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Eric Amar

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AN ANDREOTTI-GRAUERT THEOREM WITH L^r ESTIMATES.

ERIC AMAR

ABSTRACT. By a theorem of Andreotti and Grauert if ω is a (p, q) -current, $q < n$, in a Stein manifold, $\bar{\partial}$ closed and with compact support, then there is a solution u to $\bar{\partial}u = \omega$ still with compact support. The aim of this work is to show that if moreover $\omega \in L^r(dm)$, where m is a suitable Lebesgue measure on the Stein manifold, then we have a solution u with compact support and in $L^r(dm)$.

1. INTRODUCTION.

Let ω be a $\bar{\partial}$ closed (p, q) form in \mathbb{C}^n with compact support $K := \text{Supp } \omega$ and such that $\omega \in L^r(\mathbb{C}^n)$. Setting K in a ball $\mathbb{B} := B(0, R)$ with R big enough, we know, by a theorem of Ovreid [9], that we have a $(p, q - 1)$ form $u \in L^r(\mathbb{B})$ such that $\bar{\partial}u = \omega$. On the other hand we also know, at least when $q < n$, that there is a current v with compact support such that $\bar{\partial}v = \omega$, by a theorem of Andreotti-Grauert [3].

So a natural question is : may we have a solution u of $\bar{\partial}u = \omega$ with compact support and in $L^r(\mathbb{C}^n)$?

We already answered this question by the affirmative in a join work with S. Mongodi [2] explicitly by the "method of coronas".

The aim of this work is to extend this result to Stein manifolds and for it we use a completely different approach inspired by the Serre duality [10]. Because Hahn Banach theorem is used, these results are no longer so explicit and constructive as in [2]. On the other hand the control of the support is better : the support of the solution u is contained in any r' -regular domain containing the support of the data ω . The definition of r' -regular domain will be given later on, but strictly pseudo convex domains are such domains.

I am indebted to G. Tomassini who started my interest in this subject by e-mails on precisely this kind of questions and also to S. Mongodi for a lot of discussions by mails on the subject during the preparation of our join paper [2].

2. DUALITY.

We shall study a duality between currents inspired by the Serre duality [10].

Let $\Omega \subset \mathbb{C}^n$ be an open set in \mathbb{C}^n and t a (p, q) current with compact support in Ω , noted $t \in \mathcal{D}'_{c, (p,q)}(\Omega)$.

Let also $\varphi \in \mathcal{C}^\infty_{(n-p, n-q)}(\Omega)$ a $(n - p, n - q)$ form in $\mathcal{C}^\infty(\Omega)$. We have that $t \wedge \varphi$ is a (n, n) current with compact support in Ω .

As usual we use the following notation for the pairing

$$\langle t, \varphi \rangle := t(\varphi),$$

where $t(\varphi)$ is the action, as a current, of t on the smooth form φ of complementary bi-degree.

Lemma 2.1. *Let Ω be an open set in \mathbb{C}^n , $t \in \mathcal{D}'_{c, (p,q)}(\Omega)$ a (p, q) current with compact support in Ω and $u \in \mathcal{D}'_{c, (p,q-1)}(\Omega)$. Then we have $\bar{\partial}t = u$ iff*

$$(*) \quad \forall \varphi \in \mathcal{C}_{(n-p, n-q)}^\infty(\Omega), \quad \langle t, \varphi \rangle = (-1)^{p+q-1} \langle u, \bar{\partial} \varphi \rangle.$$

Proof.

