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High-frequency market-making with inventory constraints and directional bets

Pietro FODRA [12] Mauricio LABADIE [1]

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Abstract

In this paper we extend the market-making models with inventory constraints of Avellaneda and Stoikov (*High-frequency trading in a limit-order book*, Quantitative Finance Vol.8 No.3 2008) and Lehalle, Guéant and Fernandez-Tapia (*Dealing with inventory risk*, Preprint 2011) to the case of a rather general class mid-price processes, under either exponential or linear PnL utility functions, and with an inventory-risk-aversion parameter that penalises the marker-maker if she finishes her day with a non-zero inventory. This general, non-martingale framework allows a market-maker to make directional bets on market trends whilst keeping under control her inventory risk. In order to achieve this, the marker-maker places non-symmetric limit orders that favour market orders to hit her ask (resp. bid) quotes if she expects that prices will go up (resp. down).

In the case of a mean-reverting mid-price, we show numerically that the market-maker can increase her PnL between 10% and 25% depending on her buget risk on inventory and PnL distribution (especially variance, skewness, kurtosis and VaR). Moreover, with this inventory-risk-aversion parameter the market-maker has not only direct control on her inventory risk but she also has indirect control on the moments of her PnL distribution. Therefore, this parameter can be seen as a fine-tuning of the marker-maker's risk-reward profile.

Keywords: Quantitative Finance, high-frequency trading, market-making, limit-order book, inventory risk, optimisation, stochastic control, Hamilton-Jacobi-Bellman, PnL distribution.

1 Introduction

Market-makers

A market-maker is a trader who buys and sells assets in a stock exchange via make firm quotes: once she shows a buying/selling quantity at a certain price price, she is engaged to trade under those conditions. As liquidity provider, a market-maker receives a compensation: she buys at a lower price (bid) and sells at a higher price (ask). This difference is called the *spread*.

A market maker is exposed to two main risks, *adverse selection* and *inventory risk*. *Adverse selection* means that if the market-maker sells (resp. buys) an asset it is not necessarily good news, it could mean

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that her ask (resp. bid) price is lower (resp. higher) than it should on the current market conditions. *Inventory risk* means comes into play by inbalances in the arrival of buying and selling orders: since the market-maker quotes both bid and ask prices, her net position depends depends on which quotes are executed and in which quantities. A market maker uses the spread to both control her inventory and compensate herself from adverse selection. In a nutshell, a market-maker loses money against informed traders, but she covers that loss by making noise (i.e. un-informed) traders pay the spread on each transaction.

In order to create a market-making strategy, we need to consider three factors: *price*, *spread* and *inventory*. The *price* is often the mid-price, i.e. the average between the current ask and bid prices of the market. The *spread* of the market-maker is her only control on her PnL and *inventory* throughout the trading day. It is true that the market maker can affect the mid-price by improving the current ask and bid market quotes, but since that would normally trigger a market order that consumes the offer, in a first approach we can consider that the mid price cannot be affected by the quotes of the market maker.

Organisation of the study

The goal of this study is to find the optimal ask and bid quotes for a high-frequency market-maker that, under the framework of a directional bet on the market trend, simultaneously maximise her PnL and minimise her inventory risk. The main inspiration is the paper of Avellaneda and Stoikov [1], who found via stochastic control the optimal bid and ask quotes for a high-frequency market-maker. The second inspiration comes from Lehalle *et al* [4], who formalised the findings of Avellaneda and Stoikov [1]. It is also worth to mention that the Hamilton-Jacobi-Bellman framework we use was originally set by Ho and Stoll [5], but in neither of these three articles the effect of a directional market bet on the PnL distribution was considered.

In Chapter 2 we set the framework under which we will be working, which is stochastic control and Hamilton-Jacobi-Bellman equations. In Section 3 we completely solve the optimal control problem for a linear utility function and an arbitrary Markov process by finding explicitly the unique solution. In Section 4 we perturbate the linear utility function via a quadratic inventory penalty. By using first-order approximations on the arrival of orders to the limit-order book, we find explicitly the (approximate) solutions and their corresponding optimal controls, which are perturbations of the closed-form solution and controls we found in Section 3. In Section 5 we extend the results of Avellaneda and Stoikov [1] and Lehalle *et al* [4] for exponential utility functions. We show that their approach can be used to more general process than arithmetic Brownian motion and with inventory-risk aversion. In Section 6 we perform numerical simulations to show how do directional bets on the market trend affect the market-making strategies. We also assess the effect of the inventory-risk-aversion parameter on the PnL distribution and the inventory risk and show that this parameter controls directly the inventory risk and indirectly the PnL distribution.

2 Stochastic control framework

Setting of the problem

We suppose that the mid-price process $S(t)$ follows an Itô diffusion, i.e.

$$dS(t) = b(t, S(t))dt + \sigma(t, S(t))dW(t) \tag{1}$$

where $W(t)$ is a standard Brownian motion.

A market-maker can control her ask and bid quotes, which we denote $p^+(t)$ and $p^-(t)$ respectively. Instead of working with the market-maker's prices we will rather work with the market-maker's spreads, i.e

$$\delta^+(t) := p^+(t) - S(t) \geq 0, \qquad \delta^-(t) := S(t) - p^-(t) \geq 0\,.$$

The market-maker's spreads are non-negative because, as a liquidity provider to the market, she covers her inventory risk with her bid-ask spread $\delta^+ + \delta^- \geq 0$.

We assume that the market-maker has a utility function $\phi(s, q, t)$ and an associated value function

$$u(t, s, q, x) := \mathbb{E}_{t,s,q,x}\Big[\phi\Big(S(T), Q(T), X(T)\Big)\Big]$$

where $t \in [0, T]$ is the current time, $s = S(t)$ is the current mid-price of the asset, $x = X(t)$ is the current wealth in cash and $q = Q(t)$ is the current inventory level.

This value function has two components, a continuous term relative to the continuous variable s and a discrete term coming from the jump variables (q, x).

The Hamilton-Jacobi-Bellman equation

Let us give a heuristic interpretation of the Hamilton-Jacobi-Bellman equation in terms of the two infinitesimal operators, one on the continuous variable s and the other on the jump variables (q, x), as well as of the stochastic controls (δ^+, δ^-).

First, suppose q and x fixed. Since the continuous variable s follows (1), from Feynmann-Kac representation formula we have that if $(t, s) \mapsto u_C(t, s, q, x)$ satisfies

$$(\partial_t + \mathcal{L})\, u_C = 0\,, \qquad \mathcal{L} := b(t, s)\partial_s + \frac{1}{2}\sigma^2(t, s)\partial_{ss}\,, \qquad u_C(T, s, q, x) = \phi(s, q, x) \tag{2}$$

then (see e.g. Pham [6])

$$u_C(t, s, q, x) = \mathbb{E}_{t,s}\Big[\phi\Big(S(T), q, x)\Big)\Big]\,.$$

Second, suppose now that s is fixed. We model the arrival of orders to the limit-order Book as two independent Poisson Process, N^+ for the ask quotes and N^- for the bid quotes , with intensity $\lambda^\pm$ (respectively).

- For the ask quote, we assume that λ^+ depends on the distance to the mid-price s, i.e. $\lambda^+ = \lambda^+(\delta^+)$. Moreover, since buying market orders favour ask quotes with the smallest spread δ^+, it is natural to assume that $\delta^+ \mapsto \lambda^+(\delta^+)$ is increasing, i.e. the probability of execution for the market-maker decreases as she moves her ask quote further away from the mid price.
- Analogously, for the bid quote we have $\lambda^- = \lambda^-(\delta^-)$ and $\delta^- \mapsto \lambda^-(\delta^-)$ increasing.

The jump variables (q, x) are related via the arrival of market orders that hit the quotes of the market-maker:

- Suppose a buying market order of one share of S hits the ask quote of the market-maker. It follows then that $x \mapsto x + (s + \delta^+)$ and $q \mapsto q - 1$.
- Analogously, if a selling market order of one share hits her bid quote then $x \mapsto x + (s - \delta^-)$ and $q \mapsto q + 1$.

