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Eric Amar

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An Andreotti-Grauert theorem with L^r estimates.

Eric Amar

Abstract

By a theorem of Andreotti and Grauert if ω is a (p, q) current, $q < n$, in a Stein manifold, $\bar{\partial}$ closed and with compact support, then there is a solution u to $\bar{\partial}u = \omega$ still with compact support. The main result of this work is to show that if moreover $\omega \in L^r(dm)$, where m is a suitable "Lebesgue" measure on the Stein manifold, then we have a solution u with compact support *and* in $L^r(dm)$. We prove it by estimates in L^r spaces with weights.

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1 Introduction.

Let ω be a $\bar{\partial}$ closed (p, q) form in \mathbb{C}^n with compact support $K := \text{Supp}\omega$ and such that $\omega \in L^r(\mathbb{C}^n)$, the Lebesgue space in \mathbb{C}^n . Setting K in a ball $\mathbb{B} := B(0, R)$ with R big enough, we know, by a theorem of Ovreid [Ovreid, 1971], that we have a $(p, q - 1)$ form $u \in L^r(\mathbb{B})$ such that $\bar{\partial}u = \omega$. On the other hand we also know, at least when $q < n$, that there is a current v with compact support such that $\bar{\partial}v = \omega$, by a theorem of Andreotti-Grauert [Andreotti and Grauert, 1962]. So a natural question is: may we have a solution u of $\bar{\partial}u = \omega$ with compact support *and* in $L^r(\mathbb{C}^n)$?

There is a work by H. Skoda [Skoda, 1976] who proved such a result. Let Ω be a strictly pseudo-convex bounded domain in \mathbb{C}^n with smooth boundary then in [Skoda, 1976, Corollaire p. 295], H. Skoda proved that if f is a (p, q) -form with measure coefficients, $q < n$, $\bar{\partial}$ closed and with compact support in Ω , then there is a solution U to the equation $\bar{\partial}U = f$ such that $\|U\|_{L^r(\Omega)} \leq C(\Omega, r)\|f\|_1$, for any r such that $1 < r < \frac{2n+2}{2n-1}$ and U has zero boundary values in the sense of Stokes formula. This means that essentially U has compact support and, because Ω is bounded, $\|f\|_1 \lesssim \|f\|_{L^r(\Omega)}$. So he got the answer for Ω strictly pseudo-convex and $1 < r < \frac{2n+2}{2n-1}$.

We answered this question by the affirmative for any $r \in [1, \infty]$ in a joint work with S. Mongodi [Amar and Mongodi, 2014] linearly by the "method of coronas". This method asks for extra L^r conditions on derivatives of coefficients of ω , when $q < n$; we shall denote the set of ω verifying these conditions $\mathcal{W}_q^r(\Omega)$, as in [Amar and Mongodi, 2014].

The aim of this work is to extend this result to Stein manifolds and get rid of the extra L^r conditions $\mathcal{W}_q^r(\Omega)$. For it we use a completely different approach inspired by the Serre duality [Serre, 1955]. Because Hahn Banach theorem is used, this method is no longer constructive as in [Amar and Mongodi, 2014].

The basic notion we shall use here is the following.

Definition 1.1. *Let X be a complex manifold equipped with a Borel σ -finite measure dm and Ω a domain in X ; let $r \in [1, \infty]$, we shall say that Ω is r **regular** if for any $p, q \in \{0, \dots, n\}$, $q \geq 1$, there is a constant $C = C_{p,q}(\Omega)$ such that for any (p, q) form ω , $\bar{\partial}$ closed in Ω and in $L^r(\Omega, dm)$ there is a $(p, q-1)$ form $u \in L^r(\Omega, dm)$ such that $\bar{\partial}u = \omega$ and $\|u\|_{L^r(\Omega)} \leq C\|\omega\|_{L^r(\Omega)}$.*

*We shall say that Ω is **weakly r regular** if for any compact set $K \Subset \Omega$ there are 3 open sets $\Omega_1, \Omega_2, \Omega_3$ such that $K \Subset \Omega_3 \subset \Omega_2 \subset \Omega_1 \subset \Omega_0 := \Omega$ and 3 constants C_1, C_2, C_3 such that:*

$$\forall j = 0, 1, 2, \forall p, q \in \{0, \dots, n\}, q \geq 1, \forall \omega \in L_{p,q}^r(\Omega_j, dm), \bar{\partial}\omega = 0,$$

$$\exists u \in L_{p,q-1}^r(\Omega_{j+1}, dm), \bar{\partial}u = \omega$$

and $\|u\|_{L^r(\Omega_{j+1})} \leq C_{j+1}\|\omega\|_{L^r(\Omega_j)}$.

I.e. we have a 3 steps chain of resolution.

Of course the r regularity implies the weak r regularity, just taking $\Omega_1 = \Omega_2 = \Omega_3 = \Omega$.

Examples of 2 regular domains are the bounded pseudo-convex domains by Hörmander [Hörmander, 1994].

Examples of r regular domains in \mathbb{C}^n are the bounded strictly pseudo-convex (s.p.c.) domains with smooth boundary by Ovrelid [Ovrelid, 1971]; the polydiscs in \mathbb{C}^n by Charpentier [Charpentier, 1980], finite transverse intersections of strictly pseudo-convex bounded domains in \mathbb{C}^n by Menini [Menini, 1997]. A generalisation of the results by Menini was done in the nice work of Ma and Vassiliadou [Ma and Vassiliadou, 2000]: they treated also the case of intersection of q -convex sets.

Examples of r regular domains in a Stein manifold are the strictly pseudo-convex domains with smooth boundary [Amar, 2016]. (See the previous work for $(0, 1)$ forms by N. Kerzman [Kerzman, 1971] and for all (p, q) forms by J-P. Demailly and C. Laurent [Demailly and Laurent-Thiébaud, 1987, Remarque 4, page 596], but here the manifold has to be equipped with a metric with null curvature. See also [Amar, 2017] for the case of intersection of q -convex sets in a Stein manifold).

Let X be a Stein manifold and Ω a domain in X , i.e. an open connected set in X . Let $\mathcal{H}_p(\Omega)$ be the set of all $(p, 0)$ $\bar{\partial}$ closed forms in Ω . If $p = 0$, $\mathcal{H}_0(\Omega) = \mathcal{H}(\Omega)$ is the set of holomorphic functions in Ω . If $p > 0$, we have, in a chart (φ, U) , $h \in \mathcal{H}_p(\Omega) \Rightarrow h(z) = \sum_{|J|=p} a_J(z) dz^J$, where $dz^J := dz_{j_1} \wedge \dots \wedge dz_{j_p}$ and the functions $a_J(z)$ are holomorphic in $\varphi(U) \subset \mathbb{C}^n$.

We shall denote $L_{p,q}^{r,c}(\Omega)$ the set of (p, q) -forms in $L^r(\Omega)$ with compact support in Ω .

We also use the notation r' for the conjugate exponent of r , i.e. $\frac{1}{r} + \frac{1}{r'} = 1$.

Our main theorem is:

Theorem 1.2. *Let Ω be a weakly r' regular domain in a Stein manifold X . Then there is a $C > 0$ such that for any (p, q) form ω in $L^{r,c}(\Omega)$, $r > 1$ with:*

- if $1 \leq q < n$, $\bar{\partial}\omega = 0$;
- if $q = n$, $\forall V \subset \Omega$, $\text{Supp } \omega \subset V$, $\omega \perp \mathcal{H}_{n-p}(V)$;

there is a $(p, q-1)$ form u in $L^{r,c}(\Omega)$ such that $\bar{\partial}u = \omega$ as distributions and $\|u\|_{L^r(\Omega)} \leq C\|\omega\|_{L^r(\Omega)}$.