If $\bar{\partial}u = t$, let $\chi \in \mathcal{C}_c^\infty(\Omega)$ such that $\chi \equiv 1$ on the support of u , hence on the support of t . Then

$$\forall \varphi \in \mathcal{C}_{(n-p, n-q)}^\infty(\Omega), \quad \langle t, \varphi \rangle = \langle t, \chi \varphi \rangle = \langle \bar{\partial}u, \chi \varphi \rangle = (-1)^{p+q-1} \langle u, \bar{\partial}(\chi \varphi) \rangle$$

by definition of the action of derivatives on currents. Hence

$$\langle t, \varphi \rangle = (-1)^{p+q-1} \langle u, \bar{\partial} \chi \wedge \varphi \rangle + (-1)^{p+q-1} \langle u, \chi \bar{\partial} \varphi \rangle = (-1)^{p+q-1} \langle u, \chi \bar{\partial} \varphi \rangle$$

because $\bar{\partial} \chi = 0$ on the support of u ; so

$$\langle t, \varphi \rangle = (-1)^{p+q-1} \langle u, \chi \bar{\partial} \varphi \rangle = (-1)^{p+q-1} \langle u, \bar{\partial} \varphi \rangle$$

because $\chi = 1$ on the support of u .

Conversely if we have $(*)$ we take $\chi \in \mathcal{C}_c^\infty(\Omega)$ such that $\chi \equiv 1$ on union of the support of u and the support of t then

$$\forall \varphi \in \mathcal{C}_{(n-p, n-q)}^\infty(\Omega), \quad \langle t, \varphi \rangle = \langle t, \chi \varphi \rangle$$

because $\chi = 1$ on the support of t , and by $(*)$

$$\langle t, \chi \varphi \rangle = (-1)^{p+q-1} \langle u, \bar{\partial}(\chi \varphi) \rangle = \langle \bar{\partial}u, \chi \varphi \rangle = \langle \bar{\partial}u, \varphi \rangle$$

because $\chi = 1$ on the support of u hence on the support of $\bar{\partial}u$.

So

$$\forall \varphi \in \mathcal{C}_{(n-p, n-q)}^\infty(\Omega), \quad \langle t, \varphi \rangle = \langle \bar{\partial}u, \varphi \rangle,$$

which means that $\bar{\partial}u = t$ as a current. ■

3. SOLUTION OF THE $\bar{\partial}$ EQUATION WITH COMPACT SUPPORT.

3.1. Domain r regular.

We shall use the definition.

Definition 3.1. *Let X be a complex manifold and Ω a domain in X ; we shall say that Ω is r -regular, for $r \in [1, \infty]$ if for $p, q \in \{0, \dots, n\}$ there is a constant $C = C_{p,q}$ such that for any (p, q) form ω , $\bar{\partial}$ closed in Ω and in $L^r(\Omega)$ there is a $(p, q-1)$ form $u \in L^r(\Omega)$ such that $\bar{\partial}u = \omega$ and $\|u\|_{L^r(\Omega)} \leq C \|\omega\|_{L^r(\Omega)}$.*

The L^r norms for forms will be defined later on.

Examples of 2-regular domains are the bounded pseudo-convex domains by Hörmander [6].

Examples of r -regular domains in \mathbb{C}^n are the strictly pseudo-convex domains with smooth boundary by Ovreid [9]. The polydiscs in \mathbb{C}^n by Charpentier [4], finite intersections of strictly pseudo-convex bounded domains with transverse intersections by Menini [8].

Examples of r regular domains in Stein manifold are the strictly pseudo-convex domains with smooth boundary by Kerzman [7] for $(0, 1)$ forms and Demailly-Laurent [5] for the general case.

3.2. Weighted L^r spaces.

Let $\mathbb{B} := B(0, R)$, the ball of \mathbb{C}^n centered at 0 and of radius R . We shall need the following notations.

We note dm the Lebesgue measure on \mathbb{C}^n and we shall define $L^r(\mathbb{B}, w)$ to be the set of functions f defined on \mathbb{B} such that

$$\|f\|_{L^r(\mathbb{B}, w)}^r := \int_{\mathbb{B}} |f(z)|^r w(z) dm(z) < \infty,$$

with a weight $w(z) > 0$. As usual we set $L^r(\mathbb{B})$ for $L^r(\mathbb{B}, 1)$.

