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ON L^r ESTIMATES FOR SOLUTIONS OF THE $\bar{\partial}$ EQUATION WITH COMPACT SUPPORT.

ERIC AMAR

ABSTRACT. By a theorem of Andreotti and Grauert if ω is a (p, q) current, $q < n$, in a Stein manifold Ω , $\bar{\partial}$ closed and with compact support, then there is a solution u to $\bar{\partial}u = \omega$ still with compact support in Ω . The main result of this work is to show that if moreover $\omega \in L^r(m)$, where m is a suitable Lebesgue measure on the Stein manifold, then we have a solution u with compact support and, for $1 < r < 2(n+1)$, in $L^s(m)$, $\frac{1}{s} = \frac{1}{r} - \frac{1}{2(n+1)}$; because $s > r$, we have a gain of regularity. We prove it by estimates in L^r spaces with weights.

1. INTRODUCTION.

Let ω be a $\bar{\partial}$ closed (p, q) form in \mathbb{C}^n with compact support $K := \text{Supp } \omega$ and such that $\omega \in L^r(\mathbb{C}^n)$. Setting K in a ball $\mathbb{B} := B(0, R)$ with R big enough, we know, by a theorem of Ovreliid [11], that we have a $(p, q-1)$ form $u \in L^r(\mathbb{B})$ such that $\bar{\partial}u = \omega$. On the other hand we also know, at least when $q < n$, that there is a current v with compact support such that $\bar{\partial}v = \omega$, by a theorem of Andreotti-Grauert [5]. So a natural question is : may we have a solution u of $\bar{\partial}u = \omega$ with compact support and in $L^r(\mathbb{C}^n)$?

In the sequel a domain in a Stein manifold will always means a be a relatively compact open connected set with C^∞ smooth boundary. We shall denote by *s.p.c.* a strictly pseudo convex domain.

There is a work by H. Skoda [13] who proved such a result. Let Ω be a *s.p.c.* domain in \mathbb{C}^n then in [13], Proposition 1 p. 292 and proposition 2 p. 293, H. Skoda proved that if f is a (p, q) form with measure coefficients, $q < n$, $\bar{\partial}$ closed and with compact support in Ω , then there is a solution U to the equation $\bar{\partial}U = f$ such that $\|U\|_{L^r(\Omega)} \leq C(\Omega, r)\|f\|_r$, for any r such that $1 < r < \frac{2n+2}{2n-1}$ and U has zero boundary values in the sense of Stokes formula.

This means that essentially U has compact support hence H. Skoda got the answer for Ω strictly pseudo-convex and $1 < r < \frac{2n+2}{2n-1}$.

We answered this question by the affirmative for any $r \in [1, \infty]$ in a joined work with S. Mongodi [4] linearly by the "method of coronas". This method works only in \mathbb{C}^n and asks for extra L^r conditions on derivatives of coefficients of ω , when $q < n$.

The aim of this work is to extend this result to Stein manifolds and get rid of the extra L^r conditions. For it we use a completely different approach inspired by the Serre duality [12]. Because Hahn Banach theorem is used, these results are no longer linear nor constructive as in [4] but they are completely explicit.

On any complex manifold X we define first the "Lebesgue measure" as in Hörmander's book [8] section 5.2, with a hermitian metric locally equivalent to the usual one on any analytic coordinates patch. Associated to this metric there is a volume form dm and we take it for the Lebesgue measure on X .

Let Ω be a domain in a complex manifold X . We shall define $L^r(\Omega)$ to be the set of functions f defined on Ω such that

$$\|f\|_{L^r(\Omega)}^r := \int_{\Omega} |f(z)|^r dm(z) < \infty,$$

with m the Lebesgue measure on X . The same way we denote $L_{(p,q)}^r(\Omega)$ the set of (p, q) currents ω with coefficients in $L^r(\Omega)$. The set of currents in $L_{(p,q)}^r(\Omega)$ with compact support will be denoted by $L_{(p,q)}^{r,c}(\Omega)$.

Theorem 1.1. *Let Ω be a Stein manifold. Let ω be a (p, q) current with compact support in $L_{(p,q)}^r(\Omega)$, such that $\bar{\partial}\omega = 0$ if $1 \leq q < n$ and $\forall h \in \mathcal{H}_p(\Omega)$, $\langle \omega, h \rangle = 0$ if $q = n$.*

Then, for $1 \leq r < 2(n+1)$, there is a $(p, q-1)$ current u in $L^{s,c}(\Omega)$ such that $\bar{\partial}u = \omega$ with

$$\frac{1}{s} = \frac{1}{r} - \frac{1}{2(n+1)}.$$

For $r = 2(n+1)$ and any $s < \infty$, there is a $(p, q-1)$ current u in $L^{s,c}(\Omega)$ such that $\bar{\partial}u = \omega$.

For $r > 2(n+1)$, there is a $(p, q-1)$ current u in $L^{\infty,c}(\Omega)$ such that $\bar{\partial}u = \omega$.

Moreover if the support of ω is contained in a s.p.c. domain $D \subset \Omega$, then the support of u is also contained in D .

The definition of (r, s) regular domain shall be given later on, but Stein manifolds are such domains. We note r' the conjugate exponent of r , i.e. $\frac{1}{r} + \frac{1}{r'} = 1$.

We shall also get a better control on the support of a solution.

Theorem 1.2. *Let Ω be a (s', r') regular domain in a Stein manifold X . Then for any (p, q) form ω with compact support in $L^r(\Omega)$, $\bar{\partial}\omega = 0$, if $q < n$, and $\omega \perp \mathcal{H}_p(\Omega)$ if $q = n$, with support in $\Omega \setminus C$ where C is a (s, s) regular domain, there is a $u \in L^s(\Omega)$ with compact support, such that $\bar{\partial}u = \omega$ and with support in $\Omega \setminus \bar{U}$, where U is any open set relatively compact in C , provided that $q \geq 2$.*

One may observe that $\Omega \setminus C$ is not Stein in general even if Ω is, so this result is actually an improvement over the previous theorem.

There is also a result of this kind for $q = 1$, see section 2.3.

The idea is to solve $\bar{\partial}u = \omega$ in a space $L^r(\Omega)$ with a weight η "big" outside of a relatively compact domain D containing the support of ω ; this way we shall have a "small" solution u outside of D . Then, using a sequence of such weights going to infinity outside of D , we shall have a u zero outside of D .

I am indebted to G. Tomassini who started my interest in this subject. I want to thank C. Laurent who made me realise that I can use a weaker condition than my previous r regularity one to get these results, with exactly the same proofs.

2. SOLUTION OF THE $\bar{\partial}$ EQUATION WITH COMPACT SUPPORT.

2.1. Domain (r, s) regular.

Definition 2.1. *Let X be a complex manifold and Ω a domain in X ; let $r \in [1, \infty]$, we shall say that Ω is (r, s) , $s \geq r$, regular if for any compact set $K \Subset \Omega$ there are 3 open sets $\Omega_1, \Omega_2, \Omega_3$ such that $K \Subset \Omega_3 \subset \Omega_2 \subset \Omega_1 \subset \Omega_0 := \Omega$ and 3 constants C_1, C_2, C_3 such that :*

$\forall j = 0, 1, 2, \forall \omega \in L_{p,q+j}^r(\Omega_j), \bar{\partial}\omega = 0, \exists u \in L_{p,q+j-1}^s(\Omega_{j+1}), \bar{\partial}u = \omega$ in Ω_{j+1} and $\|u\|_{L^s(\Omega_{j+1})} \leq C_{j+1} \|\omega\|_{L^r(\Omega_j)}$.

I.e. regularity means that we have a 3 steps chain of resolution.

As we shall see the interesting case for us is when Ω_1 is as small as possible still such that $K \Subset \Omega_3 \subset \Omega_2 \subset \Omega_1$.

Examples of $(2, 2)$ regular domains are the relatively compact pseudo-convex domains in Stein manifold by Hörmander [8].

