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Error estimates in Approximate Deconvolution Models

Argus A. Dunca* Roger Lewandowski†

Abstract

We consider general Approximate Deconvolution Models (ADM). We estimate the error modeling as a function of the residual stress τ_N and we compute the rate of convergence to the mean Navier-Stokes Equations in terms of the deconvolution order N.

MCS Classification : 76D05, 35Q30, 76F65, 76D03

Key-words : Navier-Stokes equations, Large eddy simulation, Deconvolution models.

1 Introduction and main result

1.1 Framework of LES

Roughly speaking, Large Eddy Simulation modeling of turbulent flows can be viewed as to apply a low pass filter to the Navier-Stokes Equations (NSE) and to model the resulting residual stress [6], which yields a LES model. The filter is usually a convolution operation. In order to perform numerical simulations, we must apply a numerical operator to the LES model over a given grid, which is also a filtering operation that needs in addition a dynamical subgrid model [7], [4]. Note that those two operations may be switched, by filtering the numerical scheme derived from the (NSE) [12].

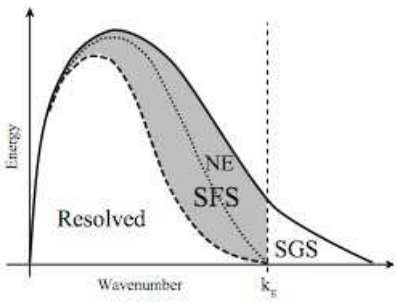

Figure 1: *From Chow et al. 2005 [4]. American Meteorological Society. Reprinted with permission.*

Whatever the order of the procedure, we distinguish three types of scales:

- the resolved scales,
- the subgrid scales (SGS),

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- the subfilter scales (SFS).

In figure 1.1, k_g denotes the cut frequency, whose choice depends on computing resources. The shaded area in figure 1.1 which describes the SFS, is divided in two regions by a dotted line. The upper part measures the numerical error (NE), while the lower part measures the error modeling. The natural issue is the reduction the SFS area.

1.2 Approximate Deconvolution Models

Starting from a Bardina's like model [1], Stolz and Adams [11] have suggested to use a deconvolution algorithm to reduce in some sense the size of the SFS area, a way based on principles from imaging [3] which has led to various versions of Approximate Deconvolution Models (ADM).

This paper is in the continuation of [2], the results of which we recall in what follows. We focus on the continuum version of the ADM in the periodic case. Let

$$G_\alpha(\mathbf{w}) = \overline{\mathbf{w}} = G_\alpha \star \mathbf{w} \tag{1.1}$$

denote the filtering operation, where $\alpha > 0$ is the filter's width, usually of same magnitude of the mesh size in a numerical simulation [9]. Notice that the convolution kernel and the corresponding operator are denoted in the same way, so far no risk of confusion occurs. The kernel G_α is assumed to converge to the dirac function in the sense of the distributions, when α goes to zero. The deconvolution operator $D_{N,\alpha}$ is defined by

$$D_{N,\alpha} = \sum_{n=0}^{N} (I - G_\alpha)^n. \tag{1.2}$$

We assume that G_α is invertible and is such that

$$D_{N,\alpha} \to A_\alpha = G_\alpha^{-1} \quad \text{as} \quad N \to \infty. \tag{1.3}$$

The ADM is specified by the initial value problem:

$$\begin{aligned} \partial_t \mathbf{w} + \nabla \cdot (\overline{D_{N,\alpha}(\mathbf{w}) \otimes D_{N,\alpha}(\mathbf{w})}) - \nu \Delta \mathbf{w} + \nabla q &= \bar{\mathbf{f}}, \\ \nabla \cdot \mathbf{w} &= 0, \\ \mathbf{w}(0, \mathbf{x}) &= \overline{\mathbf{u}_0}(\mathbf{x}), \end{aligned} \tag{1.4}$$

where $\mathbf{w}$ and q are the unknowns.

Throughout the paper, $\nu > 0$ and $\alpha > 0$ are fixed. Since we are in the periodic case, the fields involved in Problem (1.4) are defined on the torus

$$\mathbb{T}_3 = \mathbb{R}^3/\mathcal{T}_3 \quad \text{where} \quad \mathcal{T}_3 := 2\pi\mathbb{Z}^3/L, \tag{1.5}$$

for some given $L > 0$, which is the size of the computational box. All the fields have a zero mean on $\mathbb{T}_3$. We assume that

$$\mathbf{u}_0 \in \mathbf{H}_0, \quad \mathbf{f} \in L^2([0,T] \times \mathbb{T}_3)^3, \tag{1.6}$$

where (see [2])

$$\mathbf{H}_s = \left\{ \mathbf{w} : \mathbb{T}_3 \to \mathbb{R}^3,\ \mathbf{w} \in H^s(\mathbb{T}_3)^3, \quad \nabla \cdot \mathbf{w} = 0, \quad \int_{\mathbb{T}_3} \mathbf{w}\, d\mathbf{x} = \mathbf{0} \right\}, \tag{1.7}$$

