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► **To cite this version:**

Eric Amar. AN ANDREOTTI-GRAUERT THEOREM WITH  $L^r$  ESTIMATES.. 2012. hal-00676110v1

**HAL Id: hal-00676110**

**<https://hal.science/hal-00676110v1>**

Preprint submitted on 3 Mar 2012 (v1), last revised 9 Oct 2019 (v9)

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# AN ANDREOTTI-GRAUERT THEOREM WITH $L^r$ ESTIMATES.

ERIC AMAR

ABSTRACT. By a theorem of Andreotti and Grauert if  $\omega$  is a  $(p, q)$  -current,  $q < n$ , in a Stein manifold,  $\bar{\partial}$  closed and with compact support, then there is a solution  $u$  to  $\bar{\partial}u = \omega$  still with compact support. The aim of this work is to show that if moreover  $\omega \in L^r(dm)$ , where  $m$  is a suitable Lebesgue measure on the Stein manifold, then we have a solution  $u$  with compact support and in  $L^r(dm)$ .

## 1. INTRODUCTION.

Let  $\omega$  be a  $\bar{\partial}$  closed  $(p, q)$  form in  $\mathbb{C}^n$  with compact support  $K := \text{Supp } \omega$  and such that  $\omega \in L^r(\mathbb{C}^n)$ . Setting  $K$  in a ball  $\mathbb{B} := B(0, R)$  with  $R$  big enough, we know, by a theorem of Ovreid [9], that we have a  $(p, q - 1)$  form  $u \in L^r(\mathbb{B})$  such that  $\bar{\partial}u = \omega$ . On the other hand we also know, at least when  $q < n$ , that there is a current  $v$  with compact support such that  $\bar{\partial}v = \omega$ , by a theorem of Andreotti-Grauert [3].

So a natural question is : may we have a solution  $u$  of  $\bar{\partial}u = \omega$  with compact support and in  $L^r(\mathbb{C}^n)$  ?

We already answered this question by the affirmative in a join work with S. Mongodi [2] explicitly by the "method of coronas".

The aim of this work is to extend this result to Stein manifolds and for it we use a completely different approach inspired by the Serre duality [10]. Because Hahn Banach theorem is used, these results are no longer so explicit and constructive as in [2]. On the other hand the control of the support is better : the support of the solution  $u$  is contained in any  $r'$  -regular domain containing the support of the data  $\omega$ . The definition of  $r'$  -regular domain will be given later on, but strictly pseudo convex domains are such domains.

I am indebted to G. Tomassini who started my interest in this subject by e-mails on precisely this kind of questions and also to S. Mongodi for a lot of discussions by mails on the subject during the preparation of our join paper [2].

## 2. DUALITY.

We shall study a duality between currents inspired by the Serre duality [10].

Let  $\Omega \subset \mathbb{C}^n$  be an open set in  $\mathbb{C}^n$  and  $t$  a  $(p, q)$  current with compact support in  $\Omega$ , noted  $t \in \mathcal{D}'_{c, (p,q)}(\Omega)$ .

Let also  $\phi \in \mathcal{C}^\infty_{(n-p, n-q)}(\Omega)$  a  $(n - p, n - q)$  form in  $\mathcal{C}^\infty(\Omega)$ . We have that  $t \wedge \phi$  is a  $(n, n)$  current with compact support in  $\Omega$ .

As usual we use the following notation for the pairing

$$\langle t, \phi \rangle := t(\phi),$$

where  $t(\phi)$  is the action, as a current, of  $t$  on the smooth form  $\phi$  of complementary bi-degree.

**Lemma 2.1.** *Let  $\Omega$  be an open set in  $\mathbb{C}^n$ ,  $t \in \mathcal{D}'_{c, (p,q)}(\Omega)$  a  $(p, q)$  current with compact support in  $\Omega$  and  $u \in \mathcal{D}'_{c, (p,q-1)}(\Omega)$ . Then we have  $\bar{\partial}t = u$  iff*

$$(*) \quad \forall \phi \in \mathcal{C}_{(n-p, n-q)}^\infty(\Omega), \quad \langle t, \phi \rangle = (-1)^{p+q-1} \langle u, \bar{\partial} \phi \rangle.$$

Proof.