Under this framework, we have that the value function of the jump $(q, x) \mapsto u_D(t, s, q, x)$ satisfies

$$\begin{aligned} \partial_t u_D + \lambda^+(\delta^+)\left[u_D(t,s,q-1,x+(s+\delta^+)) - u_D(t,s,q,x)\right] & \\ +\lambda^+(\delta^+)\left[u_D(t,s,q+1,x-(s-\delta^-)) - u_D(t,s,q,x)\right] &= 0\,. \\ u(T,s,q,x) &= \phi(s,q,x)\,. \end{aligned} \tag{3}$$

Third, let us now put together the continuous and jump dynamics (2)-(3). Suppose that the market-maker's spread quotes (δ^+, δ^-) are known (i.e. deterministic), the value function $u(t, s, q, x)$ satisfies has the following infinitesimal generator (see e.g. Ho and Stoll [5]):

$$\begin{aligned} (\partial_t + \mathcal{L})\, u + \lambda^+(\delta^+)\left[u(t,s,q-1,x+(s+\delta^+)) - u(t,s,q,x)\right] & \\ +\lambda^+(\delta^+)\left[u(t,s,q+1,x-(s-\delta^-)) - u(t,s,q,x)\right] &= 0\,. \\ u(T,s,q,x) &= \phi(s,q,x)\,. \end{aligned} \tag{4}$$

Fourth, notice that (4) is valid only when the spread quotes (δ^+, δ^-) are known, but in the current case they are part of the set of unknowns of the problem. In consequence, from the stochastic control theory it follows that the value function $u(t, s, q, x)$ with unknown controls (δ^+, δ^-) is solution of the Hamilton-Jacobi-Bellman equation

$$\begin{aligned} (\partial_t + \mathcal{L})\, u + \max_{\delta^+} \lambda^+(\delta^+)\left[u(t,s,q-1,x+(s+\delta^+)) - u(t,s,q,x)\right] & \\ + \max_{\delta^-} \lambda^+(\delta^+)\left[u(t,s,q+1,x-(s-\delta^-)) - u(t,s,q,x)\right] &= 0\,, \\ u(T,s,q,x) &= \phi(s,q,x)\,. \end{aligned} \tag{5}$$

In general we should use *sup* instead of *max* in (5), but if we assume that $b(t, s)$ and $\sigma(t, s)$ are Lipschitz and that the jump dynamic (3) is bounded then the supremum is attained and the solution $u(t, s, q, x)$ is unique, as in the particular cases we will consider here.

Fifth, Avellaneda and Stoikov [1] showed that, given the empirical evidence provided by the current research in Econophysics e.g. Potters and Bouchaud [7], we can assume that $\lambda^{\pm}(\delta) = Ae^{-k\delta}$. Under this framework, the Hamilton-Jacobi-Bellman (5) becomes

$$
\begin{aligned}
(\partial_t + \mathcal{L})\, u + \max_{\delta^+} A e^{-k\delta^+} \left[u(t,s,q-1,x+(s+\delta^+)) - u(t,s,q,x) \right] & \\
+ \max_{\delta^-} A e^{-k\delta^-} \left[u(t,s,q+1,x-(s-\delta^-)) - u(t,s,q,x) \right] &= 0\,, \\
u(T,s,q,x) &= \phi(s,q,x)\,,
\end{aligned} \tag{6}
$$

Solving the Hamilton-Jacobi-Bellman equation

Equation (6) is the one we will consider in the rest of the present work. The steps to solve it are as follows:

1. Based on the utility function $\phi(s,q,x)$ we make an *ansatz*, i.e. we guess the general form of the solution of the on Hamilton-Jacobi-Bellman equation. For example, if

$$\phi(s,q,x) = x + \varphi(s,q)$$

we will use

$$u(t,s,q,x) = x + v(t,s,q)\,. \tag{7}$$

2. We substitute the ansatz on the HJB in order to find an easier HJB equation for v. We use this new HJB equation to find the optimal controls (δ_*^+, δ_*^-) that maximize the jump. With the ansatz (7) the optimal controls are

$$\delta_*^+ = \frac{1}{k} - s + v(t,s,q) - v(t,s,q-1)\,, \qquad \delta_*^- = \frac{1}{k} + s - v(t,s,q+1) + v(t,s,q)\,.$$

3. We substitute the optimal controls on the HJB equation: the resulting equation is called the *verification equation*. In our case it is

$$
\begin{aligned}
(\partial_t + \mathcal{L})\, v + \frac{A}{k}\left(e^{-k\delta_*^+} + e^{-k\delta_*^-} \right) &= 0\,, \\
v(T,s,q) &= \varphi(s,q)\,,
\end{aligned}
$$

which is highly nonlinear because $\delta_*^\pm = \delta_*^\pm(v)$.

4. We solve the verification equation via the Feynmann-Kac representation formula in order to find an explicit expression of the optimal controls. Indeed, for an equation of the form

$$
\begin{aligned}
(\partial_t + \mathcal{L})\, w + f(\delta_*^+, \delta_*^-, t, s, q, x) &= 0\,, \\
w|_{t=T} &= g(\delta_*^+, \delta_*^-, s, q, x)\,,
\end{aligned}
$$

where (δ_*^+, δ_*^-) do not depend on w, the (unique) solution is (see e.g. Pham [6])

$$w(t,s,q,x) = \mathbb{E}_{t,s,q,x}\left[\int_t^T f(\delta_*^+, \delta_*^-, \xi, S(\xi), q, x) d\xi + g(\delta_*^+, \delta_*^-, S(T), q, x) \right]\,,$$

where $\mathbb{E}_{t,s,q,x}$ is the conditional expectation given $S(t) = s$, $Q(t) = q$ and $X(t) = x$

5. Alternatively, we could express the optimal quotes in terms of the market-maker's bid-ask spread

$$\psi_* := \delta_*^+ + \delta_*^-$$

and the mid-point of the spread (called the *indifference price*)

$$r_* := \frac{1}{2}\left(p_*^+ + p_*^-\right) = s + \frac{1}{2}\left(\delta_*^+ - \delta_*^-\right) .$$

Notice that if $\delta^+ = \delta^-$ then $r_*(t) = s = S(t)$. Therefore, $r_* - s$ measures the level of asymmetry of the quotes with respect to the mid-price s.

3 Linear utility function

Let us suppose that the utility function is linear, i.e.

$$\phi(s, q, x) = x + qs .$$

Then the corresponding value function is

$$u(t, s, q, x) = \max_{(\delta^+,\delta^-)} \mathbb{E}_{t,s,q,x}[X(T) + Q(T)S(T)] , \tag{8}$$

where $X(t)$ is the cash process, $Q(t)$ the inventory process, $S(t)$ the price process and $\mathbb{E}_{t,s,q,x}$ is the conditional expectation given $S(t) = s$, $Q(t) = q$ and $X(t) = x$. Alternatively, this corresponds to choosing the final condition as $\phi(s, q, x) = x + qs$.

$X(T) + Q(T)S(T)$ is the final value of the market-maker's portfolio, and corresponds to the final PnL of the market-maker. Indeed, $X(T)$ the cash she holds whilst $Q(T)S(T)$ the cash value of her inventory: she holds $Q(T)$ assets and clears them at (unitary) price $S(T)$ on the close auction.

Ansatz

From the final condition (i.e. the utility function) $\phi(s, q, x) = x + qs$ we will search a solution of the form

$$u(t, s, q, x) = x + \theta_0(t, s) + q\theta_1(t, s) . \tag{9}$$

Plugging (9) into (6) yields

$$\begin{aligned}
(\partial_t + \mathcal{L})(\theta_0 + q\theta_1) + \max_{\delta^+} Ae^{-k\delta^+}\left[s + \delta^+ - \theta_1\right] + \max_{\delta^-} Ae^{-k\delta^-}\left[-s + \delta^- + \theta_1\right] &= 0 , \\
\theta_0(T, s) &= 0 , \\
\theta_1(T, s) &= s .
\end{aligned}$$

Computing the optimal controls

Define

$$f^+(\delta^+) := Ae^{-k\delta^+}\left[s + \delta^+ - \theta_1\right] .$$

Using Calculus we obtain that the maximum is attained at

$$\delta_*^+ = \frac{1}{k} - s + \theta_1 \,.$$

Analogously, if

$$f^-(\delta^-) := Ae^{-k\delta^-}\left[-s + \delta^- + \theta_1\right]$$

then

$$\delta_*^- = \frac{1}{k} + s - \theta_1 \,.$$

In consequence, the optimal quotes (δ_*^+, δ_*^-), spread ψ_* and indifference price r_* are

$$\delta_*^\pm = \frac{1}{k} \pm (\theta_1 - s)\,, \qquad \psi_* = \delta_*^+ + \delta_*^- = \frac{2}{k}\,, \qquad r_* = \theta_1 \,.$$

Solving the verification equation

Since

$$f^+(\delta_*^+) = \frac{A}{ek}e^{-k(\theta_1 - s)}\,, \qquad f^+(\delta_*^+) = \frac{A}{ek}e^{k(\theta_1 - s)}$$

it follows that the ansatz $\theta_0 + q\theta_1$ solves the verification equation

$$\begin{aligned} (\partial_t + \mathcal{L})\,(\theta_0 + q\theta_1) + \frac{2A}{ek}\cosh\left[k(\theta_1 - s)\right] &= 0 \\ \theta_0(T,s) &= 0\,. \\ \theta_1(T,s) &= s\,. \end{aligned} \tag{10}$$

We separate (10) in terms of the powers of q, one equation for $q^0 = 1$ and another for $q^1 = q$. With this procedure we obtain two coupled equations,

$$\begin{aligned} (\partial_t + \mathcal{L})\,\theta_0 + \frac{2A}{ek}\cosh\left[k(\theta_1 - s)\right] &= 0 \\ \theta_0(T,s) &= 0\,, \end{aligned} \tag{11}$$

and

$$\begin{aligned} (\partial_t + \mathcal{L})\,\theta_1 &= 0 \\ \theta_1(T,s) &= s\,. \end{aligned} \tag{12}$$

Applying the Feynman-Kac formula to (12) and recursively to (11) yields, respectively (see e.g. Pham [6]),

$$\theta_1(t,s) = \mathbb{E}_{t,s}[S(T)]\,, \qquad \theta_0(t,s) = \frac{2A}{ek}\mathbb{E}_{t,s}\left\{\int_t^T \cosh\Big[k\Big(\theta_1(\xi, S(\xi)) - S(\xi)\Big)\Big]d\xi\right\}\,.$$