Problem (1.4) was first considered in [5] and [8] when G_α is the usual Helmholtz filter, whose transfer function $\widehat{G}_\alpha$ is defined by his symbol,

$$\forall\, \mathbf{k} \in \mathcal{T}_3^\star, \quad \widehat{G}_{\alpha,\mathbf{k}} = \frac{1}{1+\alpha^2|\mathbf{k}|^2}, \quad \widehat{G}_{\alpha,\mathbf{0}} = 0. \tag{1.8}$$

One proves in [5] the existence and the uniqueness of a distributional solution that converges to the NSE in some sense when α goes to zero. One shows in [8] among other things, that the rate of convergence to the NSE is of order $\alpha^{1/3}$, under suitable regularity assumptions about the solution to the NSE. In those two works, the main constant involved in the estimates are depending on N.

In [2], one carrefully studies the problem of taking the limit as N goes to infinity for a large class of filter. One starts in establishing the existence and the uniqueness of what has been called a "regular weak solution", $(\mathbf{w}_N, q_N)$, which is between strong and weak solution because of its specific regularity (see definition 2.1 and Theorem 2.1 below). Moreover, the solution that we got in [2] satisfies estimates which do not depend on N.

We also show in [2] that the sequence $(\mathbf{w}_N, q_N)_{N\in\mathbb{N}}$ converges (up to a subsequence) to a solution $(\mathbf{w}, q) = (\overline{\mathbf{u}}, \overline{p})$ to the "Mean Navier-Stokes Equations" (2.17), when N goes to infinity and α is fixed (see also the alternative form of the MNSE (2.19)). (Theorem 2.2 below). Moreover,

$$(\mathbf{u}, p) = (A_\alpha \mathbf{w}, A_\alpha q) \tag{1.9}$$

is a dissipative solution to the NSE.

The results are firstly proved when $\widehat{G}_\alpha$ is defined by (1.8). They are then extended to the case of generalised Helmholz filters,

$$\forall\, \mathbf{k} \in \mathcal{T}_3^\star, \quad \widehat{G}_{\alpha,\mathbf{k}} = \frac{1}{1+\alpha^{2p}|\mathbf{k}|^{2p}}, \quad \widehat{G}_{\alpha,\mathbf{0}} = 0, \tag{1.10}$$

for $p > 3/4$. Existence of a unique solution to the ADM was recently extended to the case $p > 1/2$ in [**?**].
The results in [2], actually the convergence result, clearly indicate that there are chances that ADM models may help in reducing SFS, so far one knows the rate of convergence in terms of N.

We prove in this paper that this rate of convergence is of order $(pN)^{-(1/4p)}$.

1.3 Error modeling and residual stress: main results

We still denote by $(\mathbf{w}_N, q_N)$ the regular weak solution to (1.4), $(\mathbf{u}, p)$ such that $(\overline{\mathbf{u}}, \overline{p})$ is the limit of $(\mathbf{w}_N, q_N)_{N\in\mathbb{N}}$ and a solution to the "Mean Navier-Stokes Equations" (2.17). We introduce the modeling error $\boldsymbol{\varepsilon}_N$ and the residual stress $\boldsymbol{\tau}_N$ by

$$\boldsymbol{\varepsilon}_N = \overline{\mathbf{u}} - \mathbf{w}_N, \qquad \boldsymbol{\tau}_N = \mathbf{u}\otimes\mathbf{u} - D_N\overline{\mathbf{u}}\otimes D_N\overline{\mathbf{u}} \tag{1.11}$$

The study of the behavior of $\boldsymbol{\varepsilon}_N$ and $\boldsymbol{\tau}_N$ is an "analytic form of à priori testing", used to study the accurracy of general LES models [10], that we have adapted to our framework. We aim to estimate the rate of convergence to zero of some norms of $\boldsymbol{\varepsilon}$ in terms of N.

