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# New maximal regularity results for the heat equation in exterior domains, and applications

R. Danchin and P. B. Mucha

**Abstract** This paper is dedicated to the proof of new maximal regularity results involving Besov spaces for the heat equation in the half-space or in bounded or exterior domains of  $\mathbb{R}^n$ . We strive for *time independent* a priori estimates in regularity spaces of type  $L_1(0, T; X)$  where  $X$  stands for some *homogeneous* Besov space. In the case of bounded domains, the results that we get are similar to those of the whole space or of the half-space. For exterior domains, we need to use mixed Besov norms in order to get a control on the low frequencies. Those estimates are crucial for proving global-in-time results for nonlinear heat equations in a critical functional framework.

## Introduction

We are concerned with the proof of maximal regularity estimates for the heat equation with Dirichlet boundary conditions, namely,

$$\begin{aligned} u_t - \nu \Delta u &= f && \text{in } (0, T) \times \Omega, \\ u &= 0 && \text{at } (0, T) \times \partial\Omega, \\ u &= u_0 && \text{on } \Omega \end{aligned} \tag{1}$$

in various domains  $\Omega$  of  $\mathbb{R}^n$  ( $n \geq 2$ ).

We are interested in  $L_1$ -in-time estimates for the solutions to (1) with a gain of two full spatial derivatives with respect to the data, that is

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$$\|u_t, v\nabla^2 u\|_{L_1(0,T;X)} \leq C(\|u_0\|_X + \|f\|_{L_1(0,T;X)}) \quad (2)$$

with a constant  $C$  independent of  $T$ .

Such time independent estimates are of importance not only for the heat semi-group theory but also in the applications. Typically, they are crucial for proving global existence and uniqueness statements for nonlinear heat equations with small data in a critical functional framework. Moreover, the fact that two full derivatives may be gained with respect to the source term allows to consider not only the  $-\Delta$  operator but also small perturbations of it. In addition, we shall see below that it is possible to choose  $X$  in such a way that the constructed solution  $u$  is  $L_1$ -in-time with values in the set of Lipschitz functions. Hence, if the considered nonlinear heat equation determines the velocity field of some fluid then this velocity field admits a unique Lipschitzian flow for all time. The model may thus be reformulated equivalently in Lagrangian variables (see e.g. our recent work [4] in the slightly different context of incompressible flows). This is obviously of interest to investigate free boundary problems.

Let us recall however that estimates such as (2) are false if  $X$  is any reflexive Banach space, hence in particular if  $X$  is a Lebesgue or Sobolev space (see e.g. [6]). On the other hand, it is well known that (2) holds true in the whole space  $\mathbb{R}^n$  if  $X$  is a homogeneous Besov space *with third index* 1. Let us be more specific. Let us fix some homogeneous Littlewood-Paley decomposition  $(\dot{\Delta}_j)_{j \in \mathbb{Z}}$  (see the definition in the next section) and denote by  $(e^{\alpha\Delta})_{\alpha>0}$  the heat semi-group over  $\mathbb{R}^n$ . Then it is well known (see e.g. [1]) that there exist two constants  $c$  and  $C$  such that for all  $j \in \mathbb{Z}$  and  $\alpha \in \mathbb{R}^+$  one has

$$\|e^{\alpha\Delta} \dot{\Delta}_j h\|_{L_p(\mathbb{R}^n)} \leq C e^{-c\alpha 2^{2j}} \|\dot{\Delta}_j h\|_{L_p(\mathbb{R}^n)}. \quad (3)$$

Hence if  $u$  satisfies (1) then one may write

$$\dot{\Delta}_j u(t) = e^{v t \Delta} \dot{\Delta}_j u_0 + \int_0^t e^{v(t-\tau)\Delta} \dot{\Delta}_j f \, d\tau.$$

Therefore, taking advantage of (3), we discover that

$$\|\dot{\Delta}_j u(t)\|_{L_p(\mathbb{R}^n)} \leq C \left( e^{-c v t 2^{2j}} \|\dot{\Delta}_j u_0\|_{L_p(\mathbb{R}^n)} + \int_0^t e^{-c v(t-\tau) 2^{2j}} \|\dot{\Delta}_j f\|_{L_p(\mathbb{R}^n)} \, d\tau \right),$$

whence

$$\begin{aligned} \|\dot{\Delta}_j u\|_{L_\infty(0,T;L_p(\mathbb{R}^n))} + v 2^{2j} \|\dot{\Delta}_j u\|_{L_1(0,T;L_p(\mathbb{R}^n))} \\ \leq C (\|\dot{\Delta}_j u_0\|_{L_p(\mathbb{R}^n)} + \|\dot{\Delta}_j f\|_{L_1(0,T;L_p(\mathbb{R}^n))}). \end{aligned}$$

Multiplying the inequality by  $2^{js}$  and summing up over  $j$ , we thus eventually get for some absolute constant  $C$  independent of  $v$  and  $T$ ,

$$\begin{aligned} \|u\|_{L^\infty(0,T;\dot{B}_{p,1}^s(\mathbb{R}^n))} + \|u_t, \nu \nabla^2 u\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}^n))} \\ \leq C(\|f\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}^n))} + \|u_0\|_{\dot{B}_{p,1}^s(\mathbb{R}^n)}), \end{aligned} \quad (4)$$

where the homogeneous Besov semi-norm that is used in the above inequality is defined by

$$\|u\|_{\dot{B}_{p,1}^s(\mathbb{R}^n)} := \sum_{j \in \mathbb{Z}} 2^{sj} \|\dot{\Delta}_j u\|_{L_p(\mathbb{R}^n)}.$$

From this and the definition of homogeneous Besov space  $\dot{B}_{p,1}^s(\mathbb{R}^n)$  (see Section 1), we easily deduce the following classical result:

**Theorem 1.** *Let  $p \in [1, \infty]$  and  $s \in \mathbb{R}$ . Let  $f \in L_1(0, T; \dot{B}_{p,1}^s(\mathbb{R}^n))$  and  $u_0 \in \dot{B}_{p,1}^s(\mathbb{R}^n)$ . Then (1) with  $\Omega = \mathbb{R}^n$  has a unique solution  $u$  in*

$$\mathcal{C}([0, T]; \dot{B}_{p,1}^s(\mathbb{R}^n)) \quad \text{with} \quad \partial_t u, \nabla^2 u \in L_1(0, T; \dot{B}_{p,1}^s(\mathbb{R}^n))$$

and (4) is satisfied.

The present paper is mainly devoted to generalizations of Theorem 1 to the half-space, bounded or exterior domains (that is the complement of bounded simply connected domains), and applications to the global solvability of nonlinear heat equations.

Proving maximal regularity estimates for general domains essentially relies on Theorem 1 and localization techniques. More precisely, after localizing the equation thanks to a suitable resolution of unity, one has to estimate ‘‘interior terms’’ with support that do not intersect the boundary of  $\Omega$  and ‘‘boundary terms’’ the support of which meets  $\partial\Omega$ . In order to prove *interior estimates* that is bounds for the interior terms, it suffices to resort to the theorem in the whole space, Theorem 1, for those interior terms satisfy (1) (with suitable data) once extended by zero onto the whole space. In contrast, the extension of the boundary terms by zero does not satisfy (1) on  $\mathbb{R}^n$ . However, performing a change of variable reduces their study to that of (1) on the half-space  $\mathbb{R}_+^n$ . Therefore, proving maximal regularity estimates in general domains mainly relies on such estimates on  $\mathbb{R}^n$  and on  $\mathbb{R}_+^n$ . As a matter of fact, we shall see that the latter case stems from the former, by symmetrization, *provided  $s$  is close enough to 0*. In the case of a general domain, owing to change of variables and localization however, we shall obtain (4) either *up to low order terms* or with a *time-dependent* constant  $C$ . In a bounded domain, it turns out that Poincaré inequality (or equivalently the fact that the Dirichlet Laplacian operator has eigenvalues bounded away from 0) allows to prove an exponential decay which is sufficient to cancel out those lower order terms. In the case of an exterior domain, that decay turns out to be only algebraic (at most  $t^{-n/2}$  in dimension  $n$ ). As a consequence, absorbing the lower order terms will enforce us to use mixed Besov norms and to assume that  $n \geq 3$ .

The paper unfolds as follows. The basic tools for our analysis (Besov spaces on domains, product estimates, embedding results) are presented in the next section. In

Section 2 we prove maximal regularity estimates similar to those of Theorem 1 first in the half-space and next in exterior or bounded domains. As an application, in the last section, we establish global existence results for nonlinear heat equations with small data in a critical functional framework.

## 1 Tools

In this section, we introduce the main functional spaces and (harmonic analysis) tools that will be needed in this paper.

### 1.1 Besov spaces on the whole space

Throughout we fix a smooth nonincreasing radial function  $\chi : \mathbb{R}^n \rightarrow [0, 1]$  supported in  $B(0, 1)$  and such that  $\chi \equiv 1$  on  $B(0, 1/2)$ , and set  $\varphi(\xi) := \chi(\xi/2) - \chi(\xi)$ . Note that this implies that  $\varphi$  is valued in  $[0, 1]$ , supported in  $\{1/2 \leq r \leq 2\}$  and that

$$\sum_{k \in \mathbb{Z}} \varphi(2^{-k}\xi) = 1 \quad \text{for all } \xi \neq 0. \quad (5)$$

Then we introduce the homogeneous Littlewood-Paley decomposition  $(\dot{\Delta}_k)_{k \in \mathbb{Z}}$  over  $\mathbb{R}^n$  by setting

$$\dot{\Delta}_k u := \varphi(2^{-k}D)u = \mathcal{F}^{-1}(\varphi(2^{-k}\cdot)\mathcal{F}u).$$

Above  $\mathcal{F}$  stands for the Fourier transform on  $\mathbb{R}^n$ . We also define the low frequency cut-off  $\dot{S}_k := \chi(2^{-k}D)$ .

In order to define Besov spaces on  $\mathbb{R}^n$ , we first introduce the following homogeneous semi-norms and nonhomogeneous Besov norms (for all  $s \in \mathbb{R}$  and  $(p, r) \in [1, \infty]^2$ ):

$$\begin{aligned} \|u\|_{\dot{B}_{p,r}^s(\mathbb{R}^n)} &:= \left\| 2^{sk} \|\dot{\Delta}_k u\|_{L_p(\mathbb{R}^n)} \right\|_{\ell_r(\mathbb{Z})} \\ \|u\|_{B_{p,r}^s(\mathbb{R}^n)} &:= \left\| 2^{sk} \|\dot{\Delta}_k u\|_{L_p(\mathbb{R}^n)} \right\|_{\ell_r(\mathbb{N})} + \|\dot{S}_0 u\|_{L_p(\mathbb{R}^n)}. \end{aligned}$$

The nonhomogeneous Besov space  $B_{p,r}^s(\mathbb{R}^n)$  is the set of tempered distributions  $u$  such that  $\|u\|_{B_{p,r}^s(\mathbb{R}^n)}$  is finite. Following [1], we define the homogeneous Besov space  $\dot{B}_{p,r}^s(\mathbb{R}^n)$  as

$$\dot{B}_{p,r}^s(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'_h(\mathbb{R}^n) : \|u\|_{\dot{B}_{p,r}^s(\mathbb{R}^n)} < \infty \right\},$$

where  $\mathcal{S}'_h(\mathbb{R}^n)$  stands for the set of tempered distributions  $u$  over  $\mathbb{R}^n$  such that for all smooth compactly supported function  $\theta$  over  $\mathbb{R}^n$ , we have  $\lim_{\lambda \rightarrow +\infty} \theta(\lambda D)u = 0$  in  $L_\infty(\mathbb{R}^n)$ . Note that any distribution  $u \in \mathcal{S}'_h(\mathbb{R}^n)$  satisfies  $u = \sum_{k \in \mathbb{Z}} \dot{\Delta}_k u$  in  $\mathcal{S}'_h(\mathbb{R}^n)$ .

