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► **To cite this version:**

Eric Amar. AN ANDREOTTI-GRAUERT THEOREM WITH  $L^r$  ESTIMATES.. 2012. hal-00676110v4

**HAL Id: hal-00676110**

**<https://hal.science/hal-00676110v4>**

Preprint submitted on 23 Nov 2012 (v4), last revised 9 Oct 2019 (v9)

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# AN ANDREOTTI-GRAUERT THEOREM WITH $L^r$ ESTIMATES.

ERIC AMAR

ABSTRACT. By a theorem of Andreotti and Grauert if  $\omega$  is a  $(p, q)$  current,  $q < n$ , in a Stein manifold,  $\bar{\partial}$  closed and with compact support, then there is a solution  $u$  to  $\bar{\partial}u = \omega$  still with compact support. The aim of this work is to show that if moreover  $\omega \in L^r(dm)$ , where  $m$  is a suitable Lebesgue measure on the Stein manifold, then we have a solution  $u$  with compact support and in  $L^r(dm)$ .

As a consequence of the previous result and again by a duality argument we have  $L^{r,loc}(dm) - L^{r,loc}(dm)$  solutions for the  $\bar{\partial}$  equation on suitable domains.

## 1. INTRODUCTION.

Let  $\omega$  be a  $\bar{\partial}$  closed  $(p, q)$  form in  $\mathbb{C}^n$  with compact support  $K := \text{Supp}\omega$  and such that  $\omega \in L^r(\mathbb{C}^n)$ . Setting  $K$  in a ball  $\mathbb{B} := B(0, R)$  with  $R$  big enough, we know, by a theorem of Ovreid [9], that we have a  $(p, q - 1)$  form  $u \in L^r(\mathbb{B})$  such that  $\bar{\partial}u = \omega$ . On the other hand we also know, at least when  $q < n$ , that there is a current  $v$  with compact support such that  $\bar{\partial}v = \omega$ , by a theorem of Andreotti-Grauert [3].

So a natural question is : may we have a solution  $u$  of  $\bar{\partial}u = \omega$  with compact support and in  $L^r(\mathbb{C}^n)$  ?

We already answered this question by the affirmative in a join work with S. Mongodi [2] explicitly and linearly by the "method of coronas". This method asks for extra  $L^r$  conditions on derivatives of coefficients of  $\omega$ , when  $q < n$  ; we shall note the set of  $\omega$  verifying these conditions  $\mathcal{W}_q^r(\Omega)$  as in [2].

The aim of this work is to extend this result to Stein manifolds and get rid of the extra  $L^r$  conditions  $\mathcal{W}_q^r(\Omega)$ . For it we use a completely different approach inspired by the Serre duality [11]. Because Hahn Banach theorem is used, these results are no longer so explicit and constructive as in [2].

On the other hand the control of the support is better : if the support of the data  $\omega$  is contained in  $\Omega \setminus C$  where  $\Omega$  and  $C$  are  $r'$  regular domains, then the support of the solution  $u$  is contained in  $\Omega \setminus C'$ , where  $C'$  is any domain relatively compact in  $C$ , provided that  $q \geq 2$ . There is also this kind of result for  $q = 1$ , see section 5.

The definition of  $r$  regular domain will be given later on, but relatively compact strictly pseudo convex domains in Stein manifold or polydiscs in  $\mathbb{C}^n$  are such domains.

The idea is to solve  $\bar{\partial}u = \omega$  in a space  $L^r(\Omega)$  with a weigh  $\eta$  "very big" outside of the support of  $\omega$  ; this way we shall have a "very small" solution  $u$  outside of the support of  $\omega$ . Then, using a sequence of such weighs going to infinity outside of the support of  $\omega$ , we shall have a  $u$  zero outside of the support of  $\omega$ .

Of course this is a little bit more involved ...

Let  $(\Omega, \mu)$  be a measured domain with a positive measure  $\mu$ ,  $L^{r,c}(\Omega, \mu)$  the space of  $L^r(\Omega, \mu)$  functions 0  $\mu$  a.e. outside a compact set of  $\Omega$  and  $L^{r,loc}(\Omega, \mu)$  the space of  $\mu$  measurable functions

on  $\Omega$  which are in  $L^r(K, \mu)$  for  $K$  any compact set of  $\Omega$ . We prove, for  $1 \leq r < \infty$ , that the dual of  $L^{r,c}(\Omega, \mu)$  is  $L^{r',loc}(\Omega, \mu)$ , with  $r'$  the conjugate exponent of  $r$ , and that the dual of  $L^{r,loc}(\Omega, \mu)$  is  $L^{r',c}(\Omega, \mu)$ . Using this and the main result of this work, we show that if  $\omega \in L^{r,loc}_{(p,q)}(\Omega, dm)$ ,  $\bar{\partial}\omega = 0$ , there is a form  $u \in L^{r,loc}_{(p,q-1)}(\Omega, dm)$  such that  $\bar{\partial}u = \omega$ , provided that  $\Omega$  is a  $r'$  regular domain. This last result was suggested to me by a referee who thought it was already known, after the first version of this paper was already posted on HAL and ArXiv.

I am indebted to G. Tomassini who started my interest in this subject by e-mails on precisely this kind of questions and also to S. Mongodi for a lot of discussions by mails on the subject during the preparation of our join paper [2].

## 2. DUALITY.

We shall study a duality between currents inspired by the Serre duality [11].

Let  $\Omega \subset \mathbb{C}^n$  be an open set in  $\mathbb{C}^n$  and  $t$  a  $(p, q)$  current with compact support in  $\Omega$ , noted  $t \in \mathcal{D}'_{c, (p,q)}(\Omega)$ .

Let also  $\varphi \in \mathcal{C}^\infty_{(n-p, n-q)}(\Omega)$  a  $(n-p, n-q)$  form in  $\mathcal{C}^\infty(\Omega)$ . We have that  $t \wedge \varphi$  is a  $(n, n)$  current with compact support in  $\Omega$ .

As usual we use the following notation for the pairing

$$\langle t, \varphi \rangle := t(\varphi),$$

where  $t(\varphi)$  is the action, as a current, of  $t$  on the smooth form  $\varphi$  of complementary bi-degree.

**Lemma 2.1.** *Let  $\Omega$  be an open set in  $\mathbb{C}^n$ ,  $\omega \in \mathcal{D}'_{c, (p,q)}(\Omega)$  a  $(p, q)$  current with compact support in  $\Omega$  and  $u \in \mathcal{D}'_{c, (p,q-1)}(\Omega)$ . Then we have  $\bar{\partial}u = \omega$  iff*

$$(*) \quad \forall \varphi \in \mathcal{C}^\infty_{(n-p, n-q)}(\Omega), \quad \langle \omega, \varphi \rangle = (-1)^{p+q-1} \langle u, \bar{\partial}\varphi \rangle.$$

*Proof.*

If  $\bar{\partial}u = \omega$ , let  $\chi \in \mathcal{C}^\infty_c(\Omega)$  such that  $\chi \equiv 1$  on the support of  $u$ , hence on the support of  $\omega$ . Then

$$\forall \varphi \in \mathcal{C}^\infty_{(n-p, n-q)}(\Omega), \quad \langle \omega, \varphi \rangle = \langle \omega, \chi\varphi \rangle = \langle \bar{\partial}u, \chi\varphi \rangle = (-1)^{p+q-1} \langle u, \bar{\partial}(\chi\varphi) \rangle$$

by definition of the action of derivatives on currents. Hence

$$\langle \omega, \varphi \rangle = (-1)^{p+q-1} \langle u, \bar{\partial}\chi \wedge \varphi \rangle + (-1)^{p+q-1} \langle u, \chi \bar{\partial}\varphi \rangle = (-1)^{p+q-1} \langle u, \chi \bar{\partial}\varphi \rangle$$

because  $\bar{\partial}\chi = 0$  on the support of  $u$ ; so

$$\langle \omega, \varphi \rangle = (-1)^{p+q-1} \langle u, \chi \bar{\partial}\varphi \rangle = (-1)^{p+q-1} \langle u, \bar{\partial}\varphi \rangle$$

because  $\chi = 1$  on the support of  $u$ .