We start by estimating $\boldsymbol{\varepsilon}_N$ in function of $\boldsymbol{\tau}_N$. Following the way of [2], we consider $A_\alpha^{1/2} D_{N,\alpha}^{1/2} \boldsymbol{\varepsilon}_N$, which is directly estimated from the equations and whose norms controls

many norms of $\boldsymbol{\varepsilon}_N$. However, to carry the calculation out, we must assume that $\mathbf{u} \in L^4([0,T],\mathbf{H}_1)$, which in particular yields uniqueness of this solution to the NSE [13]. The first main result in this paper, section 3, Theorem 3.1 and corollary 3.1, is that for all $t \in [0,T]$,

$$||\boldsymbol{\varepsilon}_N(t,\cdot)||_0^2 + \alpha^{2p}||\boldsymbol{\varepsilon}_N(t,\cdot)||_p^2 + \nu\int_0^t (||\nabla\boldsymbol{\varepsilon}_N(s,\cdot)||_0^2 + \alpha^{2p}||\nabla\boldsymbol{\varepsilon}_N(s,\cdot)||_p^2)ds \leq \frac{8}{\nu}e^{\frac{1}{\nu^3}||\mathbf{u}||^4_{L^4([0,T],\mathbf{H}_1)}}\int_0^t ||\boldsymbol{\tau}_N(s,\cdot)||_0^2 ds, \tag{1.12}$$

regardless of N. This result shows that the modeling error is driven by the L^2 norm of the residual stress. Our second main result in this paper, section 4, Theorem 4.1, we prove that

$$||\boldsymbol{\tau}_N||_{L^2([0,T]\times\mathbb{T}_3)^9} \leq \frac{(2\alpha)^{1/2}}{\sqrt[4p]{2p(N+1)}}||\mathbf{u}||^2_{L^4(\mathbf{H}_1)}, \tag{1.13}$$

which shows that the error modeling is of order $(pN)^{-(1/4p)}$.

2 Mathematical framework

The aim of this section is to fix the mathematical framework and to recall precise statements of the main results of [2].

2.1 Fourier Series expansions

We have seen in [2] that the space $\mathbf{H}_s$ defined by (1.7) is a closed subset of the space

$$\mathbb{H}_s = \left\{ \mathbf{w} = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \widehat{\mathbf{w}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} : \quad \sum_{\mathbf{k}\in\mathcal{T}_3^\star} |\mathbf{k}|^{2s}|\widehat{\mathbf{w}}_{\mathbf{k}}|^2 < \infty, \ \mathbf{k}\cdot\widehat{\mathbf{w}}_{\mathbf{k}} = 0 \right\},$$

equipped with the Hermitian structure defined by the inner product and the associated norm

$$(\mathbf{w},\mathbf{v})_{\mathbf{H}_s} = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} |\mathbf{k}|^{2s}\widehat{\mathbf{w}}_{\mathbf{k}}\cdot\overline{\widehat{\mathbf{v}}_{\mathbf{k}}}, \quad ||\mathbf{w}||_s = \left(\sum_{\mathbf{k}\in\mathcal{T}_3^\star} |\mathbf{k}|^{2s}|\mathbf{w}_{\mathbf{k}}|^2\right)^{\frac{1}{2}}, \tag{2.1}$$

where $\forall\,\mathbf{k} = (k_1,k_2,k_3) \in \mathcal{T}_3$, $|\mathbf{k}|^2 = k_1^2 + k_2^2 + k_3^2$, and in (2.1) the overbar denotes the complex conjugate. For the simplicity, we note G instead of G_α, $A = G^{-1}$ unstead of A_α, and we write

$$G = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \widehat{G}_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{2.2}$$

Throughout the paper, we assume that $\widehat{G}_{\mathbf{k}}$ is defined by (1.10) , $p > 3/4$. Notice that one has the differential relation

$$-\alpha^{2p}\Delta^p\overline{\mathbf{u}} + \overline{\mathbf{u}} = \mathbf{u}, \quad \nabla\cdot\overline{\mathbf{u}} = 0. \tag{2.3}$$

The symbol of the deconvolution operator $D_N = \sum_{0\le n\le N} (\mathbf{I}-G)^n$ takes for values

$$\widehat{D_{N,\mathbf{k}}} = \sum_{n=0}^{N}\left(\frac{\alpha^{2p}|\mathbf{k}|^{2p}}{1+\alpha^{2p}|\mathbf{k}|^{2p}}\right)^n = (1+\alpha^{2p}|\mathbf{k}|^{2p})\rho_{N,p,\mathbf{k}}, \qquad \rho_{N,p,\mathbf{k}} = 1-\left(\frac{\alpha^{2p}|\mathbf{k}|^{2p}}{1+\alpha^{2p}|\mathbf{k}|^{2p}}\right)^{N+1}. \tag{2.4}$$

Basic calculations yield

$$1 \le \widehat{D_{N,p}}(\mathbf{k}) \le N+1, \quad \forall\, \mathbf{k}\in\mathcal{T}_3, \tag{2.5}$$

$$\widehat{D_{N,p}}(\mathbf{k}) \approx (N+1)\frac{1+\alpha^{2p}|\mathbf{k}|^{2p}}{\alpha^{2p}|\mathbf{k}|^{2p}}, \quad \text{for large } |\mathbf{k}|, \tag{2.6}$$

$$\lim_{|\mathbf{k}|\to+\infty} \widehat{D_{N,p}}(\mathbf{k}) = N+1, \tag{2.7}$$

$$\widehat{D_{N,p}}(\mathbf{k}) \le (1+\alpha^{2p}|\mathbf{k}|^{2p}) = \widehat{A}_{\mathbf{k}}, \quad \forall\, \mathbf{k}\in\mathcal{T}_3. \tag{2.8}$$

2.2 Recall of the former results

Throughout the paper, we assume that $\mathbf{u}_0$ and $\mathbf{f}$ satisfy (1.6), and $\alpha > 0$ is fixed.