Examples of (r, r) regular domains in \mathbb{C}^n are the *s.p.c.* domains by Ovreid [11] ; the polydiscs in \mathbb{C}^n by Charpentier [6], finite transverse intersections of strictly pseudo-convex bounded domains in \mathbb{C}^n by Menini [10].

In fact if Ω is a *s.p.c.* domain in \mathbb{C}^n we have a better result : Ω is (r, s) regular with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2(n+1)}$ by a more precise result we proved in [2] :

Theorem 2.2. *Let Ω be a s.p.c. domain of \mathbb{C}^n , then for $1 < r < 2n + 2$ we have*

$$\forall \omega \in L^r_{(p,q)}(\Omega), \bar{\partial}\omega = 0, \exists u \in L^s_{(p,q-1)}(\Omega) :: \bar{\partial}u = \omega$$

$$\text{with } \frac{1}{s} = \frac{1}{r} - \frac{1}{2(n+1)}.$$

For $r = 1$, we have

$$\forall \omega \in L^1_{(p,q)}(\Omega), \bar{\partial}\omega = 0, \exists u \in L^{s,\infty}_{(p,q-1)}(\Omega) :: \bar{\partial}u = \omega.$$

For $r = 2n + 2$ we have

$$\forall \omega \in L^{2n+2}_{(p,q)}(\Omega), \bar{\partial}\omega = 0, \exists u \in \bigcap_{p>1} L^p(\Omega) :: \bar{\partial}u = \omega.$$

For $r > 2n + 2$ we have

$$\forall \omega \in L^{2n+2}_{(p,q)}(\Omega), \bar{\partial}\omega = 0, \exists u \in \Lambda^\beta_{(p,0)}(\Omega) :: \bar{\partial}u = \omega,$$

where Λ^β are the Lipschitz functions with $2\beta = 1 - \frac{2(n+1)}{r}$.

For domains in a Stein manifold X we can solve the $\bar{\partial}$ equation in *s.p.c.* domains with $L^r - L^r$ estimates for $(0, 1)$ forms by use of N. Kerzman kernels [9] and the metric above.

We can solve the $\bar{\partial}$ equation in strictly pseudo-convex domains with smooth boundary in X with $L^r - L^r$ estimates for all (p, q) currents by J-P. Demailly and C. Laurent ones([7], Remarque 4, page 596) but here the manifold has to be equipped with a metric with null curvature.

In fact we proved [3] an analogous result as the one in [7], with $L^r - L^s$ estimates with $s :: \frac{1}{s} = \frac{1}{r} - \frac{1}{2(n+1)} \geq \frac{1}{2}$, i.e. $s \leq 2$, and without any restriction on the metric.

Let $\mathcal{H}_p(\Omega)$ be the set of all $(p, 0)$ $\bar{\partial}$ closed forms in Ω .

Theorem 2.3. *Let Ω be a s.p.c. domain in the Stein manifold X .*

Let s such that $\frac{1}{s} = \frac{1}{r} - \frac{1}{2(n+1)}$; if $\omega \in L^r_{p,q}(\Omega)$ with $\bar{\partial}\omega = 0$ then for $s \leq 2$, there is $u \in L^s_{p,q-1}(\Omega)$ such that $\bar{\partial}u = \omega$.

When $2 < s < \infty$ we have :

- for $p = 0, q = 1$ we still have a function $u \in L^s(\Omega)$ such that $\bar{\partial}u = \omega$.
- For $q = n, \omega \in L^r_{p,n}(\Omega)$ with $\omega \perp \mathcal{H}_p(\Omega)$ we still have a $u \in L^s_{(p,n-1)}(\Omega)$ such that $\bar{\partial}u = \omega$.
- If X can be equipped with a hermitian metric with null curvature in the sens of Demailly-Laurent [7], we still have a $u \in L^s_{(p,q-1)}(\Omega)$ such that $\bar{\partial}u = \omega$.
- Finally if ω has compact support in $\Omega, \omega \in L^{r,c}_{p,q}(\Omega)$, with $\bar{\partial}\omega = 0$ if $q < n$ and $\omega \perp \mathcal{H}_p(\Omega)$ if $q = n$, we still have a $u \in L^s_{(p,q-1)}(\Omega)$ such that $\bar{\partial}u = \omega$.

In any cases, there is a constant $C > 0$ such that $\|u\|_s \leq C\|\omega\|_r$.

Lemma 2.4. *A Stein manifold Ω is (r, s) regular with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2(n+1)}$ provided that $s \leq 2$, i.e. for $1 \leq r \leq \frac{2(n+1)}{n+2}$.*

Proof.

By theorem 5.1.6 of Hörmander [8] there exists a \mathcal{C}^∞ strictly plurisubharmonic exhausting function φ for Ω . Call $E \subset \Omega$ the set of critical points of φ , then by the Morse-Sard lemma we have that the Lebesgue measure of $\varphi(E) \subset \mathbb{R}$ is zero. Hence we can find a sequence $c_k \in \mathbb{R} \setminus \varphi(E)$, $c_k \rightarrow \infty$, such that

$$D_k := \{z \in \Omega :: \varphi(z) < c_k\}$$

make an exhaustive sequence of open relatively compact sets in Ω , $\partial\varphi \neq 0$ on ∂D_k , hence D_k is strictly pseudo-convex with \mathcal{C}^∞ smooth boundary, and finally $D_k \nearrow \Omega$.

Let $\omega \in L_{p,q}^r(\Omega)$, $\bar{\partial}\omega = 0$ then by theorem 2.3, if $\frac{1}{s} = \frac{1}{r} - \frac{1}{2(n+1)} \geq \frac{1}{2}$, we can solve $\bar{\partial}u = \omega$ in D_k with $u \in L_{p,q-1}^s(D_k)$ and

$$\|u\|_{L^s(D_k)} \leq C_k \|\omega\|_{L^r(D_k)} \leq C_k \|\omega\|_{L^r(\Omega)}.$$

Hence if K is a compact set in Ω , there is a D_k such that $K \Subset D_k$ and we can take $\Omega_1 = \Omega_2 = \Omega_3 = D_k$.

Hence the (r, s) regularity of Ω , provided $s \leq 2$. ■

Remark 2.5. *If the current $\omega \in L_{p,q}^{r,c}(\Omega)$, $\bar{\partial}\omega = 0$, i.e. ω has compact support, then we can solve $\bar{\partial}u = \omega$ in D_k with $u \in L_{p,q-1}^s(D_k)$ and no restriction on s by theorem 2.3. This will suffice to get our main theorem.*

Lemma 2.6. *A pseudo convex domain Ω in \mathbb{C}^n is (r, s) regular for $1 \leq r < 2(n+1)$ with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2(n+1)}$. If $r = 2(n+1)$, Ω is (r, s) regular for any $s < \infty$ and if $r > 2(n+1)$, Ω is (r, ∞) regular.*

Proof.

This is exactly the same proof but we use theorem 2.2 which is valid without the restriction $s \leq 2$. ■

2.2. The main result.

Let r', s' be the conjugate exponent of r, s respectively. Let Ω be a (s', r') , $r' \geq s'$, regular domain in the complex manifold X , and ω be a (p, q) form in $L^{r,c}(\Omega)$, $r \geq 1$, $\text{Supp } \omega \Subset \Omega$. Let $\Omega_1 \subset \Omega$ be a sub domain in Ω with $\text{Supp } \omega \Subset \Omega_1 \subset \Omega$ given by the regularity of Ω .

Set the weight $\eta = \eta_\epsilon := \mathbb{1}_{\Omega_1}(z) + \epsilon \mathbb{1}_{\Omega \setminus \Omega_1}(z)$ for a fixed $\epsilon > 0$. We shall define $L^r(\Omega, \eta)$ to be the set of functions f defined on Ω such that

$$\|f\|_{L^r(\Omega, \eta)}^r := \int_{\Omega} |f(z)|^r \eta(z) dm(z) < \infty,$$

with the weight $\eta(z) > 0$. The same way we denote $L_{(p,q)}^r(\Omega, \eta)$ the set of (p, q) currents ω with coefficients in $L^r(\Omega, \eta)$.