Conversely if we have  $(*)$  we take  $\chi \in \mathcal{C}^\infty_c(\Omega)$  such that  $\chi \equiv 1$  on union of the support of  $u$  and the support of  $\omega$  then

$$\forall \varphi \in \mathcal{C}^\infty_{(n-p, n-q)}(\Omega), \quad \langle \omega, \varphi \rangle = \langle \omega, \chi\varphi \rangle$$

because  $\chi = 1$  on the support of  $\omega$ , and by  $(*)$

$$\langle \omega, \chi\varphi \rangle = (-1)^{p+q-1} \langle u, \bar{\partial}(\chi\varphi) \rangle = \langle \bar{\partial}u, \chi\varphi \rangle = \langle \bar{\partial}u, \varphi \rangle$$

because  $\chi = 1$  on the support of  $u$  hence on the support of  $\bar{\partial}u$ .

So

$$\forall \varphi \in \mathcal{C}^\infty_{(n-p, n-q)}(\Omega), \quad \langle \omega, \varphi \rangle = \langle \bar{\partial}u, \varphi \rangle,$$

which means that  $\bar{\partial}u = \omega$  as a current. ■

## 3. SOLUTION OF THE $\bar{\partial}$ EQUATION WITH COMPACT SUPPORT.

### 3.1. Domain $r$ regular.

We shall use the definition.

**Definition 3.1.** Let  $X$  be a complex manifold and  $\Omega$  a domain in  $X$ ; let  $r \in [1, \infty]$ , we shall say that  $\Omega$  is  $r$  regular, if for  $p, q \in \{0, \dots, n\}$ ,  $q \geq 1$ , there is a constant  $C = C_{p,q}$  such that for any  $(p, q)$  form  $\omega$ ,  $\bar{\partial}$  closed in  $\Omega$  and in  $L^r(\Omega)$  there is a  $(p, q-1)$  form  $u \in L^r(\Omega)$  such that  $\bar{\partial}u = \omega$  and  $\|u\|_{L^r(\Omega)} \leq C\|\omega\|_{L^r(\Omega)}$ .

The  $L^r$  norms for forms will be defined later on.

Examples of 2 regular domains are the bounded pseudo-convex domains by Hörmander [6].

Examples of  $r$  regular domains in  $\mathbb{C}^n$  are the bounded strictly pseudo-convex (s.p.c.) domains with smooth boundary by Ovreid [9]; the polydiscs in  $\mathbb{C}^n$  by Charpentier [4], finite transverse intersections of strictly pseudo-convex bounded domains Menini [8].

On a Stein manifold  $X$  we define first the "Lebesgue measure" as in Hörmander's book [6] section 5.2, with a hermitian metric locally equivalent to the usual one on any analytic coordinates system. Associated to this metric there is a volume form  $dV$  and we take it for the Lebesgue measure on  $X$ .

We can solve the  $\bar{\partial}$  equation in strictly pseudo-convex domains with smooth boundary in  $X$  with  $L^r$  estimates for  $(0, 1)$  forms by use of N. Kerzman kernels [7] and the metric above.

We can solve the  $\bar{\partial}$  equation in strictly pseudo-convex domains with smooth boundary in  $X$  with  $L^r$  estimates for all  $(p, q)$  forms by J-P. Demailly and C. Laurent ones ([5], Remarque 4, page 596) but here the manifold has to be equipped with a metric with null curvature, in order to avoid parasitic terms.

So examples of  $r$  regular domains in Stein manifold are the relatively compact s.p.c. domains with smooth boundary.

### 3.2. Weighted $L^r$ spaces.

Let  $\mathbb{B} := B(0, R)$ , the ball of  $\mathbb{C}^n$  centered at 0 and of radius  $R$ . We shall need the following notations.

We note  $dm$  the Lebesgue measure on  $\mathbb{C}^n$  and we shall define  $L^r(\mathbb{B}, \eta)$  to be the set of functions  $f$  defined on  $\mathbb{B}$  such that

$$\|f\|_{L^r(\mathbb{B}, \eta)}^r := \int_{\mathbb{B}} |f(z)|^r \eta(z) dm(z) < \infty,$$

with a weight  $\eta(z) > 0$ . As usual we set  $L^r(\mathbb{B})$  for  $L^r(\mathbb{B}, 1)$ .

Let  $\mathcal{I}_p$  be the set of multi-indices of length  $p$  in  $(1, \dots, n)$ . We shall use the following measure defined on  $\Gamma := \mathbb{B} \times \mathcal{I}_p \times \mathcal{I}_q$  the following way :

$$d\mu(z, k, l) = d\mu_{\eta, p, q}(z, k, l) := \eta(z) dm(z) \otimes \sum_{|I|=p, |J|=q} \delta_I(k) \otimes \delta_J(l),$$

where  $\delta_I(k) = 1$  if the multi-index  $k$  is equal to  $I$  and  $\delta_I(k) = 0$  if not.

This means, if  $f(z, I, J)$  is a function defined on  $\Gamma$ , that

$$\int f(z, k, l) d\mu_{\eta, p, q}(z, k, l) := \sum_{|I|=p, |J|=q} \int_{\mathbb{B}} f(z, I, J) \eta(z) dm(z).$$

If  $I$  is a multi-index of length  $p$ , let  $I^c$  be the unique multi-index, ordered increasingly, such that  $I \cup I^c = (1, 2, \dots, n)$ ; then  $I^c$  is of length  $n-p$ .

To  $t = \sum_{|I|=p, |J|=q} t_{I,J}(z) dz^I \wedge d\bar{z}^J$  a  $(p, q)$  form, we associate the function on  $\Gamma$  :

$$T(z, I, J) := (-1)^{s(I,J)} t_{I,J}(z),$$

where

$$s(I, J) = 0 \text{ if } dz^I \wedge d\bar{z}^J \wedge dz^{I^c} \wedge d\bar{z}^{J^c} = dz_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_1 \wedge \dots \wedge d\bar{z}_n \text{ as a } (n, n) \text{ form}$$

and

$$s(I, J) = 1 \text{ if not.}$$

If  $\varphi = \sum_{|I|=p, |J|=q} \varphi_{I^c, J^c}(z) dz^I \wedge d\bar{z}^J$  is of complementary bi-degree, associate in the same manner :

$\Phi^*(z, I, J) := \varphi_{I^c, J^c}(z)$ . This is still a function on  $\Gamma$ .

Now we have, for  $1 < r < \infty$ , if  $T(z, I, J)$  is a function in  $\mathbb{B}$  with  $L^r(\mathbb{B})$  coefficients and with  $\mu = \mu_{\eta, p, q}$ ,

$$\|T\|_{L^r(d\mu)}^r := \int |T(z, I, J)|^r d\mu_{\eta, p, q}(x, I, J) = \sum_{|I|=p, |J|=q} \|T(\cdot, I, J)\|_{L^r(\mathbb{B}, \eta)}^r.$$

For  $1 \leq r < \infty$  the dual of  $L^r(\mu)$  is  $L^{r'}(\mu)$  where  $r'$  is the conjugate of  $r$ ,  $\frac{1}{r} + \frac{1}{r'} = 1$ , and the norm is defined analogously with  $r'$  replacing  $r$ .

We also know that, for  $p, q$  fixed,

$$(3.1) \quad \|T\|_{L^r(\mu)} = \sup_{\Phi \in L^{r'}(\mu)} \frac{|\int T\Phi d\mu|}{\|\Phi\|_{L^{r'}(\mu)}}.$$

For a  $(p, q)$  form  $t = \sum_{|J|=p, |K|=q} t_{J, K} dz^J \wedge d\bar{z}^K$ , and a weight  $\eta > 0$  we define its norm by :

$$(3.2) \quad \|t\|_{L^r(\mathbb{B}, \eta)}^r := \sum_{|J|=p, |K|=q} \|t_{J, K}\|_{L^r(\mathbb{B}, \eta)}^r = \|T\|_{L^r(\mu)}^r.$$

Now we can state

**Lemma 3.2.** *Let  $\eta > 0$  be a weight. If  $u$  is a  $(p, q)$  current defined on  $(n - p, n - q)$  forms in  $L^{r'}(\mathbb{B}, \eta)$  and such that*

$$\forall \alpha \in L^{r'}_{(n-p, n-q)}(\mathbb{B}, \eta), \quad |\langle u, \alpha \rangle| \leq C \|\alpha\|_{L^{r'}(\mathbb{B}, \eta)},$$

then  $\|u\|_{L^r(\mathbb{B}, \eta^{1-r})} \leq C$ .