We shall make an extensive use of the following result (see the proof in e.g. [1, 5]):

**Proposition 1.** *Let  $b_{p,r}^s$  denote  $\dot{B}_{p,r}^s(\mathbb{R}^n)$  or  $B_{p,r}^s(\mathbb{R}^n)$ . Then the following a priori estimates hold true:*

- For any  $s > 0$ ,

$$\|uv\|_{b_{p,r}^s} \lesssim \|u\|_{L^\infty} \|v\|_{b_{p,r}^s} + \|v\|_{L^\infty} \|u\|_{b_{p,r}^s}.$$

- For any  $s > 0$  and  $t > 0$ ,

$$\|uv\|_{b_{p,r}^s} \lesssim \|u\|_{L^\infty} \|v\|_{b_{p,r}^s} + \|v\|_{b_{\infty,r}^{-t}} \|u\|_{b_{p,\infty}^{s+t}}.$$

- For any  $t > 0$  and  $s > -n/p'$ ,

$$\|uv\|_{b_{p,r}^s} \lesssim \|u\|_{L^\infty} \|v\|_{b_{p,r}^s} + \|u\|_{b_{p',\infty}^{n/p'}} \|v\|_{b_{p,r}^s} + \|v\|_{b_{\infty,r}^{-t}} \|u\|_{b_{p,\infty}^{s+t}}.$$

- For any  $q > 1$  and  $1 - n/q \leq s \leq 1$ ,

$$\|uv\|_{b_{q,1}^0} \lesssim \|u\|_{b_{n,1}^s} \|v\|_{b_{q,1}^{1-s}}.$$

As obviously a smooth compactly supported function belongs to any space  $\dot{B}_{p,1}^{n/p}(\mathbb{R}^n)$  with  $1 \leq p \leq \infty$ , and to any Besov space  $B_{p,1}^\sigma(\mathbb{R}^n)$ , we deduce from the previous proposition and embedding that (see the proof in [5]):

**Corollary 1.** *Let  $\theta$  be in  $\mathcal{C}_c^\infty(\mathbb{R}^n)$ . Then  $u \mapsto \theta u$  is a continuous mapping of  $b_{p,r}^s(\mathbb{R}^n)$*

- for any  $s \in \mathbb{R}$  and  $1 \leq p, r \leq \infty$ , if  $b_{p,r}^s(\mathbb{R}^n) = B_{p,r}^s(\mathbb{R}^n)$ ;
- for any  $s \in \mathbb{R}$  and  $1 \leq p, r \leq \infty$  satisfying

$$-n/p' < s < n/p \quad (-n/p < s \leq n/p \text{ if } r = 1, \quad -n/p' \leq s < n/p \text{ if } r = \infty) \quad (6)$$

$$\text{if } b_{p,r}^s(\mathbb{R}^n) = \dot{B}_{p,r}^s(\mathbb{R}^n).$$

The following proposition allows us to compare the spaces  $B_{p,r}^s(\mathbb{R}^n)$  and  $\dot{B}_{p,r}^s(\mathbb{R}^n)$  for compactly supported functions<sup>1</sup> (see the proof in [5]):

**Proposition 2.** *Let  $1 \leq p, r \leq \infty$  and  $s > -n/p'$  (or  $s \geq -n/p'$  if  $r = \infty$ ). Then for any compactly supported distribution  $f$  we have*

$$f \in B_{p,r}^s(\mathbb{R}^n) \iff f \in \dot{B}_{p,r}^s(\mathbb{R}^n)$$

and there exists a constant  $C = C(s, p, r, n, K)$  (with  $K = \text{Supp } f$ ) such that

$$C^{-1} \|f\|_{\dot{B}_{p,r}^s(\mathbb{R}^n)} \leq \|f\|_{B_{p,r}^s(\mathbb{R}^n)} \leq C \|f\|_{\dot{B}_{p,r}^s(\mathbb{R}^n)}.$$

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<sup>1</sup> Without any support assumption, it is obvious that if  $s$  is positive then we have  $\|\cdot\|_{\dot{B}_{p,r}^s(\mathbb{R}^n)} \lesssim \|\cdot\|_{B_{p,r}^s(\mathbb{R}^n)}$ , and the opposite inequality holds true if  $s$  is negative.

The following lemma will be useful for boundary estimates (see the proof in [5]):

**Lemma 1.** *Let  $Z$  be a Lipschitz diffeomorphism on  $\mathbb{R}^n$  with  $DZ$  and  $DZ^{-1}$  bounded,  $(p, r) \in [1, \infty]^2$  and  $s$  a real number satisfying (6).*

- *If in addition  $s \in (-1, 1)$  and  $Z$  is measure preserving then the linear map  $u \mapsto u \circ Z$  is continuous on  $\dot{B}_{p,r}^s(\mathbb{R}^n)$ .*
- *In the general case, the map  $u \mapsto u \circ Z$  is continuous on  $\dot{B}_{p,r}^s(\mathbb{R}^n)$  provided in addition  $J_{Z^{-1}} \in \dot{B}_{p',\infty}^{n/p'} \cap L_\infty$  with  $J_{Z^{-1}} := |\det DZ^{-1}|$ .*

## 1.2 Besov spaces on domains

We aim at extending the definition of homogeneous Besov spaces to general domains. We proceed by restriction as follows<sup>2</sup>:

**Definition 1.** For  $s \in \mathbb{R}$  and  $1 \leq p, q \leq \infty$ , we define the homogeneous Besov space  $\dot{B}_{p,q}^s(\Omega)$  over  $\Omega$  as the restriction (in the distributional sense) of  $\dot{B}_{p,q}^s(\mathbb{R}^n)$  on  $\Omega$ , that is

$$\phi \in \dot{B}_{p,q}^s(\Omega) \iff \phi = \psi|_\Omega \quad \text{for some } \psi \in \dot{B}_{p,q}^s(\mathbb{R}^n).$$

We then set

$$\|\phi\|_{\dot{B}_{p,q}^s(\Omega)} := \inf_{\psi|_\Omega = \phi} \|\psi\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)}.$$

The embedding, duality and interpolation properties of these Besov spaces may be deduced from those on  $\mathbb{R}^n$ . As regards duality, we shall use repeatedly the following result:

**Proposition 3.** *If  $-1 + 1/p < s < 1/p$  (with  $1 \leq p, r < \infty$ ) then the space  $\dot{B}_{p',r}^{-s}(\Omega)$  may be identified with the dual space of  $\dot{B}_{p,r}^s(\Omega)$ ; in the limit case  $r = \infty$  then  $\dot{B}_{p',1}^{-s}(\Omega)$  may be identified with the dual space of the completion of  $\mathcal{C}_c^\infty(\mathbb{R}^n)$  for  $\|\cdot\|_{\dot{B}_{p,\infty}^s(\Omega)}$ . Furthermore, without any condition over  $(s, p, r)$ , we have*

$$\left| \int_\Omega uv dx \right| \leq C \|u\|_{\dot{B}_{p,r}^s(\Omega)} \|v\|_{\dot{B}_{p',r}^{-s}(\Omega)}.$$

Similarly, some product laws for Besov spaces on  $\mathbb{R}^n$  may be extended to the domain case. We shall use the last inequality of Proposition 1 and also the following result that is proved in [5]:

**Proposition 4.** *Let  $b_{p,r}^s(\Omega)$  denote  $\dot{B}_{p,r}^s(\Omega)$  or  $B_{p,r}^s(\Omega)$ , and  $\Omega$  be any domain of  $\mathbb{R}^n$ . Then for any  $p \in [1, \infty]$ ,  $s$  such that  $-n/p' < s < n/p$  (or  $-n/p' < s \leq n/p$  if  $r = 1$ , or  $-n/p' \leq s < n/p$  if  $r = \infty$ ), the following inequality holds true:*

$$\|uv\|_{b_{p,r}^s(\Omega)} \leq C \|u\|_{b_{q,1}^{n/q}(\Omega)} \|v\|_{b_{p,r}^s(\Omega)} \quad \text{with } q = \min(p, p').$$

<sup>2</sup> Nonhomogeneous Besov spaces on domains may be defined by the same token.

A very useful feature of Besov spaces is their interpolation properties. We refer to the books [2, 14] for the proof of the following statement.

**Proposition 5.** *Let  $b_{p,q}^s$  denote  $B_{p,q}^s$  or  $\dot{B}_{p,q}^s$ ;  $s \in \mathbb{R}$ ,  $p \in [1, \infty)$  and  $q \in [1, \infty]$ . The real interpolation of Besov spaces gives the following statement if  $s_1 \neq s_2$ :*

$$(b_{p,q_1}^{s_1}(\Omega), b_{p,q_2}^{s_2}(\Omega))_{\theta,q} = b_{p,q}^s(\Omega)$$

with  $s = \theta s_2 + (1 - \theta)s_1$ , and  $\frac{1}{p} = \frac{\theta}{p_2} + \frac{1-\theta}{p_1}$ .

Moreover, if  $s_1 \neq s_2$ ,  $t_1 \neq t_2$  and if  $T : b_{p_1,q_1}^{s_1}(\Omega) + b_{p_2,q_2}^{s_2}(\Omega) \rightarrow b_{k_1,l_1}^{t_1}(\Omega) + b_{k_2,l_2}^{t_2}(\Omega)$  is a linear map, bounded from  $b_{p_1,q_1}^{s_1}(\Omega)$  to  $b_{k_1,l_1}^{t_1}(\Omega)$  and from  $b_{p_2,q_2}^{s_2}(\Omega)$  to  $b_{k_2,l_2}^{t_2}(\Omega)$  then for any  $\theta \in (0, 1)$ , the map  $T$  is also bounded from  $b_{p,q}^s(\Omega)$  to  $b_{k,q}^t(\Omega)$  with

$$s = \theta s_2 + (1 - \theta)s_1, \quad t = \theta t_2 + (1 - \theta)t_1, \quad \frac{1}{p} = \frac{\theta}{p_2} + \frac{1-\theta}{p_1}, \quad \frac{1}{k} = \frac{\theta}{k_2} + \frac{1-\theta}{k_1}.$$

The following composition estimate will be of constant use in the last section of this paper.

**Proposition 6.** *Let  $f : \mathbb{R}^r \rightarrow \mathbb{R}$  be a  $C^1$  function such that  $f(\mathbf{0}) = 0$  and, for some  $m \geq 1$  and  $K \geq 0$ ,*

$$|df(\mathbf{u})| \leq K|\mathbf{u}|^{m-1} \quad \text{for all } \mathbf{u} \in \mathbb{R}^r. \quad (7)$$

Then for all  $s \in (0, 1)$  and  $1 \leq p, q \leq \infty$  there exists a constant  $C$  so that

$$\|f(\mathbf{u})\|_{\dot{B}_{p,q}^s(\Omega)} \leq CK\|\mathbf{u}\|_{L^\infty(\Omega)}^{m-1}\|u\|_{\dot{B}_{p,q}^s(\Omega)}. \quad (8)$$

*Proof.* The proof relies on the characterization of the norm of  $\dot{B}_{p,q}^s(\Omega)$  by finite differences, namely<sup>3</sup>

$$\|f(\mathbf{u})\|_{\dot{B}_{p,q}^s(\Omega)} = \left( \int_{\Omega} \left( \int_{\Omega} \frac{|f(\mathbf{u}(y)) - f(\mathbf{u}(x))|^p}{|y-x|^{n+sp}} dy \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}}. \quad (9)$$

Now the mean value formula implies that

$$f(\mathbf{u}(y)) - f(\mathbf{u}(x)) = \left( \int_0^1 df(\mathbf{u}(x) + t(\mathbf{u}(y) - \mathbf{u}(x))) dt \right) \cdot (\mathbf{u}(y) - \mathbf{u}(x)).$$

Hence using the growth assumption (7),

$$|f(\mathbf{u}(y)) - f(\mathbf{u}(x))| \leq K \left( \int_0^1 |\mathbf{u}(x) + t(\mathbf{u}(y) - \mathbf{u}(x))|^{m-1} dt \right) |\mathbf{u}(y) - \mathbf{u}(x)|. \quad (10)$$

Therefore we get

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<sup>3</sup> Here we just consider the case  $q < \infty$  to shorten the presentation.



$$|f(\mathbf{u}(y)) - f(\mathbf{u}(x))| \leq K \|\mathbf{u}\|_{L^\infty(\Omega)}^{m-1} |\mathbf{u}(y) - \mathbf{u}(x)|.$$

Inserting this latter inequality in (9), we readily get (8).