Suppose that ω is such that $\bar{\partial}\omega = 0$ if $1 \leq q < n$ and for any open $U \Subset \Omega$, $\text{Supp } \omega \Subset U$ we have $\forall h \in \mathcal{H}_p(U)$, $\langle \omega, h \rangle = 0$ if $q = n$. We shall use the lemma :

Lemma 2.7. *For Ω , ω , η as above, the function $\mathcal{L} = \mathcal{L}_\omega$, defined on $(n-p, n-q+1)$ form $\alpha \in L^{s'}(\Omega, \eta)$, $\bar{\partial}$ closed in Ω , as follows :*

$\mathcal{L}(\alpha) := (-1)^{p+q-1} \langle \omega, \varphi \rangle$, where $\varphi \in L^{r'}(\Omega_1)$ is such that $\bar{\partial}\varphi = \alpha$ in Ω_1 is well defined and linear.

Proof.

Because $\epsilon > 0$ we have $\alpha \in L^{s'}(\Omega, \eta) \Rightarrow \alpha \in L^{s'}(\Omega)$ and such a φ exists by the (s', r') regularity of Ω , so there is a $\Omega_1 :: \text{Supp } \omega \Subset \Omega_1 \subset \Omega$ such that $\varphi \in L^{r'}(\Omega_1)$ with $\bar{\partial}\varphi = \alpha$ in Ω_1 .

Let us see that \mathcal{L} is well defined.

Suppose first that $q < n$.

In order for \mathcal{L} to be well defined we need

$$\forall \varphi, \psi \in L^{r'}_{(n-p, n-q)}(\Omega_1), \bar{\partial}\varphi = \bar{\partial}\psi \Rightarrow \langle \omega, \varphi \rangle = \langle \omega, \psi \rangle.$$

This is meaningful because $\omega \in L^{r,c}(\Omega)$, $r > 1$, $\text{Supp } \omega \Subset \Omega_1$.

Then we have $\bar{\partial}(\varphi - \psi) = 0$ hence, because Ω is (s', r') regular, there is a Ω_2 such that $\text{Supp } \omega \Subset \Omega_2 \subset \Omega_1$ and we can solve $\bar{\partial}$ in $L^{r'}(\Omega_2)$, i.e. $\varphi - \psi \in L^{r'}_{(n-p, n-q)}(\Omega_1) \subset L^{s'}_{(n-p, n-q)}(\Omega_1)$ because $s' \leq r'$.

Hence

$$\exists \gamma \in L^{r'}_{(n-p, n-q-1)}(\Omega_2) :: \bar{\partial}\gamma = (\varphi - \psi).$$

So $\langle \omega, \varphi - \psi \rangle = \langle \omega, \bar{\partial}\gamma \rangle = (-1)^{p+q-1} \langle \bar{\partial}\omega, \gamma \rangle = 0$ because ω being compactly supported in Ω_2 there is no boundary term.

Hence \mathcal{L} is well defined in that case.

Suppose now that $q = n$.

Of course $\bar{\partial}\omega = 0$ and we have that φ, ψ are $(p, 0)$ forms hence $\bar{\partial}(\varphi - \psi) = 0$ means that $h := \varphi - \psi$ is a $\bar{\partial}$ closed $(p, 0)$ form hence $h \in \mathcal{H}_p(\Omega_1)$. Taking $U = \Omega_1$ in the hypothesis, $\omega \perp \mathcal{H}_p(U)$ gives $\langle \omega, h \rangle = 0$, and \mathcal{L} is also well defined in that case.

We remark that no compactness is required for ω in this case.

It remains to see that \mathcal{L} is linear, so let $\alpha = \alpha_1 + \alpha_2$, with $\alpha_j \in L^{s'}(\Omega, \eta)$, $\bar{\partial}\alpha_j = 0$, $j = 1, 2$; we have $\alpha = \bar{\partial}\varphi$, $\alpha_1 = \bar{\partial}\varphi_1$ and $\alpha_2 = \bar{\partial}\varphi_2$, with $\varphi, \varphi_1, \varphi_2$ in $L^{r'}(\Omega_1)$ so, because $\bar{\partial}(\varphi - \varphi_1 - \varphi_2) = 0$, we have

if $q < n$:

$$\varphi = \varphi_1 + \varphi_2 + \bar{\partial}\psi, \text{ with } \psi \text{ in } L^{r'}(\Omega_2), \text{ as before because } s' \leq r' ;$$

so

$$\begin{aligned} \mathcal{L}(\alpha) &= (-1)^{p+q-1} \langle \omega, \varphi \rangle = (-1)^{p+q-1} \langle \omega, \varphi_1 + \varphi_2 + \bar{\partial}\psi \rangle = \\ &= \mathcal{L}(\alpha_1) + \mathcal{L}(\alpha_2) + (-1)^{p+q-1} \langle \omega, \bar{\partial}\psi \rangle, \end{aligned}$$

but $\langle \omega, \bar{\partial}\psi \rangle = \langle \bar{\partial}\omega, \psi \rangle = 0$, because $\text{Supp } \omega \Subset \Omega_2$ implies there is no boundary term.

Hence $\mathcal{L}(\alpha) = \mathcal{L}(\alpha_1) + \mathcal{L}(\alpha_2)$.

If $q = n$:

because $\bar{\partial}(\varphi - \varphi_1 - \varphi_2) = 0$, we have $h := \varphi - \varphi_1 - \varphi_2 \in \mathcal{H}_p(\Omega_1)$ and the hypothesis $\omega \perp \mathcal{H}_p(\Omega_1)$ gives $\langle \omega, h \rangle = 0$, so

$$\mathcal{L}(\alpha) = (-1)^{p+q-1} \langle \omega, \varphi \rangle = (-1)^{p+q-1} \langle \omega, \varphi_1 + \varphi_2 + h \rangle = \mathcal{L}(\alpha_1) + \mathcal{L}(\alpha_2) + (-1)^{p+q-1} \langle \omega, h \rangle,$$

hence $\mathcal{L}(\alpha) = \mathcal{L}(\alpha_1) + \mathcal{L}(\alpha_2)$.

We notice again that ω with compact support is not needed in that case.

Hence in any case $\mathcal{L}(\alpha) = \mathcal{L}(\alpha_1) + \mathcal{L}(\alpha_2)$. The same for $\alpha = \lambda\alpha_1$ and the linearity. ■

Remark 2.8. *If Ω is Stein, we can take the domains Ω_1 to be s.p.c. hence also Stein. So because $K := \text{Supp } \omega \subset \Omega_1 \subset \Omega$, then the $A(\Omega_1)$ convex hull of K , \hat{K}_{Ω_1} is still in Ω_1 and any $\bar{\partial}$ closed $(p, 0)$ current in Ω_1 can be uniformly approximated on \hat{K}_{Ω_1} by $\bar{\partial}$ closed $(p, 0)$ current in Ω by the theorem 3.2 we shall prove in the appendix.*

Therefore for $q = n$ instead of asking $\omega \perp \mathcal{H}_p(\Omega_1)$ we need just $\omega \perp \mathcal{H}_p(\Omega)$.

Lemma 2.9. *Still with the same hypotheses as above there is a $(p, q - 1)$ current u such that*

$$\forall \alpha \in L_{(n-p, n-q+1)}^{s'}(\Omega, \eta), \quad \langle u, \alpha \rangle = \mathcal{L}(\alpha) = (-1)^{p+q-1} \langle \omega, \varphi \rangle,$$

and

$$\sup_{\alpha \in L^{s'}(\Omega, \eta_\epsilon), \|\alpha\|_{L^{s'}(\Omega, \eta_\epsilon)} \leq 1} |\langle u, \alpha \rangle| \leq C \|\omega\|_{L^r(\Omega)},$$

where the constant C is independent of $\epsilon > 0$.

Proof.

