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# AN ANDREOTTI-GRAUERT THEOREM WITH $L^r$ ESTIMATES.

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# An Andreotti-Grauert theorem with $L^r$ estimates.

Eric Amar

## Abstract

By a theorem of Andreotti and Grauert if  $\omega$  is a  $(p, q)$  current,  $q < n$ , in a Stein manifold,  $\bar{\partial}$  closed and with compact support, then there is a solution  $u$  to  $\bar{\partial}u = \omega$  still with compact support. The main result of this work is to show that if moreover  $\omega \in L^r(dm)$ , where  $m$  is a suitable "Lebesgue" measure on the Stein manifold, then we have a solution  $u$  with compact support *and* in  $L^r(dm)$ . We prove it by estimates in  $L^r$  spaces with weights.

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## 1 Introduction.

Let  $\omega$  be a  $\bar{\partial}$  closed  $(p, q)$  form in  $\mathbb{C}^n$  with compact support  $K := \text{Supp}\omega$  and such that  $\omega \in L^r(\mathbb{C}^n)$ , the Lebesgue space in  $\mathbb{C}^n$ . Setting  $K$  in a ball  $\mathbb{B} := B(0, R)$  with  $R$  big enough, we know, by a theorem of Ovreliid [Ovreliid, 1971], that we have a  $(p, q - 1)$  form  $u \in L^r(\mathbb{B})$  such that  $\bar{\partial}u = \omega$ . On the other hand we also know, at least when  $q < n$ , that there is a current  $v$  with compact support such that  $\bar{\partial}v = \omega$ , by a theorem of Andreotti-Grauert [Andreotti and Grauert, 1962]. So a natural question is: may we have a solution  $u$  of  $\bar{\partial}u = \omega$  with compact support *and* in  $L^r(\mathbb{C}^n)$  ?

There is a work by H. Skoda [Skoda, 1976] who proved such a result. Let  $\Omega$  be a strictly pseudo-convex bounded domain in  $\mathbb{C}^n$  with smooth boundary then in [Skoda, 1976, Corollaire p. 295], H. Skoda proved that if  $f$  is a  $(p, q)$ -form with measure coefficients,  $q < n$ ,  $\bar{\partial}$  closed and with compact support in  $\Omega$ , then there is a solution  $U$  to the equation  $\bar{\partial}U = f$  such that  $\|U\|_{L^r(\Omega)} \leq C(\Omega, r)\|f\|_1$ , for any  $r$  such that  $1 < r < \frac{2n+2}{2n-1}$  and  $U$  has zero boundary values in the sense of Stokes formula. This means that essentially  $U$  has compact support and, because  $\Omega$  is bounded,  $\|f\|_1 \lesssim \|f\|_{L^r(\Omega)}$ . So he got the answer for  $\Omega$  strictly pseudo-convex and  $1 < r < \frac{2n+2}{2n-1}$ .

We answered this question by the affirmative for any  $r \in [1, \infty]$  in a joint work with S. Mongodi [Amar and Mongodi, 2014] linearly by the "method of coronas". This method asks for extra  $L^r$  conditions on derivatives of coefficients of  $\omega$ , when  $q < n$ ; we shall denote the set of  $\omega$  verifying these conditions  $\mathcal{W}_q^r(\Omega)$ , as in [Amar and Mongodi, 2014].

The aim of this work is to extend this result to Stein manifolds and get rid of the extra  $L^r$  conditions  $\mathcal{W}_q^r(\Omega)$ . For it we use a completely different approach inspired by the Serre duality [Serre, 1955]. Because Hahn Banach theorem is used, this method is no longer constructive as in [Amar and Mongodi, 2014].

The basic notion we shall use here is the following.

**Definition 1.1.** *Let  $X$  be a complex manifold equipped with a Borel  $\sigma$ -finite measure  $dm$  and  $\Omega$  a domain in  $X$ ; let  $r \in [1, \infty]$ , we shall say that  $\Omega$  is  $r$  **regular** if for any  $p, q \in \{0, \dots, n\}$ ,  $q \geq 1$ , there is a constant  $C = C_{p,q}(\Omega)$  such that for any  $(p, q)$  form  $\omega$ ,  $\bar{\partial}$  closed in  $\Omega$  and in  $L^r(\Omega, dm)$  there is a  $(p, q-1)$  form  $u \in L^r(\Omega, dm)$  such that  $\bar{\partial}u = \omega$  and  $\|u\|_{L^r(\Omega)} \leq C\|\omega\|_{L^r(\Omega)}$ .*

*We shall say that  $\Omega$  is **weakly  $r$  regular** if for any compact set  $K \Subset \Omega$  there are 3 open sets  $\Omega_1, \Omega_2, \Omega_3$  such that  $K \Subset \Omega_3 \subset \Omega_2 \subset \Omega_1 \subset \Omega_0 := \Omega$  and 3 constants  $C_1, C_2, C_3$  such that:*

$$\forall j = 0, 1, 2, \forall p, q \in \{0, \dots, n\}, q \geq 1, \forall \omega \in L_{p,q}^r(\Omega_j, dm), \bar{\partial}\omega = 0,$$

$$\exists u \in L_{p,q-1}^r(\Omega_{j+1}, dm), \bar{\partial}u = \omega$$

and  $\|u\|_{L^r(\Omega_{j+1})} \leq C_{j+1}\|\omega\|_{L^r(\Omega_j)}$ .

*I.e. we have a 3 steps chain of resolution.*

Of course the  $r$  regularity implies the weak  $r$  regularity, just taking  $\Omega_1 = \Omega_2 = \Omega_3 = \Omega$ .

Examples of 2 regular domains are the bounded pseudo-convex domains by Hörmander [Hörmander, 1994].

Examples of  $r$  regular domains in  $\mathbb{C}^n$  are the bounded strictly pseudo-convex (s.p.c.) domains with smooth boundary by Ovrelid [Ovrelid, 1971]; the polydiscs in  $\mathbb{C}^n$  by Charpentier [Charpentier, 1980], finite transverse intersections of strictly pseudo-convex bounded domains in  $\mathbb{C}^n$  by Menini [Menini, 1997]. A generalisation of the results by Menini was done in the nice work of Ma and Vassiliadou [Ma and Vassiliadou, 2000]: they treated also the case of intersection of  $q$ -convex sets.

Examples of  $r$  regular domains in a Stein manifold are the strictly pseudo-convex domains with smooth boundary [Amar, 2016]. (See the previous work for  $(0, 1)$  forms by N. Kerzman [Kerzman, 1971] and for all  $(p, q)$  forms by J-P. Demailly and C. Laurent [Demailly and Laurent-Thiébaud, 1987, Remarque 4, page 596], but here the manifold has to be equipped with a metric with null curvature. See also [Amar, 2017] for the case of intersection of  $q$ -convex sets in a Stein manifold).

Let  $X$  be a Stein manifold and  $\Omega$  a domain in  $X$ , i.e. an open connected set in  $X$ . Let  $\mathcal{H}_p(\Omega)$  be the set of all  $(p, 0)$   $\bar{\partial}$  closed forms in  $\Omega$ . If  $p = 0$ ,  $\mathcal{H}_0(\Omega) = \mathcal{H}(\Omega)$  is the set of holomorphic functions in  $\Omega$ . If  $p > 0$ , we have, in a chart  $(\varphi, U)$ ,  $h \in \mathcal{H}_p(\Omega) \Rightarrow h(z) = \sum_{|J|=p} a_J(z) dz^J$ , where  $dz^J := dz_{j_1} \wedge \dots \wedge dz_{j_p}$  and the functions  $a_J(z)$  are holomorphic in  $\varphi(U) \subset \mathbb{C}^n$ .

We shall denote  $L_{p,q}^{r,c}(\Omega)$  the set of  $(p, q)$ -forms in  $L^r(\Omega)$  with compact support in  $\Omega$ .

