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Abstract. By a theorem of Andreotti and Grauert if ω is a (p,q) current, $q<n$, in a Stein manifold, $\bar\partial$ closed and with compact support, then there is a solution u to $\bar\partial u=\omega$ still with compact support. The aim of this work is to show that if moreover $\omega\in L^r(dm)$, where m is a suitable Lebesgue measure on the Stein manifold, then we have a solution u with compact support *and* in $L^r(dm)$.

As a consequence of the previous result and again by a duality argument we have $L^{r,loc}(dm)-L^{r,loc}(dm)$ solutions for the $\bar\partial$ equation on suitable domains.

1. Introduction.

Let ω be a $\bar\partial$ closed (p,q) form in $\mathbb{C}^n$ with compact support $K:=\operatorname{Supp}\omega$ and such that $\omega\in L^r(\mathbb{C}^n)$. Setting K in a ball $\mathbb{B}:=B(0,\ R)$ with R big enough, we know, by a theorem of Ovrelid [9], that we have a $(p,q-1)$ form $u\in L^r(\mathbb{B})$ such that $\bar\partial u=\omega$. On the other hand we also know, at least when $q<n$, that there is a current v with compact support such that $\bar\partial v=\omega$, by a theorem of Andreotti-Grauert [3].
So a natural question is : may we have a solution u of $\bar\partial u=\omega$ with compact support *and* in $L^r(\mathbb{C}^n)$?

We already answered this question by the affirmative in a join work with S. Mongodi [2] explicitly by the "method of coronas".

The aim of this work is to extend this result to Stein manifolds and for it we use a completely different approach inspired by the Serre duality [11]. Because Hahn Banach theorem is used, these results are no longer so explicit and constructive as in [2]. On the other hand the control of the support is better : the support of the solution u is contained in any r' regular domain containing the support of the data ω. The definition of r' regular domain will be given later on, but strictly pseudo convex domains are such domains.

Let (Ω,μ) be a measured domain with a positive measure μ, $L^{r,c}(\Omega,\mu)$ the space of $L^r(\Omega,\mu)$ functions 0 μ *a.e.* outside a compact set of Ω and $L^{r,loc}(\Omega,\mu)$ the space of μ measurable functions on Ω which are in $L^r(K,\mu)$ for K any compact set of Ω. We prove, for $1\leq r<\infty$, that the dual of $L^{r,c}(\Omega,\mu)$ is $L^{r',loc}(\Omega,\mu)$, with r' the conjugate exponent of r, and that the dual of $L^{r,loc}(\Omega,\mu)$ is $L^{r',c}(\Omega,\mu)$. Using this and the main result of this work, we show that if $\omega\in L^{r,loc}_{(p,q)}(\Omega,dm)$, $\bar\partial\omega=0$, there is a form $u\in L^{r,loc}_{(p,q-1)}(\Omega,dm)$ such that $\bar\partial u=\omega$, provided that Ω is a r' regular domain. This last result was suggested to me by a referee who thought it was already known, after the first version of this paper was already posted on HAL and ArXiv.

I am indebted to G. Tomassini who started my interest in this subject by e-mails on precisely this kind of questions and also to S. Mongodi for a lot of discussions by mails on the subject during the preparation of our join paper [2].

2. Duality.

We shall study a duality between currents inspired by the Serre duality [11].
Let $\Omega \subset \mathbb{C}^n$ be an open set in $\mathbb{C}^n$ and t a (p,q) current with compact support in Ω, noted $t \in \mathcal{D}'_{c,\ (p,q)}(\Omega)$.
Let also $\varphi \in \mathcal{C}^\infty_{(n-p,n-q)}(\Omega)$ a $(n-p,n-q)$ form in $\mathcal{C}^\infty(\Omega)$. We have that $t \wedge \varphi$ is a (n,n) current with compact support in Ω.

As usual we use the following notation for the pairing

$$\langle t,\ \varphi\rangle := t(\varphi),$$

where $t(\varphi)$ is the action, as a current, of t on the smooth form φ of complementary bi-degree.

Lemma 2.1. *Let Ω be an open set in $\mathbb{C}^n$, $t \in \mathcal{D}'_{c,\ (p,q)}(\Omega)$ a (p,q) current with compact support in Ω and $u \in \mathcal{D}'_{c,\ (p,q-1)}(\Omega)$. Then we have $\bar\partial t = u$ iff*

$$(*) \qquad \forall \varphi \in \mathcal{C}^\infty_{(n-p,n-q)}(\Omega),\ \langle t,\varphi\rangle = (-1)^{p+q-1}\big\langle u, \bar\partial\varphi\big\rangle.$$

Proof.
If $\bar\partial u = t$, let $\chi \in \mathcal{C}^\infty_c(\Omega)$ such that $\chi \equiv 1$ on the support of u, hence on the support of t. Then

$$\forall \varphi \in \mathcal{C}^\infty_{(n-p,n-q)}(\Omega),\ \langle t,\varphi\rangle = \langle t,\ \chi\varphi\rangle = \big\langle \bar\partial u, \chi\varphi\big\rangle = (-1)^{p+q-1}\big\langle u, \bar\partial(\chi\varphi)\big\rangle$$

by definition of the action of derivatives on currents. Hence

$$\langle t,\varphi\rangle = (-1)^{p+q-1}\big\langle u, \bar\partial\chi\wedge\varphi\big\rangle + (-1)^{p+q-1}\big\langle u, \chi\bar\partial\varphi\big\rangle = (-1)^{p+q-1}\big\langle u, \chi\bar\partial\varphi\big\rangle$$

because $\bar\partial\chi = 0$ on the support of u ; so

$$\langle t,\varphi\rangle = (-1)^{p+q-1}\big\langle u, \chi\bar\partial\varphi\big\rangle = (-1)^{p+q-1}\big\langle u, \bar\partial\varphi\big\rangle$$

because $\chi = 1$ on the support of u.