The notion of r regularity gives a good control of the support: if the support of the data ω is contained in $\Omega \setminus C$ where Ω is a weakly r' regular domain and C is a weakly r regular domain, then the support of the solution u is contained in $\Omega \setminus C'$, where C' is any relatively compact domain in C , provided that $q \geq 2$. One may observe that $\Omega \setminus C$ is *not* Stein in general even if Ω is.

There is also a result of this kind for $q = 1$, see section 3.3.

In particular the support of the solution u is contained in the intersection of all the weakly r' regular domains containing the support of ω .

The idea is to solve $\bar{\partial}u = \omega$ in a space $L^r(\Omega)$ with a "big weight η outside" of the support of ω ; this way we shall have a "small solution u outside" of the support of ω . Then, using a sequence of such weights going to infinity "outside" of the support of ω , we shall have a u zero "outside of the support" of ω .

Comparing to my previous work [Amar, 2012] the results here are improved and the proofs are much simpler by a systematic use of the Hodge $*$ operator.

I am indebted to G. Tomassini who started my interest in this subject on precisely this kind of questions and also to S. Mongodi for a lot of discussions during the preparation of our joint paper [Amar and Mongodi, 2014].

I want to thank C. Laurent for many instructive discussions on this subject.

Finally I also thank the referee for his/her careful reading of the manuscript and the incisive questions he/she asked.

2 Duality.

We shall study a duality between currents inspired by the Serre duality [Serre, 1955].

Let X be a complex manifold of dimension n . We proceed now exactly as in Hörmander [Hörmander, 1994, p. 119], by introducing a hermitian metric on differential forms locally equivalent to the usual one on any analytic coordinates system.

We define the "Lebesgue measure" still as in Hörmander's book [Hörmander, 1994, Section 5.2]: associated to this metric there is a volume measure dm and we take it for the Lebesgue measure on X . Moreover, because X is a complex manifold, it is canonically oriented.

2.1 Weighted L^r spaces.

Let Ω be a domain in X . We denote also dm the volume form on X .

We shall take the following notation from the book by C. Voisin [Voisin, 2002].

To a (p, q) -form α on Ω we associate its Hodge $*$ $(n - p, n - q)$ -form $*\alpha$. This gives us a pointwise scalar product and a pointwise modulus:

$$\langle \alpha, \beta \rangle dm := \alpha \wedge \overline{* \beta}; \quad |\alpha|^2 dm := \alpha \wedge \overline{* \alpha}, \quad (2.1)$$

because $\alpha \wedge \overline{* \beta}$ is a (n, n) -form hence is a function time the volume form dm .

We are also given a scalar product $\langle \alpha, \beta \rangle$ on (p, q) -forms such that $\int_{\Omega} |\alpha|^2 dm < \infty$ and the link between these notions is given by [Voisin, 2002, Lemme 5.8, p. 119]:

$$\langle \alpha, \beta \rangle = \int_{\Omega} \alpha \wedge \overline{* \beta}. \quad (2.2)$$

We shall define now $L^r_{p,q}(\Omega)$ to be the set of (p, q) -forms α defined on Ω such that

$$\|\alpha\|_{L^r_{p,q}(\Omega)} := \int_{\Omega} |\alpha(z)|^r dm(z) < \infty,$$

where $|\alpha|$ is defined by (2.1).

Lemma 2.1. *Let $\eta > 0$ be a weight. If u is a (p, q) -current defined on $(n - p, n - q)$ -forms α in $L^{r'}(\Omega, \eta)$ and such that*

$$\forall \alpha \in L^{r'}_{(n-p, n-q)}(\Omega, \eta), \quad |\langle u, *\alpha \rangle| \leq C \|\alpha\|_{L^{r'}(\Omega, \eta)},$$

then $\|u\|_{L^r_{p,q}(\Omega, \eta^{1-r})} \leq C$.

Proof.

We use the classical trick: set $\tilde{\alpha} := \eta^{1/r'}\alpha$; $\tilde{u} := \frac{1}{\eta^{1/r'}}u$ then we have

$$\langle u, *\alpha \rangle = \int_{\Omega} u \wedge \bar{\alpha} = \int_{\Omega} \tilde{u} \wedge \bar{\tilde{\alpha}} = \langle \tilde{u}, *\tilde{\alpha} \rangle$$

and $\|\tilde{\alpha}\|_{L^{r'}(\Omega)} = \|\alpha\|_{L^{r'}(\Omega, \eta)}$.

We notice that $\|\tilde{\alpha}\|_{L^{r'}(\Omega)} = \|\tilde{\alpha}\|_{L^{r'}(\Omega)}$ because we have $(*\tilde{\alpha}, *\tilde{\alpha})dm = *\tilde{\alpha} \wedge \overline{**\tilde{\alpha}}$ but $**\tilde{\alpha} = (-1)^{(p+q)(2n-p-q)}\tilde{\alpha}$, by [Voisin, 2002, Lemma 5.5], hence, because $(*\tilde{\alpha}, *\tilde{\alpha})$ is positive, $(*\tilde{\alpha}, *\tilde{\alpha}) = |\tilde{\alpha}|^2$.
By use of the duality $L_{p,q}^r(\Omega) - L_{n-p,n-q}^{r'}(\Omega)$, done in Lemma 4.3, we get

$$\|\tilde{u}\|_{L_{p,q}^r(\Omega)} = \sup_{\alpha \in L_{n-p,n-q}^{r'}(\Omega), \alpha \neq 0} \frac{|\langle \tilde{u}, *\tilde{\alpha} \rangle|}{\|\tilde{\alpha}\|_{L^{r'}(\Omega)}}.$$

But

$$\|\tilde{u}\|_{L_{p,q}^r(\Omega)}^r := \int_{\Omega} |u|^r \eta^{-\frac{r}{r'}} dm = \int_{\Omega} |u|^r \eta^{1-r} dm = \|u\|_{L^r(\Omega, \eta^{1-r})}^r.$$

So we get

$$\|u\|_{L_{p,q}^r(\Omega, \eta^{1-r})} = \sup_{*\alpha \in L_{p,q}^{r'}(\Omega, \eta), \alpha \neq 0} \frac{|\langle u, *\alpha \rangle|}{\|\alpha\|_{L^{r'}(\Omega, \eta)}}.$$

The proof is complete. ■

It may seem strange that we have such an estimate when the dual of $L^{r'}(\Omega, \eta)$ is $L^r(\Omega, \eta)$, but the reason is, of course, that in the duality current-form there is no weights.

The point here is that when η is small, η^{1-r} is big for $r > 1$.

3 Solution of the $\bar{\partial}$ equation with compact support.

3.1 r regular domains.

As we have seen, examples of r regular domains in Stein manifolds are the relatively compact s.p.c. domains with smooth boundary.

To prove that a Stein manifold Ω is weakly r regular we shall need the following lemma.

Lemma 3.1. *Let Ω be a Stein manifold. Then it contains an exhaustive sequence of open relatively compact strictly pseudo-convex sets $\{D_k\}_{k \in \mathbb{N}}$ with \mathcal{C}^∞ smooth boundary.*

Proof.

For the case of Ω pseudo-convex in \mathbb{C}^n , the proof was already done explicitly in the proof of [Henkin and Leiterer, 1984, Theorem 2.8.1, p. 86].

