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AN ANDREOTTI-GRAUERT THEOREM WITH L^r ESTIMATES.

ERIC AMAR

RÉSUMÉ. Par un théorème d'Andréotti-Grauert, si ω est un (p, q) courant, $q < n$, dans une variété de Stein, $\bar{\partial}$ fermé et à support compact, il existe une solution u à $\bar{\partial}u = \omega$ également à support compact. Le résultat principal de ce travail est de montrer que si, de plus, $\omega \in L^r(\Omega, dm)$, où m est une mesure de Lebesgue convenable sur Ω , alors on a une solution u à $\bar{\partial}u = \omega$ à support compact et dans $L^r(\Omega, dm)$. On le montre grâce à des estimées dans des espaces $L^r(\Omega)$ à poids.

Dans une deuxième partie on montre directement des estimations globales $L^{r,loc}(\Omega) - L^{r,loc}(\Omega)$ pour des solutions de l'équation $\bar{\partial}$ dans les variétés de Stein. Cela redonne, encore par dualité, une autre preuve du résultat principal.

ABSTRACT. By a theorem of Andreotti and Grauert if ω is a (p, q) current, $q < n$, in a Stein manifold, $\bar{\partial}$ closed and with compact support, then there is a solution u to $\bar{\partial}u = \omega$ still with compact support. The main result of this work is to show that if moreover $\omega \in L^r(dm)$, where m is a suitable Lebesgue measure on the Stein manifold, then we have a solution u with compact support and in $L^r(dm)$. We prove it by estimates in L^r spaces with weights.

In a second part, we prove directly that there are global $L^{r,loc}(dm) - L^{r,loc}(dm)$ solutions for the $\bar{\partial}$ equation on Stein manifolds. This gives, again by duality, another proof for the main result.

1. INTRODUCTION.

Let ω be a $\bar{\partial}$ closed (p, q) form in \mathbb{C}^n with compact support $K := \text{Supp } \omega$ and such that $\omega \in L^r(\mathbb{C}^n)$. Setting K in a ball $\mathbb{B} := B(0, R)$ with R big enough, we know, by a theorem of Ovreid [10], that we have a $(p, q - 1)$ form $u \in L^r(\mathbb{B})$ such that $\bar{\partial}u = \omega$. On the other hand we also know, at least when $q < n$, that there is a current v with compact support such that $\bar{\partial}v = \omega$, by a theorem of Andreotti-Grauert [3].

So a natural question is : may we have a solution u of $\bar{\partial}u = \omega$ with compact support and in $L^r(\mathbb{C}^n)$?

We already answered this question by the affirmative in a join work with S. Mongodi [2] explicitly and linearly by the "method of coronas". This method asks for extra L^r conditions on derivatives of coefficients of ω , when $q < n$; we shall note the set of ω verifying these conditions $\mathcal{W}_q^r(\Omega)$ as in [2].

The aim of this work is to extend this result to Stein manifolds and get rid of the extra L^r conditions $\mathcal{W}_q^r(\Omega)$. For it we use a completely different approach inspired by the Serre duality [12]. Because Hahn Banach theorem is used, these results are no longer so explicit and constructive as in [2].

On the other hand the control of the support is better : if the support of the data ω is contained in $\Omega \setminus C$ where Ω is a weakly r' regular domain and C is a weakly r regular domain, then the support of the solution u is contained in $\Omega \setminus C'$, where C' is any domain relatively compact in C , provided that $q \geq 2$. One may observe that $\Omega \setminus C$ is not Stein in general even if Ω is.

There is also a result of this kind for $q = 1$, see section 3.5.

The definition of weakly r regular domain will be given later on, but pseudo convex domains in Stein manifold or in \mathbb{C}^n are such domains.

The idea is to solve $\bar{\partial}u = \omega$ in a space $L^r(\Omega)$ with a weight η "big" outside of the support of ω ; this way we shall have a "small" solution u outside of the support of ω . Then, using a sequence of such weights going to infinity outside of the support of ω , we shall have a u zero outside of the support of ω .

In a second part of this work we prove that if $\omega \in L_{(p,q)}^{r,loc}(\Omega, dm)$, $\bar{\partial}\omega = 0$, there is a form $u \in L_{(p,q-1)}^{r,loc}(\Omega, dm)$ such that $\bar{\partial}u = \omega$, provided that Ω is a weakly r' regular domain.

Hence we have a *global* estimated solution. Of course for $r = 2$ this was already proved by Hörmander ([7] theorem 5.2.4, p 125).

This last result was suggested to me by a referee for our paper with S. Mongodi, who gave a sketch of a nice geometrical proof of the result on forms with compact support, using $L^{r,loc}$ estimates in *germs*, after the version two of this paper was already posted on HAL and ArXiv.

In order to get another proof for forms with compact support, we prove, for μ a positive measure on Ω and $1 \leq r < \infty$, that the dual of $L^{r,c}(\Omega, \mu)$ is $L^{r',loc}(\Omega, \mu)$, with r' the conjugate exponent of r , and that the dual of $L^{r,loc}(\Omega, \mu)$ is $L^{r',c}(\Omega, \mu)$. We prove also a theorem giving global solution to $\bar{\partial}$ equation for forms with coefficients in a dual space A' , provided that there are solutions for forms with coefficients in A . This is again a avatar of the Serre duality theorem.

Then, because $L^{r',c}(\Omega)$ is the dual of $L^{r,loc}(\Omega)$ we get another proof of the main theorem of this work.

I am indebted to G. Tomassini who started my interest in this subject by e-mails on precisely this kind of questions and also to S. Mongodi for a lot of discussions by mails on the subject during the preparation of our join paper [2].

Moreover I want to thank C. Laurent who makes me realise that a weaker condition than the r regularity was sufficient to get these results, with exactly the same proofs.

2. DUALITY.

We shall study a duality between currents inspired by the Serre duality [12].

Let $\Omega \subset \mathbb{C}^n$ be an open set in \mathbb{C}^n and t a (p, q) current with compact support in Ω , noted $t \in \mathcal{D}'_{c, (p,q)}(\Omega)$.

Let also $\varphi \in \mathcal{C}_{(n-p, n-q)}^\infty(\Omega)$ a $(n-p, n-q)$ form in $\mathcal{C}^\infty(\Omega)$. We have that $t \wedge \varphi$ is a (n, n) current with compact support in Ω .

As usual we use the following notation for the pairing

$$\langle t, \varphi \rangle := t(\varphi),$$

where $t(\varphi)$ is the action, as a current, of t on the smooth form φ of complementary bi-degree.

The point in the next lemma is that the test functions need not to be compactly supported.

Lemma 2.1. *Let Ω be an open set in \mathbb{C}^n , $\omega \in \mathcal{D}'_{c, (p,q)}(\Omega)$ a (p, q) current with compact support in Ω and $u \in \mathcal{D}'_{c, (p,q-1)}(\Omega)$. Then we have $\bar{\partial}u = \omega$ iff*

$$(*) \quad \forall \varphi \in \mathcal{C}_{(n-p, n-q)}^\infty(\Omega), \quad \langle \omega, \varphi \rangle = (-1)^{p+q-1} \langle u, \bar{\partial}\varphi \rangle.$$

Proof.

If $\bar{\partial}u = \omega$, let $\chi \in \mathcal{C}_c^\infty(\Omega)$ such that $\chi \equiv 1$ on the support of u , hence on the support of ω . Then

$$\forall \varphi \in \mathcal{C}_{(n-p, n-q)}^\infty(\Omega), \quad \langle \omega, \varphi \rangle = \langle \omega, \chi\varphi \rangle = \langle \bar{\partial}u, \chi\varphi \rangle = (-1)^{p+q-1} \langle u, \bar{\partial}(\chi\varphi) \rangle$$

by definition of the action of derivatives on currents. Hence

$$\langle \omega, \varphi \rangle = (-1)^{p+q-1} \langle u, \bar{\partial} \chi \wedge \varphi \rangle + (-1)^{p+q-1} \langle u, \chi \bar{\partial} \varphi \rangle = (-1)^{p+q-1} \langle u, \chi \bar{\partial} \varphi \rangle$$

because $\bar{\partial} \chi = 0$ on the support of u ; so

$$\langle \omega, \varphi \rangle = (-1)^{p+q-1} \langle u, \chi \bar{\partial} \varphi \rangle = (-1)^{p+q-1} \langle u, \bar{\partial} \varphi \rangle$$

because $\chi = 1$ on the support of u .

Conversely if we have (*) we take $\chi \in \mathcal{C}_c^\infty(\Omega)$ such that $\chi \equiv 1$ on union of the support of u and the support of ω then

$$\forall \varphi \in \mathcal{C}_{(n-p, n-q)}^\infty(\Omega), \quad \langle \omega, \varphi \rangle = \langle \omega, \chi \varphi \rangle$$

because $\chi = 1$ on the support of ω , and by (*)

$$\langle \omega, \chi \varphi \rangle = (-1)^{p+q-1} \langle u, \bar{\partial}(\chi \varphi) \rangle = \langle \bar{\partial} u, \chi \varphi \rangle = \langle \bar{\partial} u, \varphi \rangle$$

because $\chi = 1$ on the support of u hence on the support of $\bar{\partial} u$.