In consequence,

$$\delta_*^\pm = \frac{1}{k} \pm \left(\mathbb{E}_{t,s}[S(T)] - s\right)\,, \quad \psi_* = \frac{2}{k}\,, \quad r_* = \mathbb{E}_{t,s}[S(T)]$$

and

$$u(t,s,q,x) = x + \frac{2A}{ek}\mathbb{E}_{t,s}\left\{\int_t^T \cosh\left[k\Big(\theta_1(\xi,S(\xi)) - S(\xi)\Big)\right]d\xi\right\} + q\mathbb{E}_{t,s}[S(T)]\,.$$

In particular, since $\cosh(\alpha) \geq 1$ and $\cosh(\alpha) = 1 \iff \alpha = 0$ we obtain that

$$u(t,s,q,x) \geq \underline{u}(t,s,q,x) := x + \frac{2A}{ek}(T-t) + q\mathbb{E}_{t,s}[S(T)]$$

and $u(t,s,q,x) = \underline{u}(t,s,q,x) \iff \theta_1(\xi,S(\xi)) = S(\xi)$ for all $\xi \in [t,T]$. Since $t \in [0,T]$ is arbitrary then taking $\xi = t$ we have that $u(t,s,q,x) = \underline{u}(t,s,q,x) \iff s = \mathbb{E}_{t,s}[S(T)]$, i.e. if and only if $S(t)$ is a martingale.

Results

Let us summarise all our findings.

Theorem 1 *Consider the Hamilton-Jacobi-Bellman problem*

$$\begin{aligned}(\partial_t + \mathcal{L})\,u + \max_{\delta^+} Ae^{-k\delta^+}\left[u(t,s,q-1,x+(s+\delta^+)) - u(t,s,q,x)\right]& \\ + \max_{\delta^-} Ae^{-k\delta^-}\left[u(t,s,q+1,x-(s-\delta^-)) - u(t,s,q,x)\right] &= 0\,,\\ u(T,s,q,x) &= x + qs\,,\end{aligned}$$

which corresponds to a linear utility function $\phi(s,q,x) = x + qs$, value function

$$u(t,s,q,x) = \max_{(\delta^+,\delta^-)} \mathbb{E}_{t,s,q,x}[X(T) + Q(T)S(T)]$$

and stochastic controls (δ^+,δ^-). Then:

1. *The optimal controls* (δ_*^+, δ_*^-)*, spread* ψ_* *and indifference price* r_* *(i.e. the centre of the spread) of the market-maker are*

$$\delta_*^{\pm} = \frac{1}{k} \pm (\mathbb{E}_{t,s}[S(T)] - s) \qquad \psi_* = \delta_*^+ + \delta_*^- = \frac{2}{k}\,, \qquad r_* = \mathbb{E}_{t,s}[S(T)]\,.$$

2. *The (unique) solution of the HJB problem is*

$$u(t,s,q,x) = x + \frac{2A}{ek}\mathbb{E}_{t,s}\left\{\int_t^T \cosh\Big[k\Big(\theta_1(\xi, S(\xi)) - S(\xi)\Big)\Big]d\xi\right\} + q\mathbb{E}_{t,s}[S(T)]$$

where $\theta_1(t,s) = \mathbb{E}_{t,s}[S(T)]$.

3. *The solution* $u(t,s,q,x)$ *is bounded from below by*

$$\underline{u}(t,s,q,x) := x + \frac{2A}{ek}(T-t) + q\mathbb{E}_{t,s}[S(T)]$$

and

$$u(t,s,q,x) = \underline{u}(t,s,q,x) \iff s = \mathbb{E}_{t,s}[S(T)]\,,$$

i.e. if and only if $S(t)$ *is a martingale and* $r_*(t) = S(t)$.

Remarks

- The worst price dynamic for the PnL-based utility function (8) is a martingale, in the sense that with any other price dynamic the PnL is greater. But observe that the optimal spread ψ_* is centred around $r_* = \mathbb{E}_{t,s}[S(T)]$ and not around $s = S(t)$. In consequence, if the market-maker considers that the current mid-price s has deviated from its fundamental value r_* then she can make directional bets via her bid-ask quotes, which yields a higher PnL than the martingale assumption if the bet is correct.

- Applying perturbation methods on the variable q of the form

$$u(t,s,q,x) = x + \theta_0(t,s) + q\theta_1(t,s) + q^2\theta_2(t,s) + \cdots\,,$$

is a very rough approximation, to say the least. Indeed, as Lehalle *et al* [4] pointed out, q is an integer, i.e. discrete and not small, and as such a perturbation method on q cannot be performed. However, once the ansatz is shown to solve the verification equation, then by uniqueness it coincides with the solution of the original problem. Therefore, the separation of the equation into two terms, one with q^0 and another with q^1, is justified *a posteriori* via the maximum principle (i.e. existence and uniqueness) for the Hamilton-Jacobi-Bellman equation, and as such it does not rely at all on perturbation methods, as Avellaneda and Stoikov [1] suggested.

- In Theorem 1 we have implicitly assumed that the value function $u(t,s,q,x)$ is finite when we

applied the Fenmann-Kac formula. However, this is valid if and only if

$$\mathbb{E}_{t,s}\left\{\int_t^T \cosh\Big[k\Big(\theta_1(\xi,S(\xi))-S(\xi)\Big)\Big]d\xi\right\}<\infty\,,\qquad \theta_1(t,s)=\mathbb{E}_{t,s}[S(T)]\,. \tag{13}$$

If $S(t)$ is a martingale then (13) holds trivially. For a non-martingale mid-price process $S(t)$, two sufficient conditions for (13) to hold are (i) the difference $\mathbb{E}_{t,s}[S(T)]-s$ is affine on s and (ii) the moment-generating function $M_Z(\lambda)=\mathbb{E}\left[\exp\{\lambda S(t)\}\right]$ is finite for all $\lambda\in\mathbb{R}$. This is the case for any Gaussian Markov process, e.g. an arithmetic Brownian motion with drift and the Ornstein-Uhlenbeck process. However, (13) does not hold for the geometric Brownian motion with drift.

4 Linear utility function with inventory penalty

With the linear utility function there is no penalty if at the end of the trading day the market-maker carries a huge inventory. In order to force a liquidation of the inventory before the end of the day, we propose the following utility function,

$$\phi(s,q,x)=x+qs-\eta q^2\,,\qquad \eta\geq 0\,,$$

which is the PnL with a quadratic penalty on the inventory. The associated value function is

$$u(t,s,q,x)=\max_{(\delta^+,\delta^-)}\mathbb{E}_{t,s,q,x}[X(T)+Q(T)S(T)-\eta Q^2(T)]\,, \tag{14}$$

Ansatz

Given the form of the utility function, we will search a solution of the form

$$u(t,s,q,x)=x+\theta_0(t,s)+q\theta_1(t,s)-\eta q^2\theta_2(t,s)\,. \tag{15}$$

Plugging (15) into (6) yields

$$\begin{aligned}(\partial_t+\mathcal{L})\left(\theta_0+q\theta_1-\eta q^2\theta_2\right)+\max_{\delta^+}Ae^{-k\delta^+}\left[s+\delta^+-\theta_1-\eta(1-2q)\theta_2\right]&\\ +\max_{\delta^-}Ae^{-k\delta^-}\left[-s+\delta^-+\theta_1-\eta(1+2q)\theta_2\right]&=0\,,\\ \theta_0(T,s)&=0\,,\\ \theta_1(T,s)&=s\,.\\ \theta_2(T,s)&=1\,.\end{aligned}$$

Computing the optimal controls

As in the previous section, if

$$f^+(\delta^+):=Ae^{-k\delta^+}\left[s+\delta^+-\theta_1-\eta(1-2q)\theta_2\right]$$

then

$$\delta_*^+=\frac{1}{k}-s+\theta_1+\eta(1-2q)\theta_2\,.$$

On the other hand, if

$$f^-(\delta^-) := Ae^{-k\delta^-}\left[-s+\delta^-+\theta_1-\eta(1+2q)\theta_2\right]$$

then

$$\delta_*^- = \frac{1}{k}+s-\theta_1+\eta(1+2q)\theta_2\,.$$

In consequence, the optimal quotes (δ_*^+,δ_*^-), spread ψ_* and indifference price r_* are

$$\delta_*^\pm = \frac{1}{k}+\eta\theta_2 \pm (\theta_1-s-2q\eta\theta_2)\ , \qquad \psi_* = \delta_*^+ + \delta_*^- = \frac{2}{k}+2\eta\theta_2\,, \qquad r_* = \theta_1 - 2\eta q\theta_2\,.$$