Conversely if we have $(*)$ we take $\chi \in \mathcal{C}^\infty_c(\Omega)$ such that $\chi \equiv 1$ on union of the support of u and the support of t then

$$\forall \varphi \in \mathcal{C}^\infty_{(n-p,n-q)}(\Omega),\ \langle t,\varphi\rangle = \langle t,\chi\varphi\rangle$$

because $\chi = 1$ on the support of t, and by $(*)$

$$\langle t,\chi\varphi\rangle = (-1)^{p+q-1}\big\langle u, \bar\partial(\chi\varphi)\big\rangle = \big\langle \bar\partial u, \chi\varphi\big\rangle = \big\langle \bar\partial u, \varphi\big\rangle$$

because $\chi = 1$ on the support of u hence on the support of $\bar\partial u$.
So

$$\forall \varphi \in \mathcal{C}^\infty_{(n-p,n-q)}(\Omega),\ \langle t,\varphi\rangle = \big\langle \bar\partial u, \varphi\big\rangle,$$

which means that $\bar\partial u = t$ as a current. ■

3. Solution of the $\bar\partial$ equation with compact support.

3.1. Domain r regular.

We shall use the definition.

Definition 3.1. *Let X be a complex manifold and Ω a domain in X ; let $r \in [1,\ \infty]$, we shall say that Ω is r regular, if for $p,q \in \{0,...,n\}$, $q \geq 1$, there is a constant $C = C_{p,q}$ such that for any (p,q) form ω, $\bar\partial$ closed in Ω and in $L^r(\Omega)$ there is a $(p,q-1)$ form $u \in L^r(\Omega)$ such that $\bar\partial u = \omega$ and $\|u\|_{L^r(\Omega)} \leq C\|\omega\|_{L^r(\Omega)}$.*

The L^r norms for forms will be defined later on.

Examples of 2 regular domains are the bounded pseudo-convex domains by Hörmander [6].

Examples of r regular domains in $\mathbb{C}^n$ are the bounded strictly pseudo-convex (s.p.c.) domains with smooth boundary by Ovrelid [9] ; the polydiscs in $\mathbb{C}^n$ by Charpentier [4], finite intersections of strictly pseudo-convex bounded domains with transverse intersections by Menini [8].

On a Stein manifold X we define first the "Lebesgue measure" as in Hörmander's book [6] section 5.2, with a hermitian metric locally equivalent to the usual one on any analytic coordinates system. Associated to this metric there is a volume form dV and we take it for the Lebesgue measure on X.

We can solve the $\bar\partial$ equation in strictly pseudo-convex domains with smooth boundary in X with L^r estimates for $(0,1)$ forms by use of N. Kerzman kernels [7] and the metric above.

We can solve the $\bar\partial$ equation in strictly pseudo-convex domains with smooth boundary in X with L^r estimates for all (p,q) forms by J-P. Demailly and C. Laurent ones([5], Remarque 4, page 596) but here the manifold has to be equiped with a metric with null curvature, in order to avoid parasitic terms.

So examples of r regular domains in Stein manifold are the relatively compact s.p.c. domains with smooth boundary.

3.2. **Weighted L^r spaces.**

Let $\mathbb{B} := B(0,R)$, the ball of $\mathbb{C}^n$ centered at 0 and of radius R. We shall need the following notations.

We note dm the Lebesgue measure on $\mathbb{C}^n$ and we shall define $L^r(\mathbb{B},w)$ to be the set of functions f defined on $\mathbb{B}$ such that

$$\|f\|^r_{L^r(\mathbb{B},\ w)} := \int_{\mathbb{B}} |f(z)|^r\, w(z) dm(z) < \infty,$$

with a weight $w(z) > 0$. As usual we set $L^r(\mathbb{B})$ for $L^r(\mathbb{B},1)$.

Let $\mathcal{I}_p$ be the set of multi-indices of length p in $(1,...,n)$. We shall use the following measure defined on $\Gamma := \mathbb{B}\times\mathcal{I}_p\times\mathcal{I}_q$ the following way :

$$d\mu(z,k,l) = d\mu_{w,p,q}(z,k,l) := w(z)dm(z) \otimes \sum_{|I|=p,\ |J|=q} \delta_I(k)\otimes\delta_J(l),$$

where $\delta_I(k) = 1$ if the multi-index k is equal to I and $\delta_I(k) = 0$ if not.

This means, if $f(z,I,J)$ is a function defined on Γ, that

$$\int f(z,k,l) d\mu_{w,p,q}(z,k,l) := \sum_{|I|=p,\ |J|=q} \int_{\mathbb{B}} f(z,I,J) w(z) dm(z).$$

If I is a multi-index of length p, let I^c be the unique multi-index, ordered increasingly, such that $I\cup I^c = (1,\ 2,...,\ n)$; then I^c is of length $n-p$.

To $t = \sum_{|I|=p,\ |J|=q} t_{I,J}(z) dz^I\wedge d\bar z^J$ a (p,q) form, we associate the function on Γ :

$$T(z,I,J) := (-1)^{s(I,J)} t_{I,J}(z),$$

where

$s(I,J) = 0$ if $dz^I\wedge d\bar z^J\wedge dz^{I^c}\wedge d\bar z^{J^c} = dz_1\wedge\cdots\wedge dz_n\wedge d\bar z_1\wedge\cdots\wedge d\bar z_n$ as a (n,n) form

and

$s(I,J) = 1$ if not.

If $\varphi = \sum_{|I|=p,\ |J|=q} \varphi_{I^c,J^c}(z) dz^I\wedge d\bar z^J$ is of complementary bi-degree, associate in the same manner :

$\Phi^*(z,\ I,\ J) := \varphi_{I^c,J^c}(z)$. This is still a function on Γ.

Now we have, for $1 < r < \infty$, if $T(z,I,J)$ is a function in $\mathbb{B}$ with $L^r(\mathbb{B})$ coefficients and with $\mu = \mu_{w,p,q}$,

$$\|T\|^r_{L^r(d\mu)} := \int |T(z,I,J)|^r\, d\mu_{w,p,q}(x,\ I,\ J) = \sum_{|I|=p,\ |J|=q} \|T(\cdot,I,J)\|^r_{L^r(\mathbb{B},w)}.$$

For $1\le r<\infty$ the dual of $L^r(\mu)$ is $L^{r'}(\mu)$ where r' is the conjugate of r, $\dfrac{1}{r}+\dfrac{1}{r'} = 1$, and the norm is defined analogously with r' replacing r.