*Proof.*

Let us take the measure  $\mu = \mu_{\eta, p, q}$  as above. Let  $\Phi^*$  be the function on  $\Gamma$  associated to  $\alpha$  and  $T$  the one associated to  $u$ . We have, by definition of the measure  $\mu$  applied to the function

$$f(z, I, J) := T(z, I, J)\eta^{-1}\Phi^*(z, I, J),$$

$$\begin{aligned} \int T\eta^{-1}\Phi^* d\mu &= \int f(z, k, l) d\mu(z, k, l) := \sum_{|I|=p, |J|=q} \int_{\mathbb{B}} f(z, I, J)\eta(z) dm(z) = \\ &= \sum_{|I|=p, |J|=q} \int_{\mathbb{B}} T(z, I, J)\eta^{-1}(z)\Phi^*(z, I, J)\eta(z) dm(z) = \langle u, \alpha \rangle, \end{aligned}$$

by definition of  $T$  and  $\Phi^*$ .

Hence we have, by (3.1)

$$\|T\eta^{-1}\|_{L^r(\mu)} = \sup_{\Psi \in L^{r'}(\mu)} \frac{|\langle u, \alpha \rangle|}{\|\Psi\|_{L^{r'}(\mu)}}.$$

But  $\|T\eta^{-1}\|_{L^r(\mu)} = \|u\eta^{-1}\|_{L^r(\mathbb{B}, \eta)}$  by (3.2), and

$$\|f\eta^{-1}\|_{L^r(\mathbb{B}, \eta)}^r = \int_{\mathbb{B}} |f\eta^{-1}|^r \eta dm = \int_{\mathbb{B}} |f|^r \eta^{1-r} dm = \|f\|_{L^r(\mathbb{B}, \eta^{1-r})}^r,$$

so we get

$$\|u\|_{L^r(\mathbb{B}, \eta^{1-r})} = \sup_{\Psi \in L^{r'}(\mu)} \frac{|\langle u, \alpha \rangle|}{\|\Psi\|_{L^{r'}(\mu)}},$$

which implies the lemma because, still by (3.1), we can take  $\Psi = \Phi^*$  and  $\|\Psi\|_{L^{r'}(\mu)} = \|\alpha\|_{L^{r'}(\mathbb{B}, \eta)}$ . ■

It may seem strange that we have such an estimate when the dual of  $L^{r'}(\mathbb{B}, \eta)$  is  $L^r(\mathbb{B}, \eta)$ , but the reason is of course that in the duality forms-currents there is no weights.

Let  $\mathcal{H}_p(\Omega)$  be the set of all  $(p, 0)$   $\bar{\partial}$  closed forms in  $\Omega$ . If  $p = 0$ ,  $\mathcal{H}_0(\Omega) = \mathcal{H}(\Omega)$  is the set of holomorphic functions in  $\Omega$ . If  $p > 0$ , we have  $\varphi \in \mathcal{H}_p(\Omega) \Rightarrow \varphi(z) = \sum_{|J|=p} a_J(z) dz^J$ , where  $dz^J := dz_{j_1} \wedge \cdots \wedge dz_{j_p}$  and the functions  $a_J(z)$  are holomorphic in  $\Omega$ . Hence in  $\mathbb{C}^n$ , a  $(p, 0)$   $\bar{\partial}$  closed form is a vector of global holomorphic functions in  $\Omega$ .

Let us consider the weight  $\eta = \eta_\epsilon := \mathbb{1}_\Omega(z) + \epsilon \mathbb{1}_{\mathbb{B} \setminus \Omega}(z)$  for a fixed  $\epsilon > 0$ . We shall use the lemma :

**Lemma 3.3.** *Let  $\omega$  be a  $(p, q)$  form in  $L_c^r(\mathbb{C}^n)$ ,  $r > 1$ ,  $K := \text{Supp } \omega \subset \Omega$ . Let  $\mathbb{B}$  be a ball containing  $\Omega$ . Suppose that  $t$  is such that  $\bar{\partial}t = 0$  if  $1 \leq q < n$  and  $\forall h \in \mathcal{H}_p(\Omega)$ ,  $\langle \omega, h \rangle = 0$  if  $q = n$ .*

*The form  $\mathcal{L}$  defined on  $(n-p, n-q+1)$  form  $\alpha \in L^{r'}(\mathbb{B}, \eta)$ ,  $\bar{\partial}$  closed in  $\mathbb{B}$  as follows:*

$$\mathcal{L}(\alpha) := (-1)^{p+q-1} \langle \omega, \varphi \rangle, \text{ where } \varphi \in L^{r'}(\mathbb{B}) \text{ is such that } \bar{\partial}\varphi = \alpha,$$

*is well defined and linear.*

Proof.

Because  $\epsilon > 0$  we have  $\alpha \in L^{r'}(\mathbb{B}, \eta) \Rightarrow \alpha \in L^{r'}(\mathbb{B})$  and such a  $\varphi$  exists because  $\mathbb{B}$  being s.p.c. is  $r'$  regular so there is  $\varphi \in L^{r'}(\mathbb{B})$  with  $\bar{\partial}\varphi = \alpha$ .

Let us see that  $\mathcal{L}$  is well defined :

suppose first that  $q < n$ . In order for  $\mathcal{L}$  to be well defined we need

$$\forall \varphi, \psi \in L_{(n-p, n-q)}^{r'}(\mathbb{B}), \bar{\partial}\varphi = \bar{\partial}\psi \Rightarrow \langle \omega, \varphi \rangle = \langle \omega, \psi \rangle.$$

This is meaningful because  $\omega \in L_c^r(\mathbb{C}^n)$ ,  $r > 1$ ,  $\text{Supp } \omega \subset \mathbb{B}$ .

Then we have  $\bar{\partial}(\varphi - \psi) = 0$  hence, because  $\mathbb{B}$  is  $r'$  regular, we can solve  $\bar{\partial}$  in  $L^{r'}(\mathbb{B})$  :

$$\exists \gamma \in L_{(n-p, n-q-1)}^{r'}(\mathbb{B}) :: \bar{\partial}\gamma = (\varphi - \psi).$$

So  $\langle \omega, \varphi - \psi \rangle = \langle \omega, \bar{\partial}\gamma \rangle = (-1)^{p+q-1} \langle \bar{\partial}\omega, \gamma \rangle = 0$ . Hence  $\mathcal{L}$  is well defined in that case.

Suppose now that  $q = n$ , then of course  $\bar{\partial}\omega = 0$  and we have that  $\varphi, \psi$  are  $(p, 0)$  forms hence  $\bar{\partial}(\varphi - \psi) = 0$  means that  $h := \varphi - \psi$  is a  $\bar{\partial}$  closed  $(p, 0)$  form hence  $h \in \mathcal{H}_p(\mathbb{B}) \subset \mathcal{H}_p(\Omega)$ . But by assumption we have  $\langle \omega, h \rangle = 0$  hence  $\mathcal{L}$  is also well defined in that case.