By Theorem 5.1.6 of Hörmander [Hörmander, 1994] there exists a \mathcal{C}^∞ strictly plurisubharmonic (s.p.s.h.) exhausting function φ for Ω . Take $K \Subset \Omega$ such that $d\varphi \neq 0$ on K . Because φ is s.p.s.h. then $K \neq \emptyset$. Then we use the [Henkin and Leiterer, 1984, Lemma 2.12.2, p. 93], to get:

$\forall \epsilon > 0$, $\exists \rho_\epsilon$ s.p.s.h. \mathcal{C}^∞ -function on Ω such that:

- (i) $\varphi - \rho_\epsilon$ together with its first and second derivatives is less than ϵ on Ω .
- (ii) The set $\text{Crit}(\rho_\epsilon) := \{z \in \Omega :: d\rho_\epsilon(z) = 0\}$ is discrete in Ω . (In a formula, the notation $::$ means "such that".)
- (iii) $\rho_\epsilon = \varphi$ on K .

As stated in Lemma 2.12.2 if $\varphi \in \mathcal{C}^2$ then $\rho_\epsilon \in \mathcal{C}^2$, but in fact the proof of this Lemma 2.12.2 gives $\rho_\epsilon = \varphi + \sum \chi_j$, where $\sum \chi_j$ is \mathcal{C}^∞ . (see p. 93 in [Henkin and Leiterer, 1984]). Hence ρ_ϵ has the same \mathcal{C}^k regularity than φ .

Fix $\epsilon > 0$, then the function $\rho := \rho_\epsilon$ is also a s.p.s.h. exhausting function for Ω , because, from $-\epsilon \leq \varphi - \rho_\epsilon \leq \epsilon$, we get that, for any $\alpha \in \mathbb{R}$,

$$\{z \in \Omega :: \rho_\epsilon(z) < \alpha\} \subset \{z \in \Omega :: \varphi(z) < \epsilon + \alpha\}$$

and, because φ is an exhausting function, this set is relatively compact in Ω .

Because the set of critical points of ρ is discrete in Ω , the same way as in the proof of [Henkin and Leiterer, 1984, Theorem 2.8.1, p. 86], we can find a sequence $c_k \in \mathbb{R}$, $c_k \rightarrow \infty$, such that $D_k := \{z \in \Omega :: \rho(z) < c_k\}$ make an exhaustive sequence of open relatively compact sets in Ω , $\partial\rho \neq 0$ on ∂D_k , hence D_k is strictly pseudo-convex with C^∞ smooth boundary, and finally $D_k \nearrow \Omega$. The proof is complete. \blacksquare

Proposition 3.2. *A Stein manifold Ω is weakly r regular.*

Proof.

By Lemma 3.1 there is an exhaustive sequence of open relatively compact s.p.c. sets in Ω , $\{D_k\}_{k \in \mathbb{N}}$ with C^∞ smooth boundary.

Let $\omega \in L_{p,q}^r(\Omega)$, $\bar{\partial}\omega = 0$, by [Amar, 2016], we can solve $\bar{\partial}u = \omega$ in D_k with $u \in L_{p,q-1}^r(D_k)$ and

$$\|u\|_{L^r(D_k)} \leq C_k \|\omega\|_{L^r(D_k)} \leq C_k \|\omega\|_{L^r(\Omega)}.$$

Hence if Γ is a compact set in Ω , there is a D_k such that $\Gamma \Subset D_k$ and we can take $\Omega_1 = \Omega_2 = \Omega_3 = D_k$.

This proves the weak r regularity of Ω . \blacksquare

3.2 The main result.

Let X be a Stein manifold and Ω a domain in X .

In order to simplify notation, we set the pairing for α a (p, q) -form and β a $(n-p, n-q)$ -form: $\ll \alpha, \beta \gg := \int_{\Omega} \alpha \wedge \beta$.

With this notation we also have $\langle \alpha, \beta \rangle = \ll \alpha, \overline{*}\beta \gg$.

Let Ω be a weakly r' regular domain in X . We set $K := \text{Supp } \omega \Subset \Omega$ and, by the definition of the r' weak regularity, we get 3 open sets such that $K \Subset \Omega_3 \subset \Omega_2 \subset \Omega_1 \subset \Omega_0 = \Omega$ with: $\forall j = 0, 1, 2, \forall p, q \in \{0, \dots, n\}, q \geq 1$,

$$\forall \alpha \in L_{p,q}^r(\Omega_j), \bar{\partial}\alpha = 0, \exists \varphi \in L_{p,q-1}^r(\Omega_{j+1}), \bar{\partial}\varphi = \alpha.$$

Set the weight $\eta = \eta_\epsilon := \mathbb{1}_{\Omega_1}(z) + \epsilon \mathbb{1}_{\Omega \setminus \Omega_1}(z)$ for a fixed $\epsilon > 0$.

Let $\omega \in L_{p,q}^{r,c}(\Omega)$. Suppose moreover that ω is such that $\bar{\partial}\omega = 0$ if $1 \leq q < n$ and for any open $V \Subset \Omega$, $\text{Supp } \omega \Subset V$ we have $\omega \perp \mathcal{H}_{n-p}(V) \iff \forall h \in \mathcal{H}_{n-p}(V), \ll \omega, h \gg = 0$ if $q = n$.

We shall use the following lemma, with the previous notation:

Lemma 3.3. *Let \mathcal{E} be the set of $(n-p, n-q+1)$ forms $\alpha \in L^{r'}(\Omega, \eta)$, $\bar{\partial}$ closed in Ω . Let us define \mathcal{L}_ω on \mathcal{E} as follows:*

$$\mathcal{L}_\omega(\alpha) := (-1)^{p+q-1} \ll \varphi, \omega \gg,$$

where $\varphi \in L^{r'}(\Omega_1)$ is such that $\bar{\partial}\varphi = \alpha$ in Ω_1 . Then the form \mathcal{L}_ω is well defined and linear.

Proof.

Because $\epsilon > 0$ we have $\alpha \in L^{r'}(\Omega, \eta) \Rightarrow \alpha \in L^{r'}(\Omega)$ and the weak r' regularity of Ω gives a $\varphi \in L^{r'}(\Omega_1)$ with $\bar{\partial}\varphi = \alpha$ in Ω_1 .

Let us see that \mathcal{L}_ω is well defined.

- Suppose first that $q < n$.

In order for \mathcal{L}_ω to be well defined we need

$$\forall \varphi, \psi \in L_{(n-p, n-q)}^{r'}(\Omega_1), \bar{\partial}\varphi = \bar{\partial}\psi = \alpha \Rightarrow \ll \varphi, \omega \gg = \ll \psi, \omega \gg.$$

This is meaningful because $\omega \in L^{r,c}(\Omega)$, $r > 1$, $\text{Supp } \omega \Subset \Omega_1$.

Then we have $\bar{\partial}(\varphi - \psi) = 0$ in Ω_1 , hence, because Ω is weakly r' regular, we can solve $\bar{\partial}$ in $L^{r'}(\Omega_2)$:

$$\exists \gamma \in L_{(n-p, n-q-1)}^{r'}(\Omega_2) :: \bar{\partial}\gamma = (\varphi - \psi).$$

So $\ll \varphi - \psi, \omega \gg = \ll \bar{\partial}\gamma, \omega \gg = (-1)^{p+q-1} \ll \gamma, \bar{\partial}\omega \gg = 0$ because ω is compactly supported in Ω_2 and $\bar{\partial}$ closed.

Hence \mathcal{L}_ω is well defined in that case.

- Suppose now that $q = n$.