We also use the notation  $r'$  for the conjugate exponent of  $r$ , i.e.  $\frac{1}{r} + \frac{1}{r'} = 1$ .

Our main theorem is:

**Theorem 1.2.** *Let  $\Omega$  be a weakly  $r'$  regular domain in a Stein manifold  $X$ . Then there is a  $C > 0$  such that for any  $(p, q)$  form  $\omega$  in  $L^{r,c}(\Omega)$ ,  $r > 1$  with:*

- if  $1 \leq q < n$ ,  $\bar{\partial}\omega = 0$ ;
- if  $q = n$ ,  $\forall V \subset \Omega$ ,  $\text{Supp } \omega \subset V$ ,  $\omega \perp \mathcal{H}_{n-p}(V)$ ;

*there is a  $(p, q-1)$  form  $u$  in  $L^{r,c}(\Omega)$  such that  $\bar{\partial}u = \omega$  as distributions and  $\|u\|_{L^r(\Omega)} \leq C\|\omega\|_{L^r(\Omega)}$ .*

The notion of  $r$  regularity gives a good control of the support: if the support of the data  $\omega$  is contained in  $\Omega \setminus C$  where  $\Omega$  is a weakly  $r'$  regular domain and  $C$  is a weakly  $r$  regular domain, then the support of the solution  $u$  is contained in  $\Omega \setminus C'$ , where  $C'$  is any relatively compact domain in  $C$ , provided that  $q \geq 2$ . One may observe that  $\Omega \setminus C$  is *not* Stein in general even if  $\Omega$  is.

There is also a result of this kind for  $q = 1$ , see section 3.3.

In particular the support of the solution  $u$  is contained in the intersection of all the weakly  $r'$  regular domains containing the support of  $\omega$ .

The idea is to solve  $\bar{\partial}u = \omega$  in a space  $L^r(\Omega)$  with a "big weight  $\eta$  outside" of the support of  $\omega$ ; this way we shall have a "small solution  $u$  outside" of the support of  $\omega$ . Then, using a sequence of such weights going to infinity "outside" of the support of  $\omega$ , we shall have a  $u$  zero "outside of the support" of  $\omega$ .

Comparing to my previous work [Amar, 2012] the results here are improved and the proofs are much simpler by a systematic use of the Hodge  $*$  operator.

I am indebted to G. Tomassini who started my interest in this subject on precisely this kind of questions and also to S. Mongodi for a lot of discussions during the preparation of our joint paper [Amar and Mongodi, 2014].

I want to thank C. Laurent for many instructive discussions on this subject.

Finally I also thank the referee for his/her careful reading of the manuscript and the incisive questions he/she asked.

## 2 Duality.

We shall study a duality between currents inspired by the Serre duality [Serre, 1955].

Let  $X$  be a complex manifold of dimension  $n$ . We proceed now exactly as in Hörmander [Hörmander, 1994, p. 119], by introducing a hermitian metric on differential forms locally equivalent to the usual one on any analytic coordinates system.

We define the "Lebesgue measure" still as in Hörmander's book [Hörmander, 1994, Section 5.2]: associated to this metric there is a volume measure  $dm$  and we take it for the Lebesgue measure on  $X$ . Moreover, because  $X$  is a complex manifold, it is canonically oriented.

### 2.1 Weighted $L^r$ spaces.

Let  $\Omega$  be a domain in  $X$ . We denote also  $dm$  the volume form on  $X$ .

We shall take the following notation from the book by C. Voisin [Voisin, 2002].

To a  $(p, q)$ -form  $\alpha$  on  $\Omega$  we associate its Hodge  $*$   $(n - p, n - q)$ -form  $*\alpha$ . This gives us a pointwise scalar product and a pointwise modulus:

$$\langle \alpha, \beta \rangle dm := \alpha \wedge \overline{* \beta}; \quad |\alpha|^2 dm := \alpha \wedge \overline{* \alpha}, \quad (2.1)$$

because  $\alpha \wedge \overline{* \beta}$  is a  $(n, n)$ -form hence is a function time the volume form  $dm$ .

We are also given a scalar product  $\langle \alpha, \beta \rangle$  on  $(p, q)$ -forms such that  $\int_{\Omega} |\alpha|^2 dm < \infty$  and the link between these notions is given by [Voisin, 2002, Lemme 5.8, p. 119]:

$$\langle \alpha, \beta \rangle = \int_{\Omega} \alpha \wedge \overline{* \beta}. \quad (2.2)$$

We shall define now  $L^r_{p,q}(\Omega)$  to be the set of  $(p, q)$ -forms  $\alpha$  defined on  $\Omega$  such that

$$\|\alpha\|_{L^r_{p,q}(\Omega)} := \int_{\Omega} |\alpha(z)|^r dm(z) < \infty,$$

where  $|\alpha|$  is defined by (2.1).

**Lemma 2.1.** *Let  $\eta > 0$  be a weight. If  $u$  is a  $(p, q)$ -current defined on  $(n - p, n - q)$ -forms  $\alpha$  in  $L^{r'}(\Omega, \eta)$  and such that*

$$\forall \alpha \in L^{r'}_{(n-p, n-q)}(\Omega, \eta), \quad |\langle u, *\alpha \rangle| \leq C \|\alpha\|_{L^{r'}(\Omega, \eta)},$$

*then  $\|u\|_{L^r_{p,q}(\Omega, \eta^{1-r})} \leq C$ .*

Proof.

We use the classical trick: set  $\tilde{\alpha} := \eta^{1/r'}\alpha$ ;  $\tilde{u} := \frac{1}{\eta^{1/r'}}u$  then we have

$$\langle u, *\alpha \rangle = \int_{\Omega} u \wedge \bar{\alpha} = \int_{\Omega} \tilde{u} \wedge \bar{\tilde{\alpha}} = \langle \tilde{u}, *\tilde{\alpha} \rangle$$

and  $\|\tilde{\alpha}\|_{L^{r'}(\Omega)} = \|\alpha\|_{L^{r'}(\Omega, \eta)}$ .

We notice that  $\|\tilde{\alpha}\|_{L^{r'}(\Omega)} = \|\tilde{\alpha}\|_{L^{r'}(\Omega)}$  because we have  $(*\tilde{\alpha}, *\tilde{\alpha})dm = *\tilde{\alpha} \wedge \overline{**\tilde{\alpha}}$  but  $**\tilde{\alpha} = (-1)^{(p+q)(2n-p-q)}\tilde{\alpha}$ , by [Voisin, 2002, Lemma 5.5], hence, because  $(*\tilde{\alpha}, *\tilde{\alpha})$  is positive,  $(*\tilde{\alpha}, *\tilde{\alpha}) = |\tilde{\alpha}|^2$ .  
By use of the duality  $L_{p,q}^r(\Omega) - L_{n-p,n-q}^{r'}(\Omega)$ , done in Lemma 4.3, we get

$$\|\tilde{u}\|_{L_{p,q}^r(\Omega)} = \sup_{\alpha \in L_{n-p,n-q}^{r'}(\Omega), \alpha \neq 0} \frac{|\langle \tilde{u}, *\tilde{\alpha} \rangle|}{\|\tilde{\alpha}\|_{L^{r'}(\Omega)}}.$$

But

$$\|\tilde{u}\|_{L_{p,q}^r(\Omega)}^r := \int_{\Omega} |u|^r \eta^{-\frac{r}{r'}} dm = \int_{\Omega} |u|^r \eta^{1-r} dm = \|u\|_{L^r(\Omega, \eta^{1-r})}^r.$$

So we get

$$\|u\|_{L_{p,q}^r(\Omega, \eta^{1-r})} = \sup_{*\alpha \in L_{p,q}^{r'}(\Omega, \eta), \alpha \neq 0} \frac{|\langle u, *\alpha \rangle|}{\|\alpha\|_{L^{r'}(\Omega, \eta)}}.$$

The proof is complete. ■

It may seem strange that we have such an estimate when the dual of  $L^{r'}(\Omega, \eta)$  is  $L^r(\Omega, \eta)$ , but the reason is, of course, that in the duality current-form there is no weights.