Solving the equation with linear jumps

We fix $q\in\mathbb{Z}$ and define the jump function as $J_q:\mathbb{R}^2\to\mathbb{R}$ as

$$J_q(\delta_*^+,\delta_*^-) := \frac{A}{ek}\left(e^{1-k\delta_*^+}+e^{1-k\delta_*^-}\right)\ ,$$

whose first-order Taylor expansion of aproximation of J_q is

$$\begin{aligned} J_q(\delta_*^+,\delta_*^-) &= \frac{A}{ek}\left(4-k(\delta_*^++\delta_*^-)\right)+O\left(|\delta_*^+|^2+|\delta_*^-|^2\right) \\ &= \frac{A}{ek}\left(2-k\eta\theta_2\right)+O\left(|\delta_*^+|^2+|\delta_*^-|^2\right)\ . \end{aligned}$$

Since at first order the jumps are independent of q, it follows that

$$\theta(t,s,q) = \theta_0(t)+q\theta_1(t,s)-\eta q^2\theta_2(t)$$

solves

$$\begin{aligned} (\partial_t+\mathcal{L})\left(\theta_0+q\theta_1+q^2\theta_2\right)+\frac{A}{ek}\left(2-k\eta\theta_2\right) &= 0 \\ \theta(T,s,q) &= qs-\eta q^2\,. \end{aligned}$$

We separate (16) in terms of the powers of q, one equation for $q^0=1$ and another for $q^1=q$. With this procedure we obtain three coupled equations,

$$\begin{aligned} \partial_t\theta_0+\frac{A}{ek}\left(2-k\eta\theta_2\right) &= 0 \\ \theta_0(T) &= 0\,, \end{aligned} \tag{16}$$

$$\begin{aligned} (\partial_t+\mathcal{L})\,\theta_1 &= 0 \\ \theta_1(T,s) &= s\,, \end{aligned} \tag{17}$$

and

$$\begin{aligned} \partial_t\theta_2 &= 0 \\ \theta_2(T) &= 1\,. \end{aligned} \tag{18}$$

In consequence,

$$\theta_2 = 1\,, \qquad \theta_1(t,s) = \mathbb{E}_{t,s}[S(T)]\,, \qquad \theta_0(t) = \frac{A}{ek}\left(2-k\eta\right)(T-t)\,.$$

Finding a sub-solution

Since

$$\frac{A}{ek}\left(e^{1-k\delta_*^+}+e^{1-k\delta_*^-}\right)\geq\frac{A}{ek}\left(4-k(\delta_*^++\delta_*^-)\right)$$

then the solution of the equation with linear jump (i.e. first-order Taylor) is a sub-solution of the original problem with exponential jump. In other words, if we define

$$\underline{u}(t,s,q,x):=x+\theta_0(t)+q\theta_1(s,t)-\eta q^2\theta_2(t)\,,$$

where $\theta_0(t)$, $\theta_1(s,t)$ and $\theta_2(t)$ are defined as above then

$$\underline{u}(t,s,q,x)\leq u(t,s,q,x)\,,$$

i.e. it is a sub-solution of the HJB equation.

Results

Let us summarise all our findings.

Theorem 2 *Consider the Hamilton-Jacobi-Bellman problem*

$$\begin{aligned}\partial_t u+\mathcal{L}u+\max_{\delta^+}Ae^{-k\delta^+}&\left[u(t,s,q-1,x+(s+\delta^+))-u(t,s,q,x)\right]\\ +\max_{\delta^-}Ae^{-k\delta^-}&\left[u(t,s,q+1,x-(s-\delta^-))-u(t,s,q,x)\right]&=&\;0\,,\\ u(T,s,q,x)&&=&\;x+qs-\eta q^2\,,\end{aligned}$$

which corresponds to a linear utility function with quadratic inventory penalty $\phi(s,q,x)=x+qs-\eta q^2$, value function

$$u(t,s,q,x)=\max_{(\delta^+,\delta^-)}\mathbb{E}_{t,s,q,x}\left[X(T)+Q(T)S(T)-\eta Q^2(T)\right]$$

and stochastic controls (δ^+,δ^-). Then:

1. *If $u(t,s,q,x)$ is the (unique) solution of the HJB equation then*

$$\underline{u}(t,s,q,x):=x+\frac{A}{ek}\left(2-k\eta\right)(T-t)+q\mathbb{E}_{t,s}[S(T)]-\eta q^2$$

   is a sub-solution of the HJB equation and $\underline{u}(t,s,q,x)\leq u(t,s,q,x)$.

2. *With the linear aproximation of the jumps, the optimal controls (δ^+,δ^-), spread ψ_* and indifference price r_* are*

$$\delta_*^\pm=\frac{1}{k}+\eta\pm\left(\mathbb{E}_{t,s}[S(T)]-s-2q\eta\right)\,,\qquad\psi_*=\delta_*^++\delta_*^-=\frac{2}{k}+2\eta\,,\qquad r_*=\mathbb{E}_{t,s}[S(T)]-2\eta q\,.$$

General inventory penalties

Suppose that the utility function is now

$$\phi(s,q,x) = x + qs - \eta q^2 \pi(s)\,, \qquad \eta \geq 0\,,$$

where $s \mapsto \pi(s)$ is continuous and for $s \geq 0$ it is non-decreasing and non-negative. For example, if $\pi \equiv 1$ we recover the previous case whilst if $\pi(s) = s^2$ we recover the classical mean-variance PnL criterion. The associated value function is

$$u(t,s,q,x) = \max_{(\delta^+,\delta^-)} \mathbb{E}_{t,s,q,x}[X(T) + Q(T)S(T) - \eta Q^2(T)\pi(S(T))]\,. \tag{19}$$

With the ansatz

$$u(t,s,q,x) = x + \theta_0(t,s) + q\theta_1(t,s) - \eta q^2 \theta_2(t,s) \tag{20}$$

it can be shown that the functions $\theta_0 - \theta_2$ solve the equations

$$\begin{aligned} (\partial_t + \mathcal{L})\,\theta_0 + \frac{A}{ek}\,(2 - k\eta\theta_2) &= 0 \\ \theta_0(T) &= 0\,, \end{aligned} \tag{21}$$

$$\begin{aligned} (\partial_t + \mathcal{L})\,\theta_1 &= 0 \\ \theta_1(T,s) &= s\,, \end{aligned} \tag{22}$$

and

$$\begin{aligned} (\partial_t + \mathcal{L})\,\theta_2 &= 0 \\ \theta_2(T) &= \pi(s)\,, \end{aligned} \tag{23}$$

whose explicit solutions are

$$\theta_2(t,s) = \mathbb{E}_{t,s}[\pi(S(T))]\,, \quad \theta_1(t,s) = \mathbb{E}_{t,s}[S(T)]\,, \quad \theta_0(t,s) = \frac{2A}{ek}(T-t) - \frac{\eta A}{e}\mathbb{E}_{t,s}\left[\int_t^T \theta_2(\xi, S(\xi))d\xi\right]\,.$$

In the light of these results, we have the following extension of Theorem 2 to a general penalty function π.

Theorem 3 *Consider the Hamilton-Jacobi-Bellman problem*

$$\begin{aligned} \partial_t u + \mathcal{L}u + \max_{\delta^+} Ae^{-k\delta^+}\left[u(t,s,q-1,x+(s+\delta^+)) - u(t,s,q,x)\right] & \\ + \max_{\delta^-} Ae^{-k\delta^-}\left[u(t,s,q+1,x-(s-\delta^-)) - u(t,s,q,x)\right] &= 0\,, \\ u(T,s,q,x) &= x + qs - \eta q^2\pi(s)\,, \end{aligned}$$

which corresponds to a linear utility function with quadratic inventory penalty $\phi(s,q,x) = x + qs - \eta q^2 \pi(s)$*, value function*

$$u(t,s,q,x) = \max_{(\delta^+,\delta^-)} \mathbb{E}_{t,s,q,x}\left[X(T) + Q(T)S(T) - \eta Q^2(T)\pi(S(T))\right]$$

and stochastic controls (δ^+, δ^-)*. Then:*

1. *If* $u(t,s,q,x)$ *is the (unique) solution of the HJB equation then*

$$\underline{u}(t,s,q,x) := x + \frac{2A}{ek}(T-t) - \frac{\eta A}{e}\mathbb{E}_{t,s}\left[\int_t^T \theta_2(\xi, S(\xi)d\xi\right] + q\mathbb{E}_{t,s}[S(T)] - \eta q^2\theta_2(t,s)\,,$$

where $\theta_2(t,s) = \mathbb{E}_{t,s}[\pi(S(T))]$*, is a sub-solution of the HJB equation and* $\underline{u}(t,s,q,x) \leq u(t,s,q,x)$*.*

2. *With the linear aproximation of the jumps, the optimal controls* (δ^+, δ^-)*, spread* ψ_* *and indifference price* r_* *are*

$$\delta_*^{\pm} = \frac{1}{k} + \eta \pm \left(\mathbb{E}_{t,s}[S(T)] - s - 2q\eta\mathbb{E}_{t,s}[\pi(S(T))]\right)\,, \qquad \psi_* = \delta_*^+ + \delta_*^- = \frac{2}{k} + 2\eta\,,$$

$$r_* = \mathbb{E}_{t,s}[S(T)] - 2\eta q\mathbb{E}_{t,s}[\pi(S(T))]\,.$$

Remarks

- The optimal controls depend on the ansatz we make on the utility function $u(t,s,q,x)$, i.e. on the functions $\theta_1(t,s)$ and $\theta_2(t)$. However, without the linear aproximation of the jumps our ansatz cannot give the solution: in fact, there is no solution with the chosen ansatz. That said, given that the sub-solution $\underline{u}$ is explicit, and by definition it provides a lower bound on the real solution u, we can consider that the optimal quotes we have found maximise the sub-solution, and as such they may be seen as *conservative* estimates of the real optimal quotes.