So

$$\forall \varphi \in \mathcal{C}_{(n-p, n-q)}^\infty(\Omega), \quad \langle \omega, \varphi \rangle = \langle \bar{\partial} u, \varphi \rangle,$$

which means that $\bar{\partial} u = \omega$ as a current. ■

3. SOLUTION OF THE $\bar{\partial}$ EQUATION WITH COMPACT SUPPORT.

3.1. Weighted L^r spaces.

Let Ω be a domain in \mathbb{C}^n . We shall need the following notations.

We note dm the Lebesgue measure on \mathbb{C}^n and we shall define $L^r(\Omega, \eta)$ to be the set of functions f defined on Ω such that

$$\|f\|_{L^r(\Omega, \eta)}^r := \int_{\Omega} |f(z)|^r \eta(z) dm(z) < \infty,$$

with a weight $\eta(z) > 0$. As usual we set $L^r(\Omega)$ for $L^r(\Omega, 1)$.

Let \mathcal{I}_p be the set of multi-indices of length p in $(1, \dots, n)$. We shall use the measure defined on $\Gamma := \Omega \times \mathcal{I}_p \times \mathcal{I}_q$ the following way :

$$d\mu(z, k, l) = d\mu_{\eta, p, q}(z, k, l) := \eta(z) dm(z) \otimes \sum_{|I|=p, |J|=q} \delta_I(k) \otimes \delta_J(l),$$

where $\delta_I(k) = 1$ if the multi-index k is equal to I and $\delta_I(k) = 0$ if not.

This means, if $f(z, I, J)$ is a function defined on Γ , that

$$\int f(z, k, l) d\mu_{\eta, p, q}(z, k, l) := \sum_{|I|=p, |J|=q} \int_{\Omega} f(z, I, J) \eta(z) dm(z).$$

If I is a multi-index of length p , let I^c be the unique multi-index, ordered increasingly, such that $I \cup I^c = (1, 2, \dots, n)$; then I^c is of length $n - p$.

To $t = \sum_{|I|=p, |J|=q} t_{I, J}(z) dz^I \wedge d\bar{z}^J$ a (p, q) form, we associate the function on Γ :

$$T(z, I, J) := (-1)^{s(I, J)} t_{I, J}(z),$$

where

$$s(I, J) = 0 \text{ if } dz^I \wedge d\bar{z}^J \wedge dz^{I^c} \wedge d\bar{z}^{J^c} = dz_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_1 \wedge \dots \wedge d\bar{z}_n \text{ as a } (n, n) \text{ form}$$

and

$$s(I, J) = 1 \text{ if not.}$$

If $\varphi = \sum_{|I|=p, |J|=q} \varphi_{I^c, J^c}(z) dz^I \wedge d\bar{z}^J$ is of complementary bi-degree, associate in the same manner :

$$\Phi^*(z, I, J) := \varphi_{I^c, J^c}(z). \text{ This is still a function on } \Gamma.$$

Now we have, for $1 < r < \infty$, if $T(z, I, J)$ is a function in Ω with $L^r(\Omega)$ coefficients and with $\mu = \mu_{\eta, p, q}$,

$$\|T\|_{L^r(d\mu)}^r := \int |T(z, I, J)|^r d\mu_{\eta,p,q}(x, I, J) = \sum_{|I|=p, |J|=q} \|T(\cdot, I, J)\|_{L^r(\Omega, \eta)}^r.$$

For $1 \leq r < \infty$ the dual of $L^r(\mu)$ is $L^{r'}(\mu)$ where r' is the conjugate of r , $\frac{1}{r} + \frac{1}{r'} = 1$, and the norm is defined analogously with r' replacing r .

We also know that, for p, q fixed,

$$(3.1) \quad \|T\|_{L^r(\mu)} = \sup_{\Phi \in L^{r'}(\mu)} \frac{|\int T\Phi d\mu|}{\|\Phi\|_{L^{r'}(\mu)}}.$$

For a (p, q) form $t = \sum_{|J|=p, |K|=q} t_{J,K} dz^J \wedge d\bar{z}^K$, and a weight $\eta > 0$ we define its norm by :

$$(3.2) \quad \|t\|_{L^r(\Omega, \eta)}^r := \sum_{|J|=p, |K|=q} \|t_{J,K}\|_{L^r(\Omega, \eta)}^r = \|T\|_{L^r(\mu)}^r.$$

Now we can state

Lemma 3.1. *Let $\eta > 0$ be a weight. If u is a (p, q) current defined on $(n-p, n-q)$ forms in $L^{r'}(\Omega, \eta)$ and such that*

$$\forall \alpha \in L'_{(n-p, n-q)}(\Omega, \eta), \quad |\langle u, \alpha \rangle| \leq C \|\alpha\|_{L^{r'}(\Omega, \eta)},$$

then $\|u\|_{L^r(\Omega, \eta^{1-r})} \leq C$.

Proof.

Let us take the measure $\mu = \mu_{\eta,p,q}$ as above. Let Φ^* be the function on Γ associated to α and T the one associated to u . We have, by definition of the measure μ applied to the function

$$\begin{aligned} f(z, I, J) &:= T(z, I, J)\eta^{-1}\Phi^*(z, I, J), \\ \int T\eta^{-1}\Phi^* d\mu &= \int f(z, k, l) d\mu(z, k, l) := \sum_{|I|=p, |J|=q} \int_{\Omega} f(z, I, J)\eta(z) dm(z) = \\ &= \sum_{|I|=p, |J|=q} \int_{\Omega} T(z, I, J)\eta^{-1}(z)\Phi^*(z, I, J)\eta(z) dm(z) = \langle u, \alpha \rangle, \end{aligned}$$

by definition of T and Φ^* .

Hence we have, by (3.1)

$$\|T\eta^{-1}\|_{L^r(\mu)} = \sup_{\Psi \in L^{r'}(\mu)} \frac{|\langle u, \alpha \rangle|}{\|\Psi\|_{L^{r'}(\mu)}}.$$

But $\|T\eta^{-1}\|_{L^r(\mu)} = \|u\eta^{-1}\|_{L^r(\Omega, \eta)}$ by (3.2), and

$$\|f\eta^{-1}\|_{L^r(\Omega, \eta)}^r = \int_{\Omega} |f\eta^{-1}|^r \eta dm = \int_{\Omega} |f|^r \eta^{1-r} dm = \|f\|_{L^r(\Omega, \eta^{1-r})}^r,$$

so we get

$$\|u\|_{L^r(\Omega, \eta^{1-r})} = \sup_{\Psi \in L^{r'}(\mu)} \frac{|\langle u, \alpha \rangle|}{\|\Psi\|_{L^{r'}(\mu)}},$$

which implies the lemma because, still by (3.1), we can take $\Psi = \Phi^*$ and $\|\Psi\|_{L^{r'}(\mu)} = \|\alpha\|_{L^{r'}(\Omega, \eta)}$. ■

It may seem strange that we have such an estimate when the dual of $L^{r'}(\Omega, \eta)$ is $L^r(\Omega, \eta)$, but the reason is of course that in the duality forms-currents there is no weights.

3.2. Domain r regular.

Definition 3.2. Let X be a complex manifold and Ω a domain in X ; let $r \in [1, \infty]$, we shall say that Ω is r regular if for any $p, q \in \{0, \dots, n\}$, $q \geq 1$, there is a constant $C = C_{p,q}(\Omega)$ such that for any (p, q) form ω , $\bar{\partial}$ closed in Ω and in $L^r(\Omega)$ there is a $(p, q-1)$ form $u \in L^r(\Omega)$ such that $\bar{\partial}u = \omega$ and $\|u\|_{L^r(\Omega)} \leq C\|\omega\|_{L^r(\Omega)}$.

We shall say that Ω is weakly r regular if for any compact set $K \Subset \Omega$ there are 3 open sets $\Omega_1, \Omega_2, \Omega_3$ such that $K \Subset \Omega_3 \subset \Omega_2 \subset \Omega_1 \subset \Omega_0 := \Omega$ and 3 constants C_1, C_2, C_3 such that :

$\forall j = 0, 1, 2, \forall p, q \in \{0, \dots, n\}, q \geq 1, \forall \omega \in L^r_{p,q}(\Omega_j), \bar{\partial}\omega = 0, \exists u \in L^r_{p,q-1}(\Omega_{j+1}), \bar{\partial}u = \omega$
and $\|u\|_{L^r(\Omega_{j+1})} \leq C_{j+1}\|\omega\|_{L^r(\Omega_j)}$.

I.e. we have a 3 steps chain of resolution.

Of course r regularity implies weak r regularity, just taking $\Omega_1 = \Omega_2 = \Omega_3 = \Omega$.

Examples of 2 regular domains are the bounded pseudo-convex domains by Hörmander [7].

Examples of r regular domains in \mathbb{C}^n are the bounded strictly pseudo-convex (s.p.c.) domains with smooth boundary by Ovreid [10]; the polydiscs in \mathbb{C}^n by Charpentier [4], finite transverse intersections of strictly pseudo-convex bounded domains in \mathbb{C}^n by Menini [9].

On a Stein manifold X we define first the "Lebesgue measure" as in Hörmander's book [7] section 5.2, with a hermitian metric locally equivalent to the usual one on any analytic coordinates system. Associated to this metric there is a volume form dV and we take it for the Lebesgue measure on X .