We also know that, for $p,\ q$ fixed,

$$\|T\|_{L^r(\mu)} = \sup_{\Phi\in L^{r'}(\mu)} \frac{\left|\int T\Phi d\mu\right|}{\|\Phi\|_{L^{r'}(\mu)}}. \tag{3.1}$$

For a $(p,\ q)$ form $t = \sum_{|J|=p,\ |K|=q} t_{J,K} dz^J \wedge d\bar{z}^K$, and a weight $w > 0$ we define its norm by :

$$\|t\|^r_{L^r(\mathbb{B},w)} := \sum_{|J|=p,\ |K|=q} \|t_{J,K}\|^r_{L^r(\mathbb{B},w)} = \|T\|^r_{L^r(\mu)}. \tag{3.2}$$

Now we can state

Lemma 3.2. *Let $w > 0$ be a weight. If u is a (p,q) current defined on $(n-p, n-q)$ forms in $L^{r'}(\mathbb{B}, w)$ and such that*

$$\forall \alpha \in L^{r'}_{(n-p,n-q)}(\mathbb{B}, w),\ |\langle u,\ \alpha\rangle| \le C\|\alpha\|_{L^{r'}(\mathbb{B},w)},$$

then $\|u\|_{L^r(\mathbb{B},w^{1-r})} \le C$.

Proof.

Let us take the measure $\mu = \mu_{w,p,q}$ as above. Let Φ^* be the function on Γ associated to α and T the one associated to u. We have, by definition of the measure μ applied to the function

$$f(z,I,J) := T(z,I,J)w^{-1}\Phi^*(z,I,J),$$

$$\int Tw^{-1}\Phi^* d\mu = \int f(z,\ k,\ l) d\mu(z,k,l) := \sum_{|I|=p,\ |J|=q} \int_{\mathbb{B}} f(z,\ I,\ J) w(z) dm(z) =$$

$$= \sum_{|I|=p,\ |J|=q} \int_{\mathbb{B}} T(z,I,J) w^{-1}(z) \Phi^*(z,I,J) w(z) dm(z) = \langle u,\ \alpha\rangle,$$

by definition of T and Φ^*.

Hence we have, by (3.1)

$$\left\|Tw^{-1}\right\|_{L^r(\mu)} = \sup_{\Psi\in L^{r'}(\mu)} \frac{|\langle u,\ \alpha\rangle|}{\|\Psi\|_{L^{r'}(\mu)}}.$$

But $\|Tw^{-1}\|_{L^r(\mu)} = \|uw^{-1}\|_{L^r(\mathbb{B},\ w)}$ by (3.2), and

$$\left\|fw^{-1}\right\|^r_{L^r(\mathbb{B},w)} = \int_{\mathbb{B}} \left|f\eta^{-1}\right|^r w dm = \int_{\mathbb{B}} |f|^r\, w^{1-r} dm = \|f\|_{L^r(\mathbb{B},w^{1-r})},$$

so we get

$$\|u\|_{L^r(\mathbb{B},w^{1-r})} = \sup_{\Psi\in L^{r'}(\mu)} \frac{|\langle u,\ \alpha\rangle|}{\|\Psi\|_{L^{r'}(\mu)}},$$

which implies the lemma because, still by (3.1), we can take $\Psi = \Phi^*$ and $\|\Psi\|_{L^{r'}(\mu)} = \|\alpha\|_{L^{r'}(\mathbb{B},w)}$. ■

Let $\mathcal{H}_p(\Omega)$ be the set of all $(p,\ 0)$ $\bar\partial$ closed forms in Ω. If $p = 0$, $\mathcal{H}_0(\Omega) = \mathcal{H}(\Omega)$ is the set of holomorphic functions in Ω. If $p > 0$, we have $\varphi \in \mathcal{H}_p(\Omega) \Rightarrow \varphi(z) = \sum_{|J|=p} a_J(z) dz^J$, where $dz^J := dz_{j_1} \wedge \cdots \wedge dz_{j_p}$ and the functions $a_J(z)$ are holomorphic in Ω. Hence in $\mathbb{C}^n$, a $(p,0)$ $\bar\partial$ closed form is a vector of global holomorphic functions in Ω.

Let us consider the weight $\eta = \eta_\epsilon := \mathbb{1}_\Omega(z) + \epsilon \mathbb{1}_{\mathbb{B}\setminus\Omega}(z)$ for a fixed $\epsilon > 0$. We shall use the lemma :

Lemma 3.3. *Let t be a (p,q) form in $L^r_c(\mathbb{C}^n),\ r>1,\ K:=\operatorname{Supp} t$. Let $\mathbb{B}$ be a ball containing K. Suppose that t is such that $\bar\partial t=0$ if $1\le q<n$ and $\forall h\in\mathcal{H}_p(\Omega),\ \langle t,\ h\rangle=0$ if $q=n$.*
The form $\mathcal{L}$ defined on $(n-p,n-q+1)$ form $\alpha\in L^{r'}(\mathbb{B},\eta),\ \bar\partial$ closed in Ω as follows:

$$\mathcal{L}(\alpha):=(-1)^{p+q-1}\langle t,\varphi\rangle,\ \text{where } \varphi\in L^{r'}(\mathbb{B}) \text{ is such that } \bar\partial\varphi=\alpha,$$

is well defined and linear.

Proof.
Because $\epsilon>0$ we have $\alpha\in L^{r'}(\mathbb{B},\eta)\Rightarrow\alpha\in L^{r'}(\mathbb{B})$ and such a φ exists because $\mathbb{B}$ being s.p.c. is r' regular so there is $\varphi\in L^{r'}(\mathbb{B})$ with $\bar\partial\varphi=\alpha$.

Let us see that $\mathcal{L}$ is well defined :
suppose first that $q<n$. In order for $\mathcal{L}$ to be well defined we need

$$\forall\varphi,\psi\in L^{r'}_{(n-p,n-q)}(\mathbb{B}),\ \bar\partial\varphi=\bar\partial\psi\Rightarrow\langle t,\varphi\rangle=\langle t,\psi\rangle.$$

This is meaningful because $t\in L^r_c(\mathbb{C}^n),\ r>1,\ \operatorname{Supp} t\subset\mathbb{B}$.