If  $\bar{\partial}u = t$ , let  $\chi \in \mathcal{C}_c^\infty(\Omega)$  such that  $\chi \equiv 1$  on the support of  $u$ , hence on the support of  $t$ . Then

$$\forall \phi \in \mathcal{C}_{(n-p, n-q)}^\infty(\Omega), \quad \langle t, \phi \rangle = \langle t, \chi \phi \rangle = \langle \bar{\partial}u, \chi \phi \rangle = (-1)^{p+q-1} \langle u, \bar{\partial}(\chi \phi) \rangle$$

by definition of the action of derivatives on currents. Hence

$$\langle t, \phi \rangle = (-1)^{p+q-1} \langle u, \bar{\partial}\chi \wedge \phi \rangle + (-1)^{p+q-1} \langle u, \chi \bar{\partial}\phi \rangle = (-1)^{p+q-1} \langle u, \chi \bar{\partial}\phi \rangle$$

because  $\bar{\partial}\chi = 0$  on the support of  $u$ ; so

$$\langle t, \phi \rangle = (-1)^{p+q-1} \langle u, \chi \bar{\partial}\phi \rangle = (-1)^{p+q-1} \langle u, \bar{\partial}\phi \rangle$$

because  $\chi = 1$  on the support of  $u$ .

Conversely if we have  $(*)$  we take  $\chi \in \mathcal{C}_c^\infty(\Omega)$  such that  $\chi \equiv 1$  on union of the support of  $u$  and the support of  $t$  then

$$\forall \phi \in \mathcal{C}_{(n-p, n-q)}^\infty(\Omega), \quad \langle t, \phi \rangle = \langle t, \chi \phi \rangle$$

because  $\chi = 1$  on the support of  $t$ , and by  $(*)$

$$\langle t, \chi \phi \rangle = (-1)^{p+q-1} \langle u, \bar{\partial}(\chi \phi) \rangle = \langle \bar{\partial}u, \chi \phi \rangle = \langle \bar{\partial}u, \phi \rangle$$

because  $\chi = 1$  on the support of  $u$  hence on the support of  $\bar{\partial}u$ .

So

$$\forall \phi \in \mathcal{C}_{(n-p, n-q)}^\infty(\Omega), \quad \langle t, \phi \rangle = \langle \bar{\partial}u, \phi \rangle,$$

which means that  $\bar{\partial}u = t$  as a current. ■

### 3. SOLUTION OF THE $\bar{\partial}$ EQUATION WITH COMPACT SUPPORT.

#### 3.1. Domain $r$ regular.

We shall use the definition.

**Definition 3.1.** *Let  $X$  be a complex manifold and  $\Omega$  a domain in  $X$ ; we shall say that  $\Omega$  is  $r$ -regular, for  $r \in [1, \infty]$  if for  $p, q \in \{0, \dots, n\}$  there is a constant  $C = C_{p,q}$  such that for any  $(p, q)$  form  $\omega$ ,  $\bar{\partial}$  closed in  $\Omega$  and in  $L^r(\Omega)$  there is a  $(p, q-1)$  form  $u \in L^r(\Omega)$  such that  $\bar{\partial}u = \omega$  and  $\|u\|_{L^r(\Omega)} \leq C \|\omega\|_{L^r(\Omega)}$ .*

The  $L^r$  norms for forms will be defined later on.

Examples of 2-regular domains are the bounded pseudo-convex domains by Hörmander [6].

Examples of  $r$ -regular domains in  $\mathbb{C}^n$  are the strictly pseudo-convex domains with smooth boundary by Ovreid [9]. The polydiscs in  $\mathbb{C}^n$  by Charpentier [4], finite intersections of strictly pseudo-convex bounded domains with transverse intersections by Menini [8].

Examples of  $r$  regular domains in Stein manifold are the strictly pseudo-convex domains with smooth boundary by Kerzman [7] for  $(0, 1)$  forms and Demailly-Laurent [5] for the general case.

#### 3.2. Weighted $L^r$ spaces.

Let  $\mathbb{B} := B(0, R)$ , the ball of  $\mathbb{C}^n$  centered at 0 and of radius  $R$ . We shall need the following notations.

We note  $dm$  the Lebesgue measure on  $\mathbb{C}^n$  and we shall define  $L^r(\mathbb{B}, w)$  to be the set of functions  $f$  defined on  $\mathbb{B}$  such that

$$\|f\|_{L^r(\mathbb{B}, w)}^r := \int_{\mathbb{B}} |f(z)|^r w(z) dm(z) < \infty,$$

with a weight  $w(z) > 0$ . As usual we set  $L^r(\mathbb{B})$  for  $L^r(\mathbb{B}, 1)$ .