It remains to see that  $\mathcal{L}$  is linear, so let  $\alpha = \alpha_1 + \alpha_2$ , with  $\alpha_j \in L^{r'}(\mathbb{B}, \eta)$ ,  $\bar{\partial}\alpha_j = 0$ ,  $j = 1, 2$ ; we have  $\alpha = \bar{\partial}\varphi$ ,  $\alpha_1 = \bar{\partial}\varphi_1$  and  $\alpha_2 = \bar{\partial}\varphi_2$ , with  $\varphi, \varphi_1, \varphi_2$  in  $L^{r'}(\mathbb{B})$  so, because  $\bar{\partial}(\varphi - \varphi_1 - \varphi_2) = 0$ , we have  $\varphi = \varphi_1 + \varphi_2 + \bar{\partial}\psi$ , still with  $\psi$  in  $L^{r'}(\mathbb{B})$ , so

$$\mathcal{L}(\alpha) = (-1)^{p+q-1} \langle \omega, \varphi \rangle = (-1)^{p+q-1} \langle \omega, \varphi_1 + \varphi_2 + \bar{\partial}\psi \rangle = \mathcal{L}(\alpha_1) + \mathcal{L}(\alpha_2) + (-1)^{p+q-1} \langle \omega, \bar{\partial}\psi \rangle,$$

but  $\langle \omega, \bar{\partial}\psi \rangle = 0$ , hence  $\mathcal{L}(\alpha) = \mathcal{L}(\alpha_1) + \mathcal{L}(\alpha_2)$ . The same for  $\alpha = \lambda\alpha_1$  and the linearity. ■

**Theorem 3.4.** *Let  $\omega$  be a  $(p, q)$  form in  $L_c^r(\mathbb{C}^n)$ ,  $r > 1$ ,  $K := \text{Supp } t$ . Let  $\mathbb{B}$  be a ball containing  $K$  and let  $\Omega \subset \mathbb{B}$  be a  $r'$  regular domain containing  $K$ . Suppose that  $\omega$  is such that  $\bar{\partial}\omega = 0$  if  $1 \leq q < n$  and  $\forall h \in \mathcal{H}_p(\Omega)$ ,  $\langle \omega, h \rangle = 0$  if  $q = n$ .*

*Then there is a  $(p, q-1)$  form  $u$  in  $L_c^r(\mathbb{B})$ , 0 a.e. on  $\mathbb{B} \setminus \Omega$  and such that  $\bar{\partial}u = \omega$  as distributions and  $\|u\|_{L^r(\mathbb{B})} \leq C\|\omega\|_{L^r(\Omega)}$ .*

Proof.

Let us consider the weight  $\eta = \eta_\epsilon := \mathbb{1}_\Omega(z) + \epsilon \mathbb{1}_{\mathbb{B} \setminus \Omega}(z)$  for a fixed  $\epsilon > 0$  and the form  $\mathcal{L}$  defined in lemma 3.3. By lemma 3.3 we have that  $\mathcal{L}$  is a linear form on  $(n-p, n-q+1)$  form  $\alpha \in L^{r'}(\mathbb{B}, \eta)$ ,  $\bar{\partial}$  closed in  $\Omega$ .

If  $\alpha$  is a  $(n-p, n-q+1)$  form in  $L^{r'}(\mathbb{B}, \eta)$ , hence in  $L^r(\mathbb{B})$  because  $\epsilon > 0$ ,  $\bar{\partial}$  closed in  $\mathbb{B}$ , then,  $\alpha \in L^{r'}(\Omega)$  and is still  $\bar{\partial}$  closed in  $\Omega$ , hence there is a  $\psi \in L^{r'}(\Omega) :: \bar{\partial}\psi = \alpha$  because  $\Omega$  is  $r'$  regular. Let  $\varphi$  be a  $(n-p, n-q)$  form in  $L^{r'}(\mathbb{B})$  such that  $\bar{\partial}\varphi = \alpha$  in  $\mathbb{B}$  which can be used to define  $\mathcal{L}(\alpha)$ .

For  $q < n$ , we have  $\bar{\partial}(\varphi - \psi) = \alpha - \alpha = 0$  on  $\Omega$  and, because  $\text{Supp } t \subset \Omega \subset \mathbb{B}$ ,  $\bar{\partial}\omega = 0$  we get  $\langle \omega, \varphi - \psi \rangle = 0$  hence

$$\mathcal{L}(\alpha) = \langle \omega, \varphi \rangle = \langle \omega, \psi \rangle.$$

If  $q = n$ , we still have  $\bar{\partial}(\varphi - \psi) = \alpha - \alpha = 0$  on  $\Omega$ , and  $\varphi - \psi \in \mathcal{H}_p(\Omega)$  hence again by the hypothesis we still get

$$\mathcal{L}(\alpha) = \langle \omega, \varphi \rangle = \langle \omega, \psi \rangle.$$

In any cases, by Hölder inequalities

$$|\mathcal{L}(\alpha)| \leq \|\omega\|_{L^r(\Omega)} \|\psi\|_{L^{r'}(\Omega)}.$$

But by the  $r'$  regularity of  $\Omega$  there is a constant  $C$  such that

$$\|\psi\|_{L^{r'}(\Omega)} \leq C \|\alpha\|_{L^{r'}(\Omega)}.$$

Of course we have

$$\|\alpha\|_{L^{r'}(\Omega)} \leq \|\alpha\|_{L^{r'}(\mathbb{B}, \eta)}$$

because  $\eta = 1$  on  $\Omega$ , hence

$$|\mathcal{L}(\alpha)| \leq C \|\omega\|_{L^r(\Omega)} \|\alpha\|_{L^{r'}(\mathbb{B}, \eta)}.$$

So we have that the norm of  $\mathcal{L}$  is bounded on the subspace of  $\bar{\partial}$  closed forms in  $L^{r'}(\mathbb{B}, \eta)$  by  $C\|\omega\|_{L^r(\Omega)}$  which is *independent* of  $\epsilon$ .

We apply the Hahn-Banach theorem to extend  $\mathcal{L}$  with the *same* norm to *all*  $(n-p, n-q+1)$  forms in  $L^{r'}(\mathbb{B}, \eta)$ . As in Serre duality theorem ([11], p. 20) this is one of the main ingredient in the proof.

This means, by the definition of current, that there is a  $(p, q-1)$  current  $u$  which represents the extended form  $\mathcal{L}$  and such that

$$\sup_{\alpha \in L^{r'}(\mathbb{B}, \eta), \|\alpha\|=1} |\langle u, \alpha \rangle| \leq C \|\omega\|_{L^r(\Omega)}$$

and by lemma 3.2 with the weight  $\eta$ , this implies

$$\|u\|_{L^r(\mathbb{B}, \eta^{1-r})} \leq C \|\omega\|_{L^r(\Omega)}.$$

In particular  $\|u\|_{L^r(\mathbb{B})} \leq C \|\omega\|_{L^r(\Omega)}$  because with  $\epsilon < 1$  and  $r > 1$ , we have  $\eta^{1-r} \geq 1$ .

So applied to a  $\bar{\partial}$  closed  $(n-p, n-q+1)$  smooth form  $\alpha$  we get

$$\langle u, \alpha \rangle = \mathcal{L}(\alpha) = (-1)^{p+q-1} \langle \omega, \varphi \rangle,$$

with  $\varphi$  a smooth form such that  $\bar{\partial}\varphi = \alpha$ , i.e.  $\forall \varphi :: \bar{\partial}\varphi \in L^{r'}(\mathbb{B}, \eta)$ , we have  $\langle u, \bar{\partial}\varphi \rangle = (-1)^{p+q-1} \langle \omega, \varphi \rangle$  and this means precisely, by lemma 2.1, that  $\bar{\partial}u = \omega$ .

Now for  $\epsilon > 0$  with  $\eta_\epsilon(z) := \mathbf{1}_\Omega(z) + \epsilon \mathbf{1}_{\mathbb{B} \setminus \Omega}(z)$ , let  $u_\epsilon \in L^r(\mathbb{B}, \eta_\epsilon^{1-r})$  be the previous solution, then

$$\|u_\epsilon\|_{L^r(\mathbb{B})}^r \leq \int_{\mathbb{B}} |u_\epsilon|^r \eta_\epsilon^{1-r} dm \leq C^r \|\omega\|_{L^r(\Omega)}^r.$$

Replacing  $\eta$  by its value we get

$$\int_{\Omega} |u_\epsilon|^r dm + \int_{\mathbb{B} \setminus \Omega} |u_\epsilon|^r \epsilon^{1-r} dm \leq C^r \|\omega\|_{L^r(\Omega)}^r \Rightarrow \int_{\mathbb{B} \setminus \Omega} |u_\epsilon|^r \epsilon^{1-r} dm \leq C^r \|\omega\|_{L^r(\Omega)}^r$$

hence

$$\int_{\mathbb{B} \setminus \Omega} |u_\epsilon|^r dm \leq C^r \epsilon^{r-1} \|\omega\|_{L^r(\Omega)}^r.$$

Because  $C$  and the norm of  $\omega$  are independent of  $\epsilon$ , we have  $\|u_\epsilon\|_{L^r(\mathbb{B})}$  is uniformly bounded and  $r > 1 \Rightarrow L^r(\mathbb{B})$  is a dual, hence there is a sub-sequence  $\{u_{\epsilon_k}\}_{k \in \mathbb{N}}$  of  $\{u_\epsilon\}$  which converges weakly, when  $\epsilon_k \rightarrow 0$ , to a  $(p, q-1)$  form  $u$  in  $L^r(\mathbb{B})$ , still with  $\|u\|_{L^r(\mathbb{B})} \leq C\|\omega\|_{L^r(\Omega)}$ .