By lemma 2.7 we have that \mathcal{L} is a linear form on $(n - p, n - q + 1)$ currents $\alpha \in L^{s'}(\Omega, \eta)$, $\bar{\partial}$ closed in Ω .

We also have that $\alpha \in L^{s'}(\Omega_1)$, $\bar{\partial}\alpha = 0$ in Ω_1 , hence still with the (s', r') regularity of Ω , we have

$$\exists \psi \in L_{(n-p, n-q)}^{r'}(\Omega_2) :: \bar{\partial}\psi = \alpha, \quad \|\psi\|_{L^{r'}(\Omega_2)} \leq C_2 \|\alpha\|_{L^{s'}(\Omega_1)}.$$

• If $q < n$, we have $\bar{\partial}(\varphi - \psi) = \alpha - \alpha = 0$ on Ω_2 and, by the (s', r') regularity of Ω , there is a $\Omega_3 \subset \Omega_2$ such that $\text{Supp } \omega \subset \Omega_3 \subset \Omega_2$ and a $\gamma \in L_{(n-p, n-q-1)}^{r'}(\Omega_3)$, $\bar{\partial}\gamma = \varphi - \psi$ in Ω_3 , always because $s' \leq r'$. So we get

$$\langle \omega, \varphi - \psi \rangle = \langle \omega, \bar{\partial}\gamma \rangle = (-1)^{p+q-1} \langle \bar{\partial}\omega, \gamma \rangle = 0,$$

because $\text{Supp } \omega \subset \Omega_3$ and there is no boundary term. Hence

$$\mathcal{L}(\alpha) = \langle \omega, \varphi \rangle = \langle \omega, \psi \rangle.$$

• If $q = n$, we still have $\bar{\partial}(\varphi - \psi) = \alpha - \alpha = 0$ on Ω_2 , hence $\varphi - \psi \in \mathcal{H}_p(\Omega_2)$; this time we choose $U = \Omega_2$ and the assumption gives $\langle \omega, \varphi - \psi \rangle = 0$ hence again $\mathcal{L}(\alpha) = \langle \omega, \varphi \rangle = \langle \omega, \psi \rangle$.

In any cases, by Hölder inequalities

$$|\mathcal{L}(\alpha)| \leq \|\omega\|_{L^r(\Omega_1)} \|\psi\|_{L^{r'}(\Omega_2)} = \|\omega\|_{L^r(\Omega)} \|\psi\|_{L^{r'}(\Omega_2)},$$

because ω is compactly supported in $\Omega_1 \subset \Omega$.

By the (s', r') regularity of Ω there is a constant C_2 such that

$$\|\psi\|_{L^{r'}(\Omega_2)} \leq C_2 \|\alpha\|_{L^{s'}(\Omega_1)}.$$

Of course we have

$$\|\alpha\|_{L^{s'}(\Omega_1)} \leq \|\alpha\|_{L^{s'}(\Omega, \eta)}$$

because $\eta = 1$ on Ω_1 , $\eta > 0$ on Ω , hence

$$|\mathcal{L}(\alpha)| \leq C_2 \|\omega\|_{L^r(\Omega)} \|\alpha\|_{L^{s'}(\Omega, \eta)}.$$

So we have that the norm of \mathcal{L} is bounded on the subspace of $\bar{\partial}$ closed forms in $L^{s'}(\Omega, \eta_\epsilon)$ by $C \|\omega\|_{L^r(\Omega)}$ which is independent of ϵ .

We apply the Hahn-Banach theorem to extend \mathcal{L} with the same norm to all $(n - p, n - q + 1)$ forms in $L^{s'}(\Omega, \eta)$. As in Serre duality theorem ([12], p. 20) this is one of the main ingredient in the proof.

This means, by the definition of currents, that there is a $(p, q - 1)$ current u which represents the extended form \mathcal{L} , i.e.

$$\forall \alpha \in L_{(n-p, n-q+1)}^{s'}(\Omega, \eta), \quad \langle u, \alpha \rangle = \mathcal{L}(\alpha) = (-1)^{p+q-1} \langle \omega, \varphi \rangle,$$

and such that

$$\sup_{\alpha \in L^{s'}(\Omega, \eta), \|\alpha\|_{L^{s'}(\Omega, \eta)} \leq 1} |\langle u, \alpha \rangle| \leq C \|\omega\|_{L^r(\Omega)}.$$

■

Theorem 2.10. *Let Ω be a Stein manifold. Let ω be a (p, q) current in $L^{r,c}(\Omega)$, $r > 2$ such that $\bar{\partial}\omega = 0$ if $1 \leq q < n$ and $\forall h \in \mathcal{H}_p(\Omega)$, $\langle \omega, h \rangle = 0$ if $q = n$.*

Then, for $2 < r < 2(n+1)$, there is a $(p, q-1)$ current u in $L^{s,c}(\Omega)$ such that $\bar{\partial}u = \omega$ with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2(n+1)}$.

For $r = 2(n+1)$ and any $s < \infty$, there is a $(p, q-1)$ current u in $L^{s,c}(\Omega)$ such that $\bar{\partial}u = \omega$.

For $r > 2(n+1)$, there is a $(p, q-1)$ current u in $L^{\infty,c}(\Omega)$ such that $\bar{\partial}u = \omega$.

Moreover if the support of ω is contained in a s.p.c. domain $D \subset \Omega$, then the support of u is also contained in D .

Proof.

First we notice that $\frac{1}{s} = \frac{1}{r} - \frac{1}{2(n+1)} \Rightarrow \frac{1}{r'} = \frac{1}{s'} - \frac{1}{2(n+1)}$ where s', r' are the conjugate exponent of s, r respectively.

Because theorem 2.3 gives that a strictly pseudo convex D in Ω Stein is (s', r') regular if $r' \leq 2$, we have by lemma 2.9 with the weight $\eta = \eta_\epsilon$ and choosing $\Omega_1 = D$, a current u such that

$$\forall \alpha \in L_{(n-p, n-q+1)}^{s'}(\Omega, \eta), \quad \langle u, \alpha \rangle = \mathcal{L}(\alpha)$$

and

$$\sup_{\alpha \in L^{s'}(\Omega, \eta), \|\alpha\|_{L^{s'}(\Omega, \eta)} \leq 1} |\langle u, \alpha \rangle| \leq C \|\omega\|_{L^r(\Omega)}.$$

By lemma 3.1, which will be proved in the appendix, this implies

$$\|u\|_{L^s(\Omega, \eta^{1-s})} \leq C \|\omega\|_{L^r(\Omega)}.$$

In particular $\|u\|_{L^s(\Omega)} \leq C \|\omega\|_{L^r(\Omega)}$ because with $\epsilon < 1$ and $s > 1$, we have $\eta^{1-s} \geq 1$.

So apply this to $\varphi \in \mathcal{D}_{(n-p, n-q)}(\Omega)$, a test form, and set $\alpha = \bar{\partial}\varphi$, then $\alpha \in L^{s'}(\Omega, \eta)$ and $\varphi \in L^{r'}(\Omega, \eta)$, and we get

$$\langle u, \alpha \rangle = \mathcal{L}(\alpha) = (-1)^{p+q-1} \langle \omega, \varphi \rangle.$$

But, because φ has compact support,

$$\langle \bar{\partial}u, \varphi \rangle = \langle u, \bar{\partial}\varphi \rangle = (-1)^{p+q-1} \langle \omega, \varphi \rangle$$

and this means precisely that $\bar{\partial}u = \omega$ in the distributions sense.