For φ, ψ ($n - p, 0$) forms in Ω_1 , such that $\bar{\partial}\varphi = \bar{\partial}\psi = \alpha$, we need to have $\ll \varphi, \omega \gg = \ll \psi, \omega \gg$. But then $\bar{\partial}(\varphi - \psi) = 0$, which means that $h := \varphi - \psi$ is a $\bar{\partial}$ closed ($n - p, 0$) form, hence $h \in \mathcal{H}_{n-p}(\Omega_1)$. Taking $V = \Omega_1$ in the hypothesis $\omega \perp \mathcal{H}_{n-p}(V)$, we get $\ll h, \omega \gg = 0$, and \mathcal{L}_ω is also well defined in that case.

It remains to see that \mathcal{L}_ω is linear.

- Suppose first that $q < n$.

Let $\alpha = \alpha_1 + \alpha_2$, with $\alpha_j \in L^{r'}(\Omega, \eta)$, $\bar{\partial}\alpha_j = 0$, $j = 1, 2$; we have $\alpha = \bar{\partial}\varphi$, $\alpha_1 = \bar{\partial}\varphi_1$ and $\alpha_2 = \bar{\partial}\varphi_2$, with $\varphi, \varphi_1, \varphi_2$ in $L^{r'}(\Omega_1)$ so, because $\bar{\partial}(\varphi - \varphi_1 - \varphi_2) = 0$, we have

$$\varphi = \varphi_1 + \varphi_2 + \bar{\partial}\psi, \text{ with } \psi \text{ in } L^{r'}(\Omega_2),$$

so

$$\begin{aligned} \mathcal{L}_\omega(\alpha) &= (-1)^{p+q-1} \ll \varphi, \omega \gg = (-1)^{p+q-1} \ll \varphi_1 + \varphi_2 + \bar{\partial}\psi, \omega \gg \\ &= \mathcal{L}_\omega(\alpha_1) + \mathcal{L}_\omega(\alpha_2) + (-1)^{p+q-1} \ll \bar{\partial}\psi, \omega \gg, \end{aligned}$$

but again $\ll \bar{\partial}\psi, \omega \gg = 0$, hence $\mathcal{L}_\omega(\alpha) = \mathcal{L}_\omega(\alpha_1) + \mathcal{L}_\omega(\alpha_2)$.

The same for $\alpha = \lambda\alpha_1$.

- Suppose now that $q = n$.

We have

$$\mathcal{L}_\omega(\alpha) := (-1)^{p+n-1} \ll \varphi, \omega \gg,$$

where $\varphi \in L^{r'}(\Omega_1)$ is such that $\bar{\partial}\varphi = \alpha$ in Ω_1 .

Let $\alpha = \alpha_1 + \alpha_2$, with $\alpha_j \in L^{r'}(\Omega, \eta)$, $\bar{\partial}\alpha_j = 0$, $j = 1, 2$; we have $\alpha = \bar{\partial}\varphi$, $\alpha_1 = \bar{\partial}\varphi_1$ and $\alpha_2 = \bar{\partial}\varphi_2$, with $\varphi, \varphi_1, \varphi_2$ in $L^{r'}(\Omega_1)$ so, because $\bar{\partial}(\varphi - \varphi_1 - \varphi_2) = 0$, we have $\varphi - \varphi_1 - \varphi_2$ is a ($n - p, 0$) $\bar{\partial}$ -closed form, hence:

$$\varphi = \varphi_1 + \varphi_2 + h, \text{ with } h \in \mathcal{H}_{n-p}(\Omega_1).$$

So

$$\begin{aligned} \mathcal{L}_\omega(\alpha) &= (-1)^{p+q-1} \ll \varphi, \omega \gg = (-1)^{p+q-1} \ll \varphi_1 + \varphi_2 + h, \omega \gg \\ &= \mathcal{L}_\omega(\alpha_1) + \mathcal{L}_\omega(\alpha_2) + (-1)^{p+q-1} \ll h, \omega \gg. \end{aligned}$$

Taking $V = \Omega_1$ in the hypothesis $\omega \perp \mathcal{H}_{n-p}(V)$, we get $\ll h, \omega \gg = 0$, hence $\mathcal{L}_\omega(\alpha) = \mathcal{L}_\omega(\alpha_1) + \mathcal{L}_\omega(\alpha_2)$.

The same for $\alpha = \lambda\alpha_1$. The proof is complete. \blacksquare

Remark 3.4. *If Ω is Stein, we can take the domain Ω_1 to be s.p.c. with C^∞ smooth boundary, hence also Stein. So because $K := \text{Supp } \omega \subset \Omega_1 \subset \Omega$, the $A(\Omega_1)$ convex hull of K , \hat{K}_{Ω_1} is still in Ω_1 , and any holomorphic function in Ω_1 can be uniformly approximated on \hat{K}_{Ω_1} by holomorphic functions in Ω .*

Then for $q = n$ instead of asking $\omega \perp \mathcal{H}_{n-p}(\Omega_1)$ we need just $\omega \perp \mathcal{H}_{n-p}(\Omega)$.

Theorem 3.5. *Let Ω be a weakly r' regular domain and ω be a (p, q) form in $L^{r,c}(\Omega)$, $r > 1$. Suppose that ω is such that:*

- if $1 \leq q < n$, $\bar{\partial}\omega = 0$;
- if $q = n$, $\forall V \subset \Omega$, $\text{Supp } \omega \subset V$, $\omega \perp \mathcal{H}_{n-p}(V)$

Then there is a $C > 0$ and a $(p, q-1)$ form u in $L^{r,c}(\Omega)$ such that $\bar{\partial}u = \omega$ as distributions and $\|u\|_{L^r(\Omega)} \leq C\|\omega\|_{L^r(\Omega)}$.

Proof.

Because Ω is weakly r' regular there is a $\Omega_1 \subset \Omega$, $\Omega_1 \supset \text{Supp } \omega$ such that

$$\forall \alpha \in L^{r'}(\Omega), \bar{\partial}\alpha = 0, \exists \varphi \in L^{r'}(\Omega_1) :: \bar{\partial}\varphi = \alpha, \|\varphi\|_{L^{r'}(\Omega_1)} \leq C_1\|\alpha\|_{L^{r'}(\Omega)}.$$

There is a Ω_2 such that $\text{Supp } \omega \Subset \Omega_2 \subset \Omega_1 \subset \Omega$ with the same properties as Ω_1 .

Let us consider the weight $\eta = \eta_\epsilon := \mathbb{1}_{\Omega_1}(z) + \epsilon \mathbb{1}_{\Omega \setminus \Omega_1}(z)$ for a fixed $\epsilon > 0$ and the form \mathcal{L}_ω defined in Lemma 3.3. By Lemma 3.3 we have that \mathcal{L}_ω is a linear form on $(n - p, n - q + 1)$ -forms $\alpha \in L^{r'}(\Omega, \eta)$, $\bar{\partial}$ closed in Ω .

If α is a $(n - p, n - q + 1)$ -form in $L^{r'}(\Omega, \eta)$, then α is in $L^{r'}(\Omega)$ because $\epsilon > 0$.

The weak r' regularity of Ω gives that there is a $\varphi \in L^{r'}(\Omega_1) :: \bar{\partial}\varphi = \alpha$ which can be used to define $\mathcal{L}_\omega(\alpha)$.