We can solve the $\bar{\partial}$ equation in strictly pseudo-convex domains with smooth boundary in X with L^r estimates for $(0, 1)$ forms by use of N. Kerzman kernels [8] and the metric above.

We can solve the $\bar{\partial}$ equation in strictly pseudo-convex domains with smooth boundary in X with L^r estimates for all (p, q) forms by J-P. Demailly and C. Laurent ones ([5], Remarque 4, page 596) but here the manifold has to be equipped with a metric with null curvature, in order to avoid parasitic terms.

So examples of r regular domains in Stein manifold are the relatively compact s.p.c. domains with smooth boundary.

Lemma 3.3. A pseudo-convex domain $\Omega \subset \mathbb{C}^n$ is weakly r regular.

Proof.

By theorem 2.6.11 of Hörmander [7] there exists a \mathcal{C}^∞ strictly plurisubharmonic exhausting function φ for Ω . Call $E \subset \Omega$ the set of critical points of φ , then by the Morse-Sard lemma we have that the Lebesgue measure of $\varphi(E) \subset \mathbb{R}$ is zero. Hence we can find a sequence $c_k \in \mathbb{R} \setminus \varphi(E)$, $c_k \rightarrow \infty$, such that

$$D_k := \{z \in \Omega :: \varphi(z) < c_k\}$$

make an exhaustive sequence of open relatively compact sets in Ω , $\partial\varphi \neq 0$ on ∂D_k , hence D_k is strictly pseudo-convex with \mathcal{C}^∞ smooth boundary, and finally $D_k \nearrow \Omega$.

Let $\omega \in L^r_{p,q}(\Omega)$, $\bar{\partial}\omega = 0$ then by Ovreid [10], we can solve $\bar{\partial}u = \omega$ in D_k with $u \in L^r_{p,q-1}(D_k)$ and

$$\|u\|_{L^r(D_k)} \leq C_k\|\omega\|_{L^r(D_k)} \leq C_k\|\omega\|_{L^r(\Omega)}.$$

Hence if K is a compact set in Ω , there is a D_k such that $K \Subset D_k$ and we can take $\Omega_1 = \Omega_2 = \Omega_3 = D_k$.

Hence the weak r regularity of Ω . ■

Lemma 3.4. A Stein manifold Ω is weakly r regular.

Proof.

It is exactly the same proof as above with theorem 5.1.6 of Hörmander [7] instead of theorem 2.6.11 to get a sequence $\{D_k\}_{k \in \mathbb{N}}$ of strictly pseudo-convex sets with C^∞ smooth boundary and exhausting Ω .

Then we use Kerzman estimates for $(p, 1)$ forms [8] and Demailly-Laurent estimates for (p, q) forms [5] instead of Ovreliid estimates [10], to conclude the same way. ■

3.3. The main result.

Let $\mathcal{H}_p(\Omega)$ be the set of all $(p, 0)$ $\bar{\partial}$ closed forms in Ω . If $p = 0$, $\mathcal{H}_0(\Omega) = \mathcal{H}(\Omega)$ is the set of holomorphic functions in Ω . If $p > 0$, we have $\varphi \in \mathcal{H}_p(\Omega) \Rightarrow \varphi(z) = \sum_{|J|=p} a_J(z) dz^J$, where $dz^J := dz_{j_1} \wedge \cdots \wedge dz_{j_p}$ and the functions $a_J(z)$ are holomorphic in Ω . Hence in \mathbb{C}^n , a $(p, 0)$ $\bar{\partial}$ closed form is a vector of global holomorphic functions in Ω .

Let Ω be a weakly r' regular domain in \mathbb{C}^n and ω be a (p, q) form in $L^{r,c}(\Omega)$, $r > 1$, $\text{Supp } \omega \Subset \Omega$. Let $\Omega_1 \subset \Omega$ be a sub domain in Ω with $\text{Supp } \omega \Subset \Omega_1 \subset \Omega$ given by the weak r' regularity of Ω . Set the weight $\eta = \eta_\epsilon := \mathbb{1}_{\Omega_1}(z) + \epsilon \mathbb{1}_{\Omega \setminus \Omega_1}(z)$ for a fixed $\epsilon > 0$.

Suppose that ω is such that $\bar{\partial}\omega = 0$ if $1 \leq q < n$ and for any open $V \Subset \Omega$, $\text{Supp } \omega \Subset V$ we have $\forall h \in \mathcal{H}_p(V)$, $\langle \omega, h \rangle = 0$ if $q = n$.

We shall use the lemma :

Lemma 3.5. *The form \mathcal{L} , defined on $(n - p, n - q + 1)$ form $\alpha \in L^{r'}(\Omega, \eta)$, $\bar{\partial}$ closed in Ω , as follows:*

$$\mathcal{L}(\alpha) := (-1)^{p+q-1} \langle \omega, \varphi \rangle, \text{ where } \varphi \in L^{r'}(\Omega_1) \text{ is such that } \bar{\partial}\varphi = \alpha \text{ in } \Omega_1$$

is well defined and linear.

Proof.

Because $\epsilon > 0$ we have $\alpha \in L^{r'}(\Omega, \eta) \Rightarrow \alpha \in L^{r'}(\Omega)$ and such a φ exists by the weak r' regularity of Ω , so there is a $\Omega_1 :: \text{Supp } \omega \Subset \Omega_1 \subset \Omega$ such that $\varphi \in L^{r'}(\Omega_1)$ with $\bar{\partial}\varphi = \alpha$ in Ω_1 .

Let us see that \mathcal{L} is well defined.

Suppose first that $q < n$.

In order for \mathcal{L} to be well defined we need

$$\forall \varphi, \psi \in L^{r'}_{(n-p, n-q)}(\Omega_1), \bar{\partial}\varphi = \bar{\partial}\psi \Rightarrow \langle \omega, \varphi \rangle = \langle \omega, \psi \rangle.$$

This is meaningful because $\omega \in L^{r,c}(\Omega)$, $r > 1$, $\text{Supp } \omega \Subset \Omega_1$.

Then we have $\bar{\partial}(\varphi - \psi) = 0$ hence, because Ω is weakly r' regular, there is a Ω_2 such that $\text{Supp } \omega \Subset \Omega_2 \subset \Omega_1$ and we can solve $\bar{\partial}$ in $L^{r'}(\Omega_2)$:

$$\exists \gamma \in L^{r'}_{(n-p, n-q-1)}(\Omega_2) :: \bar{\partial}\gamma = (\varphi - \psi).$$

So $\langle \omega, \varphi - \psi \rangle = \langle \omega, \bar{\partial}\gamma \rangle = (-1)^{p+q-1} \langle \bar{\partial}\omega, \gamma \rangle = 0$ because ω is compactly supported in Ω_2 .

Hence \mathcal{L} is well defined in that case.

Suppose now that $q = n$.

Of course $\bar{\partial}\omega = 0$ and we have that φ, ψ are $(p, 0)$ forms hence $\bar{\partial}(\varphi - \psi) = 0$ means that $h := \varphi - \psi$ is a $\bar{\partial}$ closed $(p, 0)$ form hence $h \in \mathcal{H}_p(\Omega_1)$. Taking $V = \Omega_1$ in the hypothesis, $\omega \perp \mathcal{H}_p(V)$ gives $\langle \omega, h \rangle = 0$, and \mathcal{L} is also well defined in that case.

It remains to see that \mathcal{L} is linear, so let $\alpha = \alpha_1 + \alpha_2$, with $\alpha_j \in L^{r'}(\Omega, \eta)$, $\bar{\partial}\alpha_j = 0$, $j = 1, 2$; we have $\alpha = \bar{\partial}\varphi$, $\alpha_1 = \bar{\partial}\varphi_1$ and $\alpha_2 = \bar{\partial}\varphi_2$, with $\varphi, \varphi_1, \varphi_2$ in $L^{r'}(\Omega_1)$ so, because $\bar{\partial}(\varphi - \varphi_1 - \varphi_2) = 0$, we have

$$\varphi = \varphi_1 + \varphi_2 + \bar{\partial}\psi, \text{ with } \psi \text{ in } L^{r'}(\Omega_2),$$

so

$$\mathcal{L}(\alpha) = (-1)^{p+q-1} \langle \omega, \varphi \rangle = (-1)^{p+q-1} \langle \omega, \varphi_1 + \varphi_2 + \bar{\partial}\psi \rangle = \mathcal{L}(\alpha_1) + \mathcal{L}(\alpha_2) + (-1)^{p+q-1} \langle \omega, \bar{\partial}\psi \rangle,$$

but $\langle \omega, \bar{\partial}\psi \rangle = 0$, hence $\mathcal{L}(\alpha) = \mathcal{L}(\alpha_1) + \mathcal{L}(\alpha_2)$.

The same for $\alpha = \lambda\alpha_1$ and the linearity. ■

Remark 3.6. *If Ω is Stein, we can take the domains Ω_1 to be s.p.c. with C^∞ smooth boundary, hence also Stein. So because $K := \text{Supp } \omega \subset \Omega_1 \subset \Omega$, then the $A(\Omega_1)$ convex hull of K , \hat{K}_{Ω_1} is still in Ω_1 and any holomorphic function in Ω_1 can be uniformly approximated on \hat{K}_{Ω_1} by holomorphic functions in Ω .*

Then for $q = n$ instead of asking $\omega \perp \mathcal{H}_p(\Omega_1)$ we need just $\omega \perp \mathcal{H}_p(\Omega)$.