Let \mathcal{I}_p be the set of multi-indices of length p in $(1, \dots, n)$. We shall use the following measure defined on $\Gamma := \mathbb{B} \times \mathcal{I}_p \times \mathcal{I}_q$ the following way :

$$d\mu(z, k, l) = d\mu_{w,p,q}(z, k, l) := w(z) dm(z) \otimes \sum_{|I|=p, |J|=q} \delta_I(k) \otimes \delta_J(l),$$

where $\delta_I(k) = 1$ if the multi-index k is equal to I and $\delta_I(k) = 0$ if not.

This means, if $f(z, I, J)$ is a function defined on Γ , that

$$\int f(z, k, l) d\mu_{w,p,q}(z, k, l) := \sum_{|I|=p, |J|=q} \int_{\mathbb{B}} f(z, I, J) w(z) dm(z).$$

If I is a multi-index of length p , let I^c be the unique multi-index, ordered increasingly, such that $I \cup I^c = (1, 2, \dots, n)$; then I^c is of length $n - p$.

To $t = \sum_{|I|=p, |J|=q} t_{I,J}(z) dz^I \wedge d\bar{z}^J$ a (p, q) form, we associate the function on Γ :

$$T(z, I, J) := (-1)^{s(I,J)} t_{I,J}(z),$$

where

$$s(I, J) = 0 \text{ if } dz^I \wedge d\bar{z}^J \wedge dz^{I^c} \wedge d\bar{z}^{J^c} = dz_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_1 \wedge \dots \wedge d\bar{z}_n \text{ as a } (n, n) \text{ form}$$

and

$$s(I, J) = 1 \text{ if not.}$$

If $\varphi = \sum_{|I|=p, |J|=q} \varphi_{I^c, J^c}(z) dz^I \wedge d\bar{z}^J$ is of complementary bi-degree, associate in the same manner :

$$\Phi^*(z, I, J) := \varphi_{I^c, J^c}(z). \text{ This is still a function on } \Gamma.$$

Now we have, for $1 < r < \infty$, if $T(z, I, J)$ is a function in \mathbb{B} with $L^r(\mathbb{B})$ coefficients and with $\mu = \mu_{w,p,q}$,

$$\|T\|_{L^r(d\mu)}^r := \int |T(z, I, J)|^r d\mu_{w,p,q}(x, I, J) = \sum_{|I|=p, |J|=q} \|T(\cdot, I, J)\|_{L^r(\mathbb{B}, w)}^r.$$

The dual of $L^r(\mu)$ is $L^{r'}(\mu)$ where r' is the conjugate of r , $\frac{1}{r} + \frac{1}{r'} = 1$, and the norm is defined analogously with r' replacing r .

We also know that, for p, q fixed,

$$(3.1) \quad \|T\|_{L^r(\mu)} = \sup_{\Phi \in L^{r'}(\mu)} \frac{|\int T\Phi d\mu|}{\|\Phi\|_{L^{r'}(\mu)}}.$$

For a (p, q) form $t = \sum_{|J|=p, |K|=q} t_{J,K} dz^J \wedge d\bar{z}^K$, and a weight $w > 0$ we define its norm by :

$$(3.2) \quad \|t\|_{L^r(\mathbb{B}, w)}^r := \sum_{|J|=p, |K|=q} \|t_{J,K}\|_{L^r(\mathbb{B}, w)}^r = \|T\|_{L^r(\mu)}^r.$$

Now we can state

Lemma 3.2. *Let $w > 0$ be a weight. If u is a (p, q) current defined on $(n - p, n - q)$ forms in $L^r(\mathbb{B}, w)$ and such that*

$$\forall \alpha \in L^r_{(n-p, n-q)}(\mathbb{B}, w), \quad |\langle u, \alpha \rangle| \leq C \|\alpha\|_{L^r(\mathbb{B}, w)},$$

then $\|u\|_{L^r(\mathbb{B}, w^{1-r})} \leq C$.

Proof.