Let  $\mathcal{I}_p$  be the set of multi-indices of length  $p$  in  $(1, \dots, n)$ . We shall use the following measure defined on  $\Gamma := \mathbb{B} \times \mathcal{I}_p \times \mathcal{I}_q$  the following way :

$$d\mu(z, k, l) = d\mu_{w,p,q}(z, k, l) := w(z) dm(z) \otimes \sum_{|I|=p, |J|=q} \delta_I(k) \otimes \delta_J(l),$$

where  $\delta_I(k) = 1$  if the multi-index  $k$  is equal to  $I$  and  $\delta_I(k) = 0$  if not.

This means, if  $f(z, I, J)$  is a function defined on  $\Gamma$ , that

$$\int f(z, k, l) d\mu_{w,p,q}(z, k, l) := \sum_{|I|=p, |J|=q} \int_{\mathbb{B}} f(z, I, J) w(z) dm(z).$$

If  $I$  is a multi-index of length  $p$ , let  $I^c$  be the unique multi-index, ordered increasingly, such that  $I \cup I^c = (1, 2, \dots, n)$ ; then  $I^c$  is of length  $n - p$ .

To  $t = \sum_{|I|=p, |J|=q} t_{I,J}(z) dz^I \wedge d\bar{z}^J$  a  $(p, q)$  form, we associate the function on  $\Gamma$  :

$$T(z, I, J) := (-1)^{s(I,J)} t_{I,J}(z),$$

where

$$s(I, J) = 0 \text{ if } dz^I \wedge d\bar{z}^J \wedge dz^{I^c} \wedge d\bar{z}^{J^c} = dz_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_1 \wedge \dots \wedge d\bar{z}_n \text{ as a } (n, n) \text{ form}$$

and

$$s(I, J) = 1 \text{ if not.}$$

If  $\phi = \sum_{|I|=p, |J|=q} \phi_{I^c, J^c}(z) dz^{I^c} \wedge d\bar{z}^{J^c}$  is of complementary bi-degree, associate in the same manner :

$$\Phi^*(z, I, J) := \phi_{I^c, J^c}(z). \text{ This is still a function on } \Gamma.$$

Now we have, for  $1 < r < \infty$ , if  $T(z, I, J)$  is a function in  $\mathbb{B}$  with  $L^r(\mathbb{B})$  coefficients and with  $\mu = \mu_{w,p,q}$ ,

$$\|T\|_{L^r(d\mu)}^r := \int |T(z, I, J)|^r d\mu_{w,p,q}(z, I, J) = \sum_{|I|=p, |J|=q} \|T(\cdot, I, J)\|_{L^r(\mathbb{B}, w)}^r.$$

The dual of  $L^r(\mu)$  is  $L^{r'}(\mu)$  where  $r'$  is the conjugate of  $r$ ,  $\frac{1}{r} + \frac{1}{r'} = 1$ , and the norm is defined analogously with  $r'$  replacing  $r$ .

We also know that, for  $p, q$  fixed,

$$(3.1) \quad \|T\|_{L^r(\mu)} = \sup_{\Phi \in L^{r'}(\mu)} \frac{|\int T\Phi d\mu|}{\|\Phi\|_{L^{r'}(\mu)}}.$$

For a  $(p, q)$  form  $t = \sum_{|J|=p, |K|=q} t_{J,K} dz^J \wedge d\bar{z}^K$ , and a weight  $w > 0$  we define its norm by :

$$(3.2) \quad \|t\|_{L^r(\mathbb{B}, w)}^r := \sum_{|J|=p, |K|=q} \|t_{J,K}\|_{L^r(\mathbb{B}, w)}^r = \|T\|_{L^r(\mu)}^r.$$

Now we can state

**Lemma 3.2.** *Let  $w > 0$  be a weight. If  $u$  is a  $(p, q)$  current defined on  $(n - p, n - q)$  forms in  $L^r(\mathbb{B}, w)$  and such that*

$$\forall \alpha \in L^r_{(n-p, n-q)}(\mathbb{B}, w), \quad |\langle u, \alpha \rangle| \leq C \|\alpha\|_{L^r(\mathbb{B}, w)},$$

then  $\|u\|_{L^r(\mathbb{B}, w^{1-r})} \leq C$ .