To see that this form  $u$  is 0 *a.e.* on  $\mathbb{B} \setminus \Omega$  let us take a component  $u_{I,J}$  of it ; it is the weak limit of the sequence of functions  $\{u_{\epsilon_k, I, J}\}$  which means, with the notations  $v := u_{I, J}$ ,  $v_k := u_{\epsilon_k, I, J}$

$$\forall f \in L^{r'}(\mathbb{B}), \int_{\mathbb{B}} v f dm = \lim_{k \rightarrow \infty} \int_{\mathbb{B}} v_k f dm.$$

As usual take  $f := \frac{\bar{v}}{|v|} \mathbb{1}_E$  where  $E := \{|v| > 0\} \cap (\mathbb{B} \setminus \Omega)$  then we get

$$\int_{\mathbb{B}} v f dm = \int_E |v| dm = \lim_{k \rightarrow \infty} \int_{\mathbb{B}} v_k f dm = \lim_{k \rightarrow \infty} \int_E \frac{v_k \bar{v}}{|v|} dm.$$

Now we have by Hölder

$$\left| \int_E \frac{v_k \bar{v}}{|v|} dm \right| \leq \|v_k\|_{L^r(E)} \|\mathbb{1}_E\|_{L^{r'}(E)}.$$

But

$$\|v_k\|_{L^r(E)}^r \leq \int_{\mathbb{B} \setminus \Omega} |u_{\epsilon_k}|^r dm \leq (\epsilon_k)^{r-1} \|\omega\|_{L^r(\Omega)}^r \rightarrow 0, \quad k \rightarrow \infty.$$

Hence

$$\left| \int_E |v| dm \right| \leq C^r \|\mathbb{1}_E\|_{L^{r'}(E)} (\epsilon_k)^{r-1} \|\omega\|_{L^r(\Omega)}^r \rightarrow 0$$

which implies  $m(E) = 0$  because on  $E$ ,  $|v| > 0$ .

This being true for all components of  $u$ , we get that the form  $u$  is 0 *a.e.* on  $\mathbb{B} \setminus \Omega$ .

So we get

$$\forall \varphi \in \mathcal{D}_{n-p, n-q}(\mathbb{B}), (-1)^{p+q-1} \langle \omega, \varphi \rangle = \langle u_\epsilon, \bar{\partial} \varphi \rangle \rightarrow \langle u, \bar{\partial} \varphi \rangle \Rightarrow \langle u, \bar{\partial} \varphi \rangle = (-1)^{p+q-1} \langle \omega, \varphi \rangle$$

hence  $\bar{\partial} u = \omega$  as distributions. ■

**Remark 3.5.** *The condition of orthogonality to  $\mathcal{H}_p(\Omega)$  in the case  $q = n$  is necessary : suppose there is a  $(p, n-1)$  current  $u$  such that  $\bar{\partial} u = \omega$  and  $u$  with compact support in  $\Omega$ , then if  $h \in \mathcal{H}_p(\Omega)$ , we have*

$$h \in \mathcal{H}_p(\Omega), \quad \langle \omega, h \rangle = \langle \bar{\partial} u, h \rangle = (-1)^{n+p} \langle u, \bar{\partial} h \rangle = 0.$$

*This condition was already seen for extension of CR functions, see [1] and the references therein.*

#### 4. CASE OF STEIN MANIFOLD.

Let  $\mathcal{H}_p(\Omega)$  be the set of all  $(p, 0)$   $\bar{\partial}$  closed forms in  $\Omega$ .

**Theorem 4.1.** *Let  $X$  be a Stein manifold and  $\Omega$  a  $r'$  regular domain in  $X$  (a relatively compact strictly pseudo-convex domain with smooth boundary for instance) with  $r'$  the conjugate exponent for  $r$ . Let  $\omega$  be a  $(p, q)$  current in  $L_c^r(\Omega)$ ,  $r > 1$ , such that  $\bar{\partial} \omega = 0$  if  $1 \leq q < n$  and  $\forall h \in \mathcal{H}_p(\Omega)$ ,  $\langle \omega, h \rangle = 0$  if  $q = n$ .*

*Let  $B$  be another  $r'$  regular domain in  $X$  such that  $\bar{\Omega} \subset B$ .*

*Then there is a  $(p, q-1)$  current  $u$  in  $L^r(B)$  such that  $\bar{\partial} u = \omega$  and which is 0 *a.e.* in  $B \setminus \Omega$ .*

*Proof.*

On a Stein manifold  $X$  we define first the "Lebesgue measure" as in Hörmander's book [6] section 5.2, with a hermitian metric locally equivalent to the usual one on any analytic coordinates system. Associated to this metric there is a volume form  $dV$  and we take it for the Lebesgue measure on  $X$ .



The remainder of the proof is identical to the proof of theorem 3.4. ■

### 5. FINER CONTROL OF THE SUPPORT.

Here we shall get a better control on the support of a solution.

**Theorem 5.1.** *Let  $\Omega$  be a  $r'$  regular domain in a Stein manifold  $X$  which possesses an exhaustive sequence of relatively compact  $r'$  regular open subsets. Then for any  $(p, q)$  form  $\omega$  in  $L^{r,c}(\Omega, dm)$ ,  $\bar{\partial}\omega = 0$ , if  $q < n$ , and  $\omega \perp \mathcal{H}_p$  if  $q = n$ , with support in  $\Omega \setminus C$  where  $C$  is also a  $r'$  regular domain, there is a  $u \in L^{r,c}(\Omega, dm)$  such that  $\bar{\partial}u = \omega$  and with support in  $\Omega \setminus C'$ , where  $C'$  is any compact set contained in  $C$ , provided that  $q \geq 2$ .*

*Proof.*

Let  $\omega$  be a  $(p, q)$  form with compact support in  $\Omega \setminus C$  then  $\omega$  is solvable in  $L^r$  by a  $v \in L^r_{p, q-1}(\Omega)$ ,  $\bar{\partial}v = \omega$ , with compact support in  $\Omega$ , by theorem ?? or, if  $\Omega$  is a polydisc in  $\mathbb{C}^n$ , and if  $\omega \in \mathcal{W}_q^r(\Omega)$  by the theorem in [2] by the coronas method.

Because  $\omega$  has compact support outside  $C$  we have  $\omega = 0$  in  $C$ ; this means that  $\bar{\partial}v = 0$  in  $C$ . Because  $C$  is  $r$  regular and  $q \geq 2$ , we have a  $h \in L^r_{p, q-2}(\Omega)$  s.t.  $\bar{\partial}h = v$  in  $C$ . Let  $C' \Subset C$  and  $\chi$  be a smooth function such that  $\chi = 1$  in  $C'$  and  $\chi = 0$  near  $\partial C$ ; then set

$$u := v - \bar{\partial}(\chi h).$$

We have that  $u = v - \chi \bar{\partial}h - \bar{\partial}\chi \wedge h = v - \chi v - \bar{\partial}\chi \wedge h$  hence  $u$  is in  $L^r$ ; moreover  $u = 0$  in  $C'$  because  $\chi = 1$  in  $C'$  hence  $\bar{\partial}\chi = 0$  there. Finally  $\bar{\partial}u = \bar{\partial}v - \bar{\partial}^2(\chi h) = \omega$  and we are done. ■

**Remark 5.2.** *The function  $h$  is given, in the strictly pseudo-convex case by Ovreliid's kernels, hence it is linear; if the primitive solution  $v$  is also linear in  $\omega$ , then the solution  $u$  is linear too. This is the case in  $\mathbb{C}^n$  with the solution given in [2].*

This theorem cannot be true for  $q = 1$  as shown by the following example :