Let us take the measure $\mu = \mu_{w,p,q}$ as above. Let Φ^* be the function on Γ associated to α and T the one associated to u . We have, by definition of the measure μ applied to the function

$$f(z, I, J) := T(z, I, J) w^{-1} \Phi^*(z, I, J),$$

$$\int T w^{-1} \Phi^* d\mu = \int f(z, k, l) d\mu(z, k, l) := \sum_{|I|=p, |J|=q} \int_{\mathbb{B}} f(z, I, J) w(z) dm(z) =$$

$$= \sum_{|I|=p, |J|=q} \int_{\mathbb{B}} T(z, I, J) w^{-1}(z) \Phi^*(z, I, J) w(z) dm(z) = \langle u, \alpha \rangle,$$

by definition of T and Φ^* .

Hence we have, by (3.1)

$$\|Tw^{-1}\|_{L^r(\mu)} = \sup_{\Psi \in L^{r'}(\mu)} \frac{|\langle u, \alpha \rangle|}{\|\Psi\|_{L^{r'}(\mu)}}.$$

But $\|Tw^{-1}\|_{L^r(\mu)} = \|uw^{-1}\|_{L^r(\mathbb{B}, w)}$ by (3.2), and

$$\|fw^{-1}\|_{L^r(\mathbb{B}, w)}^r = \int_{\mathbb{B}} |f\eta^{-1}|^r w dm = \int_{\mathbb{B}} |f|^r w^{1-r} dm = \|f\|_{L^r(\mathbb{B}, w^{1-r})},$$

so we get

$$\|u\|_{L^r(\mathbb{B}, w^{1-r})} = \sup_{\Psi \in L^{r'}(\mu)} \frac{|\langle u, \alpha \rangle|}{\|\Psi\|_{L^{r'}(\mu)}},$$

which implies the lemma because, still by (3.1), we can take $\Psi = \Phi^*$ and $\|\Psi\|_{L^{r'}(\mu)} = \|\alpha\|_{L^{r'}(\mathbb{B}, w)}$.

■

Let us denote $\mathcal{H}_p(\Omega)$ the set of all $(p, 0)$ $\bar{\partial}$ closed forms in Ω . If $p = 0$, $\mathcal{H}_0(\Omega) = \mathcal{H}(\Omega)$ is the set of holomorphic functions in Ω . If $p > 0$, we have $\varphi \in \mathcal{H}_p(\Omega) \Rightarrow \varphi(z) = \sum_{|J|=p} a_J(z) dz^J$, where $dz^J := dz_{j_1} \wedge \cdots \wedge dz_{j_p}$ and the functions $a_J(z)$ are holomorphic in Ω . Hence in \mathbb{C}^n , a $(p, 0)$ $\bar{\partial}$ closed form is a vector of global holomorphic functions in Ω .

Theorem 3.3. *Let t be a (p, q) form in $L_c^r(\mathbb{C}^n)$, $r > 1$, $K := \text{Supp } t$. Let \mathbb{B} be a ball containing K and let $\Omega \subset \mathbb{B}$ be a r' -regular domain containing K . Suppose that t is such that $\bar{\partial}t = 0$ if $1 \leq q < n$ and $\forall h \in \mathcal{H}_p(\Omega)$, $\langle t, h \rangle = 0$ if $q = n$.*

Then there is a $(p, q-1)$ form u in $L_c^r(\mathbb{B})$, 0 a.e. on $\mathbb{B} \setminus \Omega$ and such that $\bar{\partial}u = t$ as distributions.

Proof.

Let us consider the weight $\eta = \eta_\epsilon := \mathbb{1}_\Omega(z) + \epsilon \mathbb{1}_{\mathbb{B} \setminus \Omega}(z)$ for a fixed $\epsilon > 0$.

We shall consider the linear form \mathcal{L} defined on $(n-p, n-q+1)$ form $\alpha \in L^{r'}(\mathbb{B}, \eta)$, $\bar{\partial}$ closed in Ω as follows :

$$\mathcal{L}(\alpha) := (-1)^{p+q-1} \langle t, \varphi \rangle, \text{ where } \varphi \in L^{r'}(\mathbb{B}) \text{ is such that } \bar{\partial}\varphi = \alpha.$$

Because $\epsilon > 0$ we have $\alpha \in L^{r'}(\mathbb{B}, \eta) \Rightarrow \alpha \in L^{r'}(\mathbb{B})$ and such a φ exists because \mathbb{B} being r' -regular there is $\varphi \in L^{r'}(\mathbb{B})$ with $\bar{\partial}\varphi = \alpha$.