Definition 2.1 (Regular Weak solution)**.** *We say that the couple $(\mathbf{w}, q)$ is a "regular weak solution" to system (1.4) if and only if the three following items are satisfied:*

1) Regularity

$$\mathbf{w} \in L^2([0,T];\mathbf{H}_{1+p}) \cap C([0,T];\mathbf{H}_p), \tag{2.9}$$

$$\partial_t \mathbf{w} \in L^2([0,T];\mathbf{H}_0) \tag{2.10}$$

$$q \in L^2([0,T];H^1(\mathbb{T}_3)), \tag{2.11}$$

2) Initial data

$$\lim_{t\to 0} \|\mathbf{w}(t,\cdot) - \overline{\mathbf{u}_0}\|_{\mathbf{H}_p} = 0, \tag{2.12}$$

3) Weak Formulation

$$\forall\, \mathbf{v} \in L^2([0,T];H^1(\mathbb{T}_3)^3), \tag{2.13}$$

$$\int_0^T\int_{\mathbb{T}_3} \partial_t\mathbf{w}\cdot\mathbf{v} - \int_0^T\int_{\mathbb{T}_3} \overline{D_N(\mathbf{w})\otimes D_N(\mathbf{w})} : \nabla\mathbf{v} + \nu\int_0^T\int_{\mathbb{T}_3} \nabla\mathbf{w} : \nabla\mathbf{v} + \int_0^T\int_{\mathbb{T}_3} \nabla q\cdot\mathbf{v} = \int_0^T\int_{\mathbb{T}_3} \overline{\mathbf{f}}\cdot\mathbf{v}. \tag{2.14}$$

Let us recall the main results of [2].

Theorem 2.1. *([2]) Problem (1.4) has a unique regular weak solution. Moreover, when $p \ge 1$,*

$$\partial_t\mathbf{w} \in L^2([0,T],\mathbf{H}_{p-1}), \quad q \in L^2([0,T],H^p(\mathbb{T}_3)). \tag{2.15}$$

We denote by $(\mathbf{w}_N, q_N)$ the regular weak solution to Problem (1.4).

Theorem 2.2. *([2]) From the sequence $(\mathbf{w}_N, q_N)_{N\in\mathbb{N}}$ one can extract a sub-sequence (still denoted $(\mathbf{w}_N, q_N)_{N\in\mathbb{N}}$) such that*

$$\mathbf{w}_N \to \mathbf{w} \quad \begin{cases} \text{weakly in } L^2([0,T]; H^{1+p}(\mathbb{T}_3)^3) \cap L^\infty([0,T]; H^p(\mathbb{T}_3)^3), \\ \text{strongly in } L^r([0,T]; H^p(\mathbb{T}_3)^3), \quad \forall\, 1 \le r < +\infty, \end{cases} \tag{2.16}$$

$$q_N \to q \qquad \text{weakly in } L^2([0,T]; H^1(\mathbb{T}_3) \cap L^{5/3}([0,T]; W^{2p,5/3}(\mathbb{T}_3)),$$

and such that the "Mean Navier-Stokes Equations"

$$\begin{aligned} \partial_t \mathbf{w} + \nabla\cdot(\overline{A\mathbf{w}\otimes A\mathbf{w}}) - \nu\Delta\mathbf{w} + \nabla q &= \bar{\mathbf{f}}, \\ \nabla\cdot\mathbf{w} &= 0, \\ \mathbf{w}(0,\mathbf{x}) &= \overline{\mathbf{u}_0}(\mathbf{x}), \end{aligned} \tag{2.17}$$

holds in the sense of the distributions. □

Fro a better understanding of this result, let us consider $(\mathbf{u}, p)$ defined by

$$(\mathbf{u}, p) = (A\mathbf{w}, Aq) \tag{2.18}$$

From (2.17), we deduce that $(\mathbf{u}, p)$ is a solution to the equation

$$\begin{aligned} &\partial_t \overline{\mathbf{u}} + \nabla\cdot(\overline{\mathbf{u}\otimes\mathbf{u}}) - \nu\Delta\overline{\mathbf{u}} + \overline{p} = \bar{\mathbf{f}}, \\ &\nabla\cdot\overline{\mathbf{u}} = 0. \end{aligned} \tag{2.19}$$

This is why we call the equation (2.17) the "Mean Navier-Stokes Equations". Since G is invertible and commutes with the differential operators, we deduce that $(\mathbf{u}, p)$ is a distributional solution to the Navier-Stokes Equations. One proves in addition in [2] that this solution is dissipative since it verifies an energy inequality.