The point here is that when  $\eta$  is small,  $\eta^{1-r}$  is big for  $r > 1$ .

### 3 Solution of the $\bar{\partial}$ equation with compact support.

#### 3.1 $r$ regular domains.

As we have seen, examples of  $r$  regular domains in Stein manifolds are the relatively compact s.p.c. domains with smooth boundary.

To prove that a Stein manifold  $\Omega$  is weakly  $r$  regular we shall need the following lemma.

**Lemma 3.1.** *Let  $\Omega$  be a Stein manifold. Then it contains an exhaustive sequence of open relatively compact strictly pseudo-convex sets  $\{D_k\}_{k \in \mathbb{N}}$  with  $\mathcal{C}^\infty$  smooth boundary.*

Proof.

For the case of  $\Omega$  pseudo-convex in  $\mathbb{C}^n$ , the proof was already done explicitly in the proof of [Henkin and Leiterer, 1984, Theorem 2.8.1, p. 86].

By Theorem 5.1.6 of Hörmander [Hörmander, 1994] there exists a  $\mathcal{C}^\infty$  strictly plurisubharmonic (s.p.s.h.) exhausting function  $\varphi$  for  $\Omega$ . Take  $K \Subset \Omega$  such that  $d\varphi \neq 0$  on  $K$ . Because  $\varphi$  is s.p.s.h. then  $K \neq \emptyset$ . Then we use the [Henkin and Leiterer, 1984, Lemma 2.12.2, p. 93], to get:

$\forall \epsilon > 0$ ,  $\exists \rho_\epsilon$  s.p.s.h.  $\mathcal{C}^\infty$ -function on  $\Omega$  such that:

- (i)  $\varphi - \rho_\epsilon$  together with its first and second derivatives is less than  $\epsilon$  on  $\Omega$ .
- (ii) The set  $\text{Crit}(\rho_\epsilon) := \{z \in \Omega :: d\rho_\epsilon(z) = 0\}$  is discrete in  $\Omega$ . (In a formula, the notation  $::$  means "such that".)
- (iii)  $\rho_\epsilon = \varphi$  on  $K$ .

As stated in Lemma 2.12.2 if  $\varphi \in \mathcal{C}^2$  then  $\rho_\epsilon \in \mathcal{C}^2$ , but in fact the proof of this Lemma 2.12.2 gives  $\rho_\epsilon = \varphi + \sum \chi_j$ , where  $\sum \chi_j$  is  $\mathcal{C}^\infty$ . (see p. 93 in [Henkin and Leiterer, 1984]). Hence  $\rho_\epsilon$  has the same  $\mathcal{C}^k$  regularity than  $\varphi$ .

Fix  $\epsilon > 0$ , then the function  $\rho := \rho_\epsilon$  is also a s.p.s.h. exhausting function for  $\Omega$ , because, from  $-\epsilon \leq \varphi - \rho_\epsilon \leq \epsilon$ , we get that, for any  $\alpha \in \mathbb{R}$ ,

$$\{z \in \Omega :: \rho_\epsilon(z) < \alpha\} \subset \{z \in \Omega :: \varphi(z) < \epsilon + \alpha\}$$

and, because  $\varphi$  is an exhausting function, this set is relatively compact in  $\Omega$ .

Because the set of critical points of  $\rho$  is discrete in  $\Omega$ , the same way as in the proof of [Henkin and Leiterer, 1984, Theorem 2.8.1, p. 86], we can find a sequence  $c_k \in \mathbb{R}$ ,  $c_k \rightarrow \infty$ , such that  $D_k := \{z \in \Omega :: \rho(z) < c_k\}$  make an exhaustive sequence of open relatively compact sets in  $\Omega$ ,  $\partial\rho \neq 0$  on  $\partial D_k$ , hence  $D_k$  is strictly pseudo-convex with  $C^\infty$  smooth boundary, and finally  $D_k \nearrow \Omega$ . The proof is complete.  $\blacksquare$

**Proposition 3.2.** *A Stein manifold  $\Omega$  is weakly  $r$  regular.*

Proof.

By Lemma 3.1 there is an exhaustive sequence of open relatively compact s.p.c. sets in  $\Omega$ ,  $\{D_k\}_{k \in \mathbb{N}}$  with  $C^\infty$  smooth boundary.

Let  $\omega \in L_{p,q}^r(\Omega)$ ,  $\bar{\partial}\omega = 0$ , by [Amar, 2016], we can solve  $\bar{\partial}u = \omega$  in  $D_k$  with  $u \in L_{p,q-1}^r(D_k)$  and

$$\|u\|_{L^r(D_k)} \leq C_k \|\omega\|_{L^r(D_k)} \leq C_k \|\omega\|_{L^r(\Omega)}.$$

Hence if  $\Gamma$  is a compact set in  $\Omega$ , there is a  $D_k$  such that  $\Gamma \Subset D_k$  and we can take  $\Omega_1 = \Omega_2 = \Omega_3 = D_k$ .

This proves the weak  $r$  regularity of  $\Omega$ .  $\blacksquare$

### 3.2 The main result.

Let  $X$  be a Stein manifold and  $\Omega$  a domain in  $X$ .

In order to simplify notation, we set the pairing for  $\alpha$  a  $(p, q)$ -form and  $\beta$  a  $(n-p, n-q)$ -form:  $\ll \alpha, \beta \gg := \int_{\Omega} \alpha \wedge \beta$ .

With this notation we also have  $\langle \alpha, \beta \rangle = \ll \alpha, \overline{*}\beta \gg$ .

Let  $\Omega$  be a weakly  $r'$  regular domain in  $X$ . We set  $K := \text{Supp } \omega \Subset \Omega$  and, by the definition of the  $r'$  weak regularity, we get 3 open sets such that  $K \Subset \Omega_3 \subset \Omega_2 \subset \Omega_1 \subset \Omega_0 = \Omega$  with:  $\forall j = 0, 1, 2, \forall p, q \in \{0, \dots, n\}, q \geq 1$ ,

$$\forall \alpha \in L_{p,q}^r(\Omega_j), \bar{\partial}\alpha = 0, \exists \varphi \in L_{p,q-1}^r(\Omega_{j+1}), \bar{\partial}\varphi = \alpha.$$

Set the weight  $\eta = \eta_\epsilon := \mathbb{1}_{\Omega_1}(z) + \epsilon \mathbb{1}_{\Omega \setminus \Omega_1}(z)$  for a fixed  $\epsilon > 0$ .

Let  $\omega \in L_{p,q}^{r,c}(\Omega)$ . Suppose moreover that  $\omega$  is such that  $\bar{\partial}\omega = 0$  if  $1 \leq q < n$  and for any open  $V \Subset \Omega$ ,  $\text{Supp } \omega \Subset V$  we have  $\omega \perp \mathcal{H}_{n-p}(V) \iff \forall h \in \mathcal{H}_{n-p}(V), \ll \omega, h \gg = 0$  if  $q = n$ .

We shall use the following lemma, with the previous notation:

**Lemma 3.3.** *Let  $\mathcal{E}$  be the set of  $(n-p, n-q+1)$  forms  $\alpha \in L^{r'}(\Omega, \eta)$ ,  $\bar{\partial}$  closed in  $\Omega$ . Let us define  $\mathcal{L}_\omega$  on  $\mathcal{E}$  as follows:*

$$\mathcal{L}_\omega(\alpha) := (-1)^{p+q-1} \ll \varphi, \omega \gg,$$

where  $\varphi \in L^{r'}(\Omega_1)$  is such that  $\bar{\partial}\varphi = \alpha$  in  $\Omega_1$ . Then the form  $\mathcal{L}_\omega$  is well defined and linear.

Proof.