- If $\eta = 0$ we recover the optimal controls and the sub-solution of the linear case without inventory penalty, i.e. Theorem 1. Therefore, our linear approximation of the jumps is consistent, in the sense that it provides a perturbation of the optimal quotes in terms of the "inventory-risk" or "risk-aversion" parameter η.

- When $\eta > 0$ the spread ψ_* widens and the indifference price r_* shifts downwards (resp. upwards) if the inventory is positive (resp. negative), which is in line with the intuition on the inventory risk. Indeed, if the net position of the market-maker is long (resp. short) then she will improve the current ask (resp. bid) quote to lure buyers (resp. sellers), and simultaneously she will try to hide her bid (resp. ask) quote deep into the limit-order Book to deter sellers (resp. buyers). By doing so, she favours the probability of being executed in the direction that makes her to go back to zero.

- Notice that since the linear approximation of the jumps does not depend on q, the solution $\underline{u}$ of the approximate verification equation does not rely on perturbation methods and asymptotic expansions on q. Of course, the solution u to the verification equation with exponential jumps will indeed depend on q, and as such an approach similar to Lehalle *et al* [4] is needed in order to deal with the discrete variable q.

- For Theorems 2 and 3 we are assuming that the corresponding value function $u(t,s,q,x)$ is finite in order to apply the Feynmann-Kac formula. In the current framework, given a mid-price dynamic

we choose the penalty function π such that $u(t,s,q,x)$ is finite. In that spirit, if $S(t)$ is Gaussiann (e.g. arithmetic Brownian motion or Ornstein-Ulenbeck) or a martingale (even with jumps) then $\pi \equiv 1$ suffices, whilst if $S(t)$ is a geometric Brownian motion then $\pi(s) = s^2$ is a viable candidate.

5 Exponential utility function

This case has been entirely solved by Avellaneda and Stoikov [1] and Lehalle *et al* [4] when the mid-price is a Brownian motion. In this section we show that their approach can be easily extended to several other mid-price dynamics, e.g. Ornstein-Uhlenbeck.

Let us suppose that the utility function is exponential $\phi(s,q,x) = -\exp\{-\gamma(x+qs)\}$, whose coresponding value function is

$$u(t,s,q,x) = \max_{(\delta^+,\delta^-)} \mathbb{E}_{t,s,q,x}\left[-\exp\left\{-\gamma\Big(X(T)+Q(T)S(T)\Big)\right\}\right], \tag{24}$$

Ansatz

From the form of the utility function we will search a solution of the form

$$u(t,s,q,x) = -\exp\{-\gamma(x+\theta(t,s,q)\}, \qquad \theta(t,s,q) = \theta_0(t) + q\theta_1(t,s) + q^2\theta_2(t). \tag{25}$$

Plugging (25) into (6) yields the Hamilton-Jacobi-Bellman for $\theta(t,s,q)$, i.e.

$$\begin{aligned}(\partial_t + \mathcal{L})\,\theta - \frac{1}{2}\sigma^2\gamma\,(\partial_s\theta)^2 + \frac{A}{\gamma}\max_{\delta^+} e^{-k\delta^+}\left[1-\exp\{-\gamma(s+\delta^+-\theta_1+(1-2q)\theta_2)\}\right]& \\ +\frac{A}{\gamma}\max_{\delta^-} Ae^{-k\delta^-}\left[1-\exp\{-\gamma(-s+\delta^-+\theta_1+(1+2q)\theta_2)\}\right]\Big] &= 0, \\ \theta(T,s) &= x+qs,\end{aligned} \tag{26}$$

Computing the optimal controls

For the function

$$f^+(\delta^+) := \frac{A}{\gamma}e^{-k\delta^+}\left[1-\exp\{-\gamma(s+\delta^+-\theta_1+(1-2q)\theta_2)\}\right]$$

its maximum is attained at

$$\delta^+_* = \frac{1}{\gamma}\log\left(1+\frac{\gamma}{k}\right) - s + \theta_1 - (1-2q)\theta_2.$$

Analogously, if

$$f^-(\delta^-) := \frac{A}{\gamma}\max_{\delta^-} Ae^{-k\delta^-}\left[1-\exp\{-\gamma(-s+\delta^-+\theta_1+(1+2q)\theta_2)\}\right]$$

then

$$\delta^-_* = \frac{1}{\gamma}\log\left(1+\frac{\gamma}{k}\right) + s - \theta_1 - (1+2q)\theta_2.$$

In consequence, the optimal quotes (δ_*^+, δ_*^-), spread ψ_* and indifference price r_* are

$$\delta_*^{\pm} = \frac{1}{\gamma}\log\left(1+\frac{\gamma}{k}\right) - \theta_2 \pm (\theta_1 - s + 2q\theta_2)\,, \quad \psi_* = \frac{2}{\gamma}\log\left(1+\frac{\gamma}{k}\right) - 2\theta_2\,, \quad r_* = \theta_1 + 2q\theta_2\,. \tag{27}$$

Solving the equation with linear jumps

For $q \in \mathbb{Z}$ fixed we define the jump function as $J_q : \mathbb{R}^2 \to \mathbb{R}$ as

$$J_q(\delta_*^+, \delta_*^-) := \frac{A}{k+\gamma}\left(e^{-k\delta_*^+} + e^{-k\delta_*^-}\right)\,. \tag{28}$$

The first-order Taylor expansion of aproximation of J_q is

$$\begin{aligned} J_q(\delta_*^+, \delta_*^-) &= \frac{A}{k+\gamma}\left(2 - k(\delta_*^+ + \delta_*^-)\right) + O\left(|\delta_*^+|^2 + |\delta_*^-|^2\right) \\ &= \frac{2A}{k+\gamma}\left(1 - \frac{k}{\gamma}\log\left(1+\frac{\gamma}{k}\right) + k\theta_2\right) + O\left(|\delta_*^+|^2 + |\delta_*^-|^2\right)\,. \end{aligned}$$

Therefore, at first order we have that $J_q = J_0$ for all $q \in \mathbb{Z}$, i.e. the jumps are independent of q. In consequence,

$$\theta(t,s,q) = \theta_0(t) + q\theta_1(t,s) + q^2\theta_2(t)$$

solves

$$\begin{aligned} (\partial_t + \mathcal{L})(\theta_0 + q\theta_1 + q^2\theta_2) - \frac{1}{2}q^2\sigma^2\gamma\,(\partial_s\theta_1)^2 + \frac{2A}{k+\gamma}\left(1 - \frac{k}{\gamma}\log\left(1+\frac{\gamma}{k}\right) + k\theta_2\right) &= 0 \\ \theta(T,s,q) &= qs\,. \end{aligned}$$

We separate (29) in terms of the powers of q, one equation for $q^0 = 1$ and another for $q^1 = q$. With this procedure we obtain three coupled equations,

$$\begin{aligned} \partial_t\theta_0 + \frac{2A}{k+\gamma}\left(1 - \frac{k}{\gamma}\log\left(1+\frac{\gamma}{k}\right) + k\theta_2\right) &= 0 \\ \theta_0(T) &= 0\,, \end{aligned} \tag{29}$$

$$\begin{aligned} (\partial_t + \mathcal{L})\,\theta_1 &= 0 \\ \theta_1(T,s) &= s\,, \end{aligned} \tag{30}$$

and

$$\begin{aligned} \partial_t\theta_2 - \frac{1}{2}\sigma^2\gamma\,(\partial_s\theta_1)^2 &= 0 \\ \theta_2(T) &= 0\,. \end{aligned} \tag{31}$$

Applying the Feynman-Kac formula to (30) we find

$$\theta_1(t,s) = \mathbb{E}_{t,s}[S(T)]\,.$$

Integrating (31) we obtain

$$-\theta_2(t) = \frac{1}{2}\gamma \int_t^T \sigma^2(\xi, S(\xi))\Big(\partial_s \theta_1(\xi, S(\xi))\Big)^2 d\xi\,.$$

However, the ansatz we have made implies that θ_2 is independent of s. Therefore, in order to solve (31) we need to assume the following conditions on the price process $S(t)$:

$$\sigma = \sigma(t)\,, \qquad \mathbb{E}_{t,s}[S(T)] = \alpha(t,T) + s\beta(t,T)\,.$$

In consequence,

$$\theta_2(t) = -\frac{1}{2}\gamma \int_t^T \sigma^2(\xi)\beta^2(\xi,T)d\xi\,.$$

Finally, integrating (30) yields

$$\theta_0(t) = \frac{2A}{k+\gamma}\left(1 - \frac{k}{\gamma}\log\left(1+\frac{\gamma}{k}\right)\right)(T-t) - \frac{k\gamma A}{k+\gamma}\int_t^T \left\{\int_\zeta^T \sigma^2(\xi)\beta^2(\xi,T)d\xi\right\} d\zeta\,.$$