3 Estimate of the modeling error

Let $(\mathbf{w}_N, q_N)$ be the solution of Problem (1.4). We assume in addition that

$$\mathbf{u} \in L^4([0,T], \mathbf{H}_1), \tag{3.1}$$

where $\mathbf{u}$ is specified by Theorem 2.2. Assumption (3.1) yields that $\mathbf{u}$ is defined in a unique way and among other things, that the whole sequence $(\mathbf{w}_N)_{N\in\mathbb{N}}$ converges to $\overline{\mathbf{u}}$. Recall that $\boldsymbol{\varepsilon}_N = \overline{\mathbf{u}} - \mathbf{w}_N$ and $\boldsymbol{\tau}_N = \mathbf{u}\otimes\mathbf{u} - D_N\overline{\mathbf{u}}\otimes D_N\overline{\mathbf{u}}$.

The aim of this section is to estimate $\boldsymbol{\varepsilon}_N$ in terms of $\boldsymbol{\tau}_N$.

Theorem 3.1. *The following estimate holds*

$$\begin{aligned} ||A_\alpha^{1/2} D_{N,\alpha}^{1/2} \boldsymbol{\varepsilon}_N(t,\cdot)||_0^2 + \nu \int_0^t ||A_\alpha^{1/2} D_{N,\alpha}^{1/2} \boldsymbol{\varepsilon}_N(s,\cdot)||_1^2 ds \le \\ \frac{8}{\nu} e^{\frac{1}{\nu^3}||\mathbf{u}||^4_{L^4([0,T],\mathbf{H}_1)}} \int_0^t ||\boldsymbol{\tau}_N(s,\cdot)||_0^2 ds, \end{aligned} \tag{3.2}$$

for all $N > 0$ and $t \ge 0$.

Corollary 3.1. *The following inequality holds:*

$$||\boldsymbol{\varepsilon}_N(t,\cdot)||_0^2 + \alpha^{2p}||\boldsymbol{\varepsilon}_N(t,\cdot)||_p^2 + \nu\int_0^t (||\nabla\boldsymbol{\varepsilon}_N(s,\cdot)||_0^2 + \alpha^{2p}||\nabla\boldsymbol{\varepsilon}_N(s,\cdot)||_p^2)ds \leq \frac{8}{\nu}e^{\frac{1}{\nu^3}||\mathbf{u}||^4_{L^4([0,T],\mathbf{H}_1)}}\int_0^t ||\boldsymbol{\tau}_N(s,\cdot)||_0^2 ds, \tag{3.3}$$

regardless of $t \geq 0$ and N.

Proof. We write the equation satisfied by $\boldsymbol{\varepsilon}_N$ in substracting (1.4) to (2.17). We express the right hand side in terms of $\boldsymbol{\tau}_N$, which leads to

$$\partial_t\boldsymbol{\varepsilon}_N + \nabla\cdot(\overline{D_N\boldsymbol{\varepsilon}_N \otimes D_N\mathbf{w}_N}) - \nu\Delta\boldsymbol{\varepsilon}_N + \nabla r_N = -\nabla\cdot\overline{\boldsymbol{\tau}}_N - \nabla\cdot\overline{D_N\overline{\mathbf{u}}\otimes D_N\boldsymbol{\varepsilon}_N}, \tag{3.4}$$

where $r_N = \overline{p} - q_N$. We want to use $AD_N\boldsymbol{\varepsilon}_N$ as multiplier in (3.4) and integrate by parts. We deduce from (3.1) that $\overline{\mathbf{u}} \in L^4([0,T],\mathbf{H}_{1+2p})$. Therefore, $AD_N\boldsymbol{\varepsilon}_N \in L^2([0,T],\mathbf{H}_{1-p})$. Arguing exactly like in the proof of Theorem 5.1 in [2] by distinguishing the cases $3/4 < p \leq 1$ and $1 < p$, we can show that each term in equation (3.4) is of enough regularity to make sure that $AD_N\boldsymbol{\varepsilon}_N$ can be used as test in (3.4). We skip the details for the simplicity. Since all the operators we consider are self adjoint, the following holds:

$$\begin{aligned}(\partial_t\boldsymbol{\varepsilon}_N, AD_N\boldsymbol{\varepsilon}_N) &= \frac{d}{2dt}||A^{\frac{1}{2}}D_N^{\frac{1}{2}}\boldsymbol{\varepsilon}_N||_0^2,\\ (-\Delta\boldsymbol{\varepsilon}_N, AD_N\boldsymbol{\varepsilon}_N) &= ||A^{\frac{1}{2}}D_N^{\frac{1}{2}}\boldsymbol{\varepsilon}_N||_1^2.\end{aligned} \tag{3.5}$$