Because  $\epsilon > 0$  we have  $\alpha \in L^{r'}(\Omega, \eta) \Rightarrow \alpha \in L^{r'}(\Omega)$  and the weak  $r'$  regularity of  $\Omega$  gives a  $\varphi \in L^{r'}(\Omega_1)$  with  $\bar{\partial}\varphi = \alpha$  in  $\Omega_1$ .

Let us see that  $\mathcal{L}_\omega$  is well defined.

- Suppose first that  $q < n$ .

In order for  $\mathcal{L}_\omega$  to be well defined we need

$$\forall \varphi, \psi \in L_{(n-p, n-q)}^{r'}(\Omega_1), \bar{\partial}\varphi = \bar{\partial}\psi = \alpha \Rightarrow \ll \varphi, \omega \gg = \ll \psi, \omega \gg.$$

This is meaningful because  $\omega \in L^{r,c}(\Omega)$ ,  $r > 1$ ,  $\text{Supp } \omega \Subset \Omega_1$ .

Then we have  $\bar{\partial}(\varphi - \psi) = 0$  in  $\Omega_1$ , hence, because  $\Omega$  is weakly  $r'$  regular, we can solve  $\bar{\partial}$  in  $L^{r'}(\Omega_2)$ :

$$\exists \gamma \in L_{(n-p, n-q-1)}^{r'}(\Omega_2) :: \bar{\partial}\gamma = (\varphi - \psi).$$

So  $\ll \varphi - \psi, \omega \gg = \ll \bar{\partial}\gamma, \omega \gg = (-1)^{p+q-1} \ll \gamma, \bar{\partial}\omega \gg = 0$  because  $\omega$  is compactly supported in  $\Omega_2$  and  $\bar{\partial}$  closed.

Hence  $\mathcal{L}_\omega$  is well defined in that case.

- Suppose now that  $q = n$ .

For  $\varphi, \psi$  ( $n - p, 0$ ) forms in  $\Omega_1$ , such that  $\bar{\partial}\varphi = \bar{\partial}\psi = \alpha$ , we need to have  $\ll \varphi, \omega \gg = \ll \psi, \omega \gg$ . But then  $\bar{\partial}(\varphi - \psi) = 0$ , which means that  $h := \varphi - \psi$  is a  $\bar{\partial}$  closed ( $n - p, 0$ ) form, hence  $h \in \mathcal{H}_{n-p}(\Omega_1)$ . Taking  $V = \Omega_1$  in the hypothesis  $\omega \perp \mathcal{H}_{n-p}(V)$ , we get  $\ll h, \omega \gg = 0$ , and  $\mathcal{L}_\omega$  is also well defined in that case.

It remains to see that  $\mathcal{L}_\omega$  is linear.

- Suppose first that  $q < n$ .

Let  $\alpha = \alpha_1 + \alpha_2$ , with  $\alpha_j \in L^{r'}(\Omega, \eta)$ ,  $\bar{\partial}\alpha_j = 0$ ,  $j = 1, 2$ ; we have  $\alpha = \bar{\partial}\varphi$ ,  $\alpha_1 = \bar{\partial}\varphi_1$  and  $\alpha_2 = \bar{\partial}\varphi_2$ , with  $\varphi, \varphi_1, \varphi_2$  in  $L^{r'}(\Omega_1)$  so, because  $\bar{\partial}(\varphi - \varphi_1 - \varphi_2) = 0$ , we have

$$\varphi = \varphi_1 + \varphi_2 + \bar{\partial}\psi, \text{ with } \psi \text{ in } L^{r'}(\Omega_2),$$

so

$$\begin{aligned} \mathcal{L}_\omega(\alpha) &= (-1)^{p+q-1} \ll \varphi, \omega \gg = (-1)^{p+q-1} \ll \varphi_1 + \varphi_2 + \bar{\partial}\psi, \omega \gg \\ &= \mathcal{L}_\omega(\alpha_1) + \mathcal{L}_\omega(\alpha_2) + (-1)^{p+q-1} \ll \bar{\partial}\psi, \omega \gg, \end{aligned}$$

but again  $\ll \bar{\partial}\psi, \omega \gg = 0$ , hence  $\mathcal{L}_\omega(\alpha) = \mathcal{L}_\omega(\alpha_1) + \mathcal{L}_\omega(\alpha_2)$ .

The same for  $\alpha = \lambda\alpha_1$ .

- Suppose now that  $q = n$ .

We have

$$\mathcal{L}_\omega(\alpha) := (-1)^{p+n-1} \ll \varphi, \omega \gg,$$

where  $\varphi \in L^{r'}(\Omega_1)$  is such that  $\bar{\partial}\varphi = \alpha$  in  $\Omega_1$ .

Let  $\alpha = \alpha_1 + \alpha_2$ , with  $\alpha_j \in L^{r'}(\Omega, \eta)$ ,  $\bar{\partial}\alpha_j = 0$ ,  $j = 1, 2$ ; we have  $\alpha = \bar{\partial}\varphi$ ,  $\alpha_1 = \bar{\partial}\varphi_1$  and  $\alpha_2 = \bar{\partial}\varphi_2$ , with  $\varphi, \varphi_1, \varphi_2$  in  $L^{r'}(\Omega_1)$  so, because  $\bar{\partial}(\varphi - \varphi_1 - \varphi_2) = 0$ , we have  $\varphi - \varphi_1 - \varphi_2$  is a ( $n - p, 0$ )  $\bar{\partial}$ -closed form, hence:

$$\varphi = \varphi_1 + \varphi_2 + h, \text{ with } h \in \mathcal{H}_{n-p}(\Omega_1).$$

So

$$\begin{aligned} \mathcal{L}_\omega(\alpha) &= (-1)^{p+q-1} \ll \varphi, \omega \gg = (-1)^{p+q-1} \ll \varphi_1 + \varphi_2 + h, \omega \gg \\ &= \mathcal{L}_\omega(\alpha_1) + \mathcal{L}_\omega(\alpha_2) + (-1)^{p+q-1} \ll h, \omega \gg. \end{aligned}$$

Taking  $V = \Omega_1$  in the hypothesis  $\omega \perp \mathcal{H}_{n-p}(V)$ , we get  $\ll h, \omega \gg = 0$ , hence  $\mathcal{L}_\omega(\alpha) = \mathcal{L}_\omega(\alpha_1) + \mathcal{L}_\omega(\alpha_2)$ .

The same for  $\alpha = \lambda\alpha_1$ . The proof is complete.  $\blacksquare$

**Remark 3.4.** *If  $\Omega$  is Stein, we can take the domain  $\Omega_1$  to be s.p.c. with  $C^\infty$  smooth boundary, hence also Stein. So because  $K := \text{Supp } \omega \subset \Omega_1 \subset \Omega$ , the  $A(\Omega_1)$  convex hull of  $K$ ,  $\hat{K}_{\Omega_1}$  is still in  $\Omega_1$ , and any holomorphic function in  $\Omega_1$  can be uniformly approximated on  $\hat{K}_{\Omega_1}$  by holomorphic functions in  $\Omega$ .*

*Then for  $q = n$  instead of asking  $\omega \perp \mathcal{H}_{n-p}(\Omega_1)$  we need just  $\omega \perp \mathcal{H}_{n-p}(\Omega)$ .*

**Theorem 3.5.** *Let  $\Omega$  be a weakly  $r'$  regular domain and  $\omega$  be a  $(p, q)$  form in  $L^{r,c}(\Omega)$ ,  $r > 1$ . Suppose that  $\omega$  is such that:*

- if  $1 \leq q < n$ ,  $\bar{\partial}\omega = 0$ ;
- if  $q = n$ ,  $\forall V \subset \Omega$ ,  $\text{Supp } \omega \subset V$ ,  $\omega \perp \mathcal{H}_{n-p}(V)$

*Then there is a  $C > 0$  and a  $(p, q-1)$  form  $u$  in  $L^{r,c}(\Omega)$  such that  $\bar{\partial}u = \omega$  as distributions and  $\|u\|_{L^r(\Omega)} \leq C\|\omega\|_{L^r(\Omega)}$ .*

*Proof.*

Because  $\Omega$  is weakly  $r'$  regular there is a  $\Omega_1 \subset \Omega$ ,  $\Omega_1 \supset \text{Supp } \omega$  such that

$$\forall \alpha \in L^{r'}(\Omega), \bar{\partial}\alpha = 0, \exists \varphi \in L^{r'}(\Omega_1) :: \bar{\partial}\varphi = \alpha, \|\varphi\|_{L^{r'}(\Omega_1)} \leq C_1\|\alpha\|_{L^{r'}(\Omega)}.$$

There is a  $\Omega_2$  such that  $\text{Supp } \omega \Subset \Omega_2 \subset \Omega_1 \subset \Omega$  with the same properties as  $\Omega_1$ .