*Proof.*

Let us take the measure  $\mu = \mu_{w,p,q}$  as above. Let  $\Phi^*$  be the function on  $\Gamma$  associated to  $\alpha$  and  $T$  the one associated to  $u$ . We have, by definition of the measure  $\mu$  applied to the function

$$f(z, I, J) := T(z, I, J) w^{-1} \Phi^*(z, I, J),$$

$$\int T w^{-1} \Phi^* d\mu = \int f(z, k, l) d\mu(z, k, l) := \sum_{|I|=p, |J|=q} \int_{\mathbb{B}} f(z, I, J) w(z) dm(z) =$$

$$= \sum_{|I|=p, |J|=q} \int_{\mathbb{B}} T(z, I, J) w^{-1}(z) \Phi^*(z, I, J) w(z) dm(z) = \langle u, \alpha \rangle,$$

by definition of  $T$  and  $\Phi^*$ .

Hence we have, by (3.1)

$$\|Tw^{-1}\|_{L^r(\mu)} = \sup_{\Psi \in L^{r'}(\mu)} \frac{|\langle u, \alpha \rangle|}{\|\Psi\|_{L^{r'}(\mu)}}.$$

But  $\|Tw^{-1}\|_{L^r(\mu)} = \|uw^{-1}\|_{L^r(\mathbb{B}, w)}$  by (3.2), and

$$\|fw^{-1}\|_{L^r(\mathbb{B}, w)}^r = \int_{\mathbb{B}} |f\eta^{-1}|^r w dm = \int_{\mathbb{B}} |f|^r w^{1-r} dm = \|f\|_{L^r(\mathbb{B}, w^{1-r})}^r,$$

so we get

$$\|u\|_{L^r(\mathbb{B}, w^{1-r})} = \sup_{\Psi \in L^{r'}(\mu)} \frac{|\langle u, \alpha \rangle|}{\|\Psi\|_{L^{r'}(\mu)}},$$

which implies the lemma because, still by (3.1), we can take  $\Psi = \Phi^*$  and  $\|\Psi\|_{L^{r'}(\mu)} = \|\alpha\|_{L^{r'}(\mathbb{B}, w)}$ .

■

Let us denote  $\mathcal{H}_p(\Omega)$  the set of all  $(p, 0)$   $\bar{\partial}$  closed forms in  $\Omega$ . If  $p = 0$ ,  $\mathcal{H}_0(\Omega) = \mathcal{H}(\Omega)$  is the set of holomorphic functions in  $\Omega$ . If  $p > 0$ , we have  $\phi \in \mathcal{H}_p(\Omega) \implies \phi(z) = \sum_{|J|=p} a_J(z) dz^J$ , where  $dz^J := dz_{j_1} \wedge \cdots \wedge dz_{j_p}$  and the functions  $a_J(z)$  are holomorphic in  $\Omega$ . Hence in  $\mathbb{C}^n$ , a  $(p, 0)$   $\bar{\partial}$  closed form is a vector of global holomorphic functions in  $\Omega$ .

**Theorem 3.3.** *Let  $t$  be a  $(p, q)$  form in  $L_c^r(\mathbb{C}^n)$ ,  $r > 1$ ,  $K := \text{Supp } t$ . Let  $\mathbb{B}$  be a ball containing  $K$  and let  $\Omega \subset \mathbb{B}$  be a  $r'$ -regular domain containing  $K$ . Suppose that  $t$  is such that  $\bar{\partial}t = 0$  if  $1 \leq q < n$  and  $\forall h \in \mathcal{H}_p(\Omega)$ ,  $\langle t, h \rangle = 0$  if  $q = n$ .*

*Then there is a  $(p, q-1)$  form  $u$  in  $L_c^r(\mathbb{B})$ ,  $0$  a.e. on  $\mathbb{B} \setminus \Omega$  and such that  $\bar{\partial}u = t$  as distributions.*

Proof.

Let us consider the weight  $\eta = \eta_\epsilon := \mathbb{1}_\Omega(z) + \epsilon \mathbb{1}_{\mathbb{B} \setminus \Omega}(z)$  for a fixed  $\epsilon > 0$ .