Theorem 3.7. *Let Ω be a weakly r' regular domain and ω be a (p, q) form in $L^{r,c}(\Omega)$, $r > 1$. Suppose that ω is such that $\bar{\partial}\omega = 0$ if $1 \leq q < n$ and*

$$\forall V \subset \Omega, \text{Supp } \omega \subset V, \forall h \in \mathcal{H}_p(V), \langle \omega, h \rangle = 0 \text{ if } q = n.$$

Then there is a $(p, q-1)$ form u in $L^{r,c}(\Omega)$ such that $\bar{\partial}u = \omega$ as distributions and

$$\|u\|_{L^r(\Omega)} \leq C \|\omega\|_{L^r(\Omega)}.$$

Proof.

Because Ω is weakly r' regular there is a $\Omega_1 \subset \Omega$, $\Omega_1 \supset \text{Supp } \omega$ such that

$$\forall \alpha \in L^r(\Omega), \bar{\partial}\alpha = 0, \exists \varphi \in L^r(\Omega_1) :: \bar{\partial}\varphi = \alpha, \|\varphi\|_{L^r(\Omega_1)} \leq C_1 \|\alpha\|_{L^r(\Omega)}$$

and there is Ω_2 such that $\text{Supp } \omega \Subset \Omega_2 \subset \Omega_1 \subset \Omega$ with the same properties as Ω_1 .

Let us consider the weight $\eta = \eta_\epsilon := \mathbf{1}_{\Omega_1}(z) + \epsilon \mathbb{1}_{\Omega \setminus \Omega_1}(z)$ for a fixed $\epsilon > 0$ and the form \mathcal{L} defined in lemma 3.5. By lemma 3.5 we have that \mathcal{L} is a linear form on $(n-p, n-q+1)$ form $\alpha \in L^{r'}(\Omega, \eta)$, $\bar{\partial}$ closed in Ω .

If α is a $(n-p, n-q+1)$ form in $L^{r'}(\Omega, \eta)$, then α is in $L^r(\Omega)$ because $\epsilon > 0$.

The weak r' regularity of Ω gives that there is a $\varphi \in L^{r'}(\Omega_1) :: \bar{\partial}\varphi = \alpha$ which can be used to define $\mathcal{L}(\alpha)$.

We have also that $\alpha \in L^r(\Omega_1)$, $\bar{\partial}\alpha = 0$ in Ω_1 , hence still with the weakly r' regularity of Ω , we have

$$\exists \psi \in L^r(\Omega_2) :: \bar{\partial}\psi = \alpha, \|\psi\|_{L^r(\Omega_2)} \leq C_2 \|\alpha\|_{L^r(\Omega_1)}.$$

For $q < n$, we have $\bar{\partial}(\varphi - \psi) = \alpha - \alpha = 0$ on Ω_2 and, by the weak r' regularity of Ω , then there is a $\Omega_3 \subset \Omega_2$, such that $\text{Supp } \omega \subset \Omega_3 \subset \Omega_2$, and a $\gamma \in L^{r'}(\Omega_3)$, $\bar{\partial}\gamma = \varphi - \psi$ in Ω_3 . So we get

$$\langle \omega, \varphi - \psi \rangle = \langle \omega, \bar{\partial}\gamma \rangle = (-1)^{p+q-1} \langle \bar{\partial}\omega, \gamma \rangle = 0,$$

this is meaningful because $\text{Supp } \omega \subset \Omega_3$.

Hence

$$\mathcal{L}(\alpha) = \langle \omega, \varphi \rangle = \langle \omega, \psi \rangle.$$

If $q = n$, we still have $\bar{\partial}(\varphi - \psi) = \alpha - \alpha = 0$ on Ω_2 , hence $\varphi - \psi \in \mathcal{H}_p(\Omega_2)$; this time we choose $V = \Omega_2$ and the assumption gives $\langle \omega, \varphi - \psi \rangle = 0$ hence again $\mathcal{L}(\alpha) = \langle \omega, \varphi \rangle = \langle \omega, \psi \rangle$.

In any cases, by Hölder inequalities

$$|\mathcal{L}(\alpha)| \leq \|\omega\|_{L^r(\Omega_1)} \|\psi\|_{L^{r'}(\Omega_2)} \leq \|\omega\|_{L^r(\Omega)} \|\psi\|_{L^{r'}(\Omega_2)}.$$

But by the weak r' regularity of Ω there is a constant C_2 such that

$$\|\psi\|_{L^{r'}(\Omega_2)} \leq C_2 \|\alpha\|_{L^{r'}(\Omega_1)}.$$

Of course we have

$$\|\alpha\|_{L^{r'}(\Omega_1)} \leq \|\alpha\|_{L^{r'}(\Omega, \eta)}$$

because $\eta = 1$ on Ω_1 , hence

$$|\mathcal{L}(\alpha)| \leq C_2 \|\omega\|_{L^r(\Omega)} \|\alpha\|_{L^{r'}(\Omega, \eta)}.$$

So we have that the norm of \mathcal{L} is bounded on the subspace of $\bar{\partial}$ closed forms in $L^{r'}(\Omega, \eta)$ by $C\|\omega\|_{L^r(\Omega)}$ which is *independent* of ϵ .

We apply the Hahn-Banach theorem to extend \mathcal{L} with the *same* norm to *all* $(n-p, n-q+1)$ forms in $L^{r'}(\Omega, \eta)$. As in Serre duality theorem ([12], p. 20) this is one of the main ingredient in the proof.

This means, by the definition of currents, that there is a $(p, q-1)$ current u which represents the extended form \mathcal{L} and such that

$$\sup_{\alpha \in L^{r'}(\Omega, \eta), \|\alpha\|=1} |\langle u, \alpha \rangle| \leq C \|\omega\|_{L^r(\Omega)}$$

and by lemma 3.1 with the weight η , this implies

$$\|u\|_{L^r(\Omega, \eta^{1-r})} \leq C \|\omega\|_{L^r(\Omega)}.$$

In particular $\|u\|_{L^r(\Omega)} \leq C \|\omega\|_{L^r(\Omega)}$ because with $\epsilon < 1$ and $r > 1$, we have $\eta^{1-r} \geq 1$.

So applied to a $\bar{\partial}$ closed $(n-p, n-q+1)$ smooth form α we get

$$\langle u, \alpha \rangle = \mathcal{L}(\alpha) = (-1)^{p+q-1} \langle \omega, \varphi \rangle,$$

with φ a smooth form such that $\bar{\partial}\varphi = \alpha$, i.e. $\forall \varphi :: \bar{\partial}\varphi \in L^{r'}(\Omega, \eta)$, we have $\langle u, \bar{\partial}\varphi \rangle = (-1)^{p+q-1} \langle \omega, \varphi \rangle$ and this means precisely, by lemma 2.1, that $\bar{\partial}u = \omega$.

Now for $\epsilon > 0$ with $\eta_\epsilon(z) := \mathbb{1}_{\Omega_1}(z) + \epsilon \mathbb{1}_{\Omega \setminus \Omega_1}(z)$, let $u_\epsilon \in L^r(\Omega, \eta_\epsilon^{1-r})$ be the previous solution, then

$$\|u_\epsilon\|_{L^r(\Omega)}^r \leq \int_{\Omega} |u_\epsilon|^r \eta^{1-r} dm \leq C^r \|\omega\|_{L^r(\Omega)}^r.$$

Replacing η by its value we get

$$\int_{\Omega_1} |u_\epsilon|^r dm + \int_{\Omega \setminus \Omega_1} |u_\epsilon|^r \epsilon^{1-r} dm \leq C^r \|\omega\|_{L^r(\Omega)}^r \Rightarrow \int_{\Omega \setminus \Omega_1} |u_\epsilon|^r \epsilon^{1-r} dm \leq C^r \|\omega\|_{L^r(\Omega)}^r$$

hence

$$\int_{\Omega \setminus \Omega_1} |u_\epsilon|^r dm \leq C^r \epsilon^{r-1} \|\omega\|_{L^r(\Omega)}^r.$$

Because C and the norm of ω are independent of ϵ , we have $\|u_\epsilon\|_{L^r(\Omega)}$ is uniformly bounded and $r > 1 \Rightarrow L^r(\Omega)$ is a dual, hence there is a sub-sequence $\{u_{\epsilon_k}\}_{k \in \mathbb{N}}$ of $\{u_\epsilon\}$ which converges weakly, when $\epsilon_k \rightarrow 0$, to a $(p, q-1)$ form u in $L^r(\Omega)$, still with $\|u\|_{L^r(\Omega)} \leq C \|\omega\|_{L^r(\Omega)}$.