In [3, 5], we proved that:

**Proposition 7.** *Let  $\Omega$  be the half-space, or a bounded or exterior domain with  $C^1$  boundary. For all  $1 \leq p, q < \infty$ , and  $-1 + 1/p < s < 1/p$ , we have*

$$B_{p,q}^s(\Omega) = \overline{\mathcal{C}_c^\infty(\Omega)}^{\|\cdot\|_{B_{p,q}^s(\Omega)}}. \quad (11)$$

*Remark 1.* In any  $C^1$  domain  $\Omega$  and for  $0 < s < n/p$  the space  $\dot{B}_{p,q}^s(\Omega)$  embeds in  $\dot{B}_{m,q}^0(\Omega)$  with  $1/m = 1/p - s/n$ . Therefore, if  $q \leq \min(2, m)$ , it also embeds in the Lebesgue space  $L_m(\Omega)$ . So finally if  $s \in (0, \frac{1}{p})$  and  $q \leq \min(2, m)$  with  $m$  as above then Proposition 7 allows us to redefine the space  $\dot{B}_{p,q}^s(\Omega)$  by

$$\dot{B}_{p,q}^s(\Omega) = \overline{\mathcal{C}_c^\infty(\Omega)}^{\|\cdot\|_{\dot{B}_{p,q}^s(\Omega)}}. \quad (12)$$

*Remark 2.* In particular under the above hypotheses, both classes of Besov spaces admit trivial extension by zero onto the whole space. Combining with Proposition 2, we deduce that

$$B_{p,q}^s(\Omega) = \dot{B}_{p,q}^s(\Omega) \quad \text{if } -1 + 1/p < s < 1/p \quad \text{and } \Omega \text{ is bounded.}$$

Note also that, for obvious reasons, the above density result does not hold true if  $q = \infty$ , for the strong topology. However, it holds for the weak \* topology.

## 2 A priori estimates for the heat equation

This section is the core of the paper. Here we prove generalizations of Theorem 1 to more general domains. First we consider the half-space case, then we consider the exterior and bounded cases. We shall mainly focus on the unbounded case which is more tricky and just indicate at the end of this section what has to be changed in the bounded domain case.

### 2.1 The heat equation in the half-space

The purpose of this paragraph is to extend Theorem 1 to the half-space case  $\mathbb{R}_+^n$ , namely

$$\begin{aligned}
u_t - \nu \Delta u &= f \quad \text{in } (0, T) \times \mathbb{R}_+^n, \\
u|_{x_n=0} &= 0 \quad \text{on } (0, T) \times \partial \mathbb{R}_+^n, \\
u|_{t=0} &= u_0 \quad \text{on } \mathbb{R}_+^n.
\end{aligned} \tag{13}$$

**Theorem 2.** *Let  $p \in [1, \infty)$  and  $s \in (-1 + 1/p, 1/p)$ . Assume that  $f$  belongs to  $L_1(0, T; \dot{B}_{p,1}^s(\mathbb{R}_+^n))$  and that  $u_0$  is in  $\dot{B}_{p,1}^s(\mathbb{R}_+^n)$ . Then (13) has a unique solution  $u$  satisfying*

$$u \in \mathcal{C}([0, T]; \dot{B}_{p,1}^s(\mathbb{R}_+^n)), \quad u_t, \nabla^2 u \in L_1(0, T; \dot{B}_{p,1}^s(\mathbb{R}_+^n))$$

and the following estimate is valid:

$$\begin{aligned}
&\|u\|_{L_\infty(0, T; \dot{B}_{p,1}^s(\mathbb{R}_+^n))} + \|u_t, \nu \nabla^2 u\|_{L_1(0, T; \dot{B}_{p,1}^s(\mathbb{R}_+^n))} \\
&\leq C(\|f\|_{L_1(0, T; \dot{B}_{p,1}^s(\mathbb{R}_+^n))} + \|u_0\|_{\dot{B}_{p,1}^s(\mathbb{R}_+^n)}), \tag{14}
\end{aligned}$$

where  $C$  is an absolute constant with no dependence on  $\nu$  and  $T$ .

*Proof.* We argue by symmetrization. Let  $\tilde{u}_0$  and  $\tilde{f}$  be the antisymmetric extensions over  $\mathbb{R}^n$  to the data  $u_0$  and  $f$ . Then, given our assumptions over  $s$  and Proposition 7, one may assert that  $\tilde{u}_0 \in \dot{B}_{p,1}^s(\mathbb{R}^n)$ ,  $\tilde{f} \in L_1(0, T; \dot{B}_{p,1}^s(\mathbb{R}^n))$  and that, in addition

$$\|\tilde{u}_0\|_{\dot{B}_{p,1}^s(\mathbb{R}^n)} \approx \|u_0\|_{\dot{B}_{p,1}^s(\mathbb{R}_+^n)} \quad \text{and} \quad \|\tilde{f}\|_{L_1(0, T; \dot{B}_{p,1}^s(\mathbb{R}^n))} \approx \|f\|_{L_1(0, T; \dot{B}_{p,1}^s(\mathbb{R}_+^n))}.$$

Let  $\tilde{u}$  be the solution given by Theorem 1. As this solution is unique in the corresponding functional framework, the symmetry properties of the data ensure that  $\tilde{u}$  is antisymmetric with respect to  $\{x_n = 0\}$ . As a consequence, it vanishes over  $\{x_n = 0\}$ . Hence the restriction  $u$  of  $\tilde{u}$  to the half-space satisfies (13). In addition,

- $\tilde{u}_t$  coincides with the antisymmetric extension of  $u_t$ ,
- $\nabla_{x'}^2 \tilde{u}$  coincides with the antisymmetric extension of  $\nabla_{x'}^2 u$ ,
- $\nabla_{x'} \partial_{x_n} \tilde{u}$  coincides with the symmetric extension of  $\nabla_{x'} \partial_{x_n} u$ ,
- $\partial_{x_n, x_n}^2 \tilde{u} = (\Delta - \Delta_{x'}) \tilde{u}$  hence coincides with  $\tilde{u}_t - \tilde{f} - \Delta_{x'} \tilde{u}$ .

Hence one may conclude that

$$\begin{aligned}
&\|u\|_{L_\infty(0, T; \dot{B}_{p,1}^s(\mathbb{R}_+^n))} + \|u_t, \nu \nabla^2 u\|_{L_1(0, T; \dot{B}_{p,1}^s(\mathbb{R}_+^n))} \\
&\leq \|\tilde{u}\|_{L_\infty(0, T; \dot{B}_{p,1}^s(\mathbb{R}^n))} + \|\tilde{u}_t, \nu \nabla^2 \tilde{u}\|_{L_1(0, T; \dot{B}_{p,1}^s(\mathbb{R}^n))}.
\end{aligned}$$

This implies (14).

*Remark 3.* The case of non-homogeneous boundary conditions where  $u$  equals some given  $h$  at the boundary, reduces to the homogeneous case : it is only a matter of assuming that  $h$  admits some extension  $\tilde{h}$  over  $(0, T) \times \mathbb{R}_+^n$  so that  $\tilde{h}_t - \nu \Delta \tilde{h} \in L_1(0, T; \dot{B}_{p,1}^s(\mathbb{R}_+^n))$ .

## 2.2 The exterior domain case

Here we extend Theorem 1 to the case where  $\Omega$  is an exterior domain (that is the complement of a bounded simply connected domain). Here is our main statement:

**Theorem 3.** *Let  $\Omega$  be a  $C^2$  exterior domain of  $\mathbb{R}^n$  with  $n \geq 3$ . Let  $1 < q \leq p < \infty$  with  $q < n/2$ . Let  $-1 + 1/p < s < 1/p$  and  $-1 + 1/q < s' < 1/q - 2/n$ . Let*

$$u_0 \in \dot{B}_{p,1}^s \cap \dot{B}_{q,1}^{s'}(\Omega) \quad \text{and} \quad f \in L_1(0, T; \dot{B}_{p,1}^s \cap \dot{B}_{q,1}^{s'}(\Omega)).$$

Then there exists a unique solution  $u$  to (1) such that

$$u \in \mathcal{C}([0, T]; \dot{B}_{p,1}^s \cap \dot{B}_{q,1}^{s'}(\Omega)), \quad u_t, \nabla^2 u \in L_1(0, T; \dot{B}_{p,1}^s \cap \dot{B}_{q,1}^{s'}(\Omega))$$

and the following inequality is satisfied:

$$\begin{aligned} & \|u\|_{L_\infty(0, T; \dot{B}_{p,1}^s \cap \dot{B}_{q,1}^{s'}(\Omega))} + \|u_t, \nu \nabla^2 u\|_{L_1(0, T; \dot{B}_{p,1}^s \cap \dot{B}_{q,1}^{s'}(\Omega))} \\ & \leq C (\|u_0\|_{\dot{B}_{p,1}^s \cap \dot{B}_{q,1}^{s'}(\Omega)} + \|f\|_{L_1(0, T; \dot{B}_{p,1}^s \cap \dot{B}_{q,1}^{s'}(\Omega))}), \end{aligned} \quad (15)$$

where the constant  $C$  is independent of  $T$  and  $\nu$ .

Proving this theorem relies on the following statement (that is of independent interest and holds in any dimension  $n \geq 2$ ), and on lower order estimates (see Lemma 2 below) which will enable us to remove the time dependency.

**Theorem 4.** *Let  $\Omega$  be a  $C^2$  exterior domain of  $\mathbb{R}^n$  with  $n \geq 2$ . Let  $1 < p < \infty$ ,  $-1 + 1/p < s < 1/p$ ,  $f \in L_1(0, T; \dot{B}_{p,1}^s(\Omega))$ , and  $u_0 \in \dot{B}_{p,1}^s(\Omega)$ . Then equation (1) has a unique solution  $u$  such that*

$$u \in \mathcal{C}([0, T]; \dot{B}_{p,1}^s(\Omega)), \quad \partial_t u, \nabla^2 u \in L_1(0, T; \dot{B}_{p,1}^s(\Omega))$$

and the following estimate is valid:

$$\begin{aligned} & \|u\|_{L_\infty(0, T; \dot{B}_{p,1}^s(\Omega))} + \|u_t, \nu \nabla^2 u\|_{L_1(0, T; \dot{B}_{p,1}^s(\Omega))} \\ & \leq C e^{CT\nu} (\|u_0\|_{\dot{B}_{p,1}^s(\Omega)} + \|f\|_{L_1(0, T; \dot{B}_{p,1}^s(\Omega))}), \end{aligned} \quad (16)$$

where the constant  $C$  depends only on  $s$ ,  $p$ , and  $\Omega$ .

Additionally if  $K$  is a compact subset of  $\Omega$  such that  $\text{dist}(\partial\Omega, \Omega \setminus K) > 0$ , there holds

$$\begin{aligned} & \|u\|_{L_\infty(0, T; \dot{B}_{p,1}^s(\Omega))} + \|u_t, \nu \nabla^2 u\|_{L_1(0, T; \dot{B}_{p,1}^s(\Omega))} \\ & \leq C (\|u_0\|_{\dot{B}_{p,1}^s(\Omega)} + \|f\|_{L_1(0, T; \dot{B}_{p,1}^s(\Omega))} + \nu \|u\|_{L_1(0, T; \dot{B}_{p,1}^s(K))}), \end{aligned} \quad (17)$$

where  $C$  is as above.

*Proof.* We suppose that we have a smooth enough solution and focus on the proof of the estimates. We shall do it in three steps: first we prove interior estimates, next boundary estimates and finally global estimates after summation.

Note that performing the following change of unknown and data:

$$u_{new}(t, x) = \nu u_{old}(\nu^{-1}t, x), \quad u_{0,new}(x) = \nu u_{0,old}(x), \quad f_{new}(t, x) = f_{old}(\nu^{-1}t, x)$$

reduces the study to the case  $\nu = 1$ . So we shall make this assumption in all that follows.

Throughout we fix some covering  $(B(x^\ell, \lambda))_{1 \leq \ell \leq L}$  of  $K$  by balls of radius  $\lambda$  and take some neighborhood  $\Omega^0 \subset \Omega$  of  $\mathbb{R}^n \setminus K$  such that  $d(\Omega^0, \partial\Omega) > 0$ . We assume in addition that the first  $M$  balls do not intersect  $K$  while the last  $L - M$  balls are centered at some point of  $\partial\Omega$ .

Let  $\eta^0 : \mathbb{R}^n \rightarrow [0, 1]$  be a smooth function supported in  $\Omega^0$  and with value 1 on a neighborhood of  $\Omega \setminus K$ . Then we consider a subordinate partition of unity  $(\eta^\ell)_{1 \leq \ell \leq L}$  such that:

1.  $\sum_{0 \leq \ell \leq L} \eta^\ell = 1$  on  $\Omega$ ;
2.  $\|\nabla^k \eta^\ell\|_{L^\infty(\mathbb{R}^n)} \leq C_k \lambda^{-k}$  for  $k \in \mathbb{N}$  and  $1 \leq \ell \leq L$ ;
3.  $\text{Supp } \eta^\ell \subset B(x^\ell, \lambda)$ .