Let us consider the weight  $\eta = \eta_\epsilon := \mathbb{1}_{\Omega_1}(z) + \epsilon \mathbb{1}_{\Omega \setminus \Omega_1}(z)$  for a fixed  $\epsilon > 0$  and the form  $\mathcal{L}_\omega$  defined in Lemma 3.3. By Lemma 3.3 we have that  $\mathcal{L}_\omega$  is a linear form on  $(n - p, n - q + 1)$ -forms  $\alpha \in L^{r'}(\Omega, \eta)$ ,  $\bar{\partial}$  closed in  $\Omega$ .

If  $\alpha$  is a  $(n - p, n - q + 1)$ -form in  $L^{r'}(\Omega, \eta)$ , then  $\alpha$  is in  $L^{r'}(\Omega)$  because  $\epsilon > 0$ .

The weak  $r'$  regularity of  $\Omega$  gives that there is a  $\varphi \in L^{r'}(\Omega_1) :: \bar{\partial}\varphi = \alpha$  which can be used to define  $\mathcal{L}_\omega(\alpha)$ .

We have also that  $\alpha \in L^{r'}(\Omega_1)$ ,  $\bar{\partial}\alpha = 0$  in  $\Omega_1$ , hence, still with the weak  $r'$  regularity of  $\Omega$ , we have

$$\exists \psi \in L^{r'}(\Omega_2) :: \bar{\partial}\psi = \alpha, \|\psi\|_{L^{r'}(\Omega_2)} \leq C_2 \|\alpha\|_{L^{r'}(\Omega_1)}.$$

• For  $q < n$ , we have  $\bar{\partial}(\varphi - \psi) = \alpha - \alpha = 0$  on  $\Omega_2$  and, by the weak  $r'$  regularity of  $\Omega$ , there is a  $\Omega_3 \subset \Omega_2$ , such that  $\text{Supp } \omega \subset \Omega_3 \subset \Omega_2$ , and a  $\gamma \in L^{r'}(\Omega_3)$ ,  $\bar{\partial}\gamma = \varphi - \psi$  in  $\Omega_3$ . So we get

$$\ll \varphi - \psi, \omega \gg = \ll \bar{\partial}\gamma, \omega \gg = (-1)^{p+q-1} \ll \gamma, \bar{\partial}\omega \gg = 0,$$

this is meaningful because  $\text{Supp } \omega \subset \Omega_3$ .

Hence  $\mathcal{L}_\omega(\alpha) = \ll \varphi, \omega \gg = \ll \psi, \omega \gg$ .

• For  $q = n$ , we still have  $\bar{\partial}(\varphi - \psi) = \alpha - \alpha = 0$  on  $\Omega_2$ , hence  $\varphi - \psi \in \mathcal{H}_p(\Omega_2)$ ; this time we choose  $V = \Omega_2$  and the assumption gives  $\ll \varphi - \psi, \omega \gg = 0$  hence again  $\mathcal{L}_\omega(\alpha) = \ll \varphi, \omega \gg = \ll \psi, \omega \gg$ .

In any cases, by Hölder inequalities done in Lemma 4.1,

$$|\mathcal{L}_\omega(\alpha)| \leq \|\omega\|_{L^r(\Omega_1)} \|\psi\|_{L^{r'}(\Omega_2)} \leq \|\omega\|_{L^r(\Omega)} \|\psi\|_{L^{r'}(\Omega_2)}.$$

But, by the weak  $r'$  regularity of  $\Omega$ , there is a constant  $C_2$  such that

$$\|\psi\|_{L^{r'}(\Omega_2)} \leq C_2 \|\alpha\|_{L^{r'}(\Omega_1)}.$$

Of course we have

$$\|\alpha\|_{L^{r'}(\Omega_1)} \leq \|\alpha\|_{L^{r'}(\Omega, \eta)}$$

because  $\eta = 1$  on  $\Omega_1$ , hence

$$|\mathcal{L}_\omega(\alpha)| \leq C_2 \|\omega\|_{L^r(\Omega)} \|\alpha\|_{L^{r'}(\Omega, \eta)}.$$

So we have that the norm of  $\mathcal{L}_\omega$  is bounded on the subspace of  $\bar{\partial}$  closed forms in  $L^{r'}(\Omega, \eta)$  by  $C\|\omega\|_{L^r(\Omega)}$  which is *independent* of  $\epsilon$ .

We apply the Hahn-Banach theorem to extend  $\mathcal{L}_\omega$  with the *same* norm to *all*  $(n-p, n-q+1)$  forms in  $L^{r'}(\Omega, \eta)$ . As in the Serre Duality Theorem [Serre, 1955, p. 20], this is one of the major ingredients in the proof.

This means, by the definition of currents, that there is a  $(p, q-1)$  current  $u$  which represents the extended form  $\mathcal{L}_\omega$ :  $\mathcal{L}_\omega(\alpha) = \ll \alpha, u \gg$ . So if  $\alpha := \bar{\partial}\varphi$  with  $\varphi \in \mathcal{C}^\infty(\Omega)$ , we get

$$\mathcal{L}(\alpha) = \ll \alpha, u \gg = \ll \bar{\partial}\varphi, u \gg = (-1)^{p+q-1} \ll \varphi, \omega \gg$$

hence  $\bar{\partial}u = \omega$  as distributions because  $\varphi$  is compactly supported. And we have:

$$\sup_{\alpha \in L^{r'}(\Omega, \eta), \|\alpha\|=1} |\ll \alpha, u \gg| \leq C \|\omega\|_{L^r(\Omega)}.$$

By lemma 2.1 with the weight  $\eta$ , this implies

$$\|u\|_{L^r(\Omega, \eta^{1-r})} \leq C \|\omega\|_{L^r(\Omega)}$$

because  $|\ll \alpha, u \gg| = |\langle \alpha, \overline{*u} \rangle|$  and, as already seen,  $\|u\|_{L^r(\Omega, \eta^{1-r})} = \|*u\|_{L^r(\Omega, \eta^{1-r})} = \|\overline{*u}\|_{L^r(\Omega, \eta^{1-r})}$ .

In particular  $\|u\|_{L^r(\Omega)} \leq C \|\omega\|_{L^r(\Omega)}$  because with  $\epsilon < 1$  and  $r > 1$ , we have  $\eta^{1-r} \geq 1$ .