Then we have $\bar\partial(\varphi-\psi)=0$ hence, because $\mathbb{B}$ is r' regular, we can solve $\bar\partial$ in $L^{r'}(\mathbb{B})$:

$$\exists\gamma\in L^{r'}_{(n-p,n-q-1)}(\mathbb{B})::\bar\partial\gamma=(\varphi-\psi).$$

So $\langle t,\varphi-\psi\rangle=\langle t,\bar\partial\gamma\rangle=(-1)^{p+q-1}\langle\bar\partial t,\gamma\rangle=0$. Hence $\mathcal{L}$ is well defined in that case.

Suppose now that $q=n$, then of course $\bar\partial t=0$ and we have that $\varphi,\ \psi$ are $(p,\ 0)$ forms hence $\bar\partial(\varphi-\psi)=0$ means that $h:=\varphi-\psi$ is a $\bar\partial$ closed $(p,\ 0)$ form hence $h\in\mathcal{H}_p(\mathbb{B})\subset\mathcal{H}_p(\Omega)$. But by assumption we have $\langle t,h\rangle=0$ hence $\mathcal{L}$ is also well defined in that case.

It remains to see that $\mathcal{L}$ is linear, so let $\alpha=\alpha_1+\alpha_2$, with $\alpha_j\in L^{r'}(\mathbb{B},\eta),\ \bar\partial\alpha_j=0,\ j=1,2$; we have $\alpha=\bar\partial\varphi,\ \alpha_1=\bar\partial\varphi_1$ and $\alpha_2=\bar\partial\varphi_2$, with $\varphi,\ \varphi_1,\ \varphi_2$ in $L^{r'}(\mathbb{B})$ so, because $\bar\partial(\varphi-\varphi_1-\varphi_2)=0$, we have $\varphi=\varphi_1+\varphi_2+\bar\partial\psi$, so

$$\mathcal{L}(\alpha)=(-1)^{p+q-1}\langle t,\varphi\rangle=(-1)^{p+q-1}\big\langle t,\varphi_1+\varphi_2+\bar\partial\psi\big\rangle=\ \mathcal{L}(\alpha_1)+\mathcal{L}(\alpha_2)+(-1)^{p+q-1}\big\langle t,\bar\partial\psi\big\rangle,$$

but $\big\langle t,\bar\partial\psi\big\rangle=0$, hence $\mathcal{L}(\alpha)=\mathcal{L}(\alpha_1)+\mathcal{L}(\alpha_2)$. The same for $\alpha=\lambda\alpha_1$ and the linearity. ■

Theorem 3.4. *Let t be a (p,q) form in $L^r_c(\mathbb{C}^n),\ r>1,\ K:=\operatorname{Supp} t$. Let $\mathbb{B}$ be a ball containing K and let $\Omega\subset\mathbb{B}$ be a r' regular domain containing K. Suppose that t is such that $\bar\partial t=0$ if $1\le q<n$ and $\forall h\in\mathcal{H}_p(\Omega),\ \langle t,\ h\rangle=0$ if $q=n$.*
Then there is a $(p,q-1)$ form u in $L^r_c(\mathbb{B}),\ 0$ a.e. on $\mathbb{B}\backslash\Omega$ and such that $\bar\partial u=t$ as distributions and $\|u\|_{L^r(\mathbb{B})}\le C\|t\|_{L^r(\Omega)}$.

Proof.
Let us consider the weight $\eta=\eta_\epsilon:=\mathbb{1}_\Omega(z)+\epsilon\mathbb{1}_{\mathbb{B}\backslash\Omega}(z)$ for a fixed $\epsilon>0$ and the form $\mathcal{L}$ defined in lemma 3.3. By lemma 3.3 we have that $\mathcal{L}$ is a linear form on $(n-p,n-q+1)$ form $\alpha\in L^{r'}(\mathbb{B},\eta),\ \bar\partial$ closed in Ω.

If α is a $(n-p,n-q+1)$ form in $L^{r'}(\mathbb{B},\eta),\ \bar\partial$ closed in $\mathbb{B}$, then, $\alpha\in L^{r'}(\Omega)$ and is still $\bar\partial$ closed in Ω, hence there is a $\psi\in L^{r'}(\Omega)::\bar\partial\psi=\alpha$ because Ω is r' regular.
For $q<n$, we have $\bar\partial(\varphi-\psi)=\alpha-\alpha=0$ on Ω and, because $\operatorname{Supp} t\subset\Omega\subset\mathbb{B},\ \bar\partial t=0$ we get $\langle t,\ \varphi-\psi\rangle=0$ hence

$$\mathcal{L}(\alpha)=\langle t,\ \varphi\rangle=\langle t,\ \psi\rangle.$$

If $q=n$, we still have $\bar\partial(\varphi-\psi)=\alpha-\alpha=0$ on Ω, and $\varphi-\psi\in\mathcal{H}_p(\Omega)$ hence again by the hypothesis we still get

$$\mathcal{L}(\alpha)=\langle t,\ \varphi\rangle=\langle t,\ \psi\rangle.$$

In any cases, by Hölder inequalities

$$|\mathcal{L}(\alpha)|\le\|t\|_{L^r(\Omega)}\|\psi\|_{L^{r'}(\Omega)}.$$

But by the r' regularity of Ω there is a constant C such that

$$\|\psi\|_{L^{r'}(\Omega)} \leq C\|\alpha\|_{L^{r'}(\Omega)}.$$

Of course we have

$$\|\alpha\|_{L^{r'}(\Omega)} \leq \|\alpha\|_{L^{r'}(\mathbb{B},\ \eta)}$$

because $\eta = 1$ on Ω, hence

$$|\mathcal{L}(\alpha)| \leq C\|t\|_{L^r(\Omega)}\|\alpha\|_{L^{r'}(\mathbb{B},\ \eta)}.$$

So we have that the norm of $\mathcal{L}$ is bounded on the subspace of $\bar\partial$ closed forms in $L^{r'}(\mathbb{B},\eta)$ by $C\|t\|_{L^r(\Omega)}$ which is *independent* of ϵ.