Take a holomorphic function  $\varphi$  in the open unit ball  $B(0, 1)$  in  $\mathbb{C}^n$  such that it extends to no open ball of center 0 and radius  $> 1$ . For instance  $\varphi(z) := \exp\left(-\frac{z_1 + 1}{z_1 - 1}\right)$ . Take  $R < 1$ , then  $\varphi$  is  $\mathcal{C}^\infty(\bar{B}(0, R))$  hence by a theorem of Whitney  $\varphi$  extends smoothly to  $\mathbb{C}^n$ ; call  $\varphi_R$  this extension. Let  $\chi \in \mathcal{C}_c^\infty(B(0, 2))$  such that  $\chi = 1$  in the ball  $B(0, 3/2)$  and consider the  $(0, 1)$  form  $\omega := \bar{\partial}(\chi \varphi_R)$ . Then  $\text{Supp } \omega \subset B(0, 2) \setminus B(0, R)$ ,  $\omega$  is  $\bar{\partial}$  closed and is  $\mathcal{C}^\infty$  hence in  $L^r_{0,1}(B(0, 2))$ . Moreover  $B(0, R)$  is strictly pseudo-convex hence  $r'$  regular, but there is no function  $u$  such that  $\bar{\partial}u = \omega$  and  $u$  zero near the origin because any solution  $u$  will be C.R. on  $\partial B(0, R)$  and by Hartog's phenomenon will extend holomorphically to  $B(0, R)$  hence cannot be identically null near 0.

Never the less in the case  $q = 1$ , we have :

**Theorem 5.3.** *Let  $\Omega$  be a  $r'$  regular domain in a Stein manifold  $X$  which possesses an exhaustive sequence of relatively compact  $r'$  regular open subsets. Then for any  $(p, 1)$  form  $\omega$  in  $L^{r,c}(\Omega, dm)$ ,  $\bar{\partial}\omega = 0$ , with support in  $\Omega' \setminus C$  where  $\Omega'$  is also a  $r'$  regular domain such that  $\Omega' \Subset \Omega$  and  $C$  is a domain such that  $C \Subset \Omega$  and  $C \setminus \Omega' \neq \emptyset$ ; there is a  $u \in L^{r,c}(\Omega, dm)$  such that  $\bar{\partial}u = \omega$  and with support in  $\Omega \setminus C$ .*

*Proof.*

We have that  $\omega$  is solvable in  $L^r$  by a  $u \in L^r_{p,0}(\Omega')$  with compact support in  $\Omega'$ , by theorem ?? or, if  $\Omega$  and  $\Omega'$  are polydiscs in  $\mathbb{C}^n$ , by the theorem in [2] by the coronas method.. Then  $\bar{\partial}u = 0$  in  $C$  hence  $u$  is locally holomorphic in  $C$ . Because  $C \setminus \Omega' \neq \emptyset$ , there is an open set in  $C \setminus \Omega' \subset \Omega \setminus \Omega'$  where  $u$  is 0 and holomorphic, hence  $u$  is identically 0 in  $C$ ,  $C$  being connected. ■

**Corollary 5.4.** *Let  $\Omega$  be a polydisc in  $\mathbb{C}^n$ . Then for any  $(p, q)$  form  $\omega$  in  $L^{r,c}(\Omega) \cap \mathcal{W}_q^r(\Omega)$ ,  $\bar{\partial}\omega = 0$ , if  $q < n$ , and  $\omega \perp \mathcal{H}_p$  if  $q = n$ , with compact support in  $\Omega \setminus \{f = 0\}$  where  $f$  is holomorphic in  $\Omega$ , there is a  $u \in L^{r,c}(\Omega)$  such that  $\bar{\partial}u = \omega$  and  $u$  has its support still in  $\Omega \setminus \{f = 0\}$ . The solution  $u$  is linear with respect to  $\omega$ .*

*Proof.*

Because  $\omega$  has compact support outside  $\{f = 0\}$  this means that there is a  $\epsilon > 0$  such that  $\omega = 0$  in  $\{|f| < \epsilon\}$ . We have that  $C := \Omega \cap \{|f| < \epsilon\}$  is pseudo-convex hence we can find a sequence of smoothly strictly pseudo-convex domains  $\Omega_k \subset C$  such that  $\Omega_k \nearrow C$ . Choose  $k$  big enough to be sure that  $\Omega_k \supset \{|f| < \epsilon/2\}$ .

If  $q > 1$ , apply theorem 5.1 to  $C$ ,  $C' = \Omega_k$ , to get a solution  $u$  of  $\bar{\partial}u = \omega$  in  $L_{p,q-1}^r(\Omega)$  with support in  $\Omega \setminus \{f = 0\}$ .

If  $q = 1$  we can apply theorem 5.3 because  $\{f = 0\}$  goes necessarily up to the boundary of  $\Omega$  and we can always take a  $\Omega'$  strictly pseudo-convex containing the support of  $\omega$  and relatively compact in  $\Omega$ . We can always use the linear solution given by the coronas method for the primitive solution, hence because we can solve  $\bar{\partial}$  in  $L^r$  linearly in s.p.c. domains, each step is done linearly, so the complete solution is also linear. ■

**Remark 5.5.** *If  $\Omega$  is a pseudo-convex domain in a Stein manifold  $X$ , and if  $f$  is holomorphic in  $\Omega$ , then  $\Omega \setminus \{f = 0\}$  is Stein and we have already this result by theorem ??*

## 6. FUNCTIONAL ANALYSIS.

We shall show that, for  $(\Omega, \mu)$  a measured topological space and  $1 \leq r < \infty$ , the dual of  $L^{r,loc}(\Omega, \mu)$  is  $L^{r',c}(\Omega, \mu)$  where  $r'$  is the conjugate exponent of  $r$ , and  $L^{r',c}(\Omega, \mu)$  is the space of  $L^{r'}(\Omega, \mu)$  functions 0  $\mu$  a.e. outside of a compact set. And also that the dual of  $L^{r,c}(\Omega, \mu)$  is  $L^{r',loc}(\Omega, \mu)$ .

Let  $\{\Omega_j\}_{j \in \mathbb{N}}$  be an exhaustive sequence of relatively compact open sets in  $\Omega$ . We shall equip  $L^{r,loc}(\Omega, \mu)$  with the family of semi-norms :

$$\forall f \in L^{r,loc}(\Omega), \|f\|_{r,j} := \|f\|_{L^r(\Omega_j)}.$$

This is a Fréchet space.

$$\text{For } f \in L^{r,c}(\Omega, \mu) \text{ we set } \|f\|_r := \left( \int_{\Omega} |f|^r d\mu \right)^{1/r}.$$

**Theorem 6.1.** *For  $r \in [1, \infty[$  the dual of  $L^{r,loc}(\Omega, \mu)$  is  $L^{r',c}(\Omega, \mu)$ , with  $r'$  the conjugate exponent for  $r$ .*

*Proof.*

With its Fréchet topology, a linear form  $\mathcal{L}$  on  $L^{r,loc}(\Omega, \mu)$  is continuous if and only if :

$$\exists j \in \mathbb{N}, \exists C > 0 :: \forall f \in L^{r,loc}(\Omega, \mu), |\mathcal{L}(f)| \leq C \|f\|_{r,j}.$$

First for  $g \in L^{r',c}(\Omega, \mu)$  we associate the linear form

$$\forall f \in L^{r,loc}(\Omega, \mu), \mathcal{L}_g(f) := \int_{\Omega} f \bar{g} d\mu.$$

We have  $|\mathcal{L}_g(f)| \leq \|g\|_r \|f\|_{r,j}$  for any  $j \in \mathbb{N}$  such that  $\text{Supp } g \subset \Omega_j$  hence  $\mathcal{L}_g$  is continuous so  $g \in (L^{r,loc}(\Omega, \mu))'$ .