We also introduce another smooth function  $\tilde{\eta}^0$  supported in  $K$  and with value 1 on  $\text{Supp } \nabla \eta^0$  and smooth functions  $\tilde{\eta}^1, \dots, \tilde{\eta}^L$  with compact support in  $\Omega^\ell$  and such that  $\tilde{\eta}^\ell \equiv 1$  on  $\text{Supp } \eta^\ell$ .

Note that for  $\ell \in \{1, \dots, L\}$ , the bounds for the derivatives of  $\eta^\ell$  together with the fact that  $|\text{Supp } \nabla \eta^\ell| \approx \lambda^n$  and Proposition 5 implies that for  $k = 0, 1$  and any  $q \in [1, \infty]$ , we have

$$\|\nabla \eta^\ell\|_{\dot{B}_{q,1}^{k+n/q}(\mathbb{R}^n)} \lesssim \lambda^{-1-k}. \quad (18)$$

The same holds for the functions  $\tilde{\eta}^\ell$ . Throughout, we set  $U^\ell := u \eta^\ell$ .

### First step: the interior estimate

The vector-field  $U^0$  satisfies the following modification of (1):

$$\begin{aligned} U_t^0 - \Delta U^0 &= \eta^0 f - 2\nabla \eta^0 \cdot \nabla u - u \Delta \eta^0 \text{ in } (0, T) \times \mathbb{R}^n, \\ U^0|_{t=0} &= u_0 \eta^0 \text{ on } \mathbb{R}^n. \end{aligned} \quad (19)$$

Theorem 1 thus yields the following estimate:

$$\begin{aligned} \|U^0\|_{L^\infty(0, T; \dot{B}_{p,1}^s(\mathbb{R}^n))} + \|U_t^0, \nabla^2 U^0\|_{L_1(0, T; \dot{B}_{p,1}^s(\mathbb{R}^n))} &\lesssim \|\eta^0 f\|_{L_1(0, T; \dot{B}_{p,1}^s(\mathbb{R}^n))} \\ &+ \|\nabla \eta^0 \cdot \nabla u\|_{L_1(0, T; \dot{B}_{p,1}^s(\mathbb{R}^n))} + \|u \Delta \eta^0\|_{L_1(0, T; \dot{B}_{p,1}^s(\mathbb{R}^n))} + \|\eta^0 u_0\|_{\dot{B}_{p,1}^s(\mathbb{R}^n)}. \end{aligned}$$

Let us emphasize that as  $\nabla \eta^0 \cdot \nabla u$  and  $u \Delta \eta^0$  are compactly supported, we may replace the homogeneous norms by non-homogeneous ones in the first two terms.

As a consequence, because the function  $\nabla\eta^0$  is in  $\mathcal{C}_c^\infty(\mathbb{R}^n)$  and  $\tilde{\eta}^0 \equiv 1$  on  $\text{Supp } \nabla\eta^0$ , Corollary 1 ensures that

$$\begin{aligned} & \|U^0\|_{L^\infty(0,T;\dot{B}_{p,1}^s(\mathbb{R}^n))} + \|U_t^0, \nabla^2 U^0\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}^n))} \\ & \lesssim \|\eta^0 u_0\|_{\dot{B}_{p,1}^s(\mathbb{R}^n)} + \|\eta^0 f\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}^n))} + \|\tilde{\eta}^0 u\|_{L_1(0,T;\dot{B}_{p,1}^{s+1}(\mathbb{R}^n))}. \end{aligned} \quad (20)$$

Now, by interpolation,

$$\|\tilde{\eta}^0 u\|_{\dot{B}_{p,1}^{1+s}(\Omega)} \leq C \|\tilde{\eta}^0 u\|_{\dot{B}_{p,1}^{2+s}(\Omega)}^{\frac{1}{2}} \|\tilde{\eta}^0 u\|_{\dot{B}_{p,1}^s(\Omega)}^{\frac{1}{2}}. \quad (21)$$

As  $\text{Supp } \tilde{\eta}^0 \subset K$  and as homogeneous and nonhomogeneous norms are equivalent on  $K$ , one may thus conclude that

$$\begin{aligned} & \|U^0\|_{L^\infty(0,T;\dot{B}_{p,1}^s(\mathbb{R}^n))} + \|U_t^0, \nabla^2 U^0\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}^n))} \lesssim \|f\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))} \\ & + T^{1/2} \|u\|_{L_1(0,T;\dot{B}_{p,1}^{2+s}(K)) \cap L^\infty(0,T;\dot{B}_{p,1}^s(K))} + \|u_0\|_{\dot{B}_{p,1}^s(\Omega)}. \end{aligned} \quad (22)$$

Note that starting from (21) and using Young's inequality also yields for all  $\varepsilon > 0$ :

$$\begin{aligned} & \|U^0\|_{L^\infty(0,T;\dot{B}_{p,1}^s(\mathbb{R}^n))} + \|U_t^0, \nabla^2 U^0\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}^n))} \leq C (\|u_0\|_{\dot{B}_{p,1}^s(\Omega)} \\ & + \|f\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))}) + \varepsilon \|u\|_{L_1(0,T;\dot{B}_{p,1}^{2+s}(K))} + c(\varepsilon) \|u\|_{L_1(0,T;\dot{B}_{p,1}^s(K))}. \end{aligned} \quad (23)$$

The terms  $U^\ell$  with  $1 \leq \ell \leq M$  may be bounded exactly along the same lines because their support do not meet  $\partial\Omega$ , hence their extension by 0 over  $\mathbb{R}^n$  satisfies

$$\begin{aligned} U_t^\ell - \Delta U^\ell &= f^\ell \quad \text{in } (0,T) \times \mathbb{R}^n, \\ U^\ell|_{t=0} &= u_0 \eta^\ell \quad \text{on } \mathbb{R}^n \end{aligned}$$

with

$$f^\ell := -2\nabla\eta^\ell \cdot \nabla u - u\Delta\eta^\ell + \eta^\ell f. \quad (24)$$

Arguing as above and taking advantage of the fact that the functions  $\eta^\ell$  are localized in balls of radius  $\lambda$  (that is we use (18)), we now get

$$\begin{aligned} & \|f^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))} \lesssim \|\eta^\ell f\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))} \\ & + \lambda^{-2} \|\tilde{\eta}^\ell u\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))} + \lambda^{-1} \|\tilde{\eta}^\ell \nabla u\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))}. \end{aligned} \quad (25)$$

Using again (21) (with  $\tilde{\eta}^\ell$  instead of  $\tilde{\eta}^0$ ), we get

$$\begin{aligned} & \|U^\ell\|_{L^\infty(0,T;\dot{B}_{p,1}^s(\mathbb{R}^n))} + \|U_t^\ell, \nabla^2 U^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}^n))} \lesssim \|\eta^\ell f\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))} \\ & + (\lambda^{-1} T^{1/2} + \lambda^{-2} T) \|u\|_{L_1(0,T;\dot{B}_{p,1}^{2+s}(K)) \cap L^\infty(0,T;\dot{B}_{p,1}^s(K))} + \|u_0 \eta^\ell\|_{\dot{B}_{p,1}^s(\Omega)}, \end{aligned} \quad (26)$$

$$\begin{aligned} & \|U^\ell\|_{L_\infty(0,T;\dot{B}_{p,1}^s(\mathbb{R}^n))} + \|U_t^\ell, \nabla^2 U^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}^n))} \leq C(\|u_0 \eta^\ell\|_{\dot{B}_{p,1}^s(\Omega)} \\ & + \|\eta^\ell f\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))}) + \lambda^{-1} \|u\|_{L_1(0,T;\dot{B}_{p,1}^{2+s}(\mathcal{K}))}^{1/2} \|u\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathcal{K}))}^{1/2}. \end{aligned} \quad (27)$$

### Second step: the boundary estimate

We now consider an index  $\ell \in \{L+1, \dots, M\}$  so that  $B(x^\ell, \lambda)$  is centered at a point of  $\partial\Omega$ . The localization leads to the following problem:

$$\begin{aligned} U_t^\ell - \Delta U^\ell &= f^\ell \quad \text{in } (0, T) \times \Omega, \\ U^\ell &= 0 \quad \text{on } (0, T) \times \partial\Omega, \\ U_t^\ell|_{t=0} &= u_0 \eta^\ell \quad \text{on } \Omega, \end{aligned} \quad (28)$$

with  $f^\ell$  defined by (24), hence satisfying (25).

Let us now make a change of variables so as to recast (28) in the half-space. As  $\partial\Omega$  is  $C^2$ , if  $\lambda$  has been chosen small enough then for fixed  $\ell$  we are able to find a map  $Z_\ell$  so that

- i)  $Z_\ell$  is a  $C^2$  diffeomorphism from  $B(x^\ell, \lambda)$  to  $Z_\ell(B(x^\ell, \lambda))$ ;
- ii)  $Z_\ell(x^\ell) = 0$  and  $D_x Z(x^\ell) = \text{Id}$ ;
- iii)  $Z_\ell(\Omega \cap B(x^\ell, \lambda)) \subset \mathbb{R}_+^n$ ;
- iv)  $Z_\ell(\partial\Omega \cap B(x^\ell, \lambda)) = \partial\mathbb{R}_+^n \cap Z_\ell(B(x^\ell, \lambda))$ .

Setting  $\nabla_x Z_\ell = \text{Id} + A_\ell$  then one may assume in addition that there exist constants  $C_j$  depending only on  $\Omega$  and on  $j \in \{0, 1\}$  such that

$$\|D^j A_\ell\|_{L_\infty(B(x^\ell, \lambda))} \leq C_j, \quad (29)$$

a property which implies (by the mean value formula) that

$$\|A_\ell\|_{L_\infty(B(x^\ell, \lambda))} \leq C_1 \lambda, \quad (30)$$

hence by interpolation between the spaces  $L_q(B(x^\ell, \lambda))$  and  $W_q^{r-1}(B(x^\ell, \lambda))$ ,

$$\|A_\ell\|_{B_{q,1}^{\frac{n}{q}}(B(x^\ell, \lambda))} \leq C\lambda \quad \text{for all } 1 \leq q < \infty \text{ such that } n/q < r-1. \quad (31)$$

Let  $V^\ell := Z_\ell^* U^\ell := U^\ell \circ Z_\ell^{-1}$ . The system satisfied by  $V^\ell$  reads

$$\begin{aligned} V_t^\ell - \Delta_z V^\ell &= F^\ell \quad \text{in } (0, T) \times \mathbb{R}_+^n, \\ V^\ell|_{z_n=0} &= 0 \quad \text{on } (0, T) \times \partial\mathbb{R}_+^n, \\ V^\ell|_{t=0} &= Z_\ell^*(U^\ell|_{t=0}) \quad \text{on } \partial\mathbb{R}_+^n, \end{aligned} \quad (32)$$

with

$$F^\ell := Z_\ell^* f^\ell + (\Delta_x - \Delta_z)V^\ell.$$

According to Theorem 2, we thus get

$$\begin{aligned} & \|V^\ell\|_{L^\infty(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))} + \|V_t^\ell, \nabla_z^2 V^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))} \\ & \lesssim \|Z_\ell^* f^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))} + (\Delta_x - \Delta_z)V^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))} + \|Z_\ell^*(U^\ell|_{t=0})\|_{\dot{B}_{p,1}^s(\mathbb{R}_+^n)}. \end{aligned}$$

Note that the first and last terms in the right-hand side may be dealt with thanks to Lemma 1: we have

$$\begin{aligned} \|Z_\ell^* f^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))} & \lesssim \|f^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))} \\ \|Z_\ell^*(U^\ell|_{t=0})\|_{\dot{B}_{p,1}^s(\mathbb{R}_+^n)} & \lesssim \|U^\ell|_{t=0}\|_{\dot{B}_{p,1}^s(\Omega)}. \end{aligned}$$

Compared to the first step, the only definitely new term is  $(\Delta_x - \Delta_z)V^\ell$ . Explicit computations (see e.g. [5]) show that  $(\Delta_z - \Delta_x)V^\ell$  is a linear combination of components of  $\nabla_z^2 A_\ell \otimes V^\ell$  and  $\nabla_z A_\ell \otimes \nabla_z V^\ell$ . Therefore

$$\begin{aligned} \|(\Delta_x - \Delta_z)V^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))} & \lesssim \|A_\ell \otimes \nabla_z^2 V^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))} \\ & \quad + \|\nabla_z A_\ell \otimes \nabla_z V^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))}. \end{aligned}$$