We shall consider the linear form  $\mathcal{L}$  defined on  $(n-p, n-q+1)$  form  $\alpha \in L^{r'}(\mathbb{B}, \eta)$ ,  $\bar{\partial}$  closed in  $\Omega$  as follows :

$$\mathcal{L}(\alpha) := (-1)^{p+q-1} \langle t, \phi \rangle, \text{ where } \phi \in L^{r'}(\mathbb{B}) \text{ is such that } \bar{\partial}\phi = \alpha.$$

Because  $\epsilon > 0$  we have  $\alpha \in L^{r'}(\mathbb{B}, \eta) \implies \alpha \in L^{r'}(\mathbb{B})$  and such a  $\phi$  exists because  $\mathbb{B}$  being  $r'$ -regular there is  $\phi \in L^{r'}(\mathbb{B})$  with  $\bar{\partial}\phi = \alpha$ .

Let us see that  $\mathcal{L}$  is well defined :

suppose first that  $q < n$ . In order for  $\mathcal{L}$  to be well defined we need

$$\forall \phi, \psi \in L_{(n-p, n-q)}^{r'}(\mathbb{B}), \bar{\partial}\phi = \bar{\partial}\psi \implies \langle t, \phi \rangle = \langle t, \psi \rangle.$$

This is meaningful because  $t \in L_c^r(\mathbb{C}^n)$ ,  $r > 1$ ,  $\text{Supp } t \subset \mathbb{B}$ .

Then we have  $\bar{\partial}(\phi - \psi) = 0$  hence, because  $\mathbb{B}$  is  $r'$ -regular, we can solve  $\bar{\partial}$  in  $L^{r'}(\mathbb{B})$  :

$$\exists \gamma \in L_{(n-p, n-q-1)}^{r'}(\mathbb{B}) :: \bar{\partial}\gamma = (\phi - \psi).$$

So  $\langle t, \phi - \psi \rangle = \langle t, \bar{\partial}\gamma \rangle = (-1)^{p+q-1} \langle \bar{\partial}t, \gamma \rangle = 0$ . Hence  $\mathcal{L}$  is well defined in that case.

Suppose now that  $q = n$ , then of course  $\bar{\partial}t = 0$  and we have that  $\phi, \psi$  are  $(p, 0)$  forms hence  $\bar{\partial}(\phi - \psi) = 0$  means that  $h := \phi - \psi$  is a  $\bar{\partial}$  closed  $(p, 0)$  form hence  $h \in \mathcal{H}_p(\mathbb{B}) \subset \mathcal{H}_p(\Omega)$ . But by assumption we have  $\langle t, h \rangle = 0$  hence  $\mathcal{L}$  is also well defined in that case.

If  $\alpha$  is a  $(n-p, n-q+1)$  form in  $L^{r'}(\mathbb{B}, \eta)$ ,  $\bar{\partial}$  closed in  $\mathbb{B}$ , then,  $\alpha \in L^{r'}(\Omega)$  and is still  $\bar{\partial}$  closed in  $\Omega$ , hence there is a  $\psi \in L^{r'}(\Omega) :: \bar{\partial}\psi = \alpha$  because  $\Omega$  is  $r'$ -regular.

For  $q < n$ , we have  $\bar{\partial}(\phi - \psi) = \alpha - \alpha = 0$  on  $\Omega$  and, because  $\text{Supp } t \subset \Omega \subset \mathbb{B}$ ,  $\bar{\partial}t = 0$  we get  $\langle t, \phi - \psi \rangle = 0$  hence

$$\mathcal{L}(\alpha) = \langle t, \phi \rangle = \langle t, \psi \rangle.$$

If  $q = n$ , we still have  $\bar{\partial}(\phi - \psi) = \alpha - \alpha = 0$  on  $\Omega$ , and  $\phi - \psi \in \mathcal{H}_p(\Omega)$  hence again by the hypothesis we still get

$$\mathcal{L}(\alpha) = \langle t, \phi \rangle = \langle t, \psi \rangle.$$

In any cases, by Hölder inequalities

$$|\mathcal{L}(\alpha)| \leq \|t\|_{L^r(\Omega)} \|\psi\|_{L^{r'}(\Omega)}.$$

But by the  $r'$ -regularity of  $\Omega$  there is a constant  $C$  such that

$$\|\psi\|_{L^{r'}(\Omega)} \leq C \|\alpha\|_{L^{r'}(\Omega)}.$$

Of course we have

$$\|\alpha\|_{L^{r'}(\Omega)} \leq \|\alpha\|_{L^{r'}(\mathbb{B}, \eta)}$$

because  $\eta = 1$  on  $\Omega$ , hence

$$|\mathcal{L}(\alpha)| \leq C \|t\|_{L^r(\Omega)} \|\alpha\|_{L^{r'}(\mathbb{B}, \eta)}.$$

So we have that the norm of  $\mathcal{L}$  is bounded on the subspace of  $\bar{\partial}$  closed forms in  $L^{r'}(\mathbb{B}, \eta)$  by  $C \|t\|_{L^r(\Omega)}$  which is independent of  $\epsilon$ .