Conversely let  $\mathcal{L}$  be a continuous linear form on  $L^{r,loc}(\Omega, \mu)$ , this means

$$(6.3) \quad \exists J \in \mathbb{N}, \exists C > 0 :: \forall f \in L^{r,loc}(\Omega, \mu), |\mathcal{L}(f)| \leq C \|f\|_{r,J}.$$

Because the norms are increasing, we still have

$$\forall n \geq J, \forall f \in L^{r,loc}(\Omega, \mu), \quad |\mathcal{L}(f)| \leq C \|f\|_{r,n}.$$

This implies that for  $n \geq J$ , we have that, on  $\Omega_n$ ,  $\mathcal{L}$  is associated to a unique (class of) function  $g_n \in L^{r'}(\Omega_n)$ ,  $\|g_n\|_{r',n} \leq C$ , i.e.

$$(6.4) \quad \forall n \geq J, \forall f \in L^{r,loc}(\Omega, \mu), \quad \mathcal{L}(\mathbb{1}_{\Omega_n} f) = \int_{\Omega_n} f \bar{g}_n d\mu.$$

By uniqueness we have that  $g_n = g_{n-1}$  on  $\Omega_{n-1}$ .

Consider now the function  $f_n := \mathbb{1}_{E_n} \frac{\bar{g}_n}{|g_n|^{\frac{r-2}{r-1}}}$  with  $E_n := \Omega_n \setminus \Omega_J$ ; then

$$\|f_n\|_r^r = \int_{E_n} \left| \frac{\bar{g}_n}{|g_n|^{\frac{r-2}{r-1}}} \right|^r d\mu = \int_{E_n} |g_n|^{r'} d\mu \leq \int_{\Omega_n} |g_n|^{r'} d\mu \leq C^{r'}.$$

Hence  $f_n \in L^{r,loc}(\Omega, \mu)$ , and we have by (6.4) and by (6.3)

$$|L(\mathbb{1}_{\Omega_n} f_n)| = \left| \int_{\Omega_n} f_n \bar{g}_n d\mu \right| = \left| \int_{E_n} \frac{|g_n|^2}{|g_n|^{\frac{r-2}{r-1}}} d\mu \right| = \left| \int_{E_n} |g_n|^{r'} d\mu \right| \leq C \|f_n\|_{r,J} = 0,$$

because  $f_n = 0$  on  $\Omega_J$ . Hence  $g_n = 0$   $\mu$  a.e. on  $E_n$ . We have  $g_n = g_J$   $\mu$  a.e. on  $\Omega_J$  hence setting  $g := \mathbb{1}_{\Omega_J} g_J$  we get that  $g = g_n$   $\mu$  a.e. on  $\Omega_n$ . Because  $f - \mathbb{1}_{\Omega_n} f \rightarrow 0$  in  $L^{r,loc}(\Omega, \mu)$  we have

$$\mathcal{L}(f) = \lim_{n \rightarrow \infty} \mathcal{L}(\mathbb{1}_{\Omega_n} f) = \int_{\Omega} f \bar{g} d\mu,$$

by the continuity of  $\mathcal{L}$ , hence  $\mathcal{L}$  is associated to  $g$  which is in  $L^{r',c}(\Omega, \mu)$ . ■

**Theorem 6.2.** For  $r \in [1, \infty[$  the dual of  $L^{r,c}(\Omega, \mu)$  is  $L^{r',loc}(\Omega, \mu)$ , with  $r'$  the conjugate exponent for  $r$ .

Proof.

The space  $E := L^{r,c}(\Omega, \mu)$  is endowed with its topology of inductive limit  $(L^r(\Omega_n, \mu), T_n)_{n \in \mathbb{N}}$  with  $T_n : L^r(\Omega_n, \mu) \rightarrow L^{r,c}(\Omega, \mu)$  is such that  $\forall f \in L^r(\Omega_n, \mu)$ ,  $T_n f := \mathbb{1}_{\Omega_n} f \in L^{r,c}(\Omega, \mu)$ . A linear form  $\mathcal{L}$  on  $L^{r,c}(\Omega, \mu)$  is then continuous if and only if  $\mathcal{L}$  is continuous on each  $L^r(\Omega_n, \mu)$ , i.e.

$$\forall n \in \mathbb{N}, \exists C_n > 0 :: |\mathcal{L}(T_n f)| \leq C_n \|T_n f\|_n.$$

First to  $g \in L^{r',loc}(\Omega, \mu)$  we associate the linear form

$$\forall f \in L^{r,c}(\Omega, \mu), \quad \mathcal{L}_g(f) := \int_{\Omega} f \bar{g} d\mu.$$

We have

$$|\mathcal{L}_g(T_n f)| = \left| \int_{\Omega_n} f \bar{g} d\mu \right| \leq \|g\|_{L^r(\Omega_n)} \|T_n f\|_{L^{r,c}(\Omega, \mu)},$$

hence  $\mathcal{L}_g$  is continuous on  $L^{r,c}(\Omega, \mu)$ , so  $g \in (L^{r,c}(\Omega, \mu))'$ .

Conversely let  $\mathcal{L}$  be a continuous linear form on  $L^{r,c}(\Omega, \mu)$ , this means that, if we restrict  $\mathcal{L}$  to the subspace  $L^r(\Omega_n, \mu)$  we have

$$\exists C_n > 0 :: \forall f \in L^r(\Omega_n, \mu), \quad |\mathcal{L}(T_n f)| \leq C_n \|T_n f\|_n.$$

Hence by the usual duality between  $L^r(\Omega_n, \mu)$  and  $L^{r'}(\Omega_n, \mu)$  we have

$$\exists! g_n \in L^{r'}(\Omega_n, \mu) :: \|g_n\|_{L^{r'}(\Omega_n, \mu)} \leq C_n \text{ and } \mathcal{L}(T_n f) = \langle T_n f, g_n \rangle.$$

But  $\Omega_{n-1} \subset \Omega_n$  hence  $g_n = g_{n-1}$   $\mu$  a.e. on  $\Omega_{n-1}$  so we set  $g \in L^{r',loc}(\Omega)$  such that

$$g := g_n \text{ on } \Omega_n \text{ and } \forall f \in L^{r,c}(\Omega), \quad \mathcal{L}(f) = \langle f, g \rangle_{\Omega}$$

and we have that  $(L^{r,c}(\Omega, \mu))' \subset L^{r',loc}(\Omega)$ . ■

**Remark 6.3.** We can use the fact that  $L^{r,loc}(\Omega)$  is the projective limit of the projective system  $(L^r(\Omega_j), T_{jk})$  where  $\{\Omega_j\}_{j \in \mathbb{N}}$  is an exhaustive sequence of relatively compact open sets in  $\Omega$ , and  $T_{jk} : L^r(\Omega_k) \rightarrow L^r(\Omega_j)$  is defined by :

$$T_{jk}f := \mathbb{1}_{\Omega_j \cap \Omega_k} f.$$

We also have that  $L^{r,c}(\Omega)$  is the inductive limit of  $(L^r(\Omega_j), T_{jk})$  and we have that the dual of  $L^r(\Omega_j)$  is  $L^{r'}(\Omega_j)$  and the dual of an inductive limit is the projective limit with the dual inductive system ([10], Prop. 15, p 85). But we shall need the converse also in our special case, so we give explicitly the detailed proofs.

## 7. COMPLEX ANALYSIS.

Let  $\Omega$  be an open set in  $\mathbb{C}^n$  or in a complex manifold, and  $A$  a topological space of functions defined on  $\Omega$ . Let, as usual,  $\mathcal{D}(\Omega)$  be the space of  $C^\infty$  smooth functions with compact support in  $\Omega$  and  $\mathcal{D}'(\Omega)$  be the space of distributions in  $\Omega$ . Suppose that  $\mathcal{D}(\Omega) \subset A \subset \mathcal{D}'(\Omega)$ , with  $\mathcal{D}(\Omega)$  dense in  $A$ .

Associate to  $A$  the spaces of  $(p, q)$  currents  $A_{p,q}$  with coefficients in  $A$ . Moreover suppose that  $B = A'$ , the dual of  $A$ , is still a space of distributions and that the duality bracket is compatible with the distribution one.

We say that the equation  $\bar{\partial}f = \omega$  is solvable in  $A_{p,q}$  with  $0 \leq p \leq n$ ,  $1 \leq q \leq n$ , if

$$\forall \omega \in A_{p,q}, \bar{\partial}\omega = 0, \exists f \in A_{p,q-1} :: \bar{\partial}f = \omega.$$

Now suppose  $\bar{\partial}$  is solvable in  $A_{p,q+1}$  and in  $A_{p,q}$  and let  $\omega \in B_{n-p,n-q}$ ,  $\bar{\partial}\omega = 0$  still with  $0 \leq p \leq n$ ,  $1 \leq q \leq n$ , and consider the form

$$\forall \gamma \in A_{p,q+1}, \bar{\partial}\gamma = 0, \mathcal{L}_\omega(\gamma) := \langle f, \omega \rangle,$$

with  $\bar{\partial}f = \gamma$ , which exists by hypothesis, where  $\langle \cdot, \cdot \rangle$  is the duality bracket.