Furthermore, since $AD_N\boldsymbol{\varepsilon}_N$ has zero divergence, $(\nabla r_N, AD_N\boldsymbol{\varepsilon}_N) = 0$. Finally, as the operators commute with the differential operators,

$$\begin{aligned}(\nabla\cdot(\overline{D_N\boldsymbol{\varepsilon}_N\otimes D_N\mathbf{w}_N}), AD_N\boldsymbol{\varepsilon}_N) &= (A^{-1}\nabla\cdot(D_N\boldsymbol{\varepsilon}_N\otimes D_N\mathbf{w}_N), AD_N\boldsymbol{\varepsilon}_N) =\\ (AA^{-1}\nabla\cdot(D_N\boldsymbol{\varepsilon}_N\otimes D_N\mathbf{w}_N), AD_N\boldsymbol{\varepsilon}_N) &= (\nabla\cdot(D_N\boldsymbol{\varepsilon}_N\otimes D_N\mathbf{w}_N), D_N\boldsymbol{\varepsilon}_N) =\\ &((D_N\mathbf{w}_N\cdot\nabla)D_N\boldsymbol{\varepsilon}_N, D_N\boldsymbol{\varepsilon}_N) = 0,\end{aligned} \tag{3.6}$$

because $AD_N\mathbf{w}_N$ has zero divergence. Finally, arguing as in (3.6) to eliminate the bar in the integrals of right hand side, we get

$$\frac{d}{2dt}||A^{\frac{1}{2}}D_N^{\frac{1}{2}}\boldsymbol{\varepsilon}_N||_0^2 + \nu||A^{\frac{1}{2}}D_N^{\frac{1}{2}}\boldsymbol{\varepsilon}_N||_1^2 = (\boldsymbol{\tau}_N, \nabla D_N\boldsymbol{\varepsilon}_N) - ((D_N\boldsymbol{\varepsilon}_N\cdot\nabla)D_N\overline{\mathbf{u}}, D_N\boldsymbol{\varepsilon}_N) \tag{3.7}$$

We bound each term of the right hand side of (3.7) after each other. From Cauchy-Schwarz inequality combined with Young inequality, we get

$$|(\boldsymbol{\tau}_N, \nabla D_N\boldsymbol{\varepsilon}_N)| \leq \frac{4}{\nu}||\boldsymbol{\tau}||_0 + \frac{\nu}{4}||D_N\boldsymbol{\varepsilon}_N||_1. \tag{3.8}$$

In the same way, by using the Ladyzenskaya inequality [13] we obtain

$$|((D_N\boldsymbol{\varepsilon}_N\cdot\nabla)D_N\overline{\mathbf{u}}, D_N\boldsymbol{\varepsilon}_N)| \leq ||D_N\boldsymbol{\varepsilon}_N||_{L^4}^2||D_N\overline{\mathbf{u}}||_1 \leq ||D_N\boldsymbol{\varepsilon}_N||_0^{\frac{1}{2}}||D_N\boldsymbol{\varepsilon}_N||_1^{\frac{3}{2}}||D_N\overline{\mathbf{u}}||_1. \tag{3.9}$$

The symbol of D_NG is equal to $1-(1-\widehat{G}_{\mathbf{k}})^{N+1} \in [0,1]$. Therefore, we have $||D_N\overline{\mathbf{u}}||_1 \leq ||\mathbf{u}||_1$. By Young inequality combined with (3.10), we obtain

$$|((D_N\boldsymbol{\varepsilon}_N\cdot\nabla)D_N\overline{\mathbf{u}}, D_N\boldsymbol{\varepsilon}_N)| \leq \frac{1}{\nu^3}||\mathbf{u}||_1^4||D_N\boldsymbol{\varepsilon}||_0 + \frac{\nu}{4}||D_N\boldsymbol{\varepsilon}||_1. \tag{3.10}$$