We have also that $\alpha \in L^{r'}(\Omega_1)$, $\bar{\partial}\alpha = 0$ in Ω_1 , hence, still with the weak r' regularity of Ω , we have

$$\exists \psi \in L^{r'}(\Omega_2) :: \bar{\partial}\psi = \alpha, \|\psi\|_{L^{r'}(\Omega_2)} \leq C_2 \|\alpha\|_{L^{r'}(\Omega_1)}.$$

• For $q < n$, we have $\bar{\partial}(\varphi - \psi) = \alpha - \alpha = 0$ on Ω_2 and, by the weak r' regularity of Ω , there is a $\Omega_3 \subset \Omega_2$, such that $\text{Supp } \omega \subset \Omega_3 \subset \Omega_2$, and a $\gamma \in L^{r'}(\Omega_3)$, $\bar{\partial}\gamma = \varphi - \psi$ in Ω_3 . So we get

$$\ll \varphi - \psi, \omega \gg = \ll \bar{\partial}\gamma, \omega \gg = (-1)^{p+q-1} \ll \gamma, \bar{\partial}\omega \gg = 0,$$

this is meaningful because $\text{Supp } \omega \subset \Omega_3$.

Hence $\mathcal{L}_\omega(\alpha) = \ll \varphi, \omega \gg = \ll \psi, \omega \gg$.

• For $q = n$, we still have $\bar{\partial}(\varphi - \psi) = \alpha - \alpha = 0$ on Ω_2 , hence $\varphi - \psi \in \mathcal{H}_p(\Omega_2)$; this time we choose $V = \Omega_2$ and the assumption gives $\ll \varphi - \psi, \omega \gg = 0$ hence again $\mathcal{L}_\omega(\alpha) = \ll \varphi, \omega \gg = \ll \psi, \omega \gg$.

In any cases, by Hölder inequalities done in Lemma 4.1,

$$|\mathcal{L}_\omega(\alpha)| \leq \|\omega\|_{L^r(\Omega_1)} \|\psi\|_{L^{r'}(\Omega_2)} \leq \|\omega\|_{L^r(\Omega)} \|\psi\|_{L^{r'}(\Omega_2)}.$$

But, by the weak r' regularity of Ω , there is a constant C_2 such that

$$\|\psi\|_{L^{r'}(\Omega_2)} \leq C_2 \|\alpha\|_{L^{r'}(\Omega_1)}.$$

Of course we have

$$\|\alpha\|_{L^{r'}(\Omega_1)} \leq \|\alpha\|_{L^{r'}(\Omega, \eta)}$$

because $\eta = 1$ on Ω_1 , hence

$$|\mathcal{L}_\omega(\alpha)| \leq C_2 \|\omega\|_{L^r(\Omega)} \|\alpha\|_{L^{r'}(\Omega, \eta)}.$$

So we have that the norm of \mathcal{L}_ω is bounded on the subspace of $\bar{\partial}$ closed forms in $L^{r'}(\Omega, \eta)$ by $C\|\omega\|_{L^r(\Omega)}$ which is *independent* of ϵ .

We apply the Hahn-Banach theorem to extend \mathcal{L}_ω with the *same* norm to *all* $(n-p, n-q+1)$ forms in $L^{r'}(\Omega, \eta)$. As in the Serre Duality Theorem [Serre, 1955, p. 20], this is one of the major ingredients in the proof.

This means, by the definition of currents, that there is a $(p, q-1)$ current u which represents the extended form \mathcal{L}_ω : $\mathcal{L}_\omega(\alpha) = \ll \alpha, u \gg$. So if $\alpha := \bar{\partial}\varphi$ with $\varphi \in \mathcal{C}^\infty(\Omega)$, we get

$$\mathcal{L}(\alpha) = \ll \alpha, u \gg = \ll \bar{\partial}\varphi, u \gg = (-1)^{p+q-1} \ll \varphi, \omega \gg$$

hence $\bar{\partial}u = \omega$ as distributions because φ is compactly supported. And we have:

$$\sup_{\alpha \in L^{r'}(\Omega, \eta), \|\alpha\|=1} |\ll \alpha, u \gg| \leq C \|\omega\|_{L^r(\Omega)}.$$

By lemma 2.1 with the weight η , this implies

$$\|u\|_{L^r(\Omega, \eta^{1-r})} \leq C \|\omega\|_{L^r(\Omega)}$$

because $|\ll \alpha, u \gg| = |\langle \alpha, \overline{*u} \rangle|$ and, as already seen, $\|u\|_{L^r(\Omega, \eta^{1-r})} = \|*u\|_{L^r(\Omega, \eta^{1-r})} = \|\overline{*u}\|_{L^r(\Omega, \eta^{1-r})}$.

In particular $\|u\|_{L^r(\Omega)} \leq C \|\omega\|_{L^r(\Omega)}$ because with $\epsilon < 1$ and $r > 1$, we have $\eta^{1-r} \geq 1$.

Now for $\epsilon > 0$ with $\eta_\epsilon(z) := \mathbb{1}_{\Omega_1}(z) + \epsilon \mathbb{1}_{\Omega \setminus \Omega_1}(z)$, let $u_\epsilon \in L^r(\Omega, \eta_\epsilon^{1-r})$ be the previous solution, then

$$\|u_\epsilon\|_{L^r(\Omega, \eta_\epsilon^{1-r})}^r \leq \int_{\Omega} |u_\epsilon|^r \eta_\epsilon^{1-r} dm \leq C^r \|\omega\|_{L^r(\Omega)}^r.$$

Replacing η by its value we get

$$\begin{aligned} \int_{\Omega_1} |u_\epsilon|^r dm + \int_{\Omega \setminus \Omega_1} |u_\epsilon|^r \epsilon^{1-r} dm &\leq C^r \|\omega\|_{L^r(\Omega)}^r \Rightarrow \\ \Rightarrow \int_{\Omega \setminus \Omega_1} |u_\epsilon|^r \epsilon^{1-r} dm &\leq C^r \|\omega\|_{L^r(\Omega)}^r \end{aligned}$$

hence

$$\int_{\Omega \setminus \Omega_1} |u_\epsilon|^r dm \leq C^r \epsilon^{r-1} \|\omega\|_{L^r(\Omega)}^r.$$

Because C and the norm of ω are independent of ϵ , we have that $\|u_\epsilon\|_{L^r(\Omega)}$ is uniformly bounded and $r > 1$ implies that $L^r_{p,q-1}(\Omega)$ is a dual by Lemma 4.3, hence there is a sub-sequence $\{u_{\epsilon_k}\}_{k \in \mathbb{N}}$ of $\{u_\epsilon\}$ which converges weakly, when $\epsilon_k \rightarrow 0$, to a $(p, q-1)$ form u in $L^r_{p,q-1}(\Omega)$, still with $\|u\|_{L^r_{p,q-1}(\Omega)} \leq C \|\omega\|_{L^r_{p,q}(\Omega)}$. Let us write $u_k := u_{\epsilon_k}$.

To see that this form u is 0 *a.e.* on $\Omega \setminus \Omega_1$ let us write the weak convergence:

$$\forall \alpha \in L^{r'}_{p,q-1}(\Omega), \langle u_k, \alpha \rangle = \int_{\Omega} u_k \wedge \overline{*}\alpha \rightarrow \langle u, \alpha \rangle = \int_{\Omega} u \wedge \overline{*}\alpha.$$

As usual take $\alpha := \frac{u}{|u|} \mathbb{1}_E$ where $E := \{|u| > 0\} \cap (\Omega \setminus \Omega_1)$ then we get

$$\int_{\Omega} u \wedge \overline{*}\alpha = \int_E |u| dm = \lim_{k \rightarrow \infty} \int_{\Omega} u_k \wedge \overline{*}\alpha = \lim_{k \rightarrow \infty} \int_E \frac{u_k \wedge \overline{*}u}{|u|}.$$

Now we have, by Hölder inequalities:

$$\left| \int_E \frac{u_k \wedge \overline{*}u}{|u|} \right| \leq \|u_k\|_{L^r(E)} \|\mathbb{1}_E\|_{L^{r'}(E)}.$$

But

$$\|u_k\|_{L^r(E)}^r \leq \int_{\Omega \setminus \Omega_1} |u_k|^r dm \leq (\epsilon_k)^{r-1} C \|\omega\|_{L^r(\Omega)} \rightarrow 0, \quad k \rightarrow \infty$$

and $\|\mathbb{1}_E\|_{L^{r'}(E)} = (m(E))^{1/r'}$.