We apply the Hahn-Banach theorem to extend $\mathcal{L}$ with the *same* norm to *all* $(n-p, n-q+1)$ forms in $L^{r'}(\mathbb{B},\ \eta)$. As in Serre duality theorem ([11], p. 20) this is one of the main ingredient in the proof.

This means, by the definition of current, that there is a $(p, q-1)$ current u which represents the extended form $\mathcal{L}$ and such that

$$\sup_{\alpha\in L^{r'}(\mathbb{B},\eta),\ \|\alpha\|=1} |\langle u, \alpha\rangle| \leq C\|t\|_{L^r(\Omega)}$$

and by lemma 3.2 with the weight η, this implies

$$\|u\|_{L^r(\mathbb{B},\eta^{1-r})} \leq C\|t\|_{L^r(\Omega)}.$$

In particular $\|u\|_{L^r(\mathbb{B})} \leq C\|t\|_{L^r(\Omega)}$ because with $\epsilon < 1$ and $r > 1$, we have $\eta^{1-r} \geq 1$.

So applied to a $\bar\partial$ closed $(n-p, n-q+1)$ smooth form α we get

$\langle u, \alpha\rangle = (-1)^{p+q-1}\langle t, \varphi\rangle$, with φ a smooth form such that $\bar\partial\varphi = \alpha$, i.e. $\forall\varphi :: \bar\partial\varphi \in L^{r'}(\mathbb{B},\ \eta)$, we have $\left\langle u, \bar\partial\varphi\right\rangle = (-1)^{p+q-1}\langle t, \varphi\rangle$ and this means precisely, by lemma 2.1, that $\bar\partial u = t$.

Now for $\epsilon > 0$ with $\eta_\epsilon(z) := \mathbb{1}_\Omega(z) + \epsilon\mathbb{1}_{\mathbb{B}\setminus\Omega}(z)$, let $u_\epsilon \in L^r(\mathbb{B}, \eta_\epsilon^{1-r})$ be the previous solution, then

$$\|u_\epsilon\|^r_{L^r(\mathbb{B})} \leq \int_{\mathbb{B}} |u_\epsilon|^r \eta^{1-r} dm \leq C^r\|t\|^r_{L^r(\Omega)}.$$

Replacing η by its value we get

$$\int_\Omega |u_\epsilon|^r\, dm + \int_{\mathbb{B}\setminus\Omega} |u_\epsilon|^r\, \epsilon^{1-r} dm \leq C^r\|t\|^r_{L^r(\Omega)} \Rightarrow \int_{\mathbb{B}\setminus\Omega} |u_\epsilon|^r\, \epsilon^{1-r} dm \leq C^r\|t\|^r_{L^r(\Omega)}$$

hence

$$\int_{\mathbb{B}\setminus\Omega} |u_\epsilon|^r\, dm \leq C^r\epsilon^{r-1}\|t\|^r_{L^r(\Omega)}.$$

Because C and the norm of t are independent of ϵ, we have $\|u_\epsilon\|_{L^r(\mathbb{B})}$ is uniformly bounded and $r > 1 \Rightarrow L^r(\mathbb{B})$ is a dual, hence there is a sub-sequence $\{u_{\epsilon_k}\}_{k\in\mathbb{N}}$ of $\{u_\epsilon\}$ which converges weakly, when $\epsilon_k \to 0$, to a $(p, q-1)$ form u in $L^r(\mathbb{B})$, still with $\|u\|_{L^r(\mathbb{B})} \leq C\|t\|_{L^r(\Omega)}$.

To see that this form u is 0 *a.e.* on $\mathbb{B}\setminus\Omega$ let us take a component $u_{I,J}$ of it ; it is the weak limit of the sequence of functions $\{u_{\epsilon_k,I,J}\}$ which means, with the notations $v := u_{I,J},\ v_k := u_{\epsilon_k,I,J}$

$$\forall f \in L^{r'}(\mathbb{B}),\ \int_{\mathbb{B}} vf dm = \lim_{k\to\infty}\int_{\mathbb{B}} v_k f dm.$$

As usual take $f := \dfrac{\bar v}{|v|}\mathbb{1}_E$ where $E := \{|v| > 0\} \cap (\mathbb{B}\setminus\Omega)$ then we get

$$\int_{\mathbb{B}} vf dm = \int_E |v|\, dm = \lim_{k\to\infty}\int_{\mathbb{B}} v_k f dm = \lim_{k\to\infty}\int_E \frac{v_k\bar v}{|v|} dm.$$

Now we have by Hölder

$$\left|\int_E \frac{v_k\bar v}{|v|} dm\right| \leq \|v_k\|_{L^r(E)}\|\mathbb{1}_E\|_{L^{r'}(E)}.$$

But

$$\|v_k\|^r_{L^r(E)} \leq \int_{\mathbb{B}\setminus\Omega} |u_{\epsilon_k}|^r\, dm \leq (\epsilon_k)^{r-1}\|t\|_{L^r(\Omega)} \to 0,\ k \to \infty.$$

Hence

$$\left| \int_E |v|\, dm \right| \leq \ C^r \|\mathbb{1}_E\|_{L^{r'}(E)} (\epsilon_k)^{r-1} \|t\|^r_{L^r(\Omega)} \to 0$$

which implies $m(E) = 0$ because on $E,\ |v| > 0$.
This being true for all components of u, we get that the form u is 0 *a.e.* on $\mathbb{B}\backslash\Omega$.

So we get

$$\forall \varphi \in \mathcal{D}_{n-p,n-q}(\mathbb{B}),\ (-1)^{p+q-1}\langle t,\ \varphi\rangle = \left\langle u_\epsilon,\ \bar\partial\varphi\right\rangle \to \left\langle u,\ \bar\partial\varphi\right\rangle \Rightarrow \left\langle u,\ \bar\partial\varphi\right\rangle = (-1)^{p+q-1}\langle t,\ \varphi\rangle$$

hence $\bar\partial u = t$ as distributions. ■

Remark 3.5. *The condition of orthogonality to the holomorphic functions in the case $q = n$ was already seen for extension of CR functions see* [1] *and the references therein.*

4. Case of Stein manifold.

Let $\mathcal{H}_p(\Omega)$ be the set of all $(p,\ 0)$ $\bar\partial$ closed forms in Ω. Still we have that $\mathcal{H}_0(\Omega) = \mathcal{H}(\Omega)$, the set of holomorphic functions in Ω, but if $p > 1,\ \varphi \in \mathcal{H}_p(\Omega)$ is a priori no longer a vector of global holomorphic functions in Ω.