Finding a sub-solution

Since

$$\frac{A}{k+\gamma}\left(e^{-k\delta_*^+} + e^{-k\delta_*^-}\right) \geq \frac{A}{k+\gamma}\left(2 - k(\delta_*^+ + \delta_*^-)\right)$$

then the solution of the equation with linear jump (i.e. first-order Taylor expansion)is a sub-solution of the original problem with exponential jump. In other words, if we define

$$\underline{u}(t,s,q,x) := -\exp\left\{-\gamma\left(x + \theta_0(t) + q\theta_1(s,t) + q^2\theta_2(t)\right)\right\}\,,$$

where $\theta_0(t)$, $\theta_1(s,t)$ and $\theta_2(t)$ are defined as above then

$$\underline{u}(t,s,q,x) \leq u(t,s,q,x)\,.$$

Adding a quadratic inventory penalty

We modify the exponential utility function (24) by adding a quadratic inventory penalty:

$$u(t,s,q,x) = \max_{(\delta^+,\delta^-)} \mathbb{E}_{t,s,q,x}\left[-\exp\left\{-\gamma\Big(X(T) + Q(T)S(T) - \eta Q^2(T)\Big)\right\}\right]\,. \tag{32}$$

Under this new penalty framework, the computations are exactly the same as before. The only thing that changes is the equation solved by $\theta_2(t)$, i.e.

$$\begin{aligned} \partial_t\theta_2 - \frac{1}{2}\sigma^2\gamma\left(\partial_s\theta_1\right)^2 &= 0 \\ \theta_2(T) &= -\eta\,, \end{aligned} \tag{33}$$

whose solution is

$$\theta_2(t) = -\eta - \frac{1}{2}\gamma\int_t^T \sigma^2(\xi)\beta^2(\xi,T)d\xi\,.$$

This has an impact on $\theta_0(t)$, which has the new form

$$\theta_0(t) = \frac{2A}{k+\gamma}\left(1 - \frac{k}{\gamma}\log\left(1+\frac{\gamma}{k}\right) - k\eta\right)(T-t) - \frac{k\gamma A}{k+\gamma}\int_t^T \left\{\int_\zeta^T \sigma^2(\xi)\beta^2(\xi,T)d\xi\right\} d\zeta\,,$$

as well as on the optimal quotes, spread and indifference price (27).

Results

Let us summarise all our findings.

Theorem 4 *Consider the Hamilton-Jacobi-Bellman problem*

$$\begin{aligned}\partial_t u + \mathcal{L}u + \max_{\delta^+} Ae^{-k\delta^+}\left[u(t,s,q-1,x+(s+\delta^+)) - u(t,s,q,x)\right] & \\ + \max_{\delta^-} Ae^{-k\delta^-}\left[u(t,s,q+1,x-(s-\delta^-)) - u(t,s,q,x)\right] &= 0\,, \\ u(T,s,q,x) &= -\exp\{-\gamma(x+qs-\eta q^2)\}\,,\end{aligned}$$

which corresponds to an exponential utility function $\phi(s,q,x) = -\exp\{-\gamma(x+qs-\eta q^2)\}$*, value function*

$$u(t,s,q,x) = \max_{(\delta^+,\delta^-)} \mathbb{E}_{t,s,q,x}\left[-\exp\left\{-\gamma\Big(X(T)+Q(T)S(T)-\eta Q^2(T)\Big)\right\}\right]\,,$$

and stochastic controls (δ^+,δ^-)*. Assume further that the mid-price process*

$$dS(t) = b(t,S(t))dt + \sigma(t)dW(t)$$

satisfies

$$\mathbb{E}_{t,s}[S(T)] = \alpha(t,T) + s\beta(t,T)\,, \qquad \forall(t,s)\,.$$

Then:

1. *With the linear approximation of the jumps, the optimal controls* (δ^+, δ^-)*, spread* ψ_* *and indifference price* r_* *are*

$$\delta_*^{\pm} = \frac{1}{\gamma}\log\left(1+\frac{\gamma}{k}\right) - \theta_2 \pm (\theta_1 - s + 2q\theta_2)\,, \quad \psi_* = \frac{2}{\gamma}\log\left(1+\frac{\gamma}{k}\right) - 2\theta_2\,, \quad r_* = \theta_1 + 2q\theta_2\,,$$

where

$$\begin{aligned}
\theta_0(t) &= \frac{2A}{k+\gamma}\left(1 - \frac{k}{\gamma}\log\left(1+\frac{\gamma}{k}\right) - k\eta\right)(T-t) - \frac{k\gamma A}{k+\gamma}\int_t^T \left\{\int_\zeta^T \sigma^2(\xi)\beta^2(\xi,T)d\zeta\right\} d\xi\,,\\
\theta_1(t,s) &= \mathbb{E}_{t,s}[S(T)]\,,\\
\theta_2(t) &= -\eta - \frac{1}{2}\gamma\int_t^T \sigma^2(\xi)\beta^2(\xi,T)d\xi\,.
\end{aligned}$$

2. *If* $u(t,s,q,x)$ *is the (unique) solution of the HJB equation then*

$$\underline{u}(t,s,q,x) := -\exp\left\{-\gamma\Big(x + \underline{\theta}(t,s,q)\Big)\right\}\,, \qquad \underline{\theta}(t,s,q) = \theta_0(t) + q\theta_1(t,s) + q^2\theta_2(t)$$

is a sub-solution of the HJB equation and $\underline{u}(t,s,q,x) \le u(t,s,q,x)$*.*

Remarks

- The linear approximation of the jumps turns out to be independent of q, although for higher orders this is no longer true. This means that we are not performing perturbation methods and asymptotic expansions on the (discrete) variable q when we solved the approximate verification equation. Of course, if we want to solve the real HJB problem then we need to take into account the discrete variable q, which leads to an infinite system of equations as in Lehalle *et al* [4].
- In the case of $\eta = 0$ the inventory penalty tends to zero as $t \to T$. This implies that the penalisation is not stong enough to force the market-maker to finish her day with a flat inventory, as it will be shown in the numerical simulations. Therefore, it was necesary to add an "inventory-risk" parameter $\eta > 0$ in order to ensure a flat inventory at the end of the day.

Examples

- If $S(t)$ is an **arithmetic Brownian motion with drift**, i.e.

$$dS(t) = bdt + \sigma dW(t)$$

then

$$\mathbb{E}_{t,s}[S(T)] = s + b(T-t)\,,$$

which implies that

$$\theta_1 = s + b(T-t)\,, \qquad \theta_2 = -\eta - \frac{1}{2}\gamma\sigma^2(T-t)\,.$$

In consequence, the optimal controls are

$$\begin{aligned}
\delta_*^{\pm} &= \frac{1}{\gamma}\log\left(1+\frac{\gamma}{k}\right)+\eta+\frac{1}{2}\gamma\sigma^2(T-t)\pm\Big(b(T-t)-q[2\eta+\gamma\sigma^2(T-t)]\Big)\,,\\
\psi_* &= \frac{2}{\gamma}\log\left(1+\frac{\gamma}{k}\right)+2\eta+\gamma\sigma^2(T-t)\,,\\
r_* &= s+b(T-t)-q\Big(2\eta+\gamma\sigma^2(T-t)\Big)\,.
\end{aligned}$$

In particular, if $b=\eta=0$ we recover the results of Avellaneda and Stoikov [1].

- If $S(t)$ is an **Ornstein-Uhlenbeck process**, i.e.

$$dS(t)=a(\mu-S(t))dt+\sigma dW(t)$$

then

$$\mathbb{E}_{t,s}[S(T)]=se^{-a(T-t)}+\mu\left(1-e^{-a(T-t)}\right)\,,$$

which implies that

$$\theta_1=se^{-a(T-t)}+\mu\left(1-e^{-a(T-t)}\right)\,,\qquad \theta_2=-\eta-\frac{\gamma\sigma^2}{4a}\left(1-e^{-2a(T-t)}\right)\,.$$

In consequence, the optimal controls are

$$\begin{aligned}
\delta_*^{\pm} &= \frac{1}{\gamma}\log\left(1+\frac{\gamma}{k}\right)+\eta+\frac{\gamma\sigma^2}{4a}\left(1-e^{-2a(T-t)}\right)\\
&\pm\left((\mu-s)\left(1-e^{-a(T-t)}\right)-q\left[2\eta+\frac{\gamma\sigma^2}{2a}\left(1-e^{-2a(T-t)}\right)\right]\right)\,,\\
\psi_* &= \frac{2}{\gamma}\log\left(1+\frac{\gamma}{k}\right)+2\eta+\frac{\gamma\sigma^2}{2a}\left(1-e^{-2a(T-t)}\right)\,,\\
r_* &= se^{-a(T-t)}+\mu\left(1-e^{-a(T-t)}\right)-q\left(2\eta+\frac{\gamma\sigma^2}{2a}\left(1-e^{-2a(T-t)}\right)\right)\,.
\end{aligned}$$

Solving the nonlinear equation: Lehalle's approach

Assuming that $S(t)$ is a Brownian motion, Lehalle *et al* [4] found that the solution $u(t,s,q,x)$ of the nonlinear problem can be explicitly found, but that study can be easily extended.