Now for $\epsilon > 0$ with $\eta_\epsilon(z) := \mathbb{1}_{\Omega_1}(z) + \epsilon \mathbb{1}_{\Omega \setminus \Omega_1}(z)$, let $u_\epsilon \in L^s(\Omega, \eta_\epsilon^{1-s})$ be the previous solution, then

$$\|u_\epsilon\|_{L^s(\Omega)}^s \leq \int_{\Omega} |u_\epsilon|^s \eta^{1-s} dm \leq C^s \|\omega\|_{L^r(\Omega)}^s.$$

Replacing η by its value we get

$$\int_{\Omega_1} |u_\epsilon|^s dm + \int_{\Omega \setminus \Omega_1} |u_\epsilon|^s \epsilon^{1-s} dm \leq C^s \|\omega\|_{L^r(\Omega)}^s \Rightarrow \int_{\Omega \setminus \Omega_1} |u_\epsilon|^s \epsilon^{1-s} dm \leq C^s \|\omega\|_{L^r(\Omega)}^s$$

hence

$$\int_{\Omega \setminus \Omega_1} |u_\epsilon|^s dm \leq C^s \epsilon^{s-1} \|\omega\|_{L^r(\Omega)}^s.$$

Because C and the norm of ω are independent of ϵ , we have that $\|u_\epsilon\|_{L^s(\Omega)}$ is uniformly bounded and $s > 1 \Rightarrow L^s(\Omega)$ is a dual, hence there is a sub-sequence $\{u_{\epsilon_k}\}_{k \in \mathbb{N}}$ of $\{u_\epsilon\}$ which converges weakly, when $\epsilon_k \rightarrow 0$, to a $(p, q-1)$ form u in $L^s(\Omega)$, still with $\|u\|_{L^s(\Omega)} \leq C \|\omega\|_{L^r(\Omega)}$.

To see that this form u is 0 *a.e.* on $\Omega \setminus D$ let us take a component $u_{I,J}$ of it ; it is the weak limit of the sequence of functions $\{u_{\epsilon_k, I, J}\}$ which means, with the notations $v := u_{I, J}$, $v_k := u_{\epsilon_k, I, J}$

$$\forall f \in L^{s'}(\Omega), \quad \int_{\Omega} v f dm = \lim_{k \rightarrow \infty} \int_{\Omega} v_k f dm.$$

As usual take $f := \frac{\bar{v}}{|v|} \mathbb{1}_E$ where $E := \{|v| > 0\} \cap (\Omega \setminus D)$ then we get

$$\int_{\Omega} v f dm = \int_E |v| dm = \lim_{k \rightarrow \infty} \int_{\Omega} v_k f dm = \lim_{k \rightarrow \infty} \int_E \frac{v_k \bar{v}}{|v|} dm.$$

Now we have by Hölder

$$\left| \int_E \frac{v_k \bar{v}}{|v|} dm \right| \leq \|v_k\|_{L^s(E)} \|\mathbb{1}_E\|_{L^{s'}(E)}.$$

But

$$\|v_k\|_{L^s(E)}^s \leq \int_{\Omega \setminus D} |u_{\epsilon_k}|^s dm \leq (\epsilon_k)^{s-1} \|\omega\|_{L^r(\Omega)}^s \rightarrow 0, \quad k \rightarrow \infty.$$

Hence

$$\left| \int_E |v| dm \right| \leq C^s \|\mathbb{1}_E\|_{L^{s'}(E)} (\epsilon_k)^{s-1} \|\omega\|_{L^r(\Omega)}^s \rightarrow 0$$

which implies $m(E) = 0$ because on E , $|v| > 0$.

This being true for all components of ω , we get that the form u is 0 *a.e.* on $\Omega \setminus D$ i.e. $\text{Supp } u \subset D$.

So we get

$$\forall \varphi \in \mathcal{D}_{n-p, n-q}(\Omega), \quad (-1)^{p+q-1} \langle \omega, \varphi \rangle = \langle u_{\epsilon}, \bar{\partial} \varphi \rangle \rightarrow \langle u, \bar{\partial} \varphi \rangle \Rightarrow \langle u, \bar{\partial} \varphi \rangle = (-1)^{p+q-1} \langle \omega, \varphi \rangle$$

hence $\bar{\partial} u = \omega$ as distributions. ■

Remark 2.11. *The condition of orthogonality to $\mathcal{H}_p(V)$ in the case $q = n$ is necessary : suppose there is a $(p, n-1)$ current u such that $\bar{\partial} u = \omega$ and u with compact support in an open set $V \subset \Omega$, then if $h \in \mathcal{H}_p(V)$, we have*

$$h \in \mathcal{H}_p(V), \quad \langle \omega, h \rangle = \langle \bar{\partial} u, h \rangle = (-1)^{n+p} \langle u, \bar{\partial} h \rangle = 0,$$

because, u being compactly supported, there is no boundary term and

$$\langle \bar{\partial} u, h \rangle = (-1)^{n+p} \langle u, \bar{\partial} h \rangle.$$

This condition was already seen by H. Skoda [13], Proposition 1, p. 292 and also for extension of CR functions, see [1] and the references therein.

Now we suppose that ω is a (p, q) current in $L^{r,c}(\Omega)$, $r \leq 2$ such that $\bar{\partial} \omega = 0$, we shall prove the same theorem as above.

Theorem 2.12. *Let Ω be a Stein manifold. Let ω be a (p, q) current in $L^{r,c}(\Omega)$, $r \leq 2$ such that $\bar{\partial} \omega = 0$ if $1 \leq q < n$ and $\forall h \in \mathcal{H}_p(\Omega)$, $\langle \omega, h \rangle = 0$ if $q = n$.*

Then there is a $(p, q-1)$ current u in $L^{s,c}(\Omega)$ such that $\bar{\partial} u = \omega$ with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2(n+1)}$.

Moreover if the support of ω is contained in a s.p.c. $D \subset \Omega$, then the support of u is also contained in D .

Proof.

We shall use a form of the "raising steps method" we introduced and used in [3].

Let D be a s.p.c. domain in Ω containing $\text{Supp } \omega$ and $\chi_1 \in \mathcal{C}_c^\infty(D)$ such that $\chi_1 = 1$ on $\text{Supp } \omega$. By

theorem 2.3 there is a $u_1 \in L_{(p, q-1)}^{s_1}(\Omega)$ such that $\bar{\partial} u_1 = \omega$, with $\frac{1}{s_1} = \frac{1}{r} - \frac{1}{2(n+1)}$.

Let $v_1 := \chi_1 u_1$ then

$$\bar{\partial} v_1 = \bar{\partial} \chi_1 \wedge u_1 + \chi_1 \omega = \bar{\partial} \chi_1 \wedge u_1 + \omega,$$

because $\chi_1 = 1$ on $\text{Supp } \omega$. So set $\omega_1 := \bar{\partial} \chi_1 \wedge u_1$ then $\omega_1 \in L_{(p, q)}^{s_1}(\Omega)$ and we have

$$\bar{\partial} v_1 = \omega + \omega_1 \Rightarrow \bar{\partial} \omega_1 = \bar{\partial}^2 v_1 - \bar{\partial} \omega = 0.$$

If $q = n$, then

$$\forall h \in \mathcal{H}_p(\Omega), \quad \langle \bar{\partial}(\chi_1 u_1), h \rangle = 0,$$

because $\chi_1 u_1$ has compact support, so

$$\forall h \in \mathcal{H}_p(\Omega), 0 = \langle \bar{\partial}v_1, h \rangle = \langle \omega, h \rangle + \langle \omega_1, h \rangle = \langle \omega_1, h \rangle$$

because $\langle \omega, h \rangle = 0$ hence we have $\langle \omega_1, h \rangle = 0$. Of course we have also $\text{Supp } \omega_1 \subset D$.

Again with theorem 2.3 there is a $u_2 \in L_{(p,q-1)}^{s_2}(\Omega)$ such that $\bar{\partial}u_2 = \omega_1$, with $\frac{1}{s_2} = \frac{1}{s_1} - \frac{1}{2(n+1)} = \frac{1}{r} - \frac{2}{2(n+1)}$. Take $\chi_2 \in C_c^\infty(D)$ such that $\chi_2 = 1$ on $\text{Supp } \omega_1$ and set $v_2 := \chi_2 u_2$.