To see that this form u is 0 *a.e.* on $\Omega \setminus \Omega_1$ let us take a component $u_{I,J}$ of it ; it is the weak limit of the sequence of functions $\{u_{\epsilon_k, I, J}\}$ which means, with the notations $v := u_{I, J}$, $v_k := u_{\epsilon_k, I, J}$

$$\forall f \in L^{r'}(\Omega), \int_{\Omega} v f dm = \lim_{k \rightarrow \infty} \int_{\Omega} v_k f dm.$$

As usual take $f := \frac{\bar{v}}{|v|} \mathbb{1}_E$ where $E := \{|v| > 0\} \cap (\Omega \setminus \Omega_1)$ then we get

$$\int_{\Omega} v f dm = \int_E |v| dm = \lim_{k \rightarrow \infty} \int_{\Omega} v_k f dm = \lim_{k \rightarrow \infty} \int_E \frac{v_k \bar{v}}{|v|} dm.$$

Now we have by Hölder

$$\left| \int_E \frac{v_k \bar{v}}{|v|} dm \right| \leq \|v_k\|_{L^r(E)} \|\mathbb{1}_E\|_{L^{r'}(E)}.$$

But

$$\|v_k\|_{L^r(E)}^r \leq \int_{\Omega \setminus \Omega_1} |u_{\epsilon_k}|^r dm \leq (\epsilon_k)^{r-1} \|\omega\|_{L^r(\Omega)}^r \rightarrow 0, \quad k \rightarrow \infty.$$

Hence

$$\left| \int_E |v| dm \right| \leq C^r \|\mathbb{1}_E\|_{L^{r'}(E)} (\epsilon_k)^{r-1} \|\omega\|_{L^r(\Omega)}^r \rightarrow 0$$

which implies $m(E) = 0$ because on E , $|v| > 0$.

This being true for all components of u , we get that the form u is 0 *a.e.* on $\Omega \setminus \Omega_1$.

So we get

$$\forall \varphi \in \mathcal{D}_{n-p, n-q}(\Omega), (-1)^{p+q-1} \langle \omega, \varphi \rangle = \langle u_\epsilon, \bar{\partial} \varphi \rangle \rightarrow \langle u, \bar{\partial} \varphi \rangle \Rightarrow \langle u, \bar{\partial} \varphi \rangle = (-1)^{p+q-1} \langle \omega, \varphi \rangle$$

hence $\bar{\partial} u = \omega$ as distributions. ■

Remark 3.8. *As in remark 3.6 if Ω is Stein for $q = n$ instead of asking $\omega \perp \mathcal{H}_p(\Omega_2)$ we need just $\omega \perp \mathcal{H}_p(\Omega)$.*

Remark 3.9. *The condition of orthogonality to $\mathcal{H}_p(V)$ in the case $q = n$ is necessary : suppose there is a $(p, n-1)$ current u such that $\bar{\partial} u = \omega$ and u with compact support in an open $V \subset \Omega$, then if $h \in \mathcal{H}_p(V)$, we have*

$$h \in \mathcal{H}_p(V), \langle \omega, h \rangle = \langle \bar{\partial} u, h \rangle = (-1)^{n+p} \langle u, \bar{\partial} h \rangle = 0,$$

because, u being compactly supported, there is no boundary term and

$$\langle \bar{\partial} u, h \rangle = (-1)^{n+p} \langle u, \bar{\partial} h \rangle.$$

This kind of condition was already seen for extension of CR functions, see [1] and the references therein.

3.4. Case of Stein manifold.

Theorem 3.10. *Let X be a Stein manifold and Ω a weakly r' regular domain in X with r' the conjugate exponent for r . Let ω be a (p, q) current in $L^{r,c}(\Omega)$, $r > 1$, such that $\bar{\partial} \omega = 0$ if $1 \leq q < n$ and for any open set $V \subset \Omega$, $V \supset \text{Supp } \omega$ we have $\forall h \in \mathcal{H}_p(V)$, $\langle \omega, h \rangle = 0$ if $q = n$.*

Then there is a $(p, q-1)$ current u in $L^{r,c}(\Omega)$ such that $\bar{\partial} u = \omega$.

The proof is identical to the proof of theorem 3.7.

Remark 3.11. *Another proof of this theorem, with different assumptions, was given by a referee. He sketched a completely different geometrical proof, using nice but heavy theorems.*

3.5. Finer control of the support.

Here we shall get a better control on the support of a solution.

Theorem 3.12. *Let Ω be a weakly r' regular domain in a Stein manifold X . Then for any (p, q) form ω in $L^{r,c}(\Omega, dm)$, $\bar{\partial} \omega = 0$, if $q < n$, and $\omega \perp \mathcal{H}_p(V)$ for any $V :: \text{Supp } \omega \subset V$, if $q = n$, with support in $\Omega \setminus C$ where C is a weakly r regular domain, there is a $u \in L^{r,c}(\Omega, dm)$ such that $\bar{\partial} u = \omega$ and with support in $\Omega \setminus \bar{U}$, where U is any open set relatively compact in C , provided that $q \geq 2$.*

Proof.

Let ω be a (p, q) form with compact support in $\Omega \setminus C$ then ω is solvable in L^r by a $v \in L^r_{p, q-1}(\Omega)$, $\bar{\partial} v = \omega$, with compact support in Ω , by theorem 3.10 or, if Ω is a polydisc in \mathbb{C}^n , and if $\omega \in \mathcal{W}_q^r(\Omega)$ by the theorem in [2] by the coronas method.

Because ω has compact support outside C we have $\omega = 0$ in C ; this means that $\bar{\partial} v = 0$ in C .

Because C is weakly r regular and $q \geq 2$, we have

$$\exists C' \subset C, C' \supset \bar{U}, \exists h \in L^r_{p, q-2}(C') \text{ s.t. } \bar{\partial} h = v \text{ in } C'.$$

Let χ be a smooth function such that $\chi = 1$ in U and $\chi = 0$ near $\partial C'$; then set

$$u := v - \bar{\partial}(\chi h).$$

We have that $u = v - \chi \bar{\partial} h - \bar{\partial} \chi \wedge h = v - \chi v - \bar{\partial} \chi \wedge h$ hence u is in L^r ; moreover $u = 0$ in \bar{U} because $\chi = 1$ in U hence $\bar{\partial} \chi = 0$ there. Finally $\bar{\partial} u = \bar{\partial} v - \bar{\partial}^2(\chi h) = \omega$ and we are done. ■

If Ω and C are, for instance, pseudo-convex in \mathbb{C}^n then $\Omega \setminus C$ is no longer pseudo-convex in general, so this theorem improves actually the control of the support.

Remark 3.13. *The correcting function h is given by kernels in the case of Stein domains, hence it is linear ; if the primitive solution v is also linear in ω , then the solution u is linear too. This is the case in \mathbb{C}^n with the solution given in [2].*

This theorem cannot be true for $q = 1$ as shown by the following example :
Take a holomorphic function φ in the open unit ball $B(0, 1)$ in \mathbb{C}^n such that it extends to no open ball of center 0 and radius > 1 . For instance $\varphi(z) := \exp\left(-\frac{z_1 + 1}{z_1 - 1}\right)$. Take $R < 1$, then φ is $\mathcal{C}^\infty(\bar{B}(0, R))$ hence by a theorem of Whitney φ extends \mathcal{C}^∞ to \mathbb{C}^n ; call φ_R this extension. Let $\chi \in \mathcal{C}_c^\infty(B(0, 2))$ such that $\chi = 1$ in the ball $B(0, 3/2)$ and consider the $(0, 1)$ form $\omega := \bar{\partial}(\chi\varphi_R)$. Then $\text{Supp } \omega \subset B(0, 2) \setminus B(0, R)$, ω is $\bar{\partial}$ closed and is \mathcal{C}^∞ hence in $L_{0,1}^r(B(0, 2))$. Moreover $B(0, R)$ is strictly pseudo-convex hence r' regular, but there is no function u such that $\bar{\partial}u = \omega$ and u zero near the origin because any solution u will be C.R. on $\partial B(0, R)$ and by Hartog's phenomenon will extend holomorphically to $B(0, R)$, hence cannot be identically null near 0.

Never the less in the case $q = 1$, we have :

Theorem 3.14. *Let Ω be a weakly r' regular domain in a Stein manifold X .*

Then for any $(p, 1)$ form ω in $L^{r,c}(\Omega, dm)$, $\bar{\partial}\omega = 0$, with support in $\Omega_1 \setminus C$ where Ω_1 is a weak r' regular domain in Ω and C is a domain such that $C \subset \Omega$ and $C \setminus \Omega_1 \neq \emptyset$; there is a $u \in L^{r,c}(\Omega, dm)$ such that $\bar{\partial}u = \omega$ and with support in $\Omega \setminus C$.

Proof.

We have that ω is solvable in L^r by a $u \in L_{p,0}^r(\Omega_1)$ with compact support in Ω_1 , by theorem 3.10 or, if Ω and Ω_1 are polydiscs in \mathbb{C}^n , and if ω verifies the extra L^r estimates on some derivatives of its coefficients, by the theorem in [2] with the coronas method. Then $\bar{\partial}u = 0$ in C hence u is locally holomorphic in C . Because $C \setminus \Omega_1 \neq \emptyset$, there is an open set in $C \setminus \Omega_1 \subset \Omega \setminus \Omega_1$ where u is 0 and holomorphic, hence u is identically 0 in C , C being connected. ■

Remark 3.15. *If there is a $u \in L_{p,0}^{r,c}(\Omega_1)$ which is 0 in C then we have*

$$\forall h \in L_{n-p,n-1}^{r'}(C) :: \text{Supp } \bar{\partial}h \subset C, 0 = \langle u, \bar{\partial}h \rangle = \langle \omega, h \rangle,$$

hence the necessary condition :

$$\forall h \in L_{n-p,n-1}^{r'}(C) :: \text{Supp } \bar{\partial}h \subset C, \langle \omega, h \rangle = 0.$$

Corollary 3.16. *Let Ω be a polydic in \mathbb{C}^n . Then for any (p, q) form ω in $L^{r,c}(\Omega) \cap \mathcal{W}_q^r(\Omega)$, $\bar{\partial}\omega = 0$, if $q < n$, and $\omega \perp \mathcal{H}_p(\Omega)$ if $q = n$, with compact support in $\Omega \setminus \{f = 0\}$ where f is holomorphic in Ω , there is a $(p, q - 1)$ form $u \in L^{r,c}(\Omega)$ such that $\bar{\partial}u = \omega$ and u has its support still in $\Omega \setminus \{f = 0\}$. Moreover the solution u is linear with respect to ω .*

Proof.