Let us see if this form is well defined, as in lemma 3.3 :

if  $\bar{\partial}f = \bar{\partial}g = \gamma$ , then  $\bar{\partial}(f - g) = 0$  hence  $\exists \varphi \in A_{p,q-1} :: \bar{\partial}\varphi = f - g$  so

$$\langle f - g, \omega \rangle = \langle \bar{\partial}\varphi, \omega \rangle = (-1)^{p+q} \langle \varphi, \bar{\partial}\omega \rangle = 0,$$

because we have the compatibility between the two brackets.

Hence  $\mathcal{L}_\omega$  is well defined and it is linear as in lemma 3.3.

As before set  $\mathcal{H}_p(\Omega)$  the set of  $(p, 0)$  form  $f$  such that  $\bar{\partial}f = 0$  in  $\Omega$ .

For the special case  $q = 0$ , we automatically have that  $\bar{\partial}\omega = 0$ , because  $\omega$  is a  $(n-p, n)$  form, hence to still get that  $\mathcal{L}_\omega(\gamma)$  is well defined if  $\bar{\partial}\gamma = 0$ , we need  $\langle f, \omega \rangle = 0$  as soon as  $f \in \mathcal{H}_p(\Omega)$ .

We say that the equation  $\bar{\partial}f = \omega$  is *continuously solvable* in  $A_{p,q+1}$  if it is solvable in  $A_{p,q}$  and in  $A_{p,q+1}$  and if, moreover,  $\mathcal{L}_\omega(\gamma)$  is a continuous linear form on the subspace  $\gamma \in A_{p,q+1}$ ,  $\bar{\partial}\gamma = 0$ .

**Theorem 7.1.** *If the equation  $\bar{\partial}f = \omega$  is continuously solvable in  $A_{p,q+1}$  then we can solve it in the dual space  $B_{n-p,n-q}$ . This means that  $\forall \omega \in B_{n-p,n-q}$ ,  $\bar{\partial}\omega = 0$  if  $1 \leq q \leq n-1$  and  $\langle \omega, f \rangle = 0$  if  $q = 0$  for  $f \in \mathcal{H}_p(\Omega)$ , there is a  $u \in B_{n-p,n-q-1}$  such that  $\bar{\partial}u = \omega$ .*

Proof.

Let  $\omega \in B_{n-p,n-q}$ ,  $\bar{\partial}\omega = 0$  if  $1 \leq q \leq n-1$  and  $\langle \omega, f \rangle = 0$  if  $q = n$  and  $f \in \mathcal{H}_p(\Omega)$ , and consider the form  $\mathcal{L} = \mathcal{L}_\omega$  defined above on the subspace  $\gamma \in A_{p,q+1}$ ,  $\bar{\partial}\gamma = 0$ , which exists by hypothesis in  $A_{p,q+1}$ . This linear form  $\mathcal{L}$  is well defined, linear and continuous by assumption so by Hahn-Banach, it extends to the whole  $A_{p,q+1}$  and is represented by a  $u \in B_{n-p,n-q-1}$  by duality. So we have

$$\mathcal{L}(\gamma) = \langle \gamma, u \rangle = \langle \bar{\partial}f, u \rangle = \mathcal{L}_\omega(f) = \langle f, \omega \rangle,$$

but  $\langle \bar{\partial}f, u \rangle = (-1)^{p+q} \langle f, \bar{\partial}u \rangle$  hence  $\langle f, \omega \rangle = (-1)^{p+q} \langle f, \bar{\partial}u \rangle$  provided that  $\bar{\partial}f = \gamma \in A_{p,q+1}$ .

This is true for  $\gamma = \bar{\partial}f$ , where  $f$  is a  $(p, q)$ -form with coefficients in  $\mathcal{D}(\Omega)$ , hence, by the density of  $\mathcal{D}(\Omega)$  in  $A$ , we have

$$\forall f \in A_{p,q}, \langle f, \omega \rangle = (-1)^{p+q} \langle f, \bar{\partial}u \rangle,$$

hence we find a  $u \in B_{n-p, n-q-1}$  such that  $(-1)^{p+q} \bar{\partial}u = \omega$ . ■

**Corollary 7.2.** *Let  $\Omega$  be a  $r$  regular domain in a Stein manifold  $X$  which possesses an exhaustive sequence of relatively compact  $r$  regular open subsets. Then for any  $(p, q)$  form  $\omega$  in  $L^{r,loc}(\Omega, dm)$ ,  $\bar{\partial}\omega = 0$ , if  $q < n$ , and  $\omega \perp \mathcal{H}_p$  if  $q = n$ , there is a  $u \in L^{r,loc}(\Omega, dm)$  such that  $\bar{\partial}u = \omega$ . This is the case when  $\Omega$  is a relatively compact pseudo-convex domain with smooth boundary in  $X$ .*

*Proof.*

We take for  $\mu$  the Lebesgue measure on  $\Omega$  and  $A := L^{r',c}(\Omega, \mu)$ ; we have clearly that  $\mathcal{D}(\Omega) \subset A \subset \mathcal{D}'(\Omega)$  and by theorem 4.1 we can solve the  $\bar{\partial}$  equation for  $(p, q+1)$  forms  $\bar{\partial}$  closed with coefficients in  $A$ . We have to show that the form  $\mathcal{L}$  is continuous on the subspace  $\gamma \in A_{p,q+1}$ ,  $\bar{\partial}\gamma = 0$ , but we have with  $\bar{\partial}f = \gamma$ ,

$$\mathcal{L}(\gamma) := \langle f, \omega \rangle \Rightarrow |\mathcal{L}(\gamma)| = |\langle f, \omega \rangle| \leq \|f\| \times \|\omega\|.$$

The theorem 4.1 gives also that  $\|f\|_{L^r(\Omega)} \leq C\|\gamma\|_{L^r(\Omega_j)}$ , where  $\Omega_j \subset \Omega$  is a  $r$  regular domain containing the support of  $\gamma$ . Hence

$$|\mathcal{L}(\gamma)| = |\langle f, \omega \rangle| \leq \|f\| \times \|\omega\| \leq C\|\gamma\| \times \|\omega\|,$$

and the continuity of  $\mathcal{L}$ ,  $C$  and  $\omega$  being fixed.

So by theorem 7.1 we can solve the  $\bar{\partial}$  equation for forms with coefficients in the dual space of  $L^{r',c}(\Omega, \mu)$  which is, by theorem 6.2,  $L^{r,loc}(\Omega, dm)$  and the corollary. ■

**Remark 7.3.** *If  $\Omega$  is a pseudo-convex domain in a Stein manifold  $X$ , then we can solve the  $\bar{\partial}$  equation with  $L^{2,loc}(\Omega, dm)$  estimates in this case ([6], theorem 5.2.4, p 125). So the case  $r = 2$  of the corollary 7.4 is known for a long time.*

*Moreover in this case there is no restriction for the  $(p, n)$  forms.*

**Corollary 7.4.** *Let  $\Omega$  be a domain in a Stein manifold  $X$  such that the equation  $\bar{\partial}u = f$  has a solution  $u$  in  $L_{p,q}^{r,loc}(\Omega)$  if  $f \in L_{p,q+1}^{r,loc}(\Omega)$  with the control :*

$$\forall j \in \mathbb{N}, \|u\|_j \leq C_j \|f\|_j ;$$

*then we can solve this equation with  $u \in L_{p,q-1}^{r,c}(\Omega)$  if  $f \in L_{p,q}^{r,c}(\Omega)$  and  $\bar{\partial}f = 0$  if  $q < n$ , and  $f \perp \mathcal{H}_p$  if  $q = n$ .*

*Proof.*

Exactly as above. ■

**Remark 7.5.** *A proof of this last corollary was sketched by a referee, without the condition for  $(p, n)$  forms, which, never the less, is necessary. He used a geometrical proof, with nice but heavy theorems.*

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