Suppose that the mid-price process is of the form

$$dS(t) = b(t)dt + \sigma(t)dW(t)\,,$$

where $b(t)$ and $\sigma(t)$ are uniformly bounded in $[0,T]$. Let $\eta > 0$ and make the ansatz

$$u(t,s,q,x) = -\exp\{-\gamma(x+qs-\eta q^2)\}v_q(t)^{-\gamma/k}\,; \quad v_q \in \mathbb{C}^1(0,T)\,, \quad q \in \mathbb{Z}\,. \tag{34}$$

Following Lehalle *et al* [4] it can be shown that the optimal quotes are

$$\begin{aligned}
\delta_*^+ &= \frac{1}{\gamma}\log\left(1+\frac{\gamma}{k}\right) + \frac{1}{k}\log\left(\frac{v_q(t)}{v_{q-1}(t)}\right)\,,\\
\delta_*^- &= \frac{1}{\gamma}\log\left(1+\frac{\gamma}{k}\right) - \frac{1}{k}\log\left(\frac{v_{q+1}(t)}{v_q(t)}\right)\,,
\end{aligned}$$

where $(v_q(t))_{q\in\mathbb{Z}}$ solves the (infinite) ODE system

$$\begin{aligned}
v_q'(t) &= \left(\frac{k\gamma q^2}{2}\sigma^2(t) - \gamma q b(t)\right)v_q(t) - \left(A\left(1+\frac{k}{\gamma}\right)^{-1-k/\gamma}\right)\Big(v_{q+1}(t)+v_{q-1}(t)\Big)\,, \qquad (35)\\
v_q(T) &= e^{-k\eta q^2}\,.
\end{aligned}$$

Lehalle *et al* [4] used a constructive proof to show that (35) has a unique, strictly positive solution in $C^\infty\Big([0,T));\ell^2(\mathbb{Z})\Big)$. However, this result can be easily proven in a non-constructive fashion.

Let $\mathbf{E}$ be a Banach space and consider a system on $\mathbf{E}$ of the form

$$\begin{aligned}
v_q'(t) &= F_q(t,v(t))\,,\\
v_q(T) &= v_{q,T} > 0\,,
\end{aligned} \tag{36}$$

where $F(t,v) = (F_q(t,v))_{q\in\mathbb{Z}} : [0,T]\times\mathbf{E}\to\mathbf{E}$ is Lipshitz in v uniformly in $t\in[0,T]$. Applying the Cauchy-Picard Theorem and the maximum principle for ODEs on the Banach space $C^1\Big([0,T);\mathbf{E}\Big)$ yield existence, uniqueness and positivity of the solution v of (35) (see e.g. Brézis [2]). Unfortunately, since the linear function F corresponding to (35) is proportional to q^2, we cannot apply the Cauchy-Picard Theorem directly. However, if we define

$$\mathbf{E} = \ell^\infty(\mathbb{Z})\,, \qquad \mathbf{F} := \left\{v\in\mathbf{E} : \sup_{q\in\mathbb{Z}} q^2 v_q < +\infty\right\}\,, \tag{37}$$

then for t fixed we have that $F(t,\cdot) : \mathbf{F}\to\mathbf{E}$ is linear and bounded, and the bound is uniform in t. Therefore, $F(t,v)$ is Lipschitz in v, uniformly in t. In consequence, we can now apply the Cauchy-Picard Theorem to ensure that there exists a unique positive solution of (35).

Remarks on Lehalle's approach

- If instead of the Banach spaces in (37) we use the Hilbert spaces

$$\mathbf{E} = \ell^2(\mathbb{Z}), \qquad \mathbf{F} := \left\{ v \in \mathbf{E} : \sum_{q\in\mathbb{Z}} q^2 v_q < +\infty \right\}, \tag{38}$$

  we recover the framework of Lehalle *et al* [4].

- Our proof is non-constructive, which implies that we cannot provide explicit asymptotic estimates of the solution $v = (v_q)_{q\in\mathbb{Z}}$. Lehalle *et al* [4], on the contrary, constructed the operator explicitly, and thus they were able to show the asymptotic behaviour of v based on the spectrum of the linear operator.

- From the ansatz (34) we see that the coefficients in the ODE system (35) cannot depend on s. This rules out mid-price processes whose drift and volatility depend on s, e.g. Ornstein-Uhlenbeck and geometric Brownian Motion. However, arithmetic Brownian motions with time-dependent drift and volatility can be used.

6 Numerical Simulations and sample paths

We performed several simulations of the optimal market-making strategy, i.e. the spread ψ_* and the indifference price r_*, under an Ornstein-Uhlenbeck mid-price process, i.e. a mean-reverting price dynamic of the form

$$dS(t) = a(\mu - S(t))dt + \sigma dW(t)\,.$$

We considered four strategies: linear/exponential utility and mean-reverting /martingale market-making assumption. This allows us to assess the effect of the directional bet μ on the PnL of the market-making strategy. The parameters we used are $k = 100$, $A = 1500$, $T = 1$ (which corresponds to one trading day), $n = 1000$ (which corresponds to 1000 trades per day, approx. one every 30 seconds), $S(0) = 1$, $\sigma = 0.05$ (daily volatility of 5%), $a = 1$ and $\eta = 0.0001$ (which is a very small inventory-risk aversion but enough to force the inventory to end the day flat in average). Under this parameters we have a constant linear spread of $\psi_* = 0.0202$ whilst the exponential spread is time-dependent.

We chose three different values of μ: 0.98, 1.00 and 1.02. This corresponds, respectively, to a bet that the price will go down by 2%, will oscillate around its open price or go up by 2%. We used the linear utility function with inventory penalty and two strategies, a mean-reverting strategy (Ornstein-Uhlenbeck) with the correct directional bet, and the martingale strategy (aritmetic Brownian motion) with no directional bet. The martingale strategy performs a pure market-making strategy under inventory constraints. On the other hand, the mean-reverting strategy performs the same market-market strategy than the martingale but it also places directional bets, which can be seen not only in the agressiveness of the ask and bid quotes but also in the fact that the algorithm can place market orders.

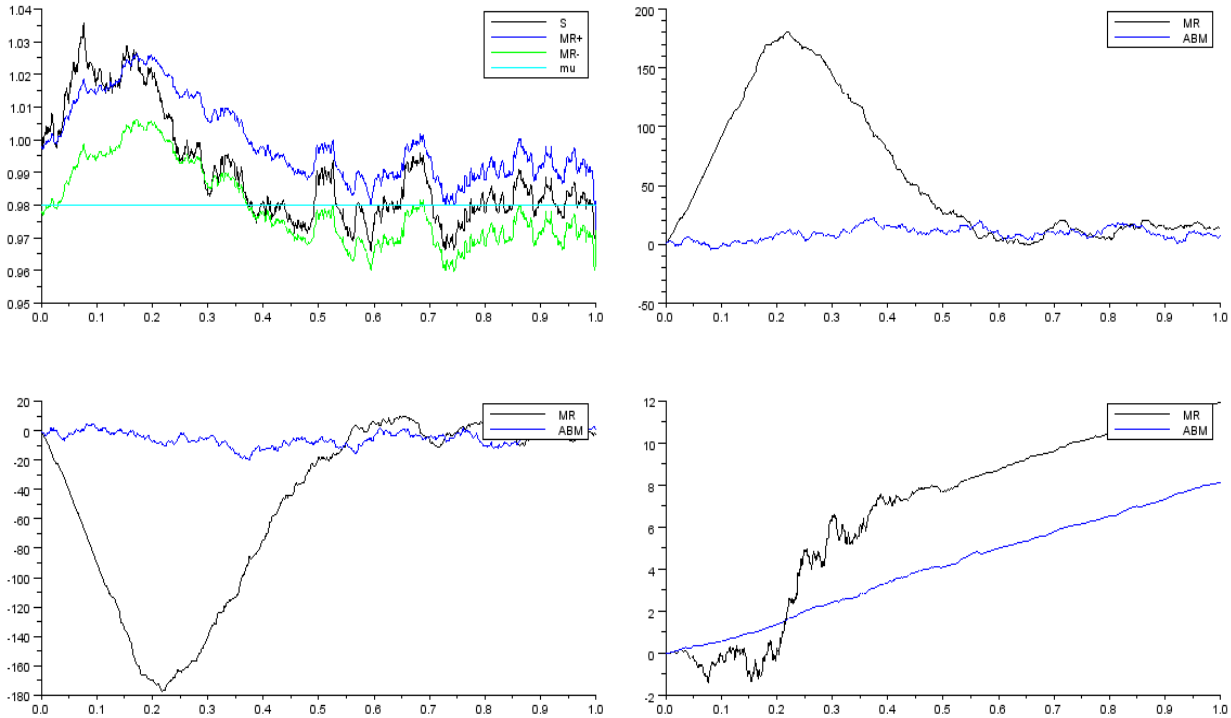

Figure 1. Simulation of the market-making strategy under a mean-reverting mid-price dynamic with (asymptotic) mean $\mu = 0.98$. **Upper-Left**: mid-price (black), optimal ask quote for the market-maker (dark blue), optimal bid quote (light green), μ (light blue). **Lower-Left**: Inventory for the mean-reverting process (black) vs the inventory for the martingale (blue). **Upper-Right**: cash. **Lower-Right**: PnL of the mean reverting process (black) compared with the benchmark, i.e. the PnL of the martingale strategy (blue).