Because ω has compact support outside $\{f = 0\}$ this means that there is a $\epsilon > 0$ such that $\omega = 0$ in $\{|f| < \epsilon\}$. We have that $C := \Omega \cap \{|f| < \epsilon\}$ is pseudo-convex hence we can find a sequence of smoothly strictly pseudo-convex domains $D_k \subset C$ such that $D_k \nearrow C$. Choose k big enough to be sure that $D_k \supset \{|f| < \epsilon/2\}$.

If $q > 1$, apply theorem 3.12 to C , $C' = D_k$, to get a solution u of $\bar{\partial}u = \omega$ in $L_{p,q-1}^r(\Omega)$ with support in $\Omega \setminus \{f = 0\}$.

If $q = 1$ we can apply theorem 3.14 because $\{f = 0\}$ goes necessarily up to the boundary of Ω and we can always take a D strictly pseudo-convex containing the support of ω and relatively compact in Ω . We can always use the linear solution given by the coronas method for the primitive solution, hence because we can solve $\bar{\partial}$ in L^r linearly in s.p.c. domains, each step is done linearly, so the complete solution is also linear. ■

Remark 3.17. *If Ω is a pseudo-convex domain in a Stein manifold X , and if f is holomorphic in Ω , then $\Omega \setminus \{f = 0\}$ is Stein and we have already this result by theorem 3.10 but not linearly with respect to ω .*

4. AN APPROACH BY $L^{r,loc}$ ESTIMATES FOR $\bar{\partial}$.

4.1. Global $L^{r,loc}(\Omega)$ estimates for $\bar{\partial}$.

We shall use again the L^r estimates to get a direct proof of *global* $L^{r,loc}$ estimates for the $\bar{\partial}$ equation in Stein domains. We shall copy the proof of theorem 2.8.1 in Henkin-Leiterer ([6] p 86).

Let Ω be a Stein manifold. Let $\{D_k\}_{k \in \mathbb{N}}$ be an increasing sequence of strictly pseudo-convex C^∞ smoothly bounded sub domains of Ω such that \bar{D}_k is $\mathcal{O}(\Omega)$ convex. Let $\omega \in L^{r,loc}_{p,q}(\Omega)$, $\bar{\partial}\omega = 0$ and first suppose that $q \geq 2$.

We have : by Ovreid results in \mathbb{C}^n and Demailly-Laurent ones in a Stein manifold that

$$\forall k \in \mathbb{N}, \exists u_k \in L^r_{p,q-1}(D_k) :: \bar{\partial}u_k = \omega, \|u_k\|_{L^r(D_k)} \leq C_k \|\omega\|_{L^r(D_k)},$$

because $\omega \in L^r_{p,q}(D_k)$ and we can solve $\bar{\partial}$ in this class. Set $v_3 = u_3$ and suppose v_4, \dots, v_k are already done with $v_k \in L^r(D_k)$, $\bar{\partial}v_k = \omega$ in D_k and $v_k = v_{k-1}$ in D_{k-2} , $\|v_k\|_{L^r(D_k)} \leq E_{k+1} \|\omega\|_{L^r(D_{k+1})}$. Let us built v_{k+1} .

We have $\exists u_{k+1} \in L^r(D_{k+1}) :: \bar{\partial}u_{k+1} = \omega$ so $\bar{\partial}(v_k - u_{k+1}) = 0$ in D_k hence

$$\exists \varphi \in L^r(D_k) :: \bar{\partial}\varphi = v_k - u_{k+1} \text{ in } D_k, \|\varphi\|_{L^r(D_k)} \text{ such that :}$$

$$\begin{aligned} \|\varphi\|_{L^r(D_k)} &\leq C_k \|v_k - u_{k+1}\|_{L^r(D_k)} \leq C_k \|v_k\|_{L^r(D_k)} + C_k \|u_{k+1}\|_{L^r(D_k)} \leq \\ &\leq (C_k E_{k+1} + C_k C_{k+1}) \|\omega\|_{L^r(D_{k+1})}. \end{aligned}$$

Let $\chi_k \in C_c^\infty(D_k)$, $0 \leq \chi_k \leq 1$, with $\chi_k \equiv 1$ in D_{k-1} and set

$$v_{k+1} = u_{k+1} + \bar{\partial}(\chi_k \varphi) = \bar{\partial}\chi_k \wedge \varphi + \chi_k (v_k - u_{k+1}) ;$$

and $\chi_k (v_k - u_{k+1}) \in L^r(D_{k+1})$ because it can be extended by 0 outside D_k and the same for $\bar{\partial}\chi_k \wedge \varphi$, hence $v_{k+1} \in L^r(D_{k+1})$. Moreover we have

$$v_{k+1} = u_{k+1} + \bar{\partial}\chi_k \wedge \varphi + \chi_k (v_k - u_{k+1}) = v_k \text{ in } D_{k-1} \text{ because there } \chi_k \equiv 1 \Rightarrow \bar{\partial}\chi_k = 0,$$

and

$$\bar{\partial}v_{k+1} = \bar{\partial}u_{k+1} + \bar{\partial}^2(\chi_k \varphi) = \bar{\partial}u_{k+1} = \omega.$$

And again we have $\|v_{k+1}\|_{L^r(D_{k+1})} \leq E_{k+2} \|\omega\|_{L^r(D_{k+2})}$, with $E_{k+2} = C_k + \|\chi_k\|_\infty (C_k E_{k+1} + C_k C_{k+1})$. So we are done, the sequence $\{D_k\}_{k \in \mathbb{N}}$ being increasing and exhausting in Ω .

If $q = 1$ we suppose that $v_2 = u_2$ and v_3, \dots, v_k are already built. Then the form $h_k := v_k - u_{k+1}$ is in $\mathcal{H}_p(D_k)$ hence

$$\exists \varphi \in \mathcal{H}_p(\Omega) :: \forall z \in D_{k-1}, |v_k(z) - u_{k+1}(z) - \varphi(z)| < 2^{-k},$$

because \bar{D}_{k-1} is $\mathcal{O}(\Omega)$ convex, $v_k - u_{k+1}$ is holomorphic in a neighbourhood of \bar{D}_{k-1} .

So let

$$v_{k+1} := u_{k+1} + \varphi,$$

then $|v_k - v_{k+1}| < 2^{-k}$ in D_{k-1} hence we have a sequence of forms which converge uniformly on compact sets in Ω to a form u . Now $u - v_{k+1}$ is holomorphic in D_k hence $\bar{\partial}u = \bar{\partial}v_{k+1} + \bar{\partial}(u - v_{k+1}) = \bar{\partial}v_{k+1} = \omega$ in D_k for all k and $u \in L^r_{p,0}(D_k)$ for all k . So we are done and we prove :

Theorem 4.1. *Let Ω be a Stein manifold and $\omega \in L_{p,q}^{r,loc}(\Omega)$, $\bar{\partial}\omega = 0$, then there is a $u \in L_{p,q-1}^{r,loc}(\Omega)$ such that $\bar{\partial}u = \omega$. Moreover we have a $C_k > 0$ such that $\|u\|_{L^r(D_k)} \leq C_k \|\omega\|_{L^r(D_{k+1})}$.*

Now in order to go back to forms with compact support, we shall use duality.

4.2. Functionnal analysis.

We shall show that, for (Ω, μ) a measured topological space and $1 \leq r < \infty$, the dual of $L^{r,loc}(\Omega, \mu)$ is $L^{r',c}(\Omega, \mu)$ where r' is the conjugate exponent of r , and $L^{r',c}(\Omega, \mu)$ is the space of $L^{r'}(\Omega, \mu)$ functions 0 μ a.e. outside of a compact set. And also that the dual of $L^{r,c}(\Omega, \mu)$ is $L^{r',loc}(\Omega, \mu)$.

Let $\{\Omega_j\}_{j \in \mathbb{N}}$ be a exhaustive sequence of relatively compact open sets in Ω . We shall equip $L^{r,loc}(\Omega, \mu)$ with the family of semi-norms :

$$\forall f \in L^{r,loc}(\Omega), \|f\|_{r,j} := \|f\|_{L^r(\Omega_j)}.$$

This is a Fréchet space.

$$\text{For } f \in L^{r,c}(\Omega, \mu) \text{ we set } \|f\|_r := \left(\int_{\Omega} |f|^r d\mu \right)^{1/r}.$$

Theorem 4.2. *For $r \in [1, \infty[$ the dual of $L^{r,loc}(\Omega, \mu)$ is $L^{r',c}(\Omega, \mu)$, with r' the conjugate exponent for r .*

Proof.