Now, according to Proposition 4 and owing to the support properties of the terms involved in the inequalities, we have

$$\|A_\ell \otimes \nabla_z^2 V^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))} \lesssim \|A_\ell\|_{\dot{B}_{q,1}^{\frac{n}{q}}(B(x^\ell, \lambda))} \|\nabla_z^2 V^\ell\|_{\dot{B}_{p,1}^s(\mathbb{R}_+^n)} \quad \text{with } q = \min(p, p').$$

Therefore we have, thanks to (30) and to (31),

$$\|A_\ell \otimes \nabla_z^2 V^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))} \lesssim \lambda \|\nabla_z^2 V^\ell\|_{\dot{B}_{p,1}^s(\mathbb{R}_+^n)}.$$

Similarly, we have

$$\|\nabla_z A_\ell \otimes \nabla_z V^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))} \lesssim \|\nabla_z V^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))}.$$

Therefore

$$\|(\Delta_x - \Delta_z)V^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))} \lesssim \lambda \|\nabla_z^2 V^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))} + \|\nabla_z V^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))}.$$

Putting together the above inequalities and remembering of (25) and Lemma 1, we finally get, taking  $\lambda$  small enough

$$\begin{aligned} & \|V^\ell\|_{L^\infty(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))} + \|V_t^\ell, \nabla_z^2 V^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))} \\ & \lesssim \|U^\ell|_{t=0}\|_{\dot{B}_{p,1}^s(\Omega)} + \|\eta^\ell f\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))} \\ & \quad + \lambda^{-2} \|\tilde{\eta}^\ell u\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))} + \lambda^{-1} \|\tilde{\eta}^\ell \nabla u\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))} + \|\nabla V^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))}. \end{aligned}$$

By interpolation, we have

$$\|\nabla V^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))} \lesssim \|\nabla^2 V^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))}^{1/2} \|V^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))}^{1/2}.$$

Therefore using Young's inequality enables us to reduce the above inequality to

$$\begin{aligned} & \|V^\ell\|_{L_\infty(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))} + \|V_t^\ell, \nabla^2 V^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))} \\ & \lesssim \|U^\ell|_{t=0}\|_{\dot{B}_{p,1}^s(\Omega)} + \|\eta^\ell f\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))} + \|V^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))} \\ & \quad + \lambda^{-2} \|\tilde{\eta}^\ell u\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))} + \lambda^{-1} \|\tilde{\eta}^\ell \nabla u\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))}. \end{aligned}$$

In order to handle the last term, there are two ways of proceeding depending on whether we want a time dependent constant or not. The first possibility is to write that, by interpolation and Hölder's inequality,

$$\|\tilde{\eta}^\ell \nabla u\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))} \leq T^{1/2} \|u\|_{L_1(0,T;\dot{B}_{p,1}^{s+2}(K)) \cap L_\infty(0,T;\dot{B}_{p,1}^s(K))}.$$

This yields

$$\begin{aligned} & \|V^\ell\|_{L_\infty(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))} + \|V_t^\ell, \nabla^2 V^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))} \\ & \lesssim \|\eta^\ell u_0\|_{\dot{B}_{p,1}^s(\Omega)} + \|\eta^\ell f\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))} + T \|V^\ell\|_{L_\infty(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))} \\ & \quad + (\lambda^{-1} T^{1/2} + \lambda^{-2} T) \|u\|_{L_1(0,T;\dot{B}_{p,1}^{s+2}(K)) \cap L_\infty(0,T;\dot{B}_{p,1}^s(K))}. \quad (33) \end{aligned}$$

The second possibility is to write that

$$\|\tilde{\eta}^\ell \nabla u\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))} \leq \|u\|_{L_1(0,T;\dot{B}_{p,1}^{s+2}(K))}^{1/2} \|u\|_{L_1(0,T;\dot{B}_{p,1}^s(K))}^{1/2}.$$

We eventually get

$$\begin{aligned} & \|V^\ell\|_{L_\infty(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))} + \|V_t^\ell, \nabla^2 V^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))} \lesssim \|\eta^\ell u_0\|_{\dot{B}_{p,1}^s(\Omega)} \\ & \quad + \|\eta^\ell f\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))} + \lambda^{-1} \|u\|_{L_1(0,T;\dot{B}_{p,1}^{2+s}(K))}^{1/2} \|u\|_{L_1(0,T;\dot{B}_{p,1}^s(K))}^{1/2} \\ & \quad + \lambda^{-2} \|u\|_{L_1(0,T;\dot{B}_{p,1}^s(K))} + \|\nabla V^\ell\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))}. \quad (34) \end{aligned}$$

### Third step: global a priori estimates

Now, in view of Lemma 1, we may write

$$\begin{aligned} \|u\|_{L_\infty(0,T;\dot{B}_{p,1}^s(\Omega))} & \leq \sum_{\ell} \|U^\ell\|_{L_\infty(0,T;\dot{B}_{p,1}^s(\Omega))} \\ & \lesssim \sum_{0 \leq \ell \leq M} \|U^\ell\|_{L_\infty(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))} + \sum_{M < \ell \leq L} \|V^\ell\|_{L_\infty(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))}, \end{aligned}$$



and similar inequalities for the other terms of the l.h.s of (33). Of course, Proposition 1 ensures that

$$\|u_0^\ell\|_{\dot{B}_{p,1}^s(\Omega)} \lesssim \|u_0\|_{\dot{B}_{p,1}^s(\Omega)} \quad \text{and} \quad \|\tilde{\eta}^\ell f\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))} \lesssim \|f\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))}.$$

So using also (22) and (26) and assuming that  $T$  is small enough, we end up with

$$\begin{aligned} \|u\|_{L_\infty(0,T;\dot{B}_{p,1}^s(\Omega))} + \|(u_t, \nabla^2 u)\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))} &\lesssim \|u_0\|_{\dot{B}_{p,1}^s(\Omega)} \\ &+ \|f\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))} + (\lambda^{-1}T^{1/2} + \lambda^{-2}T)\|u\|_{L_1(0,T;\dot{B}_{p,1}^{s+2}(\mathcal{K})) \cap L_\infty(0,T;\dot{B}_{p,1}^s(\mathcal{K}))}. \end{aligned}$$

Hence if in addition  $\lambda^{-2}T$  is small enough,

$$\|u\|_{L_\infty(0,T;\dot{B}_{p,1}^s(\Omega))} + \|(u_t, \nabla^2 u)\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))} \leq C\left(\|u_0\|_{\dot{B}_{p,1}^s(\mathbb{R}^n)} + \|f\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))}\right).$$

Repeating the argument over the interval  $[T, 2T]$  and so on, we get exactly Inequality (16).

If we want to remove the time-dependency then it is just a matter of starting from (34) and (27) instead of (33) and (22). After a few computation and thanks to Young's inequality, we get for some constant  $C$  depending on  $\lambda$ ,

$$\|u_t, \nabla^2 u\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))} \leq C(\|u_0\|_{\dot{B}_{p,1}^s(\Omega)} + \|f\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathbb{R}_+^n))} + \|u\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathcal{K}))}).$$

For completeness, let us say a few words about the existence, which is rather standard issue (see e.g. [11]). If the domain is smooth then the easiest approach is via the  $L_2$ -framework and Galerkin method. We may consider smooth approximations of data  $f$  and  $u_0$ , such that to keep them in the space  $H^m$  with sufficiently large  $m \in \mathbb{N}$ . Then the energy method provides us with approximate solutions in Sobolev spaces  $H^m$  with large  $m$ . In particular, the above a priori estimates (16) may be derived for such solutions. It is then easy to pass to the limit.

*Remark 4.* Let us emphasize that the term  $\|u\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathcal{K}))}$  may be replaced by other lower order norms such as  $\|u\|_{L_1(0,T;\dot{B}_{p,1}^{s'}(\mathcal{K}))}$  with  $s' \neq s$  close to 0. In particular  $s'$  may be put to zero.

In order to complete the proof of Theorem 3, we now have to bound the last term of (17), namely  $\|u\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathcal{K}))}$ , *independently of  $T$* . This is the goal of the next lemma (where we keep the assumption that  $\nu = 1$ ). We here adapt to the heat equation an approach that has been proposed for the Stokes system in [13].

**Lemma 2.** *Assume that  $n \geq 3$  and that  $1 < p < n/2$ . Then for any  $s \in (-1 + 1/p, 1/p - 2/n)$  sufficiently smooth solutions to (1) fulfill*

$$\|u\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathcal{K}))} \leq C(\|f\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))} + \|u_0\|_{\dot{B}_{p,1}^s(\Omega)}),$$

where  $C$  is independent of  $T$ .

*Proof.* Thanks to the linearity of the system, one may split the solution  $u$  into two parts, the first one  $u_1$  being the solution of the system with zero initial data and source term  $f$ , and the second one  $u_2$ , the solution of the system with no source term and initial data  $u_0$ . In other words,  $u = u_1 + u_2$  with  $u_1$  and  $u_2$  satisfying

$$\begin{aligned} u_{1,t} - \Delta u_1 &= f & \text{in } (0, T) \times \Omega, & & u_{2,t} - \Delta u_2 &= 0 & \text{in } (0, T) \times \Omega, \\ u_1 &= 0 & \text{on } (0, T) \times \partial\Omega, & & u_2 &= 0 & \text{on } (0, T) \times \partial\Omega, \\ u_1|_{t=0} &= 0 & \text{on } \Omega, & & u_2|_{t=0} &= u_0 & \text{on } \Omega. \end{aligned} \quad (35)$$

Let us first focus on  $u_1$ . Recall that up to a constant we have (see Proposition 3):

$$\|u_1(t)\|_{\dot{B}_{p,1}^s(K)} = \sup \int_K u_1(t, x) \eta_0(x) dx, \quad (36)$$

where the supremum is taken over all  $\eta_0 \in \dot{B}_{p',\infty}^{-s}(K)$  such that  $\|\eta_0\|_{\dot{B}_{p',\infty}^{-s}(K)} = 1$ . Of course, by virtue of Remark 2, any such function  $\eta_0$  may be extended by 0 over  $\mathbb{R}^n$ , and its extension still has a norm of order 1. So we may assume that the supremum is taken over all

$$\eta_0 \in \dot{B}_{p',\infty}^{-s}(\mathbb{R}^n) \quad \text{with } \|\eta_0\|_{\dot{B}_{p',\infty}^{-s}(\mathbb{R}^n)} = 1 \quad \text{and} \quad \text{Supp } \eta_0 \subset K. \quad (37)$$

Consider the solution  $\eta$  to the following problem:

$$\begin{aligned} \eta_t - \Delta \eta &= 0 & \text{in } (0, T) \times \Omega, \\ \eta &= 0 & \text{on } (0, T) \times \partial\Omega, \\ \eta|_{t=0} &= \eta_0 & \text{on } \Omega. \end{aligned} \quad (38)$$

Testing the equation for  $u_1$  by  $\eta(t - \cdot)$  we discover that

$$\int_{\Omega} u_1(t, x) \eta_0(x) dx = \int_0^t \int_{\Omega} f(\tau, x) \eta(t - \tau, x) dx d\tau. \quad (39)$$

The general theory for the heat operator in exterior domains implies the following estimates:

$$\|\eta(t)\|_{L_a(\Omega)} \leq C \|\eta_0\|_{L_b(\Omega)} t^{-\frac{n}{2}(\frac{1}{b} - \frac{1}{a})} \quad \text{for } 1 < b \leq a < \infty, \quad (40)$$

as well as

$$\|\Delta \eta(t)\|_{L_a(\Omega)} \leq C \|\Delta \eta_0\|_{L_b(\Omega)} t^{-\frac{n}{2}(\frac{1}{b} - \frac{1}{a})} \quad \text{for } 1 < b \leq a < \infty. \quad (41)$$

In the case  $\Omega = \mathbb{R}^n$ , those two inequalities may be derived easily from the (explicit) heat kernel. To prove (40) in the case of an exterior domain, it is enough to look at solutions to (38) as subsolutions to the problem in the whole space. More precisely, if we assume that  $\eta_0 \geq 0$  (this is not restrictive for one may consider the positive and negative part of the initial data separately), we get a solution to (38) defined over  $(0, \infty) \times \Omega$  such that  $\eta \geq 0$ . Then we consider an extension  $E\eta : \mathbb{R}^n \rightarrow \mathbb{R}$

of  $\eta$ , such that  $E\eta = \eta$  for  $x \in \Omega$  and  $E\eta = 0$  for  $x \notin \Omega$ . We claim that  $E\eta$  is a subsolution to the Cauchy problem

$$\bar{\eta}_t - \Delta \bar{\eta} = 0 \text{ in } (0, T) \times \mathbb{R}^n \text{ with } \bar{\eta}|_{t=0} = E\eta_0. \quad (42)$$

It is sufficient to show that  $\eta \leq \bar{\eta}$ , since  $\bar{\eta}$  is always nonnegative. It is clear that

$$(\eta - \bar{\eta})_t - \Delta(\eta - \bar{\eta}) = 0 \text{ in } (0, T) \times \Omega. \quad (43)$$

Consider  $(\eta - \bar{\eta})_+ := \max\{\eta - \bar{\eta}, 0\}$ . It is obvious that  $(\eta - \bar{\eta})_+$  vanishes at the boundary, because  $\eta$  is zero and  $\bar{\eta}$  is nonnegative there. Hence we conclude

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} (\eta - \bar{\eta})_+^2 dx + \int_{\Omega} |\nabla(\eta - \bar{\eta})_+|^2 dx = 0. \quad (44)$$

Thus,  $(\eta - \bar{\eta})_+ \equiv 0$ , since  $(\eta - \bar{\eta})_+|_{t=0} = 0$ . So  $\eta$  is bounded by  $\bar{\eta}$ .