We deduce from (2.8) that the symbol of D_N is less than the symbol of $A^{1/2}D_N^{1/2}$, which leads to

$$||D_N\boldsymbol{\varepsilon}_N||_0 \leq ||A^{\frac{1}{2}}D_N^{\frac{1}{2}}\boldsymbol{\varepsilon}_N||_0, \tag{3.11}$$

regardless of N. Combining (3.7), (3.8), (3.10) and (3.11) yields

$$\frac{d}{dt}||A^{\frac{1}{2}}D_N^{\frac{1}{2}}\boldsymbol{\varepsilon}_N||_0^2 + \nu||A^{\frac{1}{2}}D_N^{\frac{1}{2}}\boldsymbol{\varepsilon}_N||_1^2 \leq \frac{8}{\nu}||\boldsymbol{\tau}||_0 + \frac{1}{\nu^3}||\mathbf{u}||_1^4||A^{\frac{1}{2}}D_N^{\frac{1}{2}}\boldsymbol{\varepsilon}_N||_0^2 \tag{3.12}$$

Inequality (3.2) results from inequality (3.12) thanks to a standard generalisation of Gronwall's lemma [**?**]. □

Proof of the corollary. Let $\mathbf{v} = \sum_{\mathbf{k}\in\mathcal{T}_3} \hat{\mathbf{v}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \in \mathbf{H}_p$. We observe that

$$||A^{\frac{1}{2}}\mathbf{v}||_p^2 = \sum_{\mathbf{k}\in\mathcal{T}_3}(1+\alpha^{2p}|\mathbf{k}|^2p)|\hat{\mathbf{v}}_{\mathbf{k}}|^{2p} = ||\mathbf{v}||_0^2 + \alpha^{2p}||\mathbf{v}||_p^2. \tag{3.13}$$

Based on this remark, we first use $\mathbf{v} = D_N^{1/2}\boldsymbol{\varepsilon}_N$ and then $\mathbf{v} = \partial_i D_N^{1/2}\boldsymbol{\varepsilon}_N$ in (3.2) which yields the corresponding inequality satisfied by $D_N^{1/2}\boldsymbol{\varepsilon}_N$. We derive inequality (3.3) from that by using (2.5), which yields the general formal inequality $||\mathbf{w}||_s \leq ||D_N^{1/2}\mathbf{w}||_s$. □

4 Residual stress and rate of convergence

Now that we have shown that the error modeling $\boldsymbol{\varepsilon}_N$ is driven by the L^2 norm of the residual stress $\boldsymbol{\tau}_N$, involving the $L^4(\mathbf{H}_1)$ norm of $\mathbf{u}$, it remains estimate the L^2 norm of $\boldsymbol{\tau}_N$, which what we aim to carry out in this section. The assumption are those of section 3.

Theorem 4.1. *The following estimate holds:*

$$||\boldsymbol{\tau}_N||_{L^2([0,T]\times\mathbb{T}_3)^9} \leq \frac{(2\alpha)^{1/2}}{\sqrt[4p]{2p(N+1)}}||\mathbf{u}||_{L^4(\mathbf{H}_1)}^2, \tag{4.1}$$

Proof. We write $\boldsymbol{\tau}_N$ as

$$\boldsymbol{\tau}_N = (\mathbf{u} - D_N\overline{\mathbf{u}}) \otimes \mathbf{u} + D_N\overline{\mathbf{u}} \otimes (\mathbf{u} - D_N\overline{\mathbf{u}}). \tag{4.2}$$

Therefore, combining Hölder inequality with $1/3 + 1/6 = 1/2$ for conjugation, to the Sobolev inequality $||\mathbf{w}||_{L^6} \leq ||\mathbf{w}||_1$, we get

$$||\boldsymbol{\tau}||_0^2 \leq 2||\mathbf{u}||_1^2||\mathbf{u} - D_N\overline{\mathbf{u}}||_{L^3(\mathbb{T}_3)^3}^2, \tag{4.3}$$

We must estimate $||\mathbf{u} - D_N\overline{\mathbf{u}}||_{L^3(\mathbb{T}_3)^3}^2$. To carry this out, we use the injection of $\mathbf{H}_{1/2}$ onto $L^3(\mathbb{T}_3)^3$, which yields

$$||\mathbf{u} - D_N\overline{\mathbf{u}}||_{L^3(\mathbb{T}_3)^3}^2 \leq ||\mathbf{u} - D_N\overline{\mathbf{u}}||_{1/2}^2. \tag{4.4}$$

By using (2.4), we obtain

$$||\mathbf{u} - D_N\overline{\mathbf{u}}||_{1/2}^2 = \sum_{\mathbf{k}\in\mathcal{T}_3}\left(\frac{\alpha^{2p}|\mathbf{k}|^{2p}}{1+\alpha^{2p}|\mathbf{k}|^{2p}}\right)^{2N+2}|\mathbf{k}|\,|\hat{\mathbf{u}}_{\mathbf{k}}|^2, \tag{4.5}$$

where $\mathbf{u} = \sum_{\mathbf{k}\in\mathcal{T}_3} \hat{\mathbf{u}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$. We apply the technical inequality (5.1) proved in Appendix 5 below, with $x = \alpha^p|\mathbf{k}|^p$ and $a = 2p(N+1) > 1$. We obtain

$$\left(\frac{\alpha^{2p}|\mathbf{k}|^{2p}}{1+\alpha^{2p}|\mathbf{k}|^{2p}}\right)^{2p(N+1)} \leq \frac{\alpha^p|\mathbf{k}|^p}{\sqrt{2p(N+1)}}. \tag{4.6}$$