We apply the Hahn-Banach theorem to extend  $\mathcal{L}$  with the same norm to all  $(n-p, n-q+1)$  forms in  $L^{r'}(\mathbb{B}, \eta)$ . As in Serre duality theorem ([10], p. 20) this is one of the main ingredient in the proof.

This means, by the definition of current, that there is a  $(p, q-1)$  current  $u$  which represents the extended form  $\mathcal{L}$  and such that

$$\sup_{\alpha \in L^{r'}(\mathbb{B}, \eta), \|\alpha\|=1} |\langle u, \alpha \rangle| \leq C \|t\|_{L^r(\Omega)}$$

and by lemma 3.2 with the weight  $\eta$ , this implies

$$\|u\|_{L^r(\mathbb{B}, \eta^{1-r})} \leq C \|t\|_{L^r(\Omega)}.$$

In particular  $\|u\|_{L^r(\mathbb{B})} \leq C \|t\|_{L^r(\Omega)}$  because with  $\epsilon < 1$ ,  $r > 1$ , we have  $\eta^{1-r} \geq 1$ .

So applied to a  $\bar{\partial}$  closed  $(n-p, n-q+1)$  current  $\alpha$  we get

$\langle u, \alpha \rangle = (-1)^{p+q-1} \langle t, \phi \rangle$ , with  $\bar{\partial}\phi = \alpha$ , i.e.  $\forall \phi :: \bar{\partial}\phi \in L^{r'}(\mathbb{B}, \eta)$ , we have  $\langle u, \bar{\partial}\phi \rangle = (-1)^{p+q-1} \langle t, \phi \rangle$  and this means precisely, by lemma 2.1, that  $\bar{\partial}u = t$ .

Now for  $\epsilon > 0$  with  $\eta_\epsilon(z) := \mathbb{1}_\Omega(z) + \epsilon \mathbb{1}_{\mathbb{B} \setminus \Omega}(z)$ , let  $u_\epsilon \in L^r(\mathbb{B}, \eta_\epsilon^{1-r})$  be the previous solution, then

$$\|u_\epsilon\|_{L^r(\mathbb{B})}^r \leq \int_{\mathbb{B}} |u_\epsilon|^r \eta^{1-r} dm \leq C^r \|t\|_{L^r(\Omega)}^r.$$

Replacing  $\eta$  by its value we get

$$\int_{\Omega} |u_\epsilon|^r dm + \int_{\mathbb{B} \setminus \Omega} |u_\epsilon|^r \epsilon^{1-r} dm \leq C^r \|t\|_{L^r(\Omega)}^r \implies \int_{\mathbb{B} \setminus \Omega} |u_\epsilon|^r \epsilon^{1-r} dm \leq C^r \|t\|_{L^r(\Omega)}^r$$

hence

$$\int_{\mathbb{B} \setminus \Omega} |u_\epsilon|^r dm \leq C^r \epsilon^{r-1} \|t\|_{L^r(\Omega)}^r.$$

Because  $C$  and the norm of  $t$  are independent of  $\epsilon$ , we have  $\|u_\epsilon\|_{L^r(\mathbb{B})}$  is uniformly bounded and  $r > 1 \implies L^r(\mathbb{B})$  is a dual, hence there is a sub-sequence  $\{u_{\epsilon_k}\}_{k \in \mathbb{N}}$  of  $\{u_\epsilon\}$  which converges weakly, when  $\epsilon_k \rightarrow 0$ , to a  $(p, q-1)$  form  $u$  in  $L^r(\mathbb{B})$ .