We have :

$$\bar{\partial}v_2 = \chi_2 \omega_1 + \bar{\partial}\chi_2 \wedge u_2 = \omega_1 + \omega_2, \text{ with } \omega_2 := \bar{\partial}\chi_2 \wedge u_2 \text{ and } \omega_2 \in L_{(p,q)}^{s_2}(\Omega).$$

Then we get

$$\bar{\partial}(v_1 - v_2) = \omega + \omega_1 - (\omega_1 + \omega_2) = \omega - \omega_2$$

hence

$$\bar{\partial}\omega_2 = \bar{\partial}\omega - \bar{\partial}^2(v_1 - v_2) = 0.$$

If $q = n$, we have

$$\forall h \in \mathcal{H}_p(\Omega), 0 = \langle \bar{\partial}v_2, h \rangle = \langle \omega_1, h \rangle + \langle \omega_2, h \rangle = \langle \omega_2, h \rangle,$$

because $\langle \omega_1, h \rangle = 0$, hence $\langle \omega_2, h \rangle = 0$. And again $\text{Supp } \omega_2 \subset D$.

Now if $\frac{1}{s_2} = \frac{1}{r} - \frac{2}{2(n+1)} < \frac{1}{2} \Rightarrow s_2 > 2$ then we can apply theorem 2.10 to have a current $u_3 \in L_{(p,q-1)}^{s_3,c}(\Omega)$ with $\bar{\partial}u_3 = \omega_2$ and $\text{Supp } u_3 \subset D$. Then we set

$$U := v_1 - v_2 + u_3 ;$$

the form U has compact support in D , $U \in L_{(p,q-1)}^{s_1}(\Omega)$ and

$$\bar{\partial}U = \omega + \omega_1 - (\omega_1 + \omega_2) + \omega_2 = \omega.$$

If $s_2 \leq 2$ we continue the process by induction and after a finite number, say N , of steps we have

$$s_N = \frac{1}{r} - \frac{N}{2(n+1)} < \frac{1}{2} \Rightarrow s_N > 2$$

and by induction

$$\bar{\partial}\omega_{N-1} = 0 \text{ and if } q = n, \forall h \in \mathcal{H}_p(\Omega), \langle \omega_{N-1}, h \rangle = 0 ; \text{Supp } \omega_{N-1} \subset D.$$

So by theorem 2.10 we have a current $u_N \in L_{(p,q-1)}^{s_N,c}(\Omega)$ with $\bar{\partial}u_N = \omega_{N-1}$ and supported in D .

Setting

$$U := \sum_{j=1}^{N-1} (-1)^{j+1} v_j + (-1)^{N+1} u_N,$$

we have $U \in L_{(p,q-1)}^{s_1}(\Omega)$, U has compact support in $D \subset \Omega$ and $\bar{\partial}U = \omega$. ■

2.3. Finer control of the support.

Here we shall get a better control on the support of a solution.

Theorem 2.13. *Let Ω be a (s', r') regular domain in a Stein manifold X . Then for any (p, q) form ω in $L^{r,c}(\Omega, dm)$, $\bar{\partial}\omega = 0$, if $q < n$, and $\omega \perp \mathcal{H}_p(\Omega)$ if $q = n$, with support in $\Omega \setminus C$ where C is a (s, s) regular domain, there is a $u \in L^{s,c}(\Omega, dm)$ such that $\bar{\partial}u = \omega$ and with support in $\Omega \setminus \bar{U}$, where U is any open set relatively compact in C , provided that $q \geq 2$.*

Proof.

Let ω be a (p, q) form with compact support in $\Omega \setminus C$ then $\exists v \in L_{p,q-1}^s(\Omega)$, $\bar{\partial}v = \omega$, with compact support in Ω , by theorem 2.10.

Because ω has its support outside C we have $\omega = 0$ in C ; this means that $\bar{\partial}v = 0$ in C . Because C is (s, s) regular and $q \geq 2$, we have

$$\exists C' \subset C, C' \supset \bar{U}, \exists h \in L_{p,q-2}^s(C') \text{ s.t. } \bar{\partial}h = v \text{ in } C'.$$

Let χ be a smooth function such that $\chi = 1$ in U and $\chi = 0$ near $\partial C'$; then set

$$u := v - \bar{\partial}(\chi h).$$

We have that $u = v - \chi \bar{\partial}h - \bar{\partial}\chi \wedge h = v - \chi v - \bar{\partial}\chi \wedge h$ hence u is in L^s ; moreover $u = 0$ in \bar{U} because $\chi = 1$ in U hence $\bar{\partial}\chi = 0$ there. Finally $\bar{\partial}u = \bar{\partial}v - \bar{\partial}^2(\chi h) = \omega$ and we are done. ■

If Ω and C are, for instance, pseudo-convex in \mathbb{C}^n then $\Omega \setminus C$ is no longer pseudo-convex in general, so this theorem improves actually the control of the support.

Remark 2.14. *If the correcting function h is given by kernels, then it is linear in ω ; if the primitive solution v is also linear in ω , then the solution u is linear too. This is the case in \mathbb{C}^n if we can use the solution given in [4].*

This theorem cannot be true for $q = 1$ as shown by the following example :

Take a holomorphic function φ in the open unit ball $B(0, 1)$ in \mathbb{C}^n such that it extends to no open ball of center 0 and radius > 1 . For instance $\varphi(z) := \exp\left(-\frac{z_1 + 1}{z_1 - 1}\right)$. Take $R < 1$, then φ is $\mathcal{C}^\infty(\bar{B}(0, R))$ hence by a theorem of Whitney φ extends \mathcal{C}^∞ to \mathbb{C}^n ; call φ_R this extension. Let $\chi \in \mathcal{C}_c^\infty(B(0, 2))$ such that $\chi = 1$ in the ball $B(0, 3/2)$ and consider the $(0, 1)$ form $\omega := \bar{\partial}(\chi \varphi_R)$. Then $\text{Supp } \omega \subset B(0, 2) \setminus B(0, R)$, ω is $\bar{\partial}$ closed and is \mathcal{C}^∞ hence in $L_{0,1}^r(B(0, 2))$. Moreover $B(0, R)$ is strictly pseudo-convex hence (r', r') regular, but there is no function u such that $\bar{\partial}u = \omega$ and u zero near the origin because any solution u will be C.R. on $\partial B(0, R)$ and by Hartog's phenomenon will extend holomorphically to $B(0, R)$, hence cannot be identically null near 0.

Nevertheless in the case $q = 1$, we have :

Theorem 2.15. *Let Ω be a (r', r') regular domain in a Stein manifold X .*

Then for any $(p, 1)$ form ω in $L^{r,c}(\Omega, dm)$, $\bar{\partial}\omega = 0$, with support in $\Omega_1 \setminus C$ where Ω_1 is a (r', r') regular domain in Ω and C is a domain such that $C \subset \Omega$ and $C \setminus \Omega_1 \neq \emptyset$; there is a $u \in L^{r,c}(\Omega, dm)$ such that $\bar{\partial}u = \omega$ and with support in $\Omega \setminus C$.

Proof.

We have that ω is solvable in L^r by a $u \in L_{p,0}^r(\Omega_1)$ with compact support in Ω_1 , by theorem 2.10. Then $\bar{\partial}u = 0$ in C hence u is locally holomorphic in C . Because $C \setminus \Omega_1 \neq \emptyset$, there is an open set in $C \setminus \Omega_1 \subset \Omega \setminus \Omega_1$ read in a coordinates patch, where u is 0 and holomorphic, hence u is identically 0 in C , C being connected. ■

Remark 2.16. *If there is a $u \in L_{p,0}^{r,c}(\Omega_1)$ which is 0 in C then we have*

$$\forall h \in L_{n-p,n-1}^{r'}(C) :: \text{Supp } \bar{\partial}h \subset C, \quad 0 = \langle u, \bar{\partial}h \rangle = \langle \omega, h \rangle,$$

hence the necessary condition :

$$\forall h \in L_{n-p,n-1}^{r'}(C) :: \text{Supp } \bar{\partial}h \subset C, \quad \langle \omega, h \rangle = 0.$$

3. APPENDIX.

3.1. Duality.

Let \mathcal{I}_p be the set of multi-indices of length p in $(1, \dots, n)$. We shall use the measure defined on $\Gamma := \Omega \times \mathcal{I}_p \times \mathcal{I}_q$ the following way :

$$d\mu(z, k, l) = d\mu_{\eta,p,q}(z, k, l) := \eta(z) dm(z) \otimes \sum_{|I|=p, |J|=q} \delta_I(k) \otimes \delta_J(l),$$

where $\delta_I(k) = 1$ if the multi-index k is equal to I and $\delta_I(k) = 0$ if not.