Theorem 4.1. *Let X be a Stein manifold and Ω a r' regular domain in X (a relatively compact strictly pseudo-convex domain with smooth boundary for instance) with r' the conjugate exponent for r. Let t be a (p,q) current in $L^r_c(\Omega),\ r > 1$, such that $\bar\partial t = 0$ if $1 \leq q < n$ and $\forall h \in \mathcal{H}_p(\Omega),\ \langle t, h\rangle = 0$ if $q = n$.*
Let B be another r' regular domain in X such that $\bar\Omega \subset B$.

Then there is a $(p, q-1)$ current u in $L^r(B)$ such that $\bar\partial u = t$ and which is 0 a.e. in $B\backslash\Omega$.

Proof.
On a Stein manifold X we define first the "Lebesgue measure" as in Hörmander's book [6] section 5.2, with a hermitian metric locally equivalent to the usual one on any analytic coordinates system. Associated to this metric there is a volume form dV and we take it for the Lebesgue measure on X.

The remainder of the proof is identical to the proof of theorem 3.4. ■

5. Functionnal analysis.

We shall show that, for (Ω, μ) a measured topological space and $1 \leq r < \infty$, the dual of $L^{r,loc}(\Omega,\mu)$ is $L^{r',c}(\Omega,\mu)$ where r' is the conjugate exponent of r, and $L^{r',c}(\Omega,\mu)$ is the space of $L^{r'}(\Omega,\mu)$ functions 0 μ *a.e.* outside of a compact set. And also that the dual of $L^{r,c}(\Omega,\mu)$ is $L^{r',loc}(\Omega,\mu)$.

Let $\{\Omega_j\}_{j\in\mathbb{N}}$ be a exhaustive sequence of relatively compact open sets in Ω. We shall equip $L^{r,loc}(\Omega,\mu)$ with a a family of semi-norms :

$$\forall f \in L^{r,loc}(\Omega),\ \|f\|_{r,j} := \|f\|_{L^r(\Omega_j)}.$$

This is a Frechet space.

For $f \in L^{r,c}(\Omega,\mu)$ we set $\|f\|_r := \left(\int_\Omega |f|^r\, d\mu\right)^{1/r}$.

Theorem 5.1. *For $r \in [1, \infty[$ the dual of $L^{r,loc}(\Omega,\mu)$ is $L^{r',c}(\Omega,\mu)$, with r' the conjugate exponent for r.*

Proof.
With its Fréchet topology, a linear form $\mathcal{L}$ on $L^{r,loc}(\Omega,\mu)$ is continuous if and only if :

$$\exists j \in \mathbb{N},\ \exists C > 0 :: \forall f \in L^{r,loc}(\Omega,\mu),\ |\mathcal{L}(f)| \leq C\|f\|_{r,j}.$$

First for $g \in L^{r',c}(\Omega,\mu)$ we associate the linear form

$$\forall f \in L^{r,loc}(\Omega,\mu),\ \mathcal{L}_g(f) := \int_\Omega f\bar{g}d\mu.$$

We have $|\mathcal{L}_g(f)| \leq \|g\|_r\|f\|_{r,j}$ for any $j \in \mathbb{N}$ such that $\operatorname{Supp} g \subset \Omega_j$ hence $\mathcal{L}_g$ is continuous so $g \in (L^{r,loc}(\Omega,\mu))'$.

Conversely let $\mathcal{L}$ be a continuous linear form on $L^{r,loc}(\Omega,\mu)$, this means

$$\exists J \in \mathbb{N},\ \exists C > 0 :: \forall f \in L^{r,loc}(\Omega,\mu),\ \ |\mathcal{L}(f)| \leq C\|f\|_{r,J}. \tag{5.3}$$

Because the norms are increasing, we still have

$$\forall n \geq J,\ \forall f \in L^{r,loc}(\Omega,\mu),\ \ |\mathcal{L}(f)| \leq C\|f\|_{r,n}.$$

This implies that for $n \geq J$, we have that, on Ω_n, $\mathcal{L}$ is associated to a unique (class of) function $g_n \in L^{r'}(\Omega_n)$, $\|g_n\|_{r',n} \leq C$, i.e.

$$\forall n \geq J,\ \forall f \in L^{r,loc}(\Omega,\mu),\ \mathcal{L}(\mathbb{1}_{\Omega_n}f) = \int_{\Omega_n} f\bar{g}_n d\mu. \tag{5.4}$$

By uniqueness we have that $g_n = g_{n-1}$ on Ω_{n-1}.

Consider now the function $f_n := \mathbb{1}_{E_n}\frac{\bar{g}_n}{|g_n|^{\frac{r-2}{r-1}}}$ with $E_n := \Omega_n\backslash\Omega_J$; then

$$\|f_n\|_r^r = \int_{E_n}\left|\frac{\bar{g}_n}{|g_n|^{\frac{r-2}{r-1}}}\right|^r d\mu = \int_{E_n}|g_n|^{r'}d\mu \leq \int_{\Omega_n}|g_n|^{r'}d\mu \leq C^{r'}.$$

Hence $f_n \in L^{r,loc}(\Omega,\mu)$, and we have by (5.4) and by (5.3)

$$|L(\mathbb{1}_{\Omega_n}f_n)| = \left|\int_{\Omega_n} f_n\bar{g}_n d\mu\right| = \left|\int_{E_n}\frac{|g_n|^2}{|g_n|^{\frac{r-2}{r-1}}}d\mu\right| = \left|\int_{E_n}|g_n|^{r'}d\mu\right| \leq C\|f_n\|_{r,J} = 0,$$

because $f_n = 0$ on Ω_J. Hence $g_n = 0$ μ *a.e.* on E_n. We have $g_n = g_J$ μ *a.e.* on Ω_J hence setting $g := \mathbb{1}_{\Omega_j}g_J$ we get that $g = g_n$ μ *a.e.* on Ω_n. Because $f - \mathbb{1}_{\Omega_n}f \to 0$ in $L^{r,loc}(\Omega,\mu)$ we have

$$\mathcal{L}(f) = \lim_{n\to\infty}\mathcal{L}(\mathbb{1}_{\Omega_n}) = \int_\Omega f\bar{g}d\mu,$$

by the continuity of $\mathcal{L}$, hence $\mathcal{L}$ is associated to g which is in $L^{r',c}(\Omega,\mu)$. ■

Theorem 5.2. *For $r \in [1,\infty[$ the dual of $L^{r,c}(\Omega,\mu)$ is $L^{r',loc}(\Omega,\mu)$, with r' the conjugate exponent for r.*

Proof.