Let us see that \mathcal{L} is well defined :

suppose first that $q < n$. In order for \mathcal{L} to be well defined we need

$$\forall \varphi, \psi \in L_{(n-p, n-q)}^{r'}(\mathbb{B}), \bar{\partial}\varphi = \bar{\partial}\psi \Rightarrow \langle t, \varphi \rangle = \langle t, \psi \rangle.$$

This is meaningful because $t \in L_c^r(\mathbb{C}^n)$, $r > 1$, $\text{Supp } t \subset \mathbb{B}$.

Then we have $\bar{\partial}(\varphi - \psi) = 0$ hence, because \mathbb{B} is r' -regular, we can solve $\bar{\partial}$ in $L^{r'}(\mathbb{B})$:

$$\exists \gamma \in L_{(n-p, n-q-1)}^{r'}(\mathbb{B}) :: \bar{\partial}\gamma = (\varphi - \psi).$$

So $\langle t, \varphi - \psi \rangle = \langle t, \bar{\partial}\gamma \rangle = (-1)^{p+q-1} \langle \bar{\partial}t, \gamma \rangle = 0$. Hence \mathcal{L} is well defined in that case.

Suppose now that $q = n$, then of course $\bar{\partial}t = 0$ and we have that φ, ψ are $(p, 0)$ forms hence $\bar{\partial}(\varphi - \psi) = 0$ means that $h := \varphi - \psi$ is a $\bar{\partial}$ closed $(p, 0)$ form hence $h \in \mathcal{H}_p(\mathbb{B}) \subset \mathcal{H}_p(\Omega)$. But by assumption we have $\langle t, h \rangle = 0$ hence \mathcal{L} is also well defined in that case.

If α is a $(n-p, n-q+1)$ form in $L^{r'}(\mathbb{B}, \eta)$, $\bar{\partial}$ closed in \mathbb{B} , then, $\alpha \in L^{r'}(\Omega)$ and is still $\bar{\partial}$ closed in Ω , hence there is a $\psi \in L^{r'}(\Omega) :: \bar{\partial}\psi = \alpha$ because Ω is r' -regular.

For $q < n$, we have $\bar{\partial}(\varphi - \psi) = \alpha - \alpha = 0$ on Ω and, because $\text{Supp } t \subset \Omega \subset \mathbb{B}$, $\bar{\partial}t = 0$ we get $\langle t, \varphi - \psi \rangle = 0$ hence

$$\mathcal{L}(\alpha) = \langle t, \varphi \rangle = \langle t, \psi \rangle.$$

If $q = n$, we still have $\bar{\partial}(\varphi - \psi) = \alpha - \alpha = 0$ on Ω , and $\varphi - \psi \in \mathcal{H}_p(\Omega)$ hence again by the hypothesis we still get

$$\mathcal{L}(\alpha) = \langle t, \varphi \rangle = \langle t, \psi \rangle.$$

In any cases, by Hölder inequalities

$$|\mathcal{L}(\alpha)| \leq \|t\|_{L^r(\Omega)} \|\psi\|_{L^{r'}(\Omega)}.$$

But by the r' -regularity of Ω there is a constant C such that

$$\|\psi\|_{L^{r'}(\Omega)} \leq C \|\alpha\|_{L^{r'}(\Omega)}.$$

Of course we have

$$\|\alpha\|_{L^{r'}(\Omega)} \leq \|\alpha\|_{L^{r'}(\mathbb{B}, \eta)}$$

because $\eta = 1$ on Ω , hence

$$|\mathcal{L}(\alpha)| \leq C \|t\|_{L^r(\Omega)} \|\alpha\|_{L^{r'}(\mathbb{B}, \eta)}.$$

So we have that the norm of \mathcal{L} is bounded on the subspace of $\bar{\partial}$ closed forms in $L^{r'}(\mathbb{B}, \eta)$ by $C \|t\|_{L^r(\Omega)}$ which is independent of ϵ .