Now for  $\epsilon > 0$  with  $\eta_\epsilon(z) := \mathbb{1}_{\Omega_1}(z) + \epsilon \mathbb{1}_{\Omega \setminus \Omega_1}(z)$ , let  $u_\epsilon \in L^r(\Omega, \eta_\epsilon^{1-r})$  be the previous solution, then

$$\|u_\epsilon\|_{L^r(\Omega, \eta_\epsilon^{1-r})}^r \leq \int_{\Omega} |u_\epsilon|^r \eta_\epsilon^{1-r} dm \leq C^r \|\omega\|_{L^r(\Omega)}^r.$$

Replacing  $\eta$  by its value we get

$$\begin{aligned} \int_{\Omega_1} |u_\epsilon|^r dm + \int_{\Omega \setminus \Omega_1} |u_\epsilon|^r \epsilon^{1-r} dm &\leq C^r \|\omega\|_{L^r(\Omega)}^r \Rightarrow \\ \Rightarrow \int_{\Omega \setminus \Omega_1} |u_\epsilon|^r \epsilon^{1-r} dm &\leq C^r \|\omega\|_{L^r(\Omega)}^r \end{aligned}$$

hence

$$\int_{\Omega \setminus \Omega_1} |u_\epsilon|^r dm \leq C^r \epsilon^{r-1} \|\omega\|_{L^r(\Omega)}^r.$$

Because  $C$  and the norm of  $\omega$  are independent of  $\epsilon$ , we have that  $\|u_\epsilon\|_{L^r(\Omega)}$  is uniformly bounded and  $r > 1$  implies that  $L_{p,q-1}^r(\Omega)$  is a dual by Lemma 4.3, hence there is a sub-sequence  $\{u_{\epsilon_k}\}_{k \in \mathbb{N}}$  of  $\{u_\epsilon\}$  which converges weakly, when  $\epsilon_k \rightarrow 0$ , to a  $(p, q-1)$  form  $u$  in  $L_{p,q-1}^r(\Omega)$ , still with  $\|u\|_{L_{p,q-1}^r(\Omega)} \leq C \|\omega\|_{L_{p,q}^r(\Omega)}$ . Let us write  $u_k := u_{\epsilon_k}$ .

To see that this form  $u$  is 0 *a.e.* on  $\Omega \setminus \Omega_1$  let us write the weak convergence:

$$\forall \alpha \in L_{p,q-1}^{r'}(\Omega), \langle u_k, \alpha \rangle = \int_{\Omega} u_k \wedge \overline{*}\alpha \rightarrow \langle u, \alpha \rangle = \int_{\Omega} u \wedge \overline{*}\alpha.$$

As usual take  $\alpha := \frac{u}{|u|} \mathbb{1}_E$  where  $E := \{|u| > 0\} \cap (\Omega \setminus \Omega_1)$  then we get

$$\int_{\Omega} u \wedge \overline{*}\alpha = \int_E |u| dm = \lim_{k \rightarrow \infty} \int_{\Omega} u_k \wedge \overline{*}\alpha = \lim_{k \rightarrow \infty} \int_E \frac{u_k \wedge \overline{*}u}{|u|}.$$

Now we have, by Hölder inequalities:

$$\left| \int_E \frac{u_k \wedge \overline{*}u}{|u|} \right| \leq \|u_k\|_{L^r(E)} \|\mathbb{1}_E\|_{L^{r'}(E)}.$$

But

$$\|u_k\|_{L^r(E)}^r \leq \int_{\Omega \setminus \Omega_1} |u_k|^r dm \leq (\epsilon_k)^{r-1} C \|\omega\|_{L^r(\Omega)} \rightarrow 0, \quad k \rightarrow \infty$$

and  $\|\mathbb{1}_E\|_{L^{r'}(E)} = (m(E))^{1/r'}$ .

Hence

$$\begin{aligned} \left| \int_E |u| dm \right| &= \lim_{k \rightarrow \infty} \int_E \frac{u_k \wedge \overline{*}u}{|u|} \leq \\ &\leq \lim_{k \rightarrow \infty} C^r (m(E))^{1/r'} (\epsilon_k)^{r-1} \|\omega\|_{L^r(\Omega)}^r = 0, \end{aligned}$$

so  $\int_E |u| dm = 0$  which implies  $m(E) = 0$  because on  $E$ ,  $|u| > 0$ .

Hence we get that the form  $u$  is 0 *a.e.* on  $\Omega \setminus \Omega_1$ .

So we proved

$$\begin{aligned} \forall \varphi \in \mathcal{D}_{n-p,n-q}(\Omega), \quad (-1)^{p+q-1} \ll \varphi, \omega \gg = \ll \bar{\partial}\varphi, u_\epsilon \gg \rightarrow \ll \bar{\partial}\varphi, u \gg \\ \Rightarrow \ll \bar{\partial}\varphi, u \gg = (-1)^{p+q-1} \ll \varphi, \omega \gg \end{aligned}$$

hence  $\bar{\partial}u = \omega$  in the sense of distributions.

The proof is complete. ■

**Remark 3.6.** As in remark 3.4 if  $\Omega$  is Stein for  $q = n$  instead of asking  $\omega \perp \mathcal{H}_p(\Omega_2)$  we need just  $\omega \perp \mathcal{H}_p(\Omega)$ .

**Remark 3.7.** The condition of orthogonality to  $\mathcal{H}_p(V)$  in the case  $q = n$  is necessary: suppose there is a  $(p, n-1)$  current  $u$  such that  $\bar{\partial}u = \omega$  and  $u$  with compact support in an open set  $V \subset \Omega$ , then if  $h \in \mathcal{H}_p(V)$ , we have

$$h \in \mathcal{H}_p(V), \quad \ll \omega, h \gg = \ll \bar{\partial}u, h \gg = (-1)^{n+p} \ll u, \bar{\partial}h \gg = 0,$$

because,  $u$  being compactly supported, there is no boundary term and

$$\ll \bar{\partial}u, h \gg = (-1)^{n+p} \ll u, \bar{\partial}h \gg.$$

This kind of condition was already seen for extension of CR functions, see [Amar, 1991] and the references therein.

### 3.3 Finer control of the support.

Here we shall get a better control on the support of a solution.

**Theorem 3.8.** *Let  $\Omega$  be a weakly  $r'$  regular domain in a Stein manifold  $X$ .*

*Suppose the  $(p, q)$  form  $\omega$  is in  $L^{r,c}(\Omega, dm)$ ,  $\bar{\partial}\omega = 0$ , if  $q < n$ , and  $\omega \perp \mathcal{H}_p(V)$  for any  $V$  such that  $\text{Supp } \omega \subset V$ , if  $q = n$ , with  $\text{Supp } \omega \subset \Omega \setminus C$ , where  $C$  is a weakly  $r$  regular domain.*

*For any open relatively compact set  $U$  in  $C$ , there is a  $u \in L^{r,c}(\Omega, dm)$  such that  $\bar{\partial}u = \omega$  and with support in  $\Omega \setminus \bar{U}$ , provided that  $q \geq 2$ .*

Proof.

Let  $\omega$  be a  $(p, q)$  form with compact support in  $\Omega \setminus C$  then there is a  $v \in L^r_{p, q-1}(\Omega)$ ,  $\bar{\partial}v = \omega$ , with compact support in  $\Omega$ , by theorem 3.5 or, if  $\Omega$  is a polydisc in  $\mathbb{C}^n$  and if  $\omega \in \mathcal{W}_q^r(\Omega)$ , by the theorem in [Amar and Mongodi, 2014].

Because  $\omega$  has compact support outside  $C$  we have  $\omega = 0$  in  $C$ ; this means that  $\bar{\partial}v = 0$  in  $C$ . Because  $C$  is weakly  $r$  regular and  $q \geq 2$ , we have

$$\exists C' \subset C, C' \supset \bar{U}, \exists h \in L^r_{p, q-2}(C') \text{ s.t. } \bar{\partial}h = v \text{ in } C'.$$

Let  $\chi$  be a smooth function such that  $\chi = 1$  in  $U$  and  $\chi = 0$  near  $\partial C'$ ; then set  $u := v - \bar{\partial}(\chi h)$ .

We have that  $u = v - \chi \bar{\partial}h - \bar{\partial}\chi \wedge h = v - \chi v - \bar{\partial}\chi \wedge h$  hence  $u$  is in  $L^r(\Omega)$ ; moreover  $u = 0$  in  $\bar{U}$  because  $\chi = 1$  in  $U$  hence  $\bar{\partial}\chi = 0$  there. Finally  $\bar{\partial}u = \bar{\partial}v - \bar{\partial}^2(\chi h) = \omega$  and we are done. ■

If  $\Omega$  and  $C$  are, for instance, pseudo-convex in  $\mathbb{C}^n$  then  $\Omega \setminus C$  is no longer pseudo-convex in general, so this theorem improves actually the control of the support.