The space $E := L^{r,c}(\Omega,\mu)$ is endowed with its topology of inductive limit $(L^r(\Omega_n,\mu),\ T_n)_{n\in\mathbb{N}}$ with $T_n : L^r(\Omega_n,\mu) \to L^{r,c}(\Omega,\mu)$ is such that $\forall f \in L^r(\Omega_n,\mu),\ T_nf := \mathbb{1}_{\Omega_n}f \in L^{r,c}(\Omega,\mu)$. A linear form $\mathcal{L}$ on $L^{r,c}(\Omega,\mu)$ is then continuous if and only if $\mathcal{L}$ is continuous on each $L^r(\Omega_n,\mu)$, i.e.

$$\forall n \in \mathbb{N},\ \exists C_n > 0 :: |\mathcal{L}(T_nf)| \leq C_n\|T_nf\|_n.$$

First to $g \in L^{r',loc}(\Omega,\mu)$ we associate the linear form

$$\forall f \in L^{r,c}(\Omega,\mu),\ \mathcal{L}_g(f) := \int_\Omega f\bar{g}d\mu.$$

We have

$$|\mathcal{L}_g(T_nf)| = \left|\int_{\Omega_n} f\bar{g}d\mu\right| \leq \|g\|_{L^r(\Omega_n)}\|T_nf\|_{L^{r,c}(\Omega,\mu)},$$

hence $\mathcal{L}_g$ is continuous on $L^{r,c}(\Omega,\mu)$, so $g \in (L^{r,c}(\Omega,\mu))'$.

Conversely let $\mathcal{L}$ be a continuous linear form on $L^{r,c}(\Omega,\mu)$, this means that, if we restrict $\mathcal{L}$ to the subspace $L^r(\Omega_n,\mu)$ we have

$$\exists C_n>0 :: \forall f\in L^r(\Omega_n,\mu),\ |\mathcal{L}(T_nf)|\leq C_n\|T_nf\|_n.$$

Hence by the usual duality between $L^r(\Omega_n,\mu)$ and $L^{r'}(\Omega_n,\mu)$ we have

$$\exists! g_n\in L^{r'}(\Omega_n,\mu) :: \|g_n\|_{L^{r'}(\Omega_n,\mu)}\leq C_n \text{ and } \mathcal{L}(T_nf)=\langle T_nf,g_n\rangle.$$

But $\Omega_{n-1}\subset\Omega_n$ hence $g_n=g_{n-1}$ μ *a.e.* on Ω_{j-1} so we set $g\in L^{r',loc}(\Omega)$ such that

$$g:=g_n \text{ on } \Omega_n \text{ and } \forall f\in L^{r,c}(\Omega),\ \mathcal{L}(f)=\langle f,g\rangle_\Omega$$

and we have that $(L^{r,c}(\Omega,\mu),)'\subset L^{r',loc}(\Omega)$. ■

Remark 5.3. *We can use the fact that $L^{r,loc}(\Omega)$ is the projective limit of the projective system $(L^r(\Omega_j),\ T_{jk})$ where $\{\Omega_j\}_{j\in\mathbb{N}}$ is an exhaustive sequence of relatively compact open sets in Ω, and $T_{jk}\ :\ L^r(\Omega_k)\to L^r(\Omega_j)$ is defined by :*

$$T_{jk}f:=\mathbb{1}_{\Omega_j\cap\Omega_k}f.$$

We also have that $L^{r,c}(\Omega)$ is the inductive limit of $(L^r(\Omega_j),\ T_{jk})$ and we have that the dual of $L^r(\Omega_j)$ is $L^{r'}(\Omega_j)$ and the dual of an inductive limit is the projective limit with the dual inductive system ([10], Prop. 15, p 85). But we shall need the converse also in our special case, so we give the detailed proofs.

6. Complex analysis.

Let Ω be an open set in $\mathbb{C}^n$ or in a complex manifold, and A a topological space of functions defined on Ω. Let, as usual, $\mathcal{D}(\Omega)$ be the space of $\mathcal{C}^\infty$ smooth functions with compact support in Ω and $\mathcal{D}'(\Omega)$ be the space of distributions in Ω. Suppose that $\mathcal{D}(\Omega)\subset A\subset\mathcal{D}'(\Omega)$, with $\mathcal{D}(\Omega)$ dense in A.

Associate to A the spaces of (p,q) currents $A_{p,q}$ with coefficients in A. Moreover suppose that $B=A'$, the dual of A, is still a space of distributions and that the duality bracket is compatible with the distribution one. We say that the equation $\bar\partial f=\omega$ is solvable in $A_{p,q}$ with $0\leq p\leq n,\ 1\leq q\leq n$, if

$$\forall\omega\in A_{p,q},\bar\partial\omega=0,\ \exists f\in A_{p,q-1} :: \bar\partial f=\omega.$$

Now let $\omega\in B_{n-p,n-q},\bar\partial\omega=0$ still with $0\leq p\leq n,\ 1\leq q\leq n$, and consider the form

$$\forall f\in A_{p,q},\ \mathcal{L}_\omega(f):=\langle f,\omega\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the duality bracket.

This form is well defined and by hypothesis we have $\mathcal{L}_\omega(f)=0$ if $\bar\partial f=0$, because, if so, there is a $\varphi\in A_{p,q-1} :: \bar\partial\varphi=f$, hence $\mathcal{L}_\omega(f)=\big\langle\bar\partial\varphi,\omega\big\rangle=(-1)^{p+q}\big\langle\varphi,\bar\partial\omega\big\rangle=0$, because we have the compatibility between the two brackets.