Hence

$$\begin{aligned} \left| \int_E |u| dm \right| &= \lim_{k \rightarrow \infty} \int_E \frac{u_k \wedge \overline{*}u}{|u|} \leq \\ &\leq \lim_{k \rightarrow \infty} C^r (m(E))^{1/r'} (\epsilon_k)^{r-1} \|\omega\|_{L^r(\Omega)}^r = 0, \end{aligned}$$

so $\int_E |u| dm = 0$ which implies $m(E) = 0$ because on E , $|u| > 0$.

Hence we get that the form u is 0 *a.e.* on $\Omega \setminus \Omega_1$.

So we proved

$$\begin{aligned} \forall \varphi \in \mathcal{D}_{n-p,n-q}(\Omega), \quad (-1)^{p+q-1} \ll \varphi, \omega \gg = \ll \bar{\partial}\varphi, u_\epsilon \gg \rightarrow \ll \bar{\partial}\varphi, u \gg \\ \Rightarrow \ll \bar{\partial}\varphi, u \gg = (-1)^{p+q-1} \ll \varphi, \omega \gg \end{aligned}$$

hence $\bar{\partial}u = \omega$ in the sense of distributions.

The proof is complete. ■

Remark 3.6. As in remark 3.4 if Ω is Stein for $q = n$ instead of asking $\omega \perp \mathcal{H}_p(\Omega_2)$ we need just $\omega \perp \mathcal{H}_p(\Omega)$.

Remark 3.7. The condition of orthogonality to $\mathcal{H}_p(V)$ in the case $q = n$ is necessary: suppose there is a $(p, n-1)$ current u such that $\bar{\partial}u = \omega$ and u with compact support in an open set $V \subset \Omega$, then if $h \in \mathcal{H}_p(V)$, we have

$$h \in \mathcal{H}_p(V), \quad \ll \omega, h \gg = \ll \bar{\partial}u, h \gg = (-1)^{n+p} \ll u, \bar{\partial}h \gg = 0,$$

because, u being compactly supported, there is no boundary term and

$$\ll \bar{\partial}u, h \gg = (-1)^{n+p} \ll u, \bar{\partial}h \gg.$$

This kind of condition was already seen for extension of CR functions, see [Amar, 1991] and the references therein.

3.3 Finer control of the support.

Here we shall get a better control on the support of a solution.

Theorem 3.8. *Let Ω be a weakly r' regular domain in a Stein manifold X .*

Suppose the (p, q) form ω is in $L^{r,c}(\Omega, dm)$, $\bar{\partial}\omega = 0$, if $q < n$, and $\omega \perp \mathcal{H}_p(V)$ for any V such that $\text{Supp } \omega \subset V$, if $q = n$, with $\text{Supp } \omega \subset \Omega \setminus C$, where C is a weakly r regular domain.

For any open relatively compact set U in C , there is a $u \in L^{r,c}(\Omega, dm)$ such that $\bar{\partial}u = \omega$ and with support in $\Omega \setminus \bar{U}$, provided that $q \geq 2$.

Proof.

Let ω be a (p, q) form with compact support in $\Omega \setminus C$ then there is a $v \in L^r_{p, q-1}(\Omega)$, $\bar{\partial}v = \omega$, with compact support in Ω , by theorem 3.5 or, if Ω is a polydisc in \mathbb{C}^n and if $\omega \in \mathcal{W}_q^r(\Omega)$, by the theorem in [Amar and Mongodi, 2014].

Because ω has compact support outside C we have $\omega = 0$ in C ; this means that $\bar{\partial}v = 0$ in C . Because C is weakly r regular and $q \geq 2$, we have

$$\exists C' \subset C, C' \supset \bar{U}, \exists h \in L^r_{p, q-2}(C') \text{ s.t. } \bar{\partial}h = v \text{ in } C'.$$

Let χ be a smooth function such that $\chi = 1$ in U and $\chi = 0$ near $\partial C'$; then set $u := v - \bar{\partial}(\chi h)$.

We have that $u = v - \chi \bar{\partial}h - \bar{\partial}\chi \wedge h = v - \chi v - \bar{\partial}\chi \wedge h$ hence u is in $L^r(\Omega)$; moreover $u = 0$ in \bar{U} because $\chi = 1$ in U hence $\bar{\partial}\chi = 0$ there. Finally $\bar{\partial}u = \bar{\partial}v - \bar{\partial}^2(\chi h) = \omega$ and we are done. ■

If Ω and C are, for instance, pseudo-convex in \mathbb{C}^n then $\Omega \setminus C$ is no longer pseudo-convex in general, so this theorem improves actually the control of the support.

Remark 3.9. *The correcting function h is given by kernels in the case of Stein domains, hence it is linear; if the primitive solution v is also linear in ω , then the solution u is linear too. This is the case in \mathbb{C}^n with the solution given in [Amar and Mongodi, 2014].*

This theorem cannot be true for $q = 1$ as shown by the following example:

take a holomorphic function φ in the open unit ball $B(0, 1)$ in \mathbb{C}^n such that it extends to no open ball of center 0 and radius > 1 . For instance $\varphi(z) := \exp\left(-\frac{z_1+1}{z_1-1}\right)$. Take $R < 1$, then φ is $C^\infty(\bar{B}(0, R))$ hence by a theorem of Whitney φ extends C^∞ to \mathbb{C}^n ; call φ_R this extension. Let $\chi \in C^\infty_c(B(0, 2))$ such that $\chi = 1$ in the ball $B(0, 3/2)$ and consider the $(0, 1)$ form $\omega := \bar{\partial}(\chi \varphi_R)$. Then $\text{Supp } \omega \subset B(0, 2) \setminus B(0, R)$, ω is $\bar{\partial}$ closed and is C^∞ hence in $L^r_{0,1}(B(0, 2))$. Moreover $B(0, R)$ is strictly pseudo-convex hence r' regular, but there is no function u such that $\bar{\partial}u = \omega$ and u zero near the origin because any solution u will be C.R. on $\partial B(0, R)$ and by Hartog's phenomenon will extend holomorphically to $B(0, R)$, hence cannot be identically null near 0.

Never the less in the case $q = 1$, we have:

Theorem 3.10. *Let Ω be a weakly r' regular domain in a Stein manifold X .*

Then for any $(p, 1)$ form ω in $L^{r,c}(\Omega)$, $\bar{\partial}\omega = 0$, with support in $\Omega_1 \setminus C$ where Ω_1 is a weak r' regular domain in Ω and C is a domain such that $C \subset \Omega$ and $C \setminus \Omega_1 \neq \emptyset$; there is a $u \in L^{r,c}(\Omega)$ such that $\bar{\partial}u = \omega$ and with support in $\Omega \setminus C$.

Proof.