**Remark 3.9.** *The correcting function  $h$  is given by kernels in the case of Stein domains, hence it is linear; if the primitive solution  $v$  is also linear in  $\omega$ , then the solution  $u$  is linear too. This is the case in  $\mathbb{C}^n$  with the solution given in [Amar and Mongodi, 2014].*

This theorem cannot be true for  $q = 1$  as shown by the following example:

take a holomorphic function  $\varphi$  in the open unit ball  $B(0, 1)$  in  $\mathbb{C}^n$  such that it extends to no open ball of center 0 and radius  $> 1$ . For instance  $\varphi(z) := \exp\left(-\frac{z_1+1}{z_1-1}\right)$ . Take  $R < 1$ , then  $\varphi$  is  $C^\infty(\bar{B}(0, R))$  hence by a theorem of Whitney  $\varphi$  extends  $C^\infty$  to  $\mathbb{C}^n$ ; call  $\varphi_R$  this extension. Let  $\chi \in C^\infty_c(B(0, 2))$  such that  $\chi = 1$  in the ball  $B(0, 3/2)$  and consider the  $(0, 1)$  form  $\omega := \bar{\partial}(\chi \varphi_R)$ . Then  $\text{Supp } \omega \subset B(0, 2) \setminus B(0, R)$ ,  $\omega$  is  $\bar{\partial}$  closed and is  $C^\infty$  hence in  $L^r_{0,1}(B(0, 2))$ . Moreover  $B(0, R)$  is strictly pseudo-convex hence  $r'$  regular, but there is no function  $u$  such that  $\bar{\partial}u = \omega$  and  $u$  zero near the origin because any solution  $u$  will be C.R. on  $\partial B(0, R)$  and by Hartog's phenomenon will extend holomorphically to  $B(0, R)$ , hence cannot be identically null near 0.

Never the less in the case  $q = 1$ , we have:

**Theorem 3.10.** *Let  $\Omega$  be a weakly  $r'$  regular domain in a Stein manifold  $X$ .*

*Then for any  $(p, 1)$  form  $\omega$  in  $L^{r,c}(\Omega)$ ,  $\bar{\partial}\omega = 0$ , with support in  $\Omega_1 \setminus C$  where  $\Omega_1$  is a weak  $r'$  regular domain in  $\Omega$  and  $C$  is a domain such that  $C \subset \Omega$  and  $C \setminus \Omega_1 \neq \emptyset$ ; there is a  $u \in L^{r,c}(\Omega)$  such that  $\bar{\partial}u = \omega$  and with support in  $\Omega \setminus C$ .*

Proof.

There is  $u \in L^r_{p,0}(\Omega_1)$  such that  $\bar{\partial}u = \omega$  with compact support in  $\Omega_1$ , by theorem 3.5. Then  $\bar{\partial}u = 0$  in  $C$  hence  $u$  is locally holomorphic in  $C$ . Because  $C \setminus \Omega_1 \neq \emptyset$ , there is an open set in  $C \setminus \Omega_1 \subset \Omega \setminus \Omega_1$  where  $u$  is 0 and holomorphic, hence  $u$  is identically 0 in  $C$ ,  $C$  being connected. ■

**Remark 3.11.** *If there is a  $u \in L^{r,c}_{p,0}(\Omega_1)$  which is 0 in  $C$ , we have*

$$\forall h \in L^{r'}_{n-p, n-1}(C) :: \text{Supp } \bar{\partial}h \subset C, 0 = \lll u, \bar{\partial}h \ggg = \lll \omega, h \ggg,$$

*hence the necessary condition:*

$$\forall h \in L^{r'}_{n-p, n-1}(C) :: \text{Supp } \bar{\partial}h \subset C, \lll \omega, h \ggg = 0.$$

We proved in [Amar and Mongodi, 2014]:

**Theorem 3.12.** *Let  $f \in \mathcal{O}(\bar{\mathbb{D}}^n)$  be a holomorphic function in a neighborhood of the closed unit polydisc in  $\mathbb{C}^n$  and set  $Z := f^{-1}(0)$ . Then for any  $(0, q)$  form  $\omega$  in  $L^r(\mathbb{D}^n \setminus Z) \cap \mathcal{W}_q^r(\Omega)$ ,  $\bar{\partial}\omega = 0$ , with compact support in  $\mathbb{D}^n \setminus Z$ , for any  $k \in \mathbb{N}$ , we can find a  $(0, q-1)$ -form  $\beta \in L^{r,c}(\mathbb{D}^n)$  such that  $\bar{\partial}(f^k \beta) = \omega$ . Equivalently we can find a  $(0, q-1)$ -form  $\eta = f^k \beta$  such that  $\eta \in L^{r,c}(\mathbb{D}^n)$ ,  $\eta$  is 0 on  $Z$  up to order  $k$  and  $\bar{\partial}\eta = \omega$ .*

And by Remark 6.3 of this paper, the solutions are given by a *bounded linear operator*.

The following corollary will generalise strongly this result but at the price that *we have not the linearity, nor even the constructivity of the solution*.

**Corollary 3.13.** *Let  $\Omega$  be a Stein manifold. Let  $f$  be a holomorphic function in  $\Omega$  and set  $Z := f^{-1}(0)$ . Then for any  $(p, q)$  form  $\omega$  in  $L^{r,c}(\Omega \setminus Z)$ ,  $\bar{\partial}\omega = 0$ , if  $1 \leq q < n$ , and  $\omega \perp \mathcal{H}_p(\Omega \setminus Z)$  if  $q = n$ , there is a  $(p, q - 1)$  form  $u \in L^r(\Omega \setminus Z)$  such that  $\bar{\partial}u = \omega$  and  $u$  has its support still in  $\Omega \setminus Z$ .*

Proof.

We first show that  $\Omega \setminus Z$  is Stein. Because  $f \neq 0$  in  $\Omega \setminus Z$  we have that  $\varphi := \frac{1}{|f|^2}$  is plurisubharmonic in  $\Omega \setminus Z$  and  $\mathcal{C}^\infty(\Omega \setminus Z)$ . Because  $\Omega$  is Stein we have, by Theorem 5.1.6 of Hörmander [Hörmander, 1994], a strictly plurisubharmonic exhausting function  $\rho$  in  $\mathcal{C}^\infty(\Omega)$ . Now the function  $\gamma := \varphi + \rho$  is still strictly plurisubharmonic and  $\mathcal{C}^\infty$  in  $\Omega \setminus Z$ . Now we shall prove:

$$\forall \alpha \in \mathbb{R}, K_\alpha := \{z \in \Omega \setminus Z :: \gamma(z) < \alpha\} \text{ is relatively compact in } \Omega \setminus Z.$$

We have  $\rho(z) < \alpha - \varphi(z) < \alpha$  on  $K_\alpha$  because  $\varphi(z) \geq 0$ , hence, because  $\rho$  is exhaustive in  $\Omega$ , we have that  $K_\alpha$  is contained in a compact set  $F$  in  $\Omega$ . So on  $F$ , hence on  $K_\alpha$ , we have that  $\rho(z) \geq A > -\infty$  because  $\rho$  is continuous.

We also have  $\varphi(z) < \alpha - \rho(z)$  on  $K_\alpha$  i.e.  $|f(z)|^2 > \frac{1}{\alpha - \rho(z)}$ . So, on the set  $K_\alpha$ ,  $\alpha > \rho(z) \geq A > -\infty$ , hence  $|f(z)| > \frac{1}{\alpha - A} > 0$  on  $K_\alpha$ , so  $K_\alpha$  is far away from  $Z$ , hence  $K_\alpha$  is relatively compact in  $\Omega \setminus Z$ .