We apply the Hahn-Banach theorem to extend \mathcal{L} with the same norm to all $(n-p, n-q+1)$ forms in $L^{r'}(\mathbb{B}, \eta)$. As in Serre duality theorem ([10], p. 20) this is one of the main ingredient in the proof.

This means, by the definition of current, that there is a $(p, q-1)$ current u which represents the extended form \mathcal{L} and such that

$$\sup_{\alpha \in L^{r'}(\mathbb{B}, \eta), \|\alpha\|=1} |\langle u, \alpha \rangle| \leq C \|t\|_{L^r(\Omega)}$$

and by lemma 3.2 with the weight η , this implies

$$\|u\|_{L^r(\mathbb{B}, \eta^{1-r})} \leq C \|t\|_{L^r(\Omega)}.$$

In particular $\|u\|_{L^r(\mathbb{B})} \leq C \|t\|_{L^r(\Omega)}$ because with $\epsilon < 1$, $r > 1$, we have $\eta^{1-r} \geq 1$.

So applied to a $\bar{\partial}$ closed $(n-p, n-q+1)$ smooth form α we get

$\langle u, \alpha \rangle = (-1)^{p+q-1} \langle t, \varphi \rangle$, with φ a smooth form such that $\bar{\partial}\varphi = \alpha$, i.e. $\forall \varphi :: \bar{\partial}\varphi \in L^{r'}(\mathbb{B}, \eta)$, we have $\langle u, \bar{\partial}\varphi \rangle = (-1)^{p+q-1} \langle t, \varphi \rangle$ and this means precisely, by lemma 2.1, that $\bar{\partial}u = t$.

Now for $\epsilon > 0$ with $\eta_\epsilon(z) := \mathbb{1}_\Omega(z) + \epsilon \mathbb{1}_{\mathbb{B} \setminus \Omega}(z)$, let $u_\epsilon \in L^r(\mathbb{B}, \eta_\epsilon^{1-r})$ be the previous solution, then

$$\|u_\epsilon\|_{L^r(\mathbb{B})}^r \leq \int_{\mathbb{B}} |u_\epsilon|^r \eta_\epsilon^{1-r} dm \leq C^r \|t\|_{L^r(\Omega)}^r.$$

Replacing η by its value we get

$$\int_{\Omega} |u_\epsilon|^r dm + \int_{\mathbb{B} \setminus \Omega} |u_\epsilon|^r \epsilon^{1-r} dm \leq C^r \|t\|_{L^r(\Omega)}^r \Rightarrow \int_{\mathbb{B} \setminus \Omega} |u_\epsilon|^r \epsilon^{1-r} dm \leq C^r \|t\|_{L^r(\Omega)}^r$$

hence

$$\int_{\mathbb{B} \setminus \Omega} |u_\epsilon|^r dm \leq C^r \epsilon^{r-1} \|t\|_{L^r(\Omega)}^r.$$

Because C and the norm of t are independent of ϵ , we have $\|u_\epsilon\|_{L^r(\mathbb{B})}$ is uniformly bounded and $r > 1 \Rightarrow L^r(\mathbb{B})$ is a dual, hence there is a sub-sequence $\{u_{\epsilon_k}\}_{k \in \mathbb{N}}$ of $\{u_\epsilon\}$ which converges weakly, when $\epsilon_k \rightarrow 0$, to a $(p, q-1)$ form u in $L^r(\mathbb{B})$.