With its Fréchet topology, a linear form \mathcal{L} on $L^{r,loc}(\Omega, \mu)$ is continuous if and only if :

$$\exists j \in \mathbb{N}, \exists C > 0 :: \forall f \in L^{r,loc}(\Omega, \mu), |\mathcal{L}(f)| \leq C \|f\|_{r,j}.$$

First for $g \in L^{r',c}(\Omega, \mu)$ we associate the linear form

$$\forall f \in L^{r,loc}(\Omega, \mu), \mathcal{L}_g(f) := \int_{\Omega} f \bar{g} d\mu.$$

We have $|\mathcal{L}_g(f)| \leq \|g\|_r \|f\|_{r,j}$ for any $j \in \mathbb{N}$ such that $\text{Supp } g \subset \Omega_j$ hence \mathcal{L}_g is continuous so $g \in (L^{r,loc}(\Omega, \mu))'$.

Conversely let \mathcal{L} be a continuous linear form on $L^{r,loc}(\Omega, \mu)$, this means

$$(4.3) \quad \exists J \in \mathbb{N}, \exists C > 0 :: \forall f \in L^{r,loc}(\Omega, \mu), |\mathcal{L}(f)| \leq C \|f\|_{r,J}.$$

Because the norms are increasing, we still have

$$\forall n \geq J, \forall f \in L^{r,loc}(\Omega, \mu), |\mathcal{L}(f)| \leq C \|f\|_{r,n}.$$

This implies that for $n \geq J$, we have that, on Ω_n , \mathcal{L} is associated to a unique (class of) function $g_n \in L^{r'}(\Omega_n)$, $\|g_n\|_{r',n} \leq C$, i.e.

$$(4.4) \quad \forall n \geq J, \forall f \in L^{r,loc}(\Omega, \mu), \mathcal{L}(\mathbb{1}_{\Omega_n} f) = \int_{\Omega_n} f \bar{g}_n d\mu.$$

By uniqueness we have that $g_n = g_{n-1}$ on Ω_{n-1} .

Consider now the function $f_n := \mathbb{1}_{E_n} \frac{\bar{g}_n}{|g_n|^{\frac{r-2}{r-1}}}$ with $E_n := \Omega_n \setminus \Omega_J$; then

$$\|f_n\|_r^r = \int_{E_n} \left| \frac{\bar{g}_n}{|g_n|^{\frac{r-2}{r-1}}} \right|^r d\mu = \int_{E_n} |g_n|^{r'} d\mu \leq \int_{\Omega_n} |g_n|^{r'} d\mu \leq C^{r'}.$$

Hence $f_n \in L^{r,loc}(\Omega, \mu)$, and we have by (4.4) and by (4.3)

$$|\mathcal{L}(\mathbb{1}_{\Omega_n} f_n)| = \left| \int_{\Omega_n} f_n \bar{g}_n d\mu \right| = \left| \int_{E_n} \frac{|g_n|^2}{|g_n|^{\frac{r-2}{r-1}}} d\mu \right| = \left| \int_{E_n} |g_n|^{r'} d\mu \right| \leq C \|f_n\|_{r,J} = 0,$$

because $f_n = 0$ on Ω_J . Hence $g_n = 0$ μ a.e. on E_n . We have $g_n = g_J$ μ a.e. on Ω_J hence setting $g := \mathbb{1}_{\Omega_J} g_J$ we get that $g = g_n$ μ a.e. on Ω_n . Because $f - \mathbb{1}_{\Omega_n} f \rightarrow 0$ in $L^{r,loc}(\Omega, \mu)$ we have

$$\mathcal{L}(f) = \lim_{n \rightarrow \infty} \mathcal{L}(\mathbb{1}_{\Omega_n}) = \int_{\Omega} f \bar{g} d\mu,$$

by the continuity of \mathcal{L} , hence \mathcal{L} is associated to g which is in $L^{r',c}(\Omega, \mu)$. ■

Theorem 4.3. For $r \in [1, \infty[$ the dual of $L^{r,c}(\Omega, \mu)$ is $L^{r',loc}(\Omega, \mu)$, with r' the conjugate exponent for r .

Proof.

The space $E := L^{r,c}(\Omega, \mu)$ is endowed with its topology of inductive limit $(L^r(\Omega_n, \mu), T_n)_{n \in \mathbb{N}}$ with $T_n : L^r(\Omega_n, \mu) \rightarrow L^{r,c}(\Omega, \mu)$ is such that $\forall f \in L^r(\Omega_n, \mu)$, $T_n f := \mathbb{1}_{\Omega_n} f \in L^{r,c}(\Omega, \mu)$. A linear form \mathcal{L} on $L^{r,c}(\Omega, \mu)$ is then continuous if and only if \mathcal{L} is continuous on each $L^r(\Omega_n, \mu)$, i.e.

$$\forall n \in \mathbb{N}, \exists C_n > 0 :: |\mathcal{L}(T_n f)| \leq C_n \|T_n f\|_n.$$

First to $g \in L^{r',loc}(\Omega, \mu)$ we associate the linear form

$$\forall f \in L^{r,c}(\Omega, \mu), \mathcal{L}_g(f) := \int_{\Omega} f \bar{g} d\mu.$$

We have

$$|\mathcal{L}_g(T_n f)| = \left| \int_{\Omega_n} f \bar{g} d\mu \right| \leq \|g\|_{L^r(\Omega_n)} \|T_n f\|_{L^{r,c}(\Omega, \mu)},$$

hence \mathcal{L}_g is continuous on $L^{r,c}(\Omega, \mu)$, so $g \in (L^{r,c}(\Omega, \mu))'$.

Conversely let \mathcal{L} be a continuous linear form on $L^{r,c}(\Omega, \mu)$, this means that, if we restrict \mathcal{L} to the subspace $L^r(\Omega_n, \mu)$ we have

$$\exists C_n > 0 :: \forall f \in L^r(\Omega_n, \mu), |\mathcal{L}(T_n f)| \leq C_n \|T_n f\|_n.$$

Hence by the usual duality between $L^r(\Omega_n, \mu)$ and $L^{r'}(\Omega_n, \mu)$ we have

$$\exists! g_n \in L^{r'}(\Omega_n, \mu) :: \|g_n\|_{L^{r'}(\Omega_n, \mu)} \leq C_n \text{ and } \mathcal{L}(T_n f) = \langle T_n f, g_n \rangle.$$

But $\Omega_{n-1} \subset \Omega_n$ hence $g_n = g_{n-1}$ μ a.e. on Ω_{n-1} so we set $g \in L^{r',loc}(\Omega)$ such that

$$g := g_n \text{ on } \Omega_n \text{ and } \forall f \in L^{r,c}(\Omega), \mathcal{L}(f) = \langle f, g \rangle_{\Omega}$$

and we have that $(L^{r,c}(\Omega, \mu))' \subset L^{r',loc}(\Omega)$. ■

Remark 4.4. We can use the fact that $L^{r,loc}(\Omega)$ is the projective limit of the projective system $(L^r(\Omega_j), T_{jk})$ where $\{\Omega_j\}_{j \in \mathbb{N}}$ is an exhaustive sequence of relatively compact open sets in Ω , and $T_{jk} : L^r(\Omega_k) \rightarrow L^r(\Omega_j)$ is defined by :

$$T_{jk} f := \mathbb{1}_{\Omega_j \cap \Omega_k} f.$$

We also have that $L^{r,c}(\Omega)$ is the inductive limit of $(L^r(\Omega_j), T_{jk})$ and we have that the dual of $L^r(\Omega_j)$ is $L^{r'}(\Omega_j)$ and the dual of an inductive limit is the projective limit with the dual inductive system ([11], Prop. 15, p 85). But because the converse is also true in this case, we give explicitly the detailed proofs.

4.3. Complex analysis.

Let Ω be an open set in \mathbb{C}^n or in a complex manifold, and A a topological space of functions defined on Ω . Let, as usual, $\mathcal{D}(\Omega)$ be the space of \mathcal{C}^∞ smooth functions with compact support in Ω and $\mathcal{D}'(\Omega)$ be the space of distributions in Ω . Suppose that $\mathcal{D}(\Omega) \subset A \subset \mathcal{D}'(\Omega)$, the inclusions being continuous.

Associate to A the spaces of (p, q) currents $A_{p,q}$ with coefficients in A . Then $B = A'$, the dual of A , is still a space of distributions and we ask that the duality bracket be $\bar{\partial}$ compatible with the distribution one, i.e.

$$\forall f \in A_{p,q-1}, \forall g \in B_{n-p,n-q}, \langle \bar{\partial} f, g \rangle = \langle f, \bar{\partial} g \rangle.$$

This means that there is no boundary terms, i.e. the f or g must be "asymptotically holomorphic".

We say that the equation $\bar{\partial} f = \omega$ is solvable in $A_{p,q}$ with $0 \leq p \leq n$, $1 \leq q \leq n$, if

$$\forall \omega \in A_{p,q}, \bar{\partial}\omega = 0, \exists f \in A_{p,q-1} :: \bar{\partial}f = \omega.$$

Now suppose $\bar{\partial}$ is solvable in $A_{p,q+1}$ and in $A_{p,q}$ and let $\omega \in B_{n-p,n-q}, \bar{\partial}\omega = 0$ still with $0 \leq p \leq n, 1 \leq q \leq n$, and consider the form

$$\forall \gamma \in A_{p,q+1}, \bar{\partial}\gamma = 0, \mathcal{L}_\omega(\gamma) := \langle f, \omega \rangle,$$

with $\bar{\partial}f = \gamma$, which exists by hypothesis, where $\langle \cdot, \cdot \rangle$ is the duality bracket.