There is $u \in L^r_{p,0}(\Omega_1)$ such that $\bar{\partial}u = \omega$ with compact support in Ω_1 , by theorem 3.5. Then $\bar{\partial}u = 0$ in C hence u is locally holomorphic in C . Because $C \setminus \Omega_1 \neq \emptyset$, there is an open set in $C \setminus \Omega_1 \subset \Omega \setminus \Omega_1$ where u is 0 and holomorphic, hence u is identically 0 in C , C being connected. ■

Remark 3.11. *If there is a $u \in L^{r,c}_{p,0}(\Omega_1)$ which is 0 in C , we have*

$$\forall h \in L^{r'}_{n-p, n-1}(C) :: \text{Supp } \bar{\partial}h \subset C, 0 = \ll u, \bar{\partial}h \gg = \ll \omega, h \gg,$$

hence the necessary condition:

$$\forall h \in L^{r'}_{n-p, n-1}(C) :: \text{Supp } \bar{\partial}h \subset C, \ll \omega, h \gg = 0.$$

We proved in [Amar and Mongodi, 2014]:

Theorem 3.12. *Let $f \in \mathcal{O}(\bar{\mathbb{D}}^n)$ be a holomorphic function in a neighborhood of the closed unit polydisc in \mathbb{C}^n and set $Z := f^{-1}(0)$. Then for any $(0, q)$ form ω in $L^r(\mathbb{D}^n \setminus Z) \cap \mathcal{W}_q^r(\Omega)$, $\bar{\partial}\omega = 0$, with compact support in $\mathbb{D}^n \setminus Z$, for any $k \in \mathbb{N}$, we can find a $(0, q-1)$ -form $\beta \in L^{r,c}(\mathbb{D}^n)$ such that $\bar{\partial}(f^k \beta) = \omega$. Equivalently we can find a $(0, q-1)$ -form $\eta = f^k \beta$ such that $\eta \in L^{r,c}(\mathbb{D}^n)$, η is 0 on Z up to order k and $\bar{\partial}\eta = \omega$.*

And by Remark 6.3 of this paper, the solutions are given by a *bounded linear operator*.

The following corollary will generalise strongly this result but at the price that *we have not the linearity, nor even the constructivity of the solution*.

Corollary 3.13. *Let Ω be a Stein manifold. Let f be a holomorphic function in Ω and set $Z := f^{-1}(0)$. Then for any (p, q) form ω in $L^{r,c}(\Omega \setminus Z)$, $\bar{\partial}\omega = 0$, if $1 \leq q < n$, and $\omega \perp \mathcal{H}_p(\Omega \setminus Z)$ if $q = n$, there is a $(p, q - 1)$ form $u \in L^r(\Omega \setminus Z)$ such that $\bar{\partial}u = \omega$ and u has its support still in $\Omega \setminus Z$.*

Proof.

We first show that $\Omega \setminus Z$ is Stein. Because $f \neq 0$ in $\Omega \setminus Z$ we have that $\varphi := \frac{1}{|f|^2}$ is plurisubharmonic in $\Omega \setminus Z$ and $\mathcal{C}^\infty(\Omega \setminus Z)$. Because Ω is Stein we have, by Theorem 5.1.6 of Hörmander [Hörmander, 1994], a strictly plurisubharmonic exhausting function ρ in $\mathcal{C}^\infty(\Omega)$. Now the function $\gamma := \varphi + \rho$ is still strictly plurisubharmonic and \mathcal{C}^∞ in $\Omega \setminus Z$. Now we shall prove:

$$\forall \alpha \in \mathbb{R}, K_\alpha := \{z \in \Omega \setminus Z :: \gamma(z) < \alpha\} \text{ is relatively compact in } \Omega \setminus Z.$$

We have $\rho(z) < \alpha - \varphi(z) < \alpha$ on K_α because $\varphi(z) \geq 0$, hence, because ρ is exhaustive in Ω , we have that K_α is contained in a compact set F in Ω . So on F , hence on K_α , we have that $\rho(z) \geq A > -\infty$ because ρ is continuous.

We also have $\varphi(z) < \alpha - \rho(z)$ on K_α i.e. $|f(z)|^2 > \frac{1}{\alpha - \rho(z)}$. So, on the set K_α , $\alpha > \rho(z) \geq A > -\infty$, hence $|f(z)| > \frac{1}{\alpha - A} > 0$ on K_α , so K_α is far away from Z , hence K_α is relatively compact in $\Omega \setminus Z$.

So we can apply [Hörmander, 1994, Theorem 5.2.10, p. 127] to get that $\Omega \setminus Z$ is a Stein manifold.

Now we are in position to apply Theorem 3.5. Let ω be a (p, q) form in $L^{r,c}(\Omega \setminus Z)$, $\bar{\partial}\omega = 0$, if $1 \leq q < n$, and $\omega \perp \mathcal{H}_p(\Omega \setminus Z)$ if $q = n$, Theorem 3.5 gives a $(p, q - 1)$ form $u \in L^r(\Omega \setminus Z)$ such that $\bar{\partial}u = \omega$ and u has its compact support in $\Omega \setminus Z$.

The proof is complete. ■

Remark 3.14. *This leaves open the question to have a linear (or a constructive) solution to this problem even in the case of the polydisc.*

4 Appendix

Here we shall prove certainly known results on the duality $L^r - L^{r'}$ for (p, q) -forms in a complex manifold X . Because I was unable to find precise references for them, I prove them here.

Recall we have a pointwise scalar product and a pointwise modulus for (p, q) -forms in X :

$$\langle \alpha, \beta \rangle dm := \alpha \wedge \overline{*}\beta; \quad |\alpha|^2 dm := \alpha \wedge \overline{*}\alpha.$$

By the Cauchy-Schwarz inequality for scalar products we get:

$$\forall x \in X, \quad |\langle \alpha, \beta \rangle(x)| \leq |\alpha(x)| |\beta(x)|.$$

This gives Hölder inequalities for (p, q) -forms:

Lemma 4.1. (*Hölder inequalities*) *Let $\alpha \in L_{p,q}^r(\Omega)$ and $\beta \in L_{p,q}^{r'}(\Omega)$. We have*

$$|\langle \alpha, \beta \rangle| \leq \|\alpha\|_{L^r(\Omega)} \|\beta\|_{L^{r'}(\Omega)}.$$

Proof.

We start with $\langle \alpha, \beta \rangle = \int_\Omega \langle \alpha, \beta \rangle(x) dm(x)$ hence

$$|\langle \alpha, \beta \rangle| \leq \int_\Omega |\langle \alpha, \beta \rangle(x)| dm \leq \int_\Omega |\alpha(x)| |\beta(x)| dm(x).$$

By the usual Hölder inequalities for functions we get

$$\int_\Omega |\alpha(x)| |\beta(x)| dm(x) \leq \left(\int_\Omega |\alpha(x)|^r dm \right)^{1/r} \left(\int_\Omega |\beta(x)|^{r'} dm \right)^{1/r'}$$

which ends the proof of the lemma. ■

Lemma 4.2. *Let $\alpha \in L^r_{p,q}(\Omega)$ then*

$$\|\alpha\|_{L^r_{p,q}(\Omega)} = \sup_{\beta \in L^{r'}_{p,q}(\Omega), \beta \neq 0} \frac{|\langle \alpha, \beta \rangle|}{\|\beta\|_{L^{r'}(\Omega)}}.$$

Proof.