To prove (41) we observe that for the smooth solutions the equation implies that  $\Delta\eta|_{\partial\Omega} = 0$ , so we can consider the problem on  $\Delta\eta$  instead of  $\eta$ . Now, as  $\eta$  vanishes at the boundary, we have (see e.g. [8])

$$\|\nabla^2 \eta\|_{L_c(\Omega)} \leq \|\Delta\eta\|_{L_c(\Omega)} \text{ for all } 1 < c < \infty. \quad (45)$$

Hence, interpolating between (40) and (41) yields for  $0 < s < 1/b$ .

$$\|\eta(t)\|_{\dot{B}_{b,r}^s(\Omega)} \leq C \|\eta_0\|_{\dot{B}_{a,r}^s(\Omega)} t^{-\frac{s}{2}(\frac{1}{p} - \frac{1}{q})} \text{ for } 1 < a \leq b < \infty \text{ and } 1 \leq r \leq \infty. \quad (46)$$

In order to extend this inequality to negative indices  $s$ , we consider the following dual problem:

$$\begin{aligned} \zeta_t - \Delta \zeta &= 0 \text{ in } (0, T) \times \Omega, \\ \zeta &= 0 \text{ on } (0, T) \times \partial\Omega, \\ \zeta|_{t=0} &= \zeta_0 \text{ on } \Omega, \end{aligned} \quad (47)$$

where  $\zeta_0 \in B_{b',r'}^{-s}(\Omega)$ .

Now, testing (47) by  $\eta(t - \cdot)$  yields

$$\int_{\Omega} \eta(t, x) \zeta_0(x) dx = \int_{\Omega} \eta_0(x) \zeta(t, x) dx. \quad (48)$$

Let us observe that

$$\|\eta(t)\|_{\dot{B}_{b,r}^s(\Omega)} = \sup_{\zeta_0} \int_{\Omega} \eta(t, x) \zeta_0(x) dx, \quad (49)$$

where the supremum is taken over all  $\zeta_0 \in \dot{B}_{b',r'}^{-s}(\Omega)$  such that  $\|\zeta_0\|_{\dot{B}_{b',r'}^{-s}(\Omega)} = 1$ . Thus by virtue of (48), we get:

$$\|\eta(t)\|_{\dot{B}_{b,r}^s(\Omega)} = \sup_{\zeta_0} \int_{\Omega} \eta_0(x) \zeta(t,x) dx \leq \sup_{\zeta_0} \left( \|\eta_0\|_{\dot{B}_{a,r}^s(\Omega)} \|\zeta(t)\|_{\dot{B}_{a',r'}^{-s}(\Omega)} \right). \quad (50)$$

Since  $-s$  is positive we can apply (46) and get if  $0 < -s < 1/a'$ ,

$$\|\eta(t)\|_{\dot{B}_{b,r}^s(\Omega)} \leq C \|\eta_0\|_{\dot{B}_{a,r}^s(\Omega)} t^{-\frac{n}{2}(\frac{1}{a'} - \frac{1}{b'})} \sup_{\zeta_0} \|\zeta_0\|_{\dot{B}_{b',r'}^{-s}(\Omega)}.$$

Since  $\frac{1}{a'} - \frac{1}{b'} = \frac{1}{b} - \frac{1}{a}$ , we conclude that

$$\|\eta(t)\|_{\dot{B}_{a,r}^s(\Omega)} \leq C \|\eta_0\|_{\dot{B}_{b,r}^s(\Omega)} t^{-\frac{n}{2}(\frac{1}{b} - \frac{1}{a})} \quad \text{if } s > -1 + 1/a. \quad (51)$$

In order to get the remaining case  $s = 0$ , it suffices to argue by interpolation between (46) and (51). One can thus conclude that for all  $1 < b \leq a < \infty$ ,  $q \in [1, \infty]$  and  $-1 + 1/a < s < 1/b$ , we have

$$\|\eta(t)\|_{\dot{B}_{a,r}^s(\Omega)} \leq C \|\eta_0\|_{\dot{B}_{b,r}^s(\Omega)} t^{-\frac{n}{2}(\frac{1}{b} - \frac{1}{a})}. \quad (52)$$

Now we return to the initial problem of bounding  $u_1$ . Starting from (39) and using duality, one may write

$$\left| \int_{\Omega} u_1(t,x) \eta_0(x) dx \right| \lesssim \int_0^t \|f(\tau)\|_{\dot{B}_{p,1}^s(\Omega)} \|\eta(t-\tau)\|_{\dot{B}_{p',\infty}^{-s}(\Omega)} d\tau.$$

Hence splitting the interval  $(0, t)$  into  $(0, \max(0, t-1))$  and  $(\max(0, t-1), t)$  and applying (52) yields for any  $\varepsilon \in (0, 1+s)$ ,

$$\begin{aligned} \left| \int_{\Omega} u_1(t,x) \eta_0(x) dx \right| &\lesssim \int_{\max(0, t-1)}^t \|f(\tau)\|_{\dot{B}_{p,1}^s(\Omega)} \|\eta_0\|_{\dot{B}_{p',\infty}^{-s}(\Omega)} d\tau \\ &\quad + \int_0^{\max(0, t-1)} \|f(\tau)\|_{\dot{B}_{p,1}^s(\Omega)} \|\eta_0\|_{\dot{B}_{\frac{1}{1-\varepsilon}, \infty}^{-s}(\Omega)} (t-\tau)^{-\frac{n}{2}(\frac{1}{p} - \varepsilon)} d\tau. \end{aligned}$$

Now, as  $\eta_0$  is supported in  $K$ , one has  $\|\eta_0\|_{\dot{B}_{a,\infty}^{-s}(\Omega)} \leq C|K|^{\frac{1}{p} + \frac{1}{a} - 1} \|\eta_0\|_{\dot{B}_{p',\infty}^{-s}(\Omega)}$ . This may easily be proved by introducing a suitable smooth cut-off function with value 1 over  $K$  and taking advantage of Proposition 1. A scaling argument yields the dependency of the norm of the embedding with respect to  $|K|$ . Hence we have for some constant  $C$  depending on  $K$ :

$$\|\eta_0\|_{\dot{B}_{\frac{1}{1-\varepsilon}, \infty}^{-s}} \leq C \|\eta_0\|_{\dot{B}_{p',\infty}^{-s}(\Omega)}.$$

So, keeping in mind (39) and the fact that the supremum is taken over all the functions  $\eta_0$  satisfying (37), we deduce that

$$\|u_1(t)\|_{\dot{B}_{p,1}^s(K)} \leq C \left( \int_{\max(0,t-1)}^t \|f(\tau)\|_{\dot{B}_{p,1}^s(\Omega)} d\tau + \int_0^{\max(0,t-1)} (t-\tau)^{-\frac{n}{2}(\frac{1}{p}-\varepsilon)} \|f(\tau)\|_{\dot{B}_{p,1}^s(\Omega)} d\tau \right).$$

Therefore,

$$\int_1^T \|u_1\|_{\dot{B}_{p,1}^s(K)} dt \leq C \left( 1 + \int_1^T \tau^{-\frac{n}{2}(\frac{1}{p}-\varepsilon)} d\tau \right) \int_0^T \|f\|_{\dot{B}_{p,1}^s(\Omega)} dt. \quad (53)$$

For the time interval  $[0, 1]$ , we merely have

$$\int_0^1 \|u_1\|_{\dot{B}_{p,1}^s(K)} dt \leq C \int_0^1 \|f\|_{\dot{B}_{p,1}^s(\Omega)} dt.$$

Now, provided that one may find some  $\varepsilon > 0$  such that

$$\frac{n}{2} \left( \frac{1}{p} - \varepsilon \right) > 1, \quad (54)$$

a condition which is equivalent to  $p < n/2$ , the constant in (53) may be made independent of  $T$ . Hence we conclude that

$$\int_0^T \|u_1\|_{\dot{B}_{p,1}^s(K)} dt \leq C \int_0^T \|f\|_{\dot{B}_{p,1}^s(\Omega)} dt \quad (55)$$

with  $C$  independent of  $T$ .

Let us now bound  $u_2$ . We first write that

$$\|u_2(t)\|_{\dot{B}_{p,1}^s(K)} \leq C \|u_0\|_{\dot{B}_{p,1}^s(\Omega)} \quad (56)$$

and, if  $-1 + \varepsilon < s < 1/p$ ,

$$\|u_2(t)\|_{\dot{B}_{p,1}^s(K)} \leq C |K|^{\frac{1}{p}-\varepsilon} \|u_2(t)\|_{\dot{B}_{\frac{1}{2},1}^s(K)} \leq C |K|^{\frac{1}{p}-\varepsilon} \|u_0\|_{\dot{B}_{p,1}^s(\Omega)} t^{-\frac{n}{2}(\frac{1}{p}-\varepsilon)}.$$

Then decomposing the integral over  $[0, T]$  into an integral over  $[0, \min(1, T)]$  and  $[\min(1, T), T]$ , we easily get

$$\int_0^T \|u_2(t)\|_{\dot{B}_{p,1}^s(K)} dt \leq C \left( 1 + \int_{\min(1,T)}^T t^{-\frac{n}{2}(\frac{1}{p}-\varepsilon)} dt \right) \|u_0\|_{\dot{B}_{p,1}^s(\Omega)}. \quad (57)$$

The integrant in the r.h.s. of (57) is finite whenever (54) holds. Hence,

$$\int_0^T \|u_2(t)\|_{\dot{B}_{p,1}^s(K)} dt \leq C \|u_0\|_{\dot{B}_{p,1}^s(\Omega)}. \quad (58)$$

Putting this together with (53) and (2.2) completes the proof of the lemma.

We are now ready to prove Theorem 3. Granted with Theorem 4, it is enough to show that  $\|u\|_{L_1(0,T;\dot{B}_{p,1}^s(K)\cap\dot{B}_{q,1}^{s'}(K))}$  may be bounded by the right-hand side of (15).

As a matter of fact  $\|u\|_{L_1(0,T;\dot{B}_{q,1}^{s'}(K))}$  may be directly bounded from Lemma 2, and the same holds for  $\|u\|_{L_1(0,T;\dot{B}_{p,1}^s(K))}$  if  $p < n/2$ .

If  $p \geq n/2$ , then we use the fact so that

$$\dot{B}_{q,1}^{s+2}(\Omega) \subset \dot{B}_{q^*,1}^s(\Omega) \quad \text{with} \quad \frac{1}{q^*} = \frac{1}{q} - \frac{2}{n}.$$

Therefore, if  $q < n/2 \leq p < q^*$  then one may combine interpolation and Lemma 2 so as to absorb  $\|u\|_{L_1(0,T;\dot{B}_{p,1}^s(K))}$  by the left-hand side of (15), changing the constant  $C$  if necessary.

If  $p \geq q^*$  then one may repeat the argument again and again until the all possible values of  $p$  in  $(n/2, \infty)$  are exhausted. Theorem 3 is proved.

### 2.3 The bounded domain case

We end this section with a few remarks concerning the case where  $\Omega$  is a bounded domain of  $\mathbb{R}^n$  with  $n \geq 2$ . Then the proof of Theorem 4 is similar : we still have to introduce some suitable resolution of unity  $(\eta^\ell)_{0 \leq \ell \leq L}$ . The only difference is that, now,  $\eta^0$  has compact support. Hence Theorem 4 holds true with  $K = \overline{\Omega}$ .