To see that this form u is 0 *a.e.* on $\mathbb{B} \setminus \Omega$ let us take a component $u_{I,J}$ of it ; it is the weak limit of the sequence of functions $\{u_{\epsilon_k, I, J}\}$ which means, with the notations $v := u_{I, J}$, $v_k := u_{\epsilon_k, I, J}$

$$\forall f \in L^{r'}(\mathbb{B}), \int_{\mathbb{B}} v f dm = \lim_{k \rightarrow \infty} \int_{\mathbb{B}} v_k f dm.$$

As usual take $f := \frac{\bar{v}}{|v|} \mathbb{1}_E$ where $E := \{|v| > 0\} \cap (\mathbb{B} \setminus \Omega)$ then we get

$$\int_{\mathbb{B}} v f dm = \int_E |v| dm = \lim_{k \rightarrow \infty} \int_{\mathbb{B}} v_k f dm = \lim_{k \rightarrow \infty} \int_E \frac{v_k \bar{v}}{|v|} dm.$$

Now we have by Hölder

$$\left| \int_E \frac{v_k \bar{v}}{|v|} dm \right| \leq \|v_k\|_{L^r(E)} \|\mathbb{1}_E\|_{L^{r'}(E)}.$$

But

$$\|v_k\|_{L^r(E)}^r \leq \int_{\mathbb{B} \setminus \Omega} |u_{\epsilon_k}|^r dm \leq (\epsilon_k)^{r-1} \|t\|_{L^r(\Omega)} \rightarrow 0, \quad k \rightarrow \infty.$$

Hence

$$\left| \int_E |v| dm \right| \leq C^r \|\mathbb{1}_E\|_{L^{r'}(E)} (\epsilon_k)^{r-1} \|t\|_{L^r(\Omega)}^r \rightarrow 0$$

which implies $m(E) = 0$ because on E , $|v| > 0$.

This being true for all components of u , we get that the form u is 0 *a.e.* on $\mathbb{B} \setminus \Omega$.

So we get

$$\forall \varphi \in \mathcal{D}_{n-p, n-q}(\mathbb{B}), \quad (-1)^{p+q-1} \langle t, \varphi \rangle = \langle u_{\epsilon}, \bar{\partial} \varphi \rangle \rightarrow \langle u, \bar{\partial} \varphi \rangle \Rightarrow \langle u, \bar{\partial} \varphi \rangle = (-1)^{p+q-1} \langle t, \varphi \rangle$$

hence $\bar{\partial} u = t$ as distributions. ■

Remark 3.4. *The condition of orthogonality to the holomorphic functions in case $q = n$ was already seen for extension of CR functions see [1] and the references therein.*

4. CASE OF STEIN MANIFOLD.

Let us denote $\mathcal{H}_p(\Omega)$ the set of all $(p, 0)$ $\bar{\partial}$ closed forms in Ω . Still we have that $\mathcal{H}_0(\Omega) = \mathcal{H}(\Omega)$, the set of holomorphic functions in Ω , but if $p > 1$, $\varphi \in \mathcal{H}_p(\Omega)$ is a priori no longer a vector of global holomorphic functions in Ω .

Theorem 4.1. *Let X be a Stein manifold and Ω a bounded strictly pseudo-convex domain with smooth boundary of X . Let t be a (p, q) current in $L_c^r(\Omega)$, $r > 1$, such that $\bar{\partial} t = 0$ if $1 \leq q < n$ and $\forall h \in \mathcal{H}_p(\Omega)$, $\langle t, h \rangle = 0$ if $q = n$.*

Let B be another bounded strictly pseudo-convex domain with smooth boundary in X such that $\bar{\Omega} \subset B$.

*Then there is a $(p, q-1)$ current u in $L^r(B)$ such that $\bar{\partial} u = t$ and which is 0 *a.e.* in $B \setminus \Omega$.*

Proof.

First we define the "Lebesgue measure" on a complex manifold X as in Hörmander's book [6] section 5.2, with a hermitian metric locally equivalent to the usual one on any analytic coordinates system. Associate to this metric there is a volume form dV and we take it for the Lebesgue measure on X .

We can solve the $\bar{\partial}$ equation in B and Ω with L^r estimates for $(0, 1)$ forms by use of N. Kerzman kernels [7] and the metric above.

We can solve the $\bar{\partial}$ equation in B and Ω with L^r estimates for all (p, q) forms by J-P. Demailly and C. Laurent ones ([5], Remarque 4, page 596) but here we have to equip the manifold with a metric with null curvature, in order to avoid parasitic terms.

The remainder of the proof is identical to the proof of theorem 3.3. ■

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