We choose $\beta := \alpha |\alpha|^{r-2}$, then:

$$|\beta|^{r'} = |\alpha|^{r'(r-1)} = |\alpha|^r \Rightarrow \|\beta\|_{L^{r'}(\Omega)}^{r'} = \|\alpha\|_{L^r(\Omega)}^r.$$

Hence

$$\langle \alpha, \beta \rangle = \left\langle \alpha, \alpha |\alpha|^{r-2} \right\rangle = \int_{\Omega} (\alpha, \alpha) |\alpha|^{r-2} dm = \|\alpha\|_{L^r(\Omega)}^r.$$

On the other hand we have

$$\|\beta\|_{L^{r'}(\Omega)} = \|\alpha\|_{L^r(\Omega)}^{r/r'} = \|\alpha\|_{L^r(\Omega)}^{r-1},$$

so

$$\|\alpha\|_{L^r(\Omega)} \times \|\beta\|_{L^{r'}(\Omega)} = \|\alpha\|_{L^r(\Omega)}^r = \langle \alpha, \beta \rangle.$$

Hence $\|\alpha\|_{L^r(\Omega)} = \frac{|\langle \alpha, \beta \rangle|}{\|\beta\|_{L^{r'}(\Omega)}}.$

A fortiori for any choice of β :

$$\|\alpha\|_{L^r(\Omega)} \leq \sup_{\beta \in L^{r'}(\Omega)} \frac{|\langle \alpha, \beta \rangle|}{\|\beta\|_{L^{r'}(\Omega)}}.$$

To prove the other direction, we use the Hölder inequalities, Lemma 4.1:

$$\forall \beta \in L^{r'}_{p,q}(\Omega), \frac{|\langle \alpha, \beta \rangle|}{\|\beta\|_{L^{r'}(\Omega)}} \leq \|\alpha\|_{L^r(\Omega)}.$$

The proof is complete. ■

Now we are in position to state:

Lemma 4.3. *The dual space of the Banach space $L^r_{p,q}(\Omega)$ is $L^{r'}_{n-p,n-q}(\Omega)$.*

Proof.

Suppose first that $u \in L^{r'}_{n-p,n-q}(\Omega)$. Then consider:

$$\forall \alpha \in L^r_{p,q}(\Omega), \mathcal{L}(\alpha) := \int_{\Omega} \alpha \wedge u = \langle \alpha, \overline{*u} \rangle.$$

This is a linear form on $L^r_{p,q}(\Omega)$ and its norm, by definition, is

$$\|\mathcal{L}\| = \sup_{\alpha \in L^r(\Omega)} \frac{|\langle \alpha, \overline{*u} \rangle|}{\|\alpha\|_{L^r(\Omega)}}.$$

By use of Lemma 4.2 we get

$$\|\mathcal{L}\| = \|\overline{*u}\|_{L^{r'}_{p,q}(\Omega)} = \|u\|_{L^{r'}_{n-p,n-q}(\Omega)}.$$

So we have $(L^r_{p,q}(\Omega))^* \supset L^{r'}_{n-p,n-q}(\Omega)$ with the same norm.

Conversely take a continuous linear form \mathcal{L} on $L^r_{p,q}(\Omega)$. We have, again by definition, that:

$$\|\mathcal{L}\| = \sup_{\alpha \in L^r(\Omega)} \frac{|\mathcal{L}(\alpha)|}{\|\alpha\|_{L^r(\Omega)}}.$$

Because $\mathcal{D}_{p,q}(\Omega) \subset L_{p,q}^r(\Omega)$, \mathcal{L} is a continuous linear form on $\mathcal{D}_{p,q}(\Omega)$, hence, by definition, \mathcal{L} can be represented by a $(n-p, n-q)$ -current u . So we have:

$$\forall \alpha \in \mathcal{D}_{p,q}(\Omega), \mathcal{L}(\alpha) := \int_{\Omega} \alpha \wedge u = \langle \alpha, \overline{*u} \rangle.$$

Moreover we have, by Lemma 4.2,

$$\|\mathcal{L}\| = \sup_{\alpha \in \mathcal{D}_{p,q}(\Omega)} \frac{|\langle \alpha, \overline{*u} \rangle|}{\|\alpha\|_{L^r(\Omega)}} = \|*u\|_{L^{r'}(\Omega)}$$

because $\mathcal{D}_{p,q}(\Omega)$ is dense in $L_{p,q}^r(\Omega)$. So we proved $(L_{p,q}^r(\Omega))^* \subset L_{n-p,n-q}^{r'}(\Omega)$ with the same norm. The proof is complete. ■

References

- [Amar, 1991] Amar, E. (1991). On the extension of c.r. functions. *Math. Z.*, 206:89–102. [8](#)
- [Amar, 2012] Amar, E. (2012). An Andreotti-Grauert theorem with L^r estimates. *arXiv:1203.0759v1*. [3](#)
- [Amar, 2016] Amar, E. (2016). The raising steps method. Application to the $\bar{\partial}$ equation in Stein manifolds. *J. Geometric Analysis*, 26(2):898–913. [2](#), [5](#)
- [Amar, 2017] Amar, E. (2017). On estimates for the $\bar{\partial}$ equation in Stein manifolds. *J. London Math. Soc.*, 49(3):519–533. [2](#)
- [Amar and Mongodi, 2014] Amar, E. and Mongodi, S. (2014). On L^r hypoellipticity of solutions with compact support of the Cauchy-Riemann equation. *Annali di Matematica Pura ed Applicata*, 193(4):999–1018. [1](#), [2](#), [3](#), [9](#)
- [Andreotti and Grauert, 1962] Andreotti, A. and Grauert, G. (1962). Théorèmes de finitude pour la cohomologie des espaces complexes. *Bulletin de la Société Mathématique de France*, 90:193–259. [1](#)
- [Charpentier, 1980] Charpentier, P. (1980). Formules explicites pour les solutions minimales de l'équation $\bar{\partial}u = f$ dans la boule et dans le polydisque de \mathbb{C}^n . *Annales de l'institut Fourier*, 30(4):121–154. [2](#)
- [Demailly and Laurent-Thiébaud, 1987] Demailly, J.-P. and Laurent-Thiébaud, C. (1987). Formules intégrales pour les formes différentielles de type (p, q) dans les variétés de Stein. *Ann. Sci. Ecole Norm. Sup.*, 4(4):579–598. [2](#)
- [Henkin and Leiterer, 1984] Henkin, G. and Leiterer, J. (1984). *Theory of functions on complex manifolds*. Mathematische Monographien. Akademie-Verlag Berlin. [4](#), [5](#)
- [Hörmander, 1994] Hörmander, L. (1994). *An introduction to complex analysis in several variables*. North-Holland/American Elsevier. [2](#), [3](#), [4](#), [10](#)
- [Kerzman, 1971] Kerzman, N. (1971). Hölder and L^p estimates for solutions of $\bar{\partial}u = f$ in strongly pseudoconvex domains. *Comm. Pure. Appl. Math.*, 24:301–379. [2](#)
- [Ma and Vassiliadou, 2000] Ma, L. and Vassiliadou, S. (2000). L^p estimates for Cauchy-Riemann operator on q -convex intersections in \mathbb{C}^n . *Manuscripta math.*, 103:413–433. [2](#)
- [Menini, 1997] Menini, C. (1997). Estimations pour la résolution du $\bar{\partial}$ sur une intersection d'ouverts strictement pseudoconvexes. *Math. Z.*, 1:87–93. [2](#)
- [Ovrelid, 1971] Ovrelid, N. (1971). Integral representation formulas and L^p estimates for the $\bar{\partial}$ equation. *Math. Scand.*, 29:137–160. [1](#), [2](#)
- [Serre, 1955] Serre, J.-P. (1955). Un théorème de dualité. *Comment. Math. Helv.*, 29:9–26. [2](#), [3](#), [7](#)
- [Skoda, 1976] Skoda, H. (1976). Valeurs au bord pour les solutions de l'opérateur d'' et caractérisation des zéros de la classe de Nevanlinna. *Bull. Soc. Math. France*, 104:225–299. [1](#)

[Voisin, 2002] Voisin, C. (2002). *Théorie de Hodge et géométrie algébrique complexe.*, volume 10 of *Cours spécialisés*. S.M.F. 3, 4