We raise both sides of (4.6) to the power $1/p$, we multiply the result by $|\mathbf{k}||\hat{\mathbf{u}}_{\mathbf{k}}|^2$ and get

$$\left(\frac{\alpha^{2p}|\mathbf{k}|^{2p}}{1+\alpha^{2p}|\mathbf{k}|^{2p}}\right)^{2N+2} |\mathbf{k}||\hat{\mathbf{u}}_{\mathbf{k}}|^2 \leq \frac{\alpha}{\sqrt[2p]{2p(N+1)}}|\mathbf{k}|^2|\hat{\mathbf{u}}_{\mathbf{k}}|^2. \tag{4.7}$$

Using the relation $||\mathbf{u}||_1^2 = \sum_{\mathbf{k}\in\mathcal{T}_3} |\mathbf{k}|^2|\hat{\mathbf{u}}_{\mathbf{k}}|^2$ and (4.5) we finally obtain

$$||\mathbf{u} - D_N\overline{\mathbf{u}}||_{1/2}^2 \leq \frac{2\alpha}{\sqrt[2p]{2p(N+1)}}||\mathbf{u}||_1^4 \tag{4.8}$$

Inequality (4.1) results from (4.8) combined to (4.3), after integrating with respect to time over $[0, T]$. □

4.1 Conclusions and perspectives

The dependence of the rate of convergence in p shows that it deacreases as p increases. This is not suprising. Indeed, as p increases, the regularity of the solution to the ADM increases likewise, which makes rise the SFS area. Therefore, one needs more iterations in N to reconstruct the true numerical solution.

To conclude this study in order to fix optimal p and optimal N in terms of computer ressources, it remains to compute the complexity of the ADM, which means to compute of the dimension of the attractor. Our conjecture is that this dimension does depend on N.

5 Appendix

This technical aims in proving a general technical inequality that has been used in the proof of the estimate (4.1). The result is the following.

Theorem 5.1. *The scalar inequality*

$$\left(\frac{x^2}{1+x^2}\right)^a \leq \frac{x}{\sqrt{a}} \tag{5.1}$$

holds true for any $x > 0$, $a \geq 1$. Furthermore, the inequality is optimal in the sense that for any exponent $\beta > \frac{1}{2}$ and any $K > 0$ there exists $x > 0$ and $a > 1$ such that

$$\left(\frac{x^2}{1+x^2}\right)^a > K\frac{x}{a^\beta}.$$

Proof:
Let's assume first that a is an integer, $a = n \in \mathbb{N}^*$. Using Newton's binomial formula we have

$$(1+x^2)^n \geq x^{2n} + nx^{2n-2} \geq 2\sqrt{n}x^{2n-1}$$

for any $x > 0$. The middle term contains the first two terms in the binomial expansion. The second inequality is due to the general scalar inequality $s^2 + t^2 \geq 2st$.

Therefore dividing by $(1+x^2)^n$ and arranging terms gives

$$\left(\frac{x^2}{1+x^2}\right)^n \leq \frac{x}{2\sqrt{n}}.$$

For general a we let $n = [a] \geq 1$ be the integer part of a. It follows that for any $x > 0$ we have that

$$\left(\frac{x^2}{1+x^2}\right)^a \leq \left(\frac{x^2}{1+x^2}\right)^n \leq \frac{x}{2\sqrt{n}} \leq \frac{x}{\sqrt{a}}.$$

The last inequality is due to $4[a] \geq a$ whenever $a \geq 1$. Therefore

$$\left(\frac{x^2}{1+x^2}\right)^a \leq \frac{x}{\sqrt{a}}.$$

To prove the optimality of the inequality (in the sense explained in the theorem) we assume there exists $\beta > 0.5$ and $K > 0$ such that

$$\left(\frac{x^2}{1+x^2}\right)^a \leq K\frac{x}{a^\beta}$$

for any $x > 0, a \geq 1$.

Letting $x > 1$ and $a = x^2$ it follows that

$$\left(\frac{x^2}{1+x^2}\right)^{x^2} \leq Ka^{1/2-\beta}.$$

Now take the limit as $x \to \infty$ and obtain $e^{-1} \leq 0$, which is a contradiction. □

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