This means, if $f(z, I, J)$ is a function defined on Γ , that

$$\int f(z, k, l) d\mu_{\eta, p, q}(z, k, l) := \sum_{|I|=p, |J|=q} \int_{\Omega} f(z, I, J) \eta(z) dm(z).$$

If I is a multi-index of length p , let I^c be the unique multi-index, ordered increasingly, such that $I \cup I^c = (1, 2, \dots, n)$; then I^c is of length $n - p$.

To $t = \sum_{|I|=p, |J|=q} t_{I, J}(z) dz^I \wedge d\bar{z}^J$ a (p, q) form, we associate the function on Γ :

$$T(z, I, J) := (-1)^{s(I, J)} t_{I, J}(z),$$

where

$$s(I, J) = 0 \text{ if } dz^I \wedge d\bar{z}^J \wedge dz^{I^c} \wedge d\bar{z}^{J^c} = dz_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_1 \wedge \dots \wedge d\bar{z}_n \text{ as a } (n, n) \text{ form}$$

and

$$s(I, J) = 1 \text{ if not.}$$

If $\varphi = \sum_{|I|=p, |J|=q} \varphi_{I^c, J^c}(z) dz^I \wedge d\bar{z}^J$ is of complementary bi-degree, associate in the same manner :

$$\Phi^*(z, I, J) := \varphi_{I^c, J^c}(z). \text{ This is still a function on } \Gamma.$$

Now we have, for $1 < r < \infty$, if $T(z, I, J)$ is a function in Ω with $L^r(\Omega)$ coefficients and with $\mu = \mu_{\eta, p, q}$,

$$\|T\|_{L^r(d\mu)}^r := \int |T(z, I, J)|^r d\mu_{\eta, p, q}(z, I, J) = \sum_{|I|=p, |J|=q} \|T(\cdot, I, J)\|_{L^r(\Omega, \eta)}^r.$$

For $1 \leq r < \infty$ the dual of $L^r(\mu)$ is $L^{r'}(\mu)$ where r' is the conjugate of r , $\frac{1}{r} + \frac{1}{r'} = 1$, and the norm is defined analogously with r' replacing r .

We also know that, for p, q fixed,

$$(3.1) \quad \|T\|_{L^r(\mu)} = \sup_{\Phi \in L^{r'}(\mu)} \frac{|\int T \Phi d\mu|}{\|\Phi\|_{L^{r'}(\mu)}}.$$

For a (p, q) form $t = \sum_{|J|=p, |K|=q} t_{J, K} dz^J \wedge d\bar{z}^K$, and a weight $\eta > 0$ we define its norm by :

$$(3.2) \quad \|t\|_{L^r(\Omega, \eta)}^r := \sum_{|J|=p, |K|=q} \|t_{J, K}\|_{L^r(\Omega, \eta)}^r = \|T\|_{L^r(\mu)}^r.$$

Now we can state

Lemma 3.1. *Let $\eta > 0$ be a weight. If u is a (p, q) current defined on $(n - p, n - q)$ forms in $L^{r'}(\Omega, \eta)$ and such that*

$$\forall \alpha \in L^r_{(n-p, n-q)}(\Omega, \eta), |\langle u, \alpha \rangle| \leq C \|\alpha\|_{L^{r'}(\Omega, \eta)},$$

then $\|u\|_{L^r(\Omega, \eta^{1-r})} \leq C$.

Proof.

Let us take the measure $\mu = \mu_{\eta, p, q}$ as above. Let Φ^* be the function on Γ associated to α and T the one associated to u . We have, by definition of the measure μ applied to the function

$$\begin{aligned} f(z, I, J) &:= T(z, I, J) \eta^{-1} \Phi^*(z, I, J), \\ \int T \eta^{-1} \Phi^* d\mu &= \int f(z, k, l) d\mu(z, k, l) := \sum_{|I|=p, |J|=q} \int_{\Omega} f(z, I, J) \eta(z) dm(z) = \\ &= \sum_{|I|=p, |J|=q} \int_{\Omega} T(z, I, J) \eta^{-1}(z) \Phi^*(z, I, J) \eta(z) dm(z) = \langle u, \alpha \rangle, \end{aligned}$$

by definition of T and Φ^* .

Hence we have, by (3.1)

$$\|T\eta^{-1}\|_{L^r(\mu)} = \sup_{\Psi \in L^{r'}(\mu)} \frac{|\langle u, \alpha \rangle|}{\|\Psi\|_{L^{r'}(\mu)}}.$$

But $\|T\eta^{-1}\|_{L^r(\mu)} = \|u\eta^{-1}\|_{L^r(\Omega, \eta)}$ by (3.2), and

$$\|f\eta^{-1}\|_{L^r(\Omega, \eta)}^r = \int_{\Omega} |f\eta^{-1}|^r \eta dm = \int_{\Omega} |f|^r \eta^{1-r} dm = \|f\|_{L^r(\Omega, \eta^{1-r})}^r,$$

so we get

$$\|u\|_{L^r(\Omega, \eta^{1-r})} = \sup_{\Psi \in L^{r'}(\mu)} \frac{|\langle u, \alpha \rangle|}{\|\Psi\|_{L^{r'}(\mu)}},$$

which implies the lemma because, still by (3.1), we can take $\Psi = \Phi^*$ and $\|\Psi\|_{L^{r'}(\mu)} = \|\alpha\|_{L^{r'}(\Omega, \eta)}$.

■

It may seem strange that we have such an estimate when the dual of $L^{r'}(\Omega, \eta)$ is $L^r(\Omega, \eta)$, but the reason is, of course, that in the duality forms-currents there is no weights.

3.2. Approximation.

We shall copy Hörmander in his proof of the approximation of $\bar{\partial}$ closed $(0, 0)$ forms to get the corresponding results for $\bar{\partial}$ closed $(p, 0)$ forms. Denote $\mathcal{H}_p(V)$ the set of $(p, 0)$ $\bar{\partial}$ closed forms in the open set V .

Theorem 3.2. *Let Ω be a complex manifold and φ a strictly plurisubharmonic function in Ω such that*

$$K_c := \{z \in \Omega :: \varphi(z) \leq c\} \Subset \Omega$$

for every real number c . Every $(p, 0)$ form which is $\bar{\partial}$ closed in a neighbourhood of K_0 can then be approximated uniformly on K_0 by forms in $\mathcal{H}_p(\Omega)$.

Proof.

By the Hahn Banach theorem it suffices to prove that if $v \in L^2_{(p,0)}(K_0)$ and if

$$(3.3) \quad \langle u, v \rangle_{L^2_{(p,0)}(\Omega)} = 0$$

for all $u \in \mathcal{H}_p(\Omega)$ then (3.3) is true for every u in $\mathcal{H}_p(V)$ where V is a neighbourhood of K_0 eventually depending on u .

Set $\psi := \chi(\varphi)$ and extend the definition of v by setting $v = 0$ outside K_0 . Then (3.3) implies that ve^ψ is orthogonal to the null space N_T of the operator $T = \bar{\partial}$ operating from $L^2_{(p,0)}(\Omega, e^{-\psi})$ to $L^2_{(p,1)}(\Omega, e^{-\psi})$ because

$$u \in \mathcal{H}_p(\Omega) \Rightarrow \bar{\partial}u = 0$$

and then $u \in C^\infty(\Omega)$ because in each coordinates path u has its coefficients holomorphic, hence is in the domain of T .