In order to remove the time dependency in the estimates, we use the fact (see e.g. [7]) that the solution  $\eta$  to (38) satisfies for some  $c > 0$ ,

$$\|\eta(t)\|_{L_p(\Omega)} \leq C e^{-ct} \|\eta_0\|_{L_p(\Omega)},$$

which also implies that

$$\|\nabla^2 \eta(t)\|_{L_p(\Omega)} \leq C e^{-ct} \|\nabla^2 \eta_0\|_{L_p(\Omega)}.$$

Hence we have for any  $1 < p < \infty$  and  $-1 + 1/p < s < 1/p$ ,

$$\|\eta(t)\|_{\dot{B}_{p,1}^s(\Omega)} \leq C e^{-ct} \|\eta_0\|_{\dot{B}_{p,1}^s(\Omega)}. \quad (59)$$

Defining  $u_1$  and  $u_2$  as in (35), one may thus write

$$\left| \int_{\Omega} u_1(t,x) \eta_0(x) dx \right| \lesssim \|\eta_0\|_{\dot{B}_{p',\infty}^{-s}(\Omega)} \int_0^t \|f(\tau)\|_{\dot{B}_{p,1}^s(\Omega)} e^{-c(t-\tau)} d\tau,$$

thus giving

$$\|u_1\|_{L_1(0,T;\dot{B}_{p,1}^s(K))} \lesssim \|f\|_{L_1(0,T;\dot{B}_{p,1}^s(\Omega))}.$$

Of course, we also have

$$\|u_2\|_{L_1(0,T;\dot{B}_{p,1}^s(\mathcal{K}))} \lesssim \|u_0\|_{\dot{B}_{p,1}^s(\Omega)}.$$

So one may conclude that Lemma 2 holds true for *any*  $1 < p < \infty$  and  $-1 + 1/p < s < 1/p$ . Consequently, we get:

**Theorem 5.** *If  $1 < p < \infty$  and  $-1 + 1/p < s < 1/p$  then the statement of Theorem 2 remains true in any  $C^2$  bounded domain.*

### 3 Applications

In this last section, we give some application of the maximal regularity estimates that have been proved hitherto. As an example, we prove global stability results (in a critical functional framework) for trivial/constant solutions to the following system:

$$\begin{aligned} \mathbf{u}_t - \nu \Delta \mathbf{u} + P \cdot \nabla^2 \mathbf{u} &= f_0(\mathbf{u}) + f_1(\mathbf{u}) \cdot \nabla \mathbf{u} && \text{in } (0, T) \times \Omega, \\ \mathbf{u} &= 0 && \text{at } (0, T) \times \partial \Omega, \\ \mathbf{u}|_{t=0} &= u_0 && \text{on } \Omega. \end{aligned} \quad (60)$$

Above,  $\nu$  is a positive parameter,  $\mathbf{u}$  stands for a  $r$ -dimensional vector and  $P = (P_1, \dots, P_r)$  where the  $P_k$ 's are  $n \times n$  matrices with suitably smooth coefficients. The nonlinearities  $f_0 : \mathbb{R}^r \rightarrow \mathbb{R}^r$  and  $f_1 : \mathbb{R}^r \rightarrow \mathcal{M}_{r,n}(\mathbb{R})$  are  $C^1$  and satisfy

$$f_0(0) = 0, \quad df_0(0) = 0 \quad \text{and} \quad f_1(0) = 0, \quad (61)$$

together with some growths conditions that will be detailed below.

As we have in mind applications to Theorem 3, we focus on the case where  $\Omega$  is a smooth exterior domain of  $\mathbb{R}^n$  with  $n \geq 3$ . Of course, based on our other maximal regularity results, similar (and somewhat easier) statements may be proved for bounded domains,  $\mathbb{R}_+^n$  or  $\mathbb{R}^n$ .

Here are two important examples entering in the class of equations (60). The first one is *the nonlinear heat transfer equation* (see [16] and the references therein):

$$u_t - \nu \Delta u = f(u). \quad (62)$$

A classical form of the nonlinearity is  $f(u) = Ku^2(u - u^*)$ . However one may consider more complex models describing a flame propagation like in [12].

The second example is the *viscous Burgers equation* [9, 10]

$$u_t + u \partial_{x_1} u - \nu \Delta u = 0. \quad (63)$$

which enters in the class of models like

$$\mathbf{u}_t - \nu \Delta \mathbf{u} = B(\mathbf{u}, \nabla \mathbf{u}). \quad (64)$$

In the case where  $B(\mathbf{u}, \nabla \mathbf{u}) = -\mathbf{u} \cdot \nabla \mathbf{u}$ , this is just the equation for pressureless viscous gases with constant density.

Below, based on Theorem 3, we shall prove two global-in-time results concerning the stability of the trivial solution of System (60). In the first statement, to simplify the presentation, we only consider the case where the data belong to spaces with regularity index equals to 0. To simplify the notation, we omit the dependency with respect to the domain  $\Omega$  in all that follows.

**Theorem 6.** *Let  $1 < q < n/2$  and  $\Omega$  be an exterior domain of  $\mathbb{R}^n$  ( $n \geq 3$ ). There exist two positive constants  $\eta$  and  $c_V$  such that for all  $P : [0, \infty) \times \Omega \rightarrow \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k$  satisfying<sup>4</sup>:*

$$\|P\|_{L^\infty(0, \infty; \mathcal{M}(\dot{B}_{n,1}^0 \cap \dot{B}_{q,1}^0))} \leq \eta \nu, \quad (65)$$

for all nonlinearities  $f_0$  and  $f_1$  fulfilling (61) and

$$|df_0(\mathbf{w})| \leq C|\mathbf{w}|, \quad |df_1(\mathbf{w})| \leq C, \quad (66)$$

and for all  $\mathbf{u}_0 \in \dot{B}_{n,1}^0 \cap \dot{B}_{q,1}^0$  such that

$$\|\mathbf{u}_0\|_{\dot{B}_{n,1}^0 \cap \dot{B}_{q,1}^0} \leq c_V, \quad (67)$$

System (60) admits a unique global solution  $\mathbf{u}$  in the space

$$\mathcal{C}_b(0, \infty; \dot{B}_{n,1}^0 \cap \dot{B}_{q,1}^0) \cap L_1(0, \infty; \dot{B}_{n,1}^2 \cap \dot{B}_{q,1}^2). \quad (68)$$

*Proof.* Granted with Theorem 3, the result mainly relies on embedding, composition and product estimates in Besov spaces. We focus on the proof of a priori estimates for a global solution  $\mathbf{u}$  to (60). First, applying Theorem 3 yields

$$\begin{aligned} \|\mathbf{u}\|_{L^\infty(0, \infty; \dot{B}_{n,1}^0 \cap \dot{B}_{q,1}^0) \cap L_1(0, \infty; \dot{B}_{n,1}^2 \cap \dot{B}_{q,1}^2)} &\lesssim \|P\|_{L^\infty(0, \infty; \mathcal{M}(\dot{B}_{n,1}^0 \cap \dot{B}_{q,1}^0))} \|\mathbf{u}\|_{L_1(0, \infty; \dot{B}_{n,1}^2 \cap \dot{B}_{q,1}^2)} \\ &+ \|\mathbf{u}_0\|_{\dot{B}_{n,1}^0 \cap \dot{B}_{q,1}^0} + \|f_0(\mathbf{u})\|_{L_1(0, \infty; \dot{B}_{n,1}^0 \cap \dot{B}_{q,1}^0)} + \|f_1(\mathbf{u}) \cdot \nabla \mathbf{u}\|_{L_1(0, \infty; \dot{B}_{n,1}^0 \cap \dot{B}_{q,1}^0)}. \end{aligned} \quad (69)$$

Bounding the last two terms follows from Propositions 1 and 6. More precisely, for  $p = q, n$ , we have

$$\begin{aligned} \|f_1(\mathbf{u}) \cdot \nabla \mathbf{u}\|_{\dot{B}_{p,1}^0} &\lesssim \|f_1(\mathbf{u})\|_{\dot{B}_{n,1}^{1/2}} \|\nabla \mathbf{u}\|_{\dot{B}_{p,1}^{1/2}} \\ &\lesssim \|\mathbf{u}\|_{\dot{B}_{n,1}^{1/2}} \|\nabla \mathbf{u}\|_{\dot{B}_{p,1}^{1/2}}. \end{aligned}$$

Therefore, applying Hölder inequality,

$$\|f_1(\mathbf{u}) \cdot \nabla \mathbf{u}\|_{L_1(0, \infty; \dot{B}_{p,1}^0)} \lesssim \|\mathbf{u}\|_{L_4(0, \infty; \dot{B}_{n,1}^{1/2})} \|\nabla \mathbf{u}\|_{L_{4/3}(0, \infty; \dot{B}_{p,1}^{1/2})},$$

<sup>4</sup> Below  $\mathcal{M}(X)$  denotes the *multiplier space* associated to the Banach space  $X$ , that is the set of those functions  $f$  such that  $fg \in X$  whenever  $g$  is in  $X$  endowed with the norm  $\|f\|_{\mathcal{M}(X)} := \inf_g \|fg\|_X$  where the infimum is taken over all  $g \in X$  with norm 1.



whence, using elementary interpolation,

$$\|f_1(\mathbf{u}) \cdot \nabla \mathbf{u}\|_{L_1(0,\infty;\dot{B}_{n,1}^0 \cap \dot{B}_{q,1}^0)} \lesssim \|\mathbf{u}\|_{L_\infty(0,\infty;\dot{B}_{n,1}^0 \cap \dot{B}_{q,1}^0) \cap L_1(0,\infty;\dot{B}_{n,1}^2 \cap \dot{B}_{q,1}^2)}^2. \quad (70)$$

Bounding  $f_0(\mathbf{u})$  is slightly more involved. To handle the norm in  $L_1(0,\infty;\dot{B}_{n,1}^0(\Omega))$ , we use the following critical embedding:

$$\dot{B}_{n/2,1}^1 \hookrightarrow \dot{B}_{n^-,1}^{0+} \hookrightarrow \dot{B}_{n,1}^0.$$

Hence Proposition 1 enables us to write that

$$\begin{aligned} \|f_0(\mathbf{u})\|_{\dot{B}_{n,1}^0} &\lesssim \|f_0(\mathbf{u})\|_{\dot{B}_{n^-,1}^{0+}}, \\ &\lesssim \|\mathbf{u}\|_{L_\infty} \|\mathbf{u}\|_{\dot{B}_{n^-,1}^{0+}}, \\ &\lesssim \|\mathbf{u}\|_{L_\infty} \|\mathbf{u}\|_{\dot{B}_{n/2,1}^1}, \\ &\lesssim \|\mathbf{u}\|_{\dot{B}_{n,1}^1} \|\mathbf{u}\|_{\dot{B}_{q,1}^1 \cap \dot{B}_{n,1}^1}. \end{aligned}$$

The last inequality stems from the embedding  $\dot{B}_{n,1}^1 \hookrightarrow L_\infty$  and from the fact that  $q < n/2 < n$ , whence

$$\dot{B}_{q,1}^1 \cap \dot{B}_{n,1}^1 \hookrightarrow \dot{B}_{n/2,1}^1.$$

Therefore, using Hölder inequality and elementary interpolation, we deduce that

$$\|f_0(\mathbf{u})\|_{L_1(0,\infty;\dot{B}_{n,1}^0)} \lesssim \|\mathbf{u}\|_{L_\infty(0,\infty;\dot{B}_{n,1}^0 \cap \dot{B}_{q,1}^0) \cap L_1(0,\infty;\dot{B}_{n,1}^2 \cap \dot{B}_{q,1}^2)}^2. \quad (71)$$

Finally we have to bound  $f_0(\mathbf{u})$  in  $L_1(0,\infty;\dot{B}_{q,1}^0)$ . For that it suffices to estimate it in  $L_1(0,\infty;\dot{B}_{q,1}^{0+})$  and in  $L_1(0,\infty;L_{q^-})$ . Indeed we observe that  $L_{q^-} \hookrightarrow \dot{B}_{q,\infty}^{0-}$ , and thus

$$L_1(0,\infty;\dot{B}_{q,1}^{0+}) \cap L_1(0,\infty;L_{q^-}) \hookrightarrow L_1(0,\infty;\dot{B}_{q,1}^0). \quad (72)$$

Now, on the one hand, according to Proposition 6 and Hölder inequality we have

$$\|f_0(\mathbf{u})\|_{L_1(0,\infty;\dot{B}_{q,1}^{0+})} \lesssim \|\mathbf{u}\|_{L_{1+}(0,\infty;L_\infty)} \|\mathbf{u}\|_{L_{\infty-}(0,\infty;\dot{B}_{q,1}^{0+})}.$$