To see that this form  $u$  is 0 *a.e.* on  $\mathbb{B} \setminus \Omega$  let us take a component  $u_{I,J}$  of it ; it is the weak limit of the sequence of functions  $\{u_{\epsilon_k, I, J}\}$  which means, with the notations  $v := u_{I, J}$ ,  $v_k := u_{\epsilon_k, I, J}$

$$\forall f \in L^{r'}(\mathbb{B}), \int_{\mathbb{B}} v f dm = \lim_{k \rightarrow \infty} \int_{\mathbb{B}} v_k f dm.$$

As usual take  $f := \frac{\bar{v}}{|v|} \mathbb{1}_E$  where  $E := \{|v| > 0\} \cap (\mathbb{B} \setminus \Omega)$  then we get

$$\int_{\mathbb{B}} v f dm = \int_E |v| dm = \lim_{k \rightarrow \infty} \int_{\mathbb{B}} v_k f dm = \lim_{k \rightarrow \infty} \int_E \frac{v_k \bar{v}}{|v|} dm.$$

Now we have by Hölder

$$\left| \int_E \frac{v_k \bar{v}}{|v|} dm \right| \leq \|v_k\|_{L^r(E)} \|\mathbb{1}_E\|_{L^{r'}(E)}.$$

But

$$\|v_k\|_{L^r(E)}^r \leq \int_{\mathbb{B} \setminus \Omega} |u_{\epsilon_k}|^r dm \leq (\epsilon_k)^{r-1} \|t\|_{L^r(\Omega)} \rightarrow 0, \quad k \rightarrow \infty.$$

Hence

$$\left| \int_E |v| dm \right| \leq C^r \|\mathbb{1}_E\|_{L^{r'}(E)} (\epsilon_k)^{r-1} \|t\|_{L^r(\Omega)}^r \rightarrow 0$$

which implies  $m(E) = 0$  because on  $E$ ,  $|v| > 0$ .

This being true for all components of  $\Omega$ , we get that the form  $u$  is 0 *a.e.* on  $\mathbb{B} \setminus \Omega$ .

So we get

$$\forall \phi \in \mathcal{D}_{n-p, n-q}(\mathbb{B}), \quad (-1)^{p+q-1} \langle t, \phi \rangle = \langle u_\epsilon, \bar{\partial} \phi \rangle \rightarrow \langle u, \bar{\partial} \phi \rangle \implies \langle u, \bar{\partial} \phi \rangle = (-1)^{p+q-1} \langle t, \phi \rangle$$

hence  $\bar{\partial} u = t$  as distributions. ■

**Remark 3.4.** *The condition of orthogonality to the holomorphic functions in case  $q = n$  was already seen for extension of CR functions see [1] and the references therein.*

#### 4. CASE OF STEIN MANIFOLD.

First we define the "Lebesgue measure" on a complex manifold  $X$  as in Hörmander's book [6] section 5.2, with a hermitian metric locally equivalent to the usual one on any analytic coordinates system. Associate to this metric there is a volume form  $dV$  and we take it for the Lebesgue measure on  $X$ .

Let us denote  $\mathcal{H}_p(\Omega)$  the set of all  $(p, 0)$   $\bar{\partial}$  closed forms in  $\Omega$ . Still we have that  $\mathcal{H}_0(\Omega) = \mathcal{H}(\Omega)$ , the set of holomorphic functions in  $\Omega$ , but if  $p > 1$ ,  $\phi \in \mathcal{H}_p(\Omega)$  is a priori no longer a vector of global holomorphic functions in  $\Omega$ .

**Theorem 4.1.** *Let  $X$  be a Stein manifold and  $\Omega$  a bounded strictly pseudo-convex domain with smooth boundary of  $X$ . Let  $t$  be a  $(p, q)$  current in  $L_c^r(\Omega)$ ,  $r > 1$ , such that  $\bar{\partial} t = 0$  if  $1 \leq q < n$  and  $\forall h \in \mathcal{H}_p(\Omega)$ ,  $\langle t, h \rangle = 0$  if  $q = n$ .*

*Let  $B$  be another bounded strictly pseudo-convex domain with smooth boundary in  $X$  such that  $\Omega \subset B$ .*

*Then there is a  $(p, q-1)$  current  $u$  in  $L^r(B)$  such that  $\bar{\partial} u = t$  and which is 0 *a.e.* in  $B \setminus \Omega$ .*

Proof.

We can solve the  $\bar{\partial}$  equation in  $B$  and  $\Omega$  with  $L^r$  estimates for  $(0, 1)$  forms by use of N. Kerzman kernels [7] and for all  $(p, q)$  forms by J-P. Demailly and C. Laurent ones ([5], Remarque 4, page 596).

The remainder of the proof is identical to the proof of theorem 3.3. ■

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