So we have, by inequality (5.2.14) on page 125 of [8], that, because $Sf = 0$,

$$\exists f \in L^2_{p,1}(\Omega, e^{-\psi}) :: T^*f = ve^\psi \text{ and } \|f\|_\psi \leq \|e^\psi\|_\psi.$$

Using a partition of unity $\{\gamma_l\}_{l \in \mathbb{N}}$ we can suppose that $f_l := \gamma_l f$ has its support in a coordinates path with a C^∞ orthonormal basis $\omega_1, \dots, \omega_n$ for forms of type $(1, 0)$.

So by (5.2.3) p. 122 of [8], we have, with $\delta_j w := e^\psi \frac{\partial(we^{-\psi})}{\partial \omega_j}$,

$$e^\psi v = T^* f = \sum_l T^* f_l,$$

so, setting $v_l := e^{-\psi} T^* f_l$, we get

$$v = \sum_{l \in \mathbb{N}} v_l.$$

Now in the coordinates path U_l we have

$$T^* f_l = (-1)^{p-1} \sum_{I, |I|=p} \sum_{j=1}^n \delta_j f_{l,j} \omega^I + \dots = Bf + \dots$$

where the dots indicate terms where no $f_{l,j}$ is differentiated and which do not involve ψ , hence

$$(3.4) \quad |T^* f_l - Bf_l| \leq C |f_l|.$$

Now on we erase the subscript l on f to clarify the notations.

Setting $g_{l,j} := e^{-\psi} \delta_j f_{l,j}$, we get

$$v_l = (-1)^{p-1} \sum_{I, |I|=p} \sum_{j=1}^n g_{l,j} \omega^I + \dots$$

and

$$\int |T^* f|^2 e^{-\psi} \geq \int |Bf|^2 e^{-\psi} - C \int |f|^2 e^{-\psi}$$

So

$$\int \sum_{I, |I|=p, j=1, \dots, n} |g_{l,j}|^2 e^\psi \leq \int |v_l|^2 e^\psi + C \int |f_l|^2 e^{-\psi}.$$

So back to Ω and summing over l , we get

$$\int |g|^2 e^\psi \leq \int |v|^2 e^\psi + C \int |f|^2 e^{-\psi}.$$

But $\int |f|^2 e^{-\psi} \leq \int |ve^\psi|^2 e^{-\psi} = \int |v|^2 e^\psi$, hence

$$\int |g|^2 e^\psi \leq (C+1) \int |v|^2 e^\psi.$$

Now recall that $v = 0$ outside K_0 then we can choose a sequence χ_ν of convex functions so that $\chi_\nu(t)$ is independent of ν if $t \leq 0$ but $\chi_\nu \nearrow \infty$ when $\nu \rightarrow \infty$ if $t > 0$ and choose $\psi = \chi_\nu(\varphi)$ where $K_c := \{z \in \Omega :: \varphi(z) \leq c\} \Subset \Omega$.

Then we have a sequence g^ν such that

$$\int |g^\nu|^2 e^{\chi_\nu(\varphi)} \leq (C+1) \int |v|^2 e^{\chi_\nu(\varphi)} = (C+1) \int |v|^2 e^{\chi_1(\varphi)} = Cste,$$

because the right hand side is an integration only on K_0 and there $\varphi \leq 0 \Rightarrow \chi_\nu(\varphi) = \chi_1(\varphi)$.

So

$$(3.5) \quad \int |g^\nu|^2 e^{\chi_\nu(\varphi)} dV \leq Cste,$$

and because $\chi_\nu \geq \chi_1$, we have a subsequence of the sequence g^ν which converges weakly in the Hilbert space $L^2_{(p,1)}(\Omega, e^{\chi_1(\varphi)})$ to a limit g .

By (3.5) we obtain that $g = 0$ where $\varphi > 0$ and $T^* f = e^{\chi_1(\varphi)} v$, which means

$$T^*f - g = 0 \text{ outside } K_0$$

and, using (3.4),

$$|T^*f - g| \leq C |f| \mathbb{1}_{K_0}.$$

We get by Cauchy-Schwarz inequality

$$\forall u \in \mathcal{C}_{(p,0)}^\infty(\Omega), \left| \int_{\Omega} \langle T^*f, u \rangle e^{-\psi} - \int_{\Omega} \langle g, u \rangle e^{-\psi} \right| \leq C \|f\|_{K_0} \|u\|_{K_0}.$$

Suppose now that $u \in \mathcal{C}_{(p,0)}^\infty$ is such that $\bar{\partial}u = 0$ in a neighbourhood V of K_0 and take a \mathcal{C}^∞ function $\gamma :: \gamma = 0$ on K_0 and $\gamma = 1$ on V^c . Then $\gamma u \in \mathcal{C}_{(p,0)}^\infty(\Omega)$ and $\gamma u \equiv 0$ on K_0 , so

$$\left| \int_{\Omega} \langle T^*f, \gamma u \rangle e^{-\psi} - \int_{\Omega} \langle g, \gamma u \rangle e^{-\psi} \right| \leq C \|f\|_{K_0} \|u\|_{K_0} = 0,$$

and, because g is supported by K_0 ,

$$\int_{\Omega} \langle g, \gamma u \rangle e^{-\psi} = 0,$$

hence we get

$$0 = \int_{\Omega} \langle T^*f, \gamma u \rangle e^{-\psi} = \int_{\Omega} \langle f, \bar{\partial}(\gamma u) \rangle e^{-\psi}.$$

But $\bar{\partial}(\gamma u) = \bar{\partial}\gamma \wedge u + \gamma \bar{\partial}u = \bar{\partial}\gamma \wedge u$, hence, because $\bar{\partial}\gamma = 0$ in K_0 ,

$$(3.6) \quad \int_{\Omega} \langle f, \bar{\partial}(\gamma u) \rangle e^{-\psi} = \int_{\Omega \setminus K_0} \langle f, \bar{\partial}\gamma \wedge u \rangle e^{-\psi} = 0.$$

Now set $\tilde{u} := (1 - \gamma)u$; we have that \tilde{u} is an extension of u outside K_0 , i.e. $\forall z \in K_0, \tilde{u}(z) = u(z)$ and we have

$$\int_{K_0} \langle v, u \rangle = \int_{\Omega} \langle v, \tilde{u} \rangle = (-1)^{p-1} \int_{\Omega} \langle T^*f, \tilde{u} \rangle e^{-\psi} = (-1)^{p-1} \int_{\Omega} \langle f, \bar{\partial}\tilde{u} \rangle e^{-\psi}$$

because $v e^\psi = T^*f$, hence

$$\int_{K_0} \langle v, u \rangle = (-1)^{p-1} \int_{K_0} \langle f, \bar{\partial}\tilde{u} \rangle e^{-\psi} + (-1)^{p-1} \int_{\Omega \setminus K_0} \langle f, \bar{\partial}\tilde{u} \rangle e^{-\psi}.$$

But $\bar{\partial}\tilde{u} = \bar{\partial}(1 - \gamma) \wedge u = -\bar{\partial}\gamma \wedge u$ hence using (3.6) we get

$$\int_{\Omega \setminus K_0} \langle f, \bar{\partial}\tilde{u} \rangle e^{-\psi} = 0$$

hence finally

$$\int_{K_0} \langle v, u \rangle = (-1)^{p-1} \int_{K_0} \langle f, \bar{\partial}\tilde{u} \rangle e^{-\psi} = 0$$

because again $\bar{\partial}\tilde{u} = \bar{\partial}(1 - \gamma) \wedge u = -\bar{\partial}\gamma \wedge u = 0$ on K_0 .

So we prove that for any $u \in \mathcal{C}_{(p,0)}^\infty(\Omega) :: \bar{\partial}u = 0$ in a neighbourhood of K_0 , we have that

$$\int_{K_0} \langle v, u \rangle = 0 \text{ which proves the theorem. } \blacksquare$$

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