Let us see if this form is well defined, as in lemma 3.5 :

$$\text{if } \bar{\partial}f = \bar{\partial}g = \gamma, \text{ then } \bar{\partial}(f - g) = 0 \text{ hence } \exists \varphi \in A_{p,q-1} :: \bar{\partial}\varphi = f - g \text{ so}$$

$$\langle f - g, \omega \rangle = \langle \bar{\partial}\varphi, \omega \rangle = (-1)^{p+q} \langle \varphi, \bar{\partial}\omega \rangle = 0,$$

because we have the compatibility between the two brackets.

Hence \mathcal{L}_ω is well defined and it is linear as in lemma 3.5.

As before set $\mathcal{H}_p(\Omega)$ the set of $(p, 0)$ form f such that $\bar{\partial}f = 0$ in Ω .

For the special case $q = 0$, we automatically have that $\bar{\partial}\omega = 0$, because ω is a $(n-p, n)$ form, hence to still get that $\mathcal{L}_\omega(\gamma)$ is well defined if $\bar{\partial}\gamma = 0$, we need $\langle f, \omega \rangle = 0$ as soon as $f \in \mathcal{H}_p(\Omega)$.

We say that the equation $\bar{\partial}f = \omega$ is *continuously solvable* in $A_{p,q+1}$ if it is solvable in $A_{p,q}$ and in $A_{p,q+1}$ and if, moreover, $\mathcal{L}_\omega(\gamma)$ is a continuous linear form on the subspace $\gamma \in A_{p,q+1}, \bar{\partial}\gamma = 0$.

Theorem 4.5. *If the equation $\bar{\partial}f = \omega$ is continuously solvable in $A_{p,q+1}$ then we can solve it in the dual space $B_{n-p,n-q}$. This means that $\forall \omega \in B_{n-p,n-q}, \bar{\partial}\omega = 0$ if $1 \leq q \leq n-1$ and $\langle \omega, f \rangle = 0$ if $q = 0$ for $f \in \mathcal{H}_p(\Omega)$, there is a $u \in B_{n-p,n-q-1}$ such that $\bar{\partial}u = \omega$.*

Proof.

Let $\omega \in B_{n-p,n-q}, \bar{\partial}\omega = 0$ if $1 \leq q \leq n-1$ and $\langle \omega, f \rangle = 0$ if $q = n$ and $f \in \mathcal{H}_p(\Omega)$, and consider the form $\mathcal{L} = \mathcal{L}_\omega$ defined above on the subspace $\gamma \in A_{p,q+1}, \bar{\partial}\gamma = 0$, which exists by hypothesis in $A_{p,q+1}$. This linear form \mathcal{L} is well defined, linear and continuous by assumption, so by Hahn-Banach, it extends to the whole $A_{p,q+1}$ and is represented by a $u \in B_{n-p,n-q-1}$ by duality. So we have

$$\mathcal{L}(\gamma) = \langle \gamma, u \rangle = \langle \bar{\partial}f, u \rangle = \mathcal{L}_\omega(f) = \langle f, \omega \rangle,$$

but $\langle \bar{\partial}f, u \rangle = (-1)^{p+q} \langle f, \bar{\partial}u \rangle$ hence $\langle f, \omega \rangle = (-1)^{p+q} \langle f, \bar{\partial}u \rangle$ provided that $\bar{\partial}f = \gamma \in A_{p,q+1}$.

This is true for $\gamma = \bar{\partial}f$, where f is a (p, q) -form with coefficients in $\mathcal{D}(\Omega)$, because $\mathcal{D}(\Omega) \subset A$ and if $f \in \mathcal{D}(\Omega), \bar{\partial}f \in \mathcal{D}(\Omega) \Rightarrow \bar{\partial}f \in A_{p,q+1}$;

hence we have

$$\forall f \in \mathcal{D}(\Omega)_{p,q}, \langle f, \omega \rangle = (-1)^{p+q} \langle f, \bar{\partial}u \rangle,$$

so we find a $u \in B_{n-p,n-q-1}$ such that $(-1)^{p+q} \bar{\partial}u = \omega$. ■

This is again a avatar of Serre duality but with explicit *global* estimates.

Corollary 4.6. *Let Ω be a domain in a Stein manifold X such that the equation $\bar{\partial}u = f$ has a solution u in $L_{p,q}^{r,loc}(\Omega)$ if $f \in L_{p,q+1}^{r,loc}(\Omega)$ with the control :*

$$\forall j \in \mathbb{N}, \|u\|_j \leq C_j \|f\|_j ;$$

then we can solve this equation with $u \in L_{p,q-1}^{r,c}(\Omega)$ if $f \in L_{p,q}^{r,c}(\Omega)$ and $\bar{\partial}f = 0$ if $q < n$, and $f \perp \mathcal{H}_p$ if $q = n$.

In particular this is true if Ω is Stein in X .

Proof.

We take for μ the Lebesgue measure on Ω and $A := L^{r,loc}(\Omega)$; we have clearly that $\mathcal{D}(\Omega) \subset A \subset \mathcal{D}'(\Omega)$ and by theorem 4.1 we can solve the $\bar{\partial}$ equation for $(p, q+1)$ forms $\bar{\partial}$ closed with coefficients in A .

Let $\omega \in L_{n-p,n-q}^{r,c}(\Omega)$; we have to show that the form $\mathcal{L} = \mathcal{L}_\omega$ is continuous on the subspace $\gamma \in A_{p,q+1}, \bar{\partial}\gamma = 0$, but we have with $\bar{\partial}f = \gamma$,

$$\mathcal{L}(\gamma) := \langle f, \omega \rangle \Rightarrow |\mathcal{L}(\gamma)| = |\langle f, \omega \rangle| \leq \|f\|_{L^r(\Omega_j)} \times \|\omega\|_{L^r(\Omega)}.$$

The theorem 4.1 gives also that $\|f\|_{L^r(\Omega_j)} \leq C_j \|\gamma\|_{L^r(\Omega_{j+1})}$, where $\Omega_j \subset \Omega$ is a s.p.c. domain containing the support of γ . Hence

$$|\mathcal{L}(\gamma)| = |\langle f, \omega \rangle| \leq \|f\|_{L^r(\Omega_j)} \times \|\omega\|_{L^r(\Omega)} \leq C_j \|\gamma\|_{L^r(\Omega_{j+1})} \times \|\omega\|_{L^r(\Omega)},$$

and the continuity of \mathcal{L} , C_j and ω being fixed.

The $\bar{\partial}$ compatibility is clear because $B = A'$ is made of forms with compact support by theorem 4.3.

So by theorem 4.5 we can solve the $\bar{\partial}$ equation for forms with coefficients in the dual space of $L^{r',loc}(\Omega, \mu)$ which is, by theorem 4.3, $L^{r,c}(\Omega, dm)$.

By theorem 4.1 if Ω is Stein, then we can solve $\bar{\partial}$ in $L^{r,loc}(\Omega)$ hence in $L^{r,c}(\Omega)$, and the corollary. ■

REFERENCES

- [1] E. Amar. On the extension of c.r. functions. *Math. Z.*, 206:89–102, 1991.
- [2] E. Amar and S. Mongodi. On L^r hypoellipticity of solutions with compact support of the Cauchy-Riemann equation. *To appear in Annali di Matematica Pura e Applicata.*, 2013.
- [3] A. Andreotti and G. Grauert. Théorèmes de finitude pour la cohomologie des espaces complexes. *Bulletin de la Société Mathématique de France*, 90:193–259, 1962.
- [4] P. Charpentier. Formules explicites pour les solutions minimales de l'équation $\bar{\partial}u = f$ dans la boule et dans le polydisque de \mathbb{C}^n . *Annales de l'institut Fourier*, 30(4):121–154, 1980.
- [5] J-P. Demailly and C. Laurent-Thiébaud. Formules intégrales pour les formes différentielles de type (p, q) dans les variétés de Stein. *Ann. Sci. Ecole Norm. Sup.*, 4(4):579–598, 1987.
- [6] G. Henkin and J. Leiterer. *Theory of functions on complex manifolds*. Mathematische Monographien. Akademie-Verlag Berlin, 1984.
- [7] L. Hörmander. *An introduction to complex analysis in several variables*. North-Holland/American Elsevier, 1994.
- [8] N. Kerzman. Hölder and L^p estimates for solutions of $\bar{\partial}u = f$ in strongly pseudoconvex domains. *Comm. Pure. Appl. Math.*, 24:301–379, 1971.
- [9] C. Menini. Estimations pour la résolution du $\bar{\partial}$ sur une intersection d'ouverts strictement pseudoconvexes. *Math. Z.*, 1:87–93, 1997.
- [10] N. Ovreid. Integral representation formulas and L^p estimates for the $\bar{\partial}$ equation. *Math. Scand.*, 29:137–160, 1971.
- [11] A.P. Robertson and Wendy Robertson. *Topological vector spaces.*, volume 53 of *Cambridge Tracts in Mathematics and Mathematical physics*. 1966.
- [12] J-P. Serre. Un théorème de dualité. *Comment. Math. Helv.*, 29:9–26, 1955.