In **Figure 1** we have plotted a realisation of the market-making strategy for $\mu = 0.98$, i.e. assuming that the price will go down by 2% at the end of the day. If the market mid-price –black line– is above (resp. below) the optimal ask quote –dark blue– (resp. the optimal bid quote –light green–) then the market-maker sells (resp. buys) at market price, which we assume to coincide with the mid-price.

- The mid-price starts at $s = 1.00$, it goes up to $s = 1.03$ at $t = 0.1$ and stays above $s = 1.01$ up to $t = 0.2$. Since the bet is that the price will converge down to $\mu = 0.98$, the market-maker sells the asset at market price. At $t = 0.2$ the market-maker has an inventory of $q = -170$ and a PnL below the martingale benchmark: as she sold her assets at the market price (mid-price), she paid the spread to mount her directional bet.

- During the time interval $(0.20, 0.50)$ the market-maker buys back her position via limit-orders. In order to favour the arrival of selling orders and deter buying orders she plays very aggressive bid quotes and very conservative ask quotes, hence the mid-price is closer to her bid quote than her ask quotes. The strategy paid well because at $t = 0.5$ the mid-price converged to $\mu = 0.98$, her inventory went back to zero and her PnL outperformed the martingale benchmark.

- On $(0.50, 1.00)$ the market-maker does not make any directional bet, she only plays the bid-ask spread because the mid-price oscillates around $\mu = 0.98$. As it can be seen, her bid and ask quotes are rather symmetric with respect to the mid-price, i.e. during all this non-directional period the strategy makes the same PnL than the martingale benchmark because both lines are almost parallel.

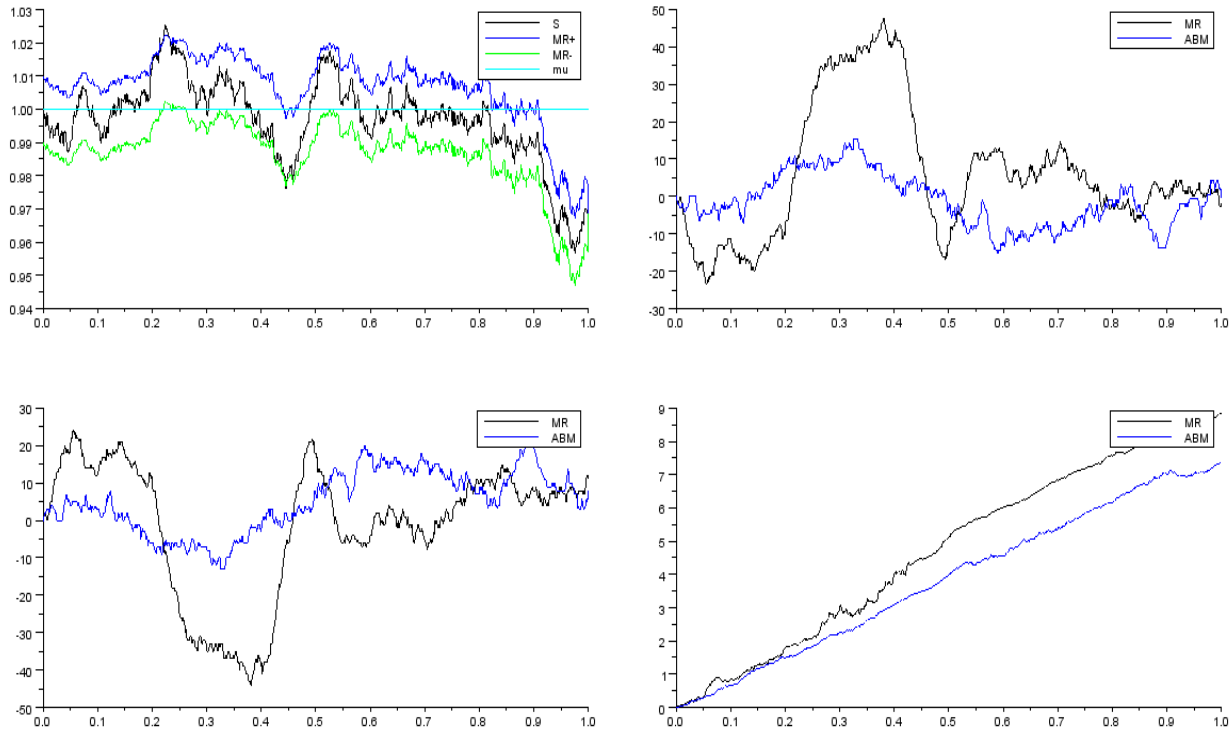

Figure 2. Simulation for $\mu = 1.00$.

In **Figure 2** the market-maker assumes that the price will oscillate around $\mu = 1.00$. Therefore, her quotes are symmetric near this threshold and are tilted when the prices wander far from it (i.e. she makes mean-reverting bets). Therefore, her inventory oscillates from positive on $(0.00, 0.2)$ to negative on $(0.25, 0.45)$, then back to positive and negative again. On $(0.85, 1.00)$ there is a huge drop in the mid-price, which in absence of inventory risk would imply a consequent positive inventory due to the mean-reverting dynamic. However, the inventory-risk-aversion forces her to avoid a directional bet at the end of the day, and as such her strategy is similar to the martingale case. It is worth to mention that the mid-price finished the day below the target of 1.00. In consequence, the market-making strategy is insensitive to peaks at the end of the day.

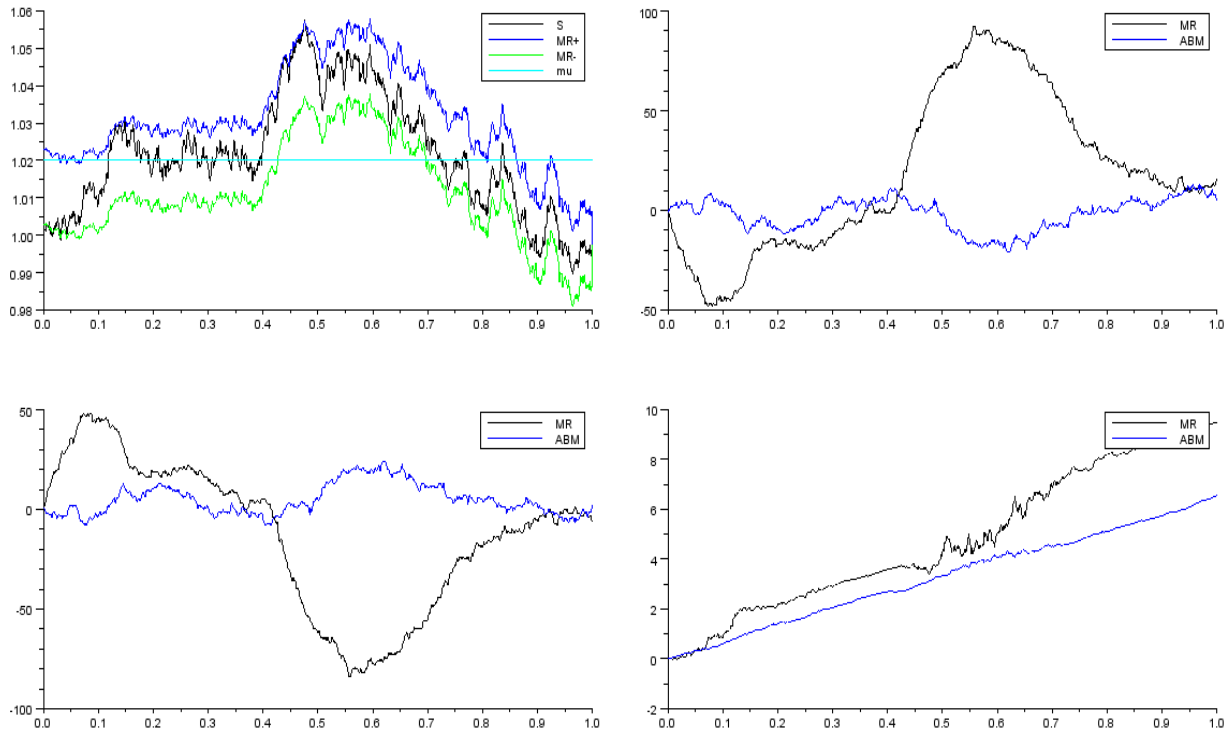

Figure 3. Simulation for $\mu = 1.02$.

In **Figure 3** the market-maker assumes that the price will converge to $\mu = 1.02$. On $(0.00, 0.10)$ she mounts an inventory of $q = 50$ because she bets the mid-price will hit $\mu = 1.02$. She eliminates her inventory during $(0.20, 0.40)$ using limit-orders with generous ask prices and not-very competitive bid prices, which translates into a greater flow of buying orders than selling orders. At $t = 0.20$ we can see that her directional bet has beaten the martingale benchmark. On $(0.40, 0.70)$ the mid-price rises from $\mu = 1.02$ and comes back, and since the market-maker is betting for a mean-reverting dynamic she builds up an inventory of $q = -90$ at $t = 0.55$. On $(0.85, 1.00)$ the mid-price falls, but instead of making a U-turn in her inventory and turn it positive, as a mean-reverting dynamic suