}_M^N E_N^{\hat{\alpha}}, \\ \tilde{\Omega}_{M\alpha}^\beta &= B_M^N \tilde{\Omega}_{N\alpha}^\beta, & \tilde{\Omega}_{M\hat{\alpha}}^{\hat{\beta}} &= \hat{B}_M^N \hat{\Omega}_{N\hat{\alpha}}^{\hat{\beta}}, \end{aligned} \quad (4.14)$$

where

$$B_M^N = \begin{pmatrix} B_{11}^{-1} & 0 \\ -B_{11}^{-1}L_{1\mu} & \delta_\mu^\nu \end{pmatrix}, \quad \hat{B}_M^N = \begin{pmatrix} B_{11}^{-1} & 0 \\ (-1)^\mu B_{11}^{-1}L_{\mu 1} & \delta_\mu^\nu \end{pmatrix} \quad (4.15)$$

and

$$\begin{aligned} \tilde{C}_\beta^{\alpha\hat{\gamma}} &= C_\beta^{\alpha\hat{\gamma}} + 2B_{11}^{-1}E_1^{\hat{\gamma}}\Omega_{1\beta}^\alpha, & \tilde{\hat{C}}_{\hat{\beta}}^{\hat{\alpha}\gamma} &= \hat{C}_{\hat{\beta}}^{\hat{\alpha}\gamma} - 2B_{11}^{-1}\Omega_{1\hat{\beta}}^{\hat{\alpha}}E_1^\gamma \\ \tilde{S}_{\beta\hat{\delta}}^{\alpha\hat{\gamma}} &= S_{\beta\hat{\delta}}^{\alpha\hat{\gamma}} + 2B_{11}^{-1}\Omega_{1\beta}^\alpha\hat{\Omega}_{1\hat{\delta}}^{\hat{\gamma}}. \end{aligned} \quad (4.16)$$

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³Note that some of signs in the expressions presented here differ from those in equation 2.20 of [2]. In addition we have corrected a small typographical error in the transformation of $S_{\beta\hat{\delta}}^{\alpha\hat{\gamma}}$.

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