By interpolation, we easily get

$$\|\mathbf{u}\|_{L_{\infty-}(0,\infty;\dot{B}_{q,1}^{0+})} \lesssim \|\mathbf{u}\|_{L_\infty(0,\infty;\dot{B}_{q,1}^0) \cap L_1(0,\infty;\dot{B}_{q,1}^2)}$$

and because  $q < n/2$ ,

$$\begin{aligned} \|\mathbf{u}\|_{L_{1+}(0,\infty;L_\infty)} &\lesssim \|\mathbf{u}\|_{L_1(0,\infty;\dot{B}_{\infty,1}^0) \cap L_2(0,\infty;\dot{B}_{\infty,1}^0)}, \\ &\lesssim \|\mathbf{u}\|_{L_1(0,\infty;\dot{B}_{n/2,1}^2) \cap L_2(0,\infty;\dot{B}_{n,1}^1)}, \\ &\lesssim \|\mathbf{u}\|_{L_1(0,\infty;\dot{B}_{q,1}^2 \cap \dot{B}_{n,1}^2) \cap L_1(0,\infty;\dot{B}_{n,1}^2) \cap L_\infty(0,\infty;\dot{B}_{n,1}^0)}. \end{aligned}$$

Therefore we have,

$$\|f_0(\mathbf{u})\|_{L_1(0,\infty;\dot{B}_{q,1}^{0+})} \lesssim \|\mathbf{u}\|_{L_1(0,\infty;\dot{B}_{q,1}^2 \cap \dot{B}_{n,1}^2) \cap L_\infty(0,\infty;\dot{B}_{q,1}^0 \cap \dot{B}_{n,1}^0)}^2. \quad (73)$$

On the other hand, using the fact that  $|f_0(\mathbf{u})| \leq C|\mathbf{u}|^2$  and Hölder inequality, we may write

$$\|f_0(\mathbf{u})\|_{L_1(0,\infty;L_{q^-})} \leq \|\mathbf{u}\|_{L_\infty(0,\infty;L_q)} \|\mathbf{u}\|_{L_1(0,\infty;L_{\infty^-})}.$$

We obviously have  $\dot{B}_{q,1}^0 \hookrightarrow L_q$  and, because  $2 - n/q < 0$ ,

$$\dot{B}_{q,1}^2 \cap \dot{B}_{n,1}^2 \hookrightarrow L_{\infty^-}.$$

Therefore

$$\|f_0(\mathbf{u})\|_{L_1(0,\infty;L_{q^-})} \lesssim \|\mathbf{u}\|_{L_\infty(0,\infty;\dot{B}_{q,1}^0)} \|\mathbf{u}\|_{L_1(\dot{B}_{q,1}^2 \cap \dot{B}_{n,1}^2)}. \quad (74)$$

So putting (73) and (74) together and taking advantage of (72), we end up with

$$\|f_0(\mathbf{u})\|_{L_1(0,\infty;\dot{B}_{q,1}^0)} \lesssim \|\mathbf{u}\|_{L_\infty(0,\infty;\dot{B}_{n,1}^0 \cap \dot{B}_{q,1}^0) \cap L_1(0,\infty;\dot{B}_{n,1}^2 \cap \dot{B}_{q,1}^2)}. \quad (75)$$

It is now time to plug (70), (71) and (75) in (69). We get

$$\begin{aligned} & \|\mathbf{u}\|_{L_\infty(0,\infty;\dot{B}_{n,1}^0 \cap \dot{B}_{q,1}^0) \cap L_1(0,\infty;\dot{B}_{n,1}^2 \cap \dot{B}_{q,1}^2)} \\ & \leq C(\|P\|_{L_\infty(0,\infty;\mathcal{M}(\dot{B}_{n,1}^0 \cap \dot{B}_{q,1}^0))} \|\mathbf{u}\|_{L_1(0,\infty;\dot{B}_{n,1}^2 \cap \dot{B}_{q,1}^2)} \\ & \quad + \|\mathbf{u}\|_{L_\infty(0,\infty;\dot{B}_{n,1}^0 \cap \dot{B}_{q,1}^0) \cap L_1(0,\infty;\dot{B}_{n,1}^2 \cap \dot{B}_{q,1}^2)}^2 + \|\mathbf{u}_0\|_{\dot{B}_{n,1}^0 \cap \dot{B}_{q,1}^0}). \end{aligned} \quad (76)$$

Obviously, the above estimate enables us to get a *global-in-time* control of the solution in the desired functional space whenever (65) and (67) are satisfied. Starting from this observation and using the existence part of Theorem 3, it is easy to prove Theorem 6 by means of Banach fixed point theorem as in [3] for instance. The details are left to the reader.

**Theorem 7.** *Assume that  $P \equiv 0$  and that  $f_1 \equiv 0$ . Suppose that  $f_0$  satisfies (61) and*

$$|df_0(\mathbf{w})| \leq C(|\mathbf{w}|^{m-1} + |\mathbf{w}|) \text{ for some } m \geq 2.$$

Let  $1 < q < \frac{n}{2}$  and  $q \leq p < \infty$ . Assume that

$$s_p := \frac{n}{p} - \frac{2}{m-1} \in \left(0, \frac{1}{p}\right) \quad \text{and} \quad 0 < s_q < \frac{1}{q} - \frac{2}{n}.$$

Then there exists a constant  $c_V$  such that if

$$\|\mathbf{u}_0\|_{\dot{B}_{p,1}^{s_p} \cap \dot{B}_{q,1}^{s_q}} \leq c_V \quad (77)$$

then System (60) admits a unique *global-in-time* solution  $\mathbf{u}$  such that

$$\mathbf{u} \in \mathcal{C}_b(0, \infty; \dot{B}_{p,1}^{s_p} \cap \dot{B}_{q,1}^{s_q}) \cap L_1(0, \infty; \dot{B}_{p,1}^{2+s_p} \cap \dot{B}_{q,1}^{2+s_q}). \quad (78)$$

*Proof.* Once again, we start from Theorem 3 which implies the following inequality:

$$\|\mathbf{u}\|_{L_\infty(0, \infty; \dot{B}_{p,1}^{s_p} \cap \dot{B}_{q,1}^{s_q}) \cap L_1(0, \infty; \dot{B}_{p,1}^{2+s_p} \cap \dot{B}_{q,1}^{2+s_q})} \lesssim \|f_0(\mathbf{u})\|_{L_1(0, \infty; \dot{B}_{p,1}^{s_p} \cap \dot{B}_{q,1}^{s_q})} + \|\mathbf{u}_0\|_{\dot{B}_{p,1}^{s_p} \cap \dot{B}_{q,1}^{s_q}}. \quad (79)$$

Now (a slight generalization of) Proposition 6 ensures that for  $s = s_p, s_q$  and for  $r = p, q$ ,

$$\|f_0(\mathbf{u})\|_{\dot{B}_{r,1}^s} \lesssim (\|\mathbf{u}\|_{L_\infty} + \|\mathbf{u}\|_{L_\infty}^{m-1}) \|\mathbf{u}\|_{\dot{B}_{r,1}^s}.$$

Therefore,

$$\begin{aligned} \|f_0(\mathbf{u})\|_{L_1(0, \infty; \dot{B}_{p,1}^{s_p} \cap \dot{B}_{q,1}^{s_q})} &\leq C(\|\mathbf{u}\|_{L_1(0, \infty; L_\infty)} \\ &\quad + \|\mathbf{u}\|_{L_{m-1}(0, \infty; L_\infty)}^{m-1}) \|\mathbf{u}\|_{L_\infty(0, \infty; \dot{B}_{p,1}^{s_p} \cap \dot{B}_{q,1}^{s_q})}. \end{aligned} \quad (80)$$

Hence it is only a matter of proving that the norm of  $\mathbf{u}$  in  $L_1(0, \infty; L_\infty)$  and in  $L_{m-1}(0, \infty; L_\infty)$  may be bounded by means of the norm in  $L_1(0, \infty; \dot{B}_{q,1}^2 \cap \dot{B}_{n,1}^2) \cap L_\infty(0, \infty; \dot{B}_{q,1}^0 \cap \dot{B}_{n,1}^0)$ . Now, we notice that  $\dot{B}_{p,1}^{s_p+2/(m-1)}$  embeds continuously in  $L_\infty$  and that, by interpolation,

$$\|\mathbf{u}\|_{L_{m-1}(0, \infty; \dot{B}_{p,1}^{s_p+2/(m-1)})} \leq \|\mathbf{u}\|_{L_\infty(0, \infty; \dot{B}_{p,1}^{s_p}) \cap L_1(0, \infty; \dot{B}_{p,1}^{s_p+2})}.$$

Hence we do have

$$\|\mathbf{u}\|_{L_{m-1}(0, \infty; L_\infty)} \lesssim \|\mathbf{u}\|_{L_\infty(0, \infty; \dot{B}_{p,1}^{s_p}) \cap L_1(0, \infty; \dot{B}_{p,1}^{s_p+2})}. \quad (81)$$

Finally, we notice that  $\dot{B}_{s_q+2,1}^2 \hookrightarrow \dot{B}_{\infty,1}^{2+s_q-n/q}$  and that  $2+s_q-n/q < 0$ . At the same time  $\dot{B}_{p,1}^{s_p+2} \hookrightarrow \dot{B}_{\infty,1}^1$ , therefore

$$\dot{B}_{q,1}^{s_q+2} \cap \dot{B}_{p,1}^{s_p+2} \hookrightarrow L_\infty.$$

Hence we have

$$\|\mathbf{u}\|_{L_1(0, \infty; L_\infty)} \lesssim \|\mathbf{u}\|_{L_1(0, \infty; \dot{B}_{p,1}^{s_p+2}) \cap L_1(0, \infty; \dot{B}_{q,1}^{s_q+2})}. \quad (82)$$

Putting (81) and (82) into (80) and then into (79) we get

$$\begin{aligned} \|\mathbf{u}\|_{L_\infty(0, \infty; \dot{B}_{p,1}^{s_p} \cap \dot{B}_{q,1}^{s_q}) \cap L_1(0, \infty; \dot{B}_{p,1}^{2+s_p} \cap \dot{B}_{q,1}^{2+s_q})} &\lesssim \|\mathbf{u}_0\|_{\dot{B}_{p,1}^{s_p} \cap \dot{B}_{q,1}^{s_q}} \\ &\quad + (\|\mathbf{u}\|_{L_\infty(0, \infty; \dot{B}_{p,1}^{s_p} \cap \dot{B}_{q,1}^{s_q}) \cap L_1(0, \infty; \dot{B}_{p,1}^{2+s_p} \cap \dot{B}_{q,1}^{2+s_q})}^{m-2} \\ &\quad + 1) \|\mathbf{u}\|_{L_\infty(0, \infty; \dot{B}_{p,1}^{s_p} \cap \dot{B}_{q,1}^{s_q}) \cap L_1(0, \infty; \dot{B}_{p,1}^{2+s_p} \cap \dot{B}_{q,1}^{2+s_q})}^2. \end{aligned}$$

The smallness of the initial data in (69) enables to close the estimate for the left-hand side of the above inequality. The existence issue is just a consequence of Banach fixed point theorem. This completes the proof of the theorem.

*Remark 5.* Even though System (60) does not have any scaling invariance in general, our two statements are somewhat critical from the regularity point of view. Indeed, in the functional framework used in Theorem 6 and under the growth condition (66), the nonlinearity  $f_0(\mathbf{u})$  is lower order compared to  $f_1(\mathbf{u}) \cdot \nabla \mathbf{u}$ . Now, we notice that if  $f_0 \equiv 0$  and  $P \equiv 0$  then the initial value problem for System (60) (in the  $\mathbb{R}^n$  case) is invariant for all  $\lambda > 0$  under the transform:

$$(u(t, x), u_0(x)) \longrightarrow \lambda(u(\lambda^2 t, \lambda x), u_0(\lambda x)).$$

At the same time, the norm  $\|\cdot\|_{\dot{B}_{n,1}^0(\mathbb{R}^n)}$  is invariant by the above rescaling for  $u_0$ .

As regards Theorem 7, the nonlinearity  $f_0(\mathbf{w})$  is at most of order  $m$ . Now, if (the coefficients of)  $f_0(\mathbf{w})$  are homogeneous polynomials of degree  $m$  then the system is invariant by

$$(u(t, x), u_0(x)) \longrightarrow \lambda^{\frac{2}{m-1}}(u(\lambda^2 t, \lambda x), u_0(\lambda x)).$$

Hence the regularity  $\dot{B}_{p,1}^{s_p}$  is critical.

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