So we can apply [Hörmander, 1994, Theorem 5.2.10, p. 127] to get that  $\Omega \setminus Z$  is a Stein manifold.

Now we are in position to apply Theorem 3.5. Let  $\omega$  be a  $(p, q)$  form in  $L^{r,c}(\Omega \setminus Z)$ ,  $\bar{\partial}\omega = 0$ , if  $1 \leq q < n$ , and  $\omega \perp \mathcal{H}_p(\Omega \setminus Z)$  if  $q = n$ , Theorem 3.5 gives a  $(p, q - 1)$  form  $u \in L^r(\Omega \setminus Z)$  such that  $\bar{\partial}u = \omega$  and  $u$  has its compact support in  $\Omega \setminus Z$ .

The proof is complete. ■

**Remark 3.14.** *This leaves open the question to have a linear (or a constructive) solution to this problem even in the case of the polydisc.*

## 4 Appendix

Here we shall prove certainly known results on the duality  $L^r - L^{r'}$  for  $(p, q)$ -forms in a complex manifold  $X$ . Because I was unable to find precise references for them, I prove them here.

Recall we have a pointwise scalar product and a pointwise modulus for  $(p, q)$ -forms in  $X$ :

$$\langle \alpha, \beta \rangle dm := \alpha \wedge \overline{*}\beta; \quad |\alpha|^2 dm := \alpha \wedge \overline{*}\alpha.$$

By the Cauchy-Schwarz inequality for scalar products we get:

$$\forall x \in X, \quad |\langle \alpha, \beta \rangle(x)| \leq |\alpha(x)| |\beta(x)|.$$

This gives Hölder inequalities for  $(p, q)$ -forms:

**Lemma 4.1.** *(Hölder inequalities) Let  $\alpha \in L_{p,q}^r(\Omega)$  and  $\beta \in L_{p,q}^{r'}(\Omega)$ . We have*

$$|\langle \alpha, \beta \rangle| \leq \|\alpha\|_{L^r(\Omega)} \|\beta\|_{L^{r'}(\Omega)}.$$

Proof.

We start with  $\langle \alpha, \beta \rangle = \int_\Omega \langle \alpha, \beta \rangle(x) dm(x)$  hence

$$|\langle \alpha, \beta \rangle| \leq \int_\Omega |\langle \alpha, \beta \rangle(x)| dm \leq \int_\Omega |\alpha(x)| |\beta(x)| dm(x).$$

By the usual Hölder inequalities for functions we get

$$\int_\Omega |\alpha(x)| |\beta(x)| dm(x) \leq \left( \int_\Omega |\alpha(x)|^r dm \right)^{1/r} \left( \int_\Omega |\beta(x)|^{r'} dm \right)^{1/r'}$$

which ends the proof of the lemma. ■

**Lemma 4.2.** *Let  $\alpha \in L^r_{p,q}(\Omega)$  then*

$$\|\alpha\|_{L^r_{p,q}(\Omega)} = \sup_{\beta \in L^{r'}_{p,q}(\Omega), \beta \neq 0} \frac{|\langle \alpha, \beta \rangle|}{\|\beta\|_{L^{r'}(\Omega)}}.$$

Proof.

We choose  $\beta := \alpha |\alpha|^{r-2}$ , then:

$$|\beta|^{r'} = |\alpha|^{r'(r-1)} = |\alpha|^r \Rightarrow \|\beta\|_{L^{r'}(\Omega)}^{r'} = \|\alpha\|_{L^r(\Omega)}^r.$$

Hence

$$\langle \alpha, \beta \rangle = \left\langle \alpha, \alpha |\alpha|^{r-2} \right\rangle = \int_{\Omega} (\alpha, \alpha) |\alpha|^{r-2} dm = \|\alpha\|_{L^r(\Omega)}^r.$$

On the other hand we have

$$\|\beta\|_{L^{r'}(\Omega)} = \|\alpha\|_{L^r(\Omega)}^{r/r'} = \|\alpha\|_{L^r(\Omega)}^{r-1},$$

so

$$\|\alpha\|_{L^r(\Omega)} \times \|\beta\|_{L^{r'}(\Omega)} = \|\alpha\|_{L^r(\Omega)}^r = \langle \alpha, \beta \rangle.$$

$$\text{Hence } \|\alpha\|_{L^r(\Omega)} = \frac{|\langle \alpha, \beta \rangle|}{\|\beta\|_{L^{r'}(\Omega)}}.$$

A fortiori for any choice of  $\beta$ :

$$\|\alpha\|_{L^r(\Omega)} \leq \sup_{\beta \in L^{r'}(\Omega)} \frac{|\langle \alpha, \beta \rangle|}{\|\beta\|_{L^{r'}(\Omega)}}.$$

To prove the other direction, we use the Hölder inequalities, Lemma 4.1:

$$\forall \beta \in L^{r'}_{p,q}(\Omega), \frac{|\langle \alpha, \beta \rangle|}{\|\beta\|_{L^{r'}(\Omega)}} \leq \|\alpha\|_{L^r(\Omega)}.$$

The proof is complete. ■

Now we are in position to state:

**Lemma 4.3.** *The dual space of the Banach space  $L^r_{p,q}(\Omega)$  is  $L^{r'}_{n-p,n-q}(\Omega)$ .*

Proof.

Suppose first that  $u \in L^{r'}_{n-p,n-q}(\Omega)$ . Then consider:

$$\forall \alpha \in L^r_{p,q}(\Omega), \mathcal{L}(\alpha) := \int_{\Omega} \alpha \wedge u = \langle \alpha, \overline{*u} \rangle.$$

This is a linear form on  $L^r_{p,q}(\Omega)$  and its norm, by definition, is

$$\|\mathcal{L}\| = \sup_{\alpha \in L^r(\Omega)} \frac{|\langle \alpha, \overline{*u} \rangle|}{\|\alpha\|_{L^r(\Omega)}}.$$

By use of Lemma 4.2 we get

$$\|\mathcal{L}\| = \|\overline{*u}\|_{L^{r'}_{p,q}(\Omega)} = \|u\|_{L^{r'}_{n-p,n-q}(\Omega)}.$$

So we have  $(L^r_{p,q}(\Omega))^* \supset L^{r'}_{n-p,n-q}(\Omega)$  with the same norm.

Conversely take a continuous linear form  $\mathcal{L}$  on  $L^r_{p,q}(\Omega)$ . We have, again by definition, that:

$$\|\mathcal{L}\| = \sup_{\alpha \in L^r(\Omega)} \frac{|\mathcal{L}(\alpha)|}{\|\alpha\|_{L^r(\Omega)}}.$$

Because  $\mathcal{D}_{p,q}(\Omega) \subset L^r_{p,q}(\Omega)$ ,  $\mathcal{L}$  is a continuous linear form on  $\mathcal{D}_{p,q}(\Omega)$ , hence, by definition,  $\mathcal{L}$  can be represented by a  $(n-p, n-q)$ -current  $u$ . So we have:

$$\forall \alpha \in \mathcal{D}_{p,q}(\Omega), \mathcal{L}(\alpha) := \int_{\Omega} \alpha \wedge u = \langle \alpha, \overline{*u} \rangle.$$

Moreover we have, by Lemma 4.2,

$$\|\mathcal{L}\| = \sup_{\alpha \in \mathcal{D}_{p,q}(\Omega)} \frac{|\langle \alpha, \overline{*u} \rangle|}{\|\alpha\|_{L^r(\Omega)}} = \|*u\|_{L^{r'}(\Omega)}$$

because  $\mathcal{D}_{p,q}(\Omega)$  is dense in  $L^r_{p,q}(\Omega)$ . So we proved  $(L^r_{p,q}(\Omega))^* \subset L^{r'}_{n-p, n-q}(\Omega)$  with the same norm. The proof is complete. ■

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