For the special case $q=0$, we automatically have that $\bar\partial\omega=0$, because ω is a $(n-p,n)$ form, hence to still get that $\mathcal{L}_\omega(f)=0$ if $\bar\partial f=0$, we need $\langle f,\omega\rangle=0$ as soon as the $(p,0)$ form f verifies $\bar\partial f=0$.

So $\mathcal{L}_\omega(f)$ depends only on $\bar\partial f$. Now suppose that $\bar\partial$ is solvable in $A_{p,q+1}$ and consider the form $\mathcal{F}$ defined on the subspace $\gamma\in A_{p,q+1},\bar\partial\gamma=0$ by $\mathcal{F}(\gamma):=\mathcal{L}_\omega(f)$, with $\bar\partial f=\gamma$, which exists by hypothesis in $A_{p,q+1}$. As in lemma 3.3, $\mathcal{F}$ is well defined and linear. We say that the equation $\bar\partial f=\omega$ is *continuously* solvable in $A_{p,q+1}$ if it is solvable in $A_{p,q+1}$ and if, moreover, $\mathcal{F}(\gamma)$ is a continuous linear form on the subspace $\gamma\in A_{p,q+1},\bar\partial\gamma=0$.

Theorem 6.1. *If the equation $\bar\partial f=\omega$ is continuously solvable in $A_{p,q+1}$ then we can solve it in its dual $B_{n-p,n-q}$.*

Proof.

Consider the form $\mathcal{F}$ defined above on the subspace $\gamma \in A_{p,q+1}, \bar\partial\gamma = 0$, which exists by hypothesis in $A_{p,q+1}$. This linear form $\mathcal{F}$ is continuous by assumption so by Hahn-Banach, it extends to the whole $A_{p,q+1}$ and is represented by a $g \in B_{n-p,n-q-1}$ by duality. So we have

$$\mathcal{F}(\gamma) = \langle \gamma, g\rangle = \big\langle \bar\partial f, g\big\rangle = \mathcal{L}_\omega(f) = \langle f, \omega\rangle,$$

but $\big\langle \bar\partial f, g\big\rangle = (-1)^{p+q}\big\langle f, \bar\partial g\big\rangle$ hence $\langle f, \omega\rangle = (-1)^{p+q}\big\langle f, \bar\partial g\big\rangle$ provided that $\bar\partial f = \gamma \in A_{p,q+1}$.
This is true for f a (p,q) -form with coefficients in $\mathcal{D}(\Omega)$, hence, by the density of $\mathcal{D}(\Omega)$ in A, we have

$$\forall f \in A_{p,q},\ \langle f, \omega\rangle = (-1)^{p+q}\big\langle f, \bar\partial g\big\rangle,$$

hence we find a $g \in B_{n-p,n-q-1}$ such that $(-1)^{p+q}\bar\partial g = \omega$. ■

Corollary 6.2. *Let Ω be a r regular domain in a Stein manifold X which possesses an exhaustive sequence of relatively compact r regular open subsets. Then for any (p,q) form ω in $L^{r,loc}(\Omega, dm)$, $\bar\partial\omega = 0$, if $q < n$, and $\omega \perp \mathcal{H}_p$ if $q = n$, there is a $u \in L^{r,loc}(\Omega, dm)$ such that $\bar\partial u = \omega$. This is the case when Ω is a relatively compact pseudo-convex domain with smooth boundary in X.*

Proof.
We take for μ the Lebesgue measure on Ω and $A := L^{r',c}(\Omega, \mu)$; we have clearly that $\mathcal{D}(\Omega) \subset A \subset \mathcal{D}'(\Omega)$ and by theorem 4.1 we can solve the $\bar\partial$ equation for $(p, q+1)$ forms $\bar\partial$ closed with coefficients in A. We have to show that the form $\mathcal{F}$ is continuous on the subspace $\gamma \in A_{p,q+1}, \bar\partial\gamma = 0$, but we have with $\bar\partial f = \gamma$,

$$\mathcal{F}(\gamma) := \mathcal{L}_\omega(f) = \langle f, \omega\rangle \Rightarrow |\mathcal{F}(\gamma)| = |\langle f, \omega\rangle| \le \|f\|\times\|\omega\|.$$

The theorem 4.1 gives also that $\|f\|_{L^r(\Omega)} \le C\|\gamma\|_{L^r(\Omega_j)}$, where $\Omega_j \subset \Omega$ is a r regular domain containing the support of γ. Hence

$$|\mathcal{F}(\gamma)| = |\langle f, \omega\rangle| \le \|f\|\times\|\omega\| \le C\|\gamma\|\times\|\omega\|,$$

and the continuity of $\mathcal{F}$, C and ω being fixed.

So by theorem 6.1 we can solve the $\bar\partial$ equation for forms with coefficients in the dual space of $L^{r',c}(\Omega, \mu)$ which is, by theorem 5.2, $L^{r,loc}(\Omega, dm)$ and the corollary. ■

Remark 6.3. *If Ω is a pseudo-convex domain in a Stein manifold X, then we can solve the $\bar\partial$ equation with $L^{2,loc}(\Omega, dm)$ estimates in this case ([6], theorem 5.2.4, p 118). So the case $r = 2$ of the corollary 6.4 is known for a long time.*

In $\mathbb{C}^n$, then it works also for Ω a polydisc, Ω a bounded strictly pseudo-convex and for Ω finite transverse intersection of bounded strictly pseudo-convex domains.

Corollary 6.4. *Let Ω be a domain in a Stein manifold X such that the equation $\bar\partial u = f$ has a solution u in $L^{r,loc}_{p,q}(\Omega)$ if $f \in L^{r,loc}_{p,q+1}(\Omega)$ with the control :*

$$\forall j \in \mathbb{N},\ \|u\|_j \le C_j\|f\|_j\ ;$$

then we can solve this equation with $u \in L^{r,c}_{p,q-1}(\Omega)$ if $f \in L^{r,loc}_{p,q}(\Omega)$ and $\bar\partial f = 0$ if $q < n$, and $f \perp \mathcal{H}_p$ if $q = n$.

Proof.

Exactly as above. ■

Remark 6.5. *A proof of this last corollary was sketched by a referee : he used a geometrical proof, with nice but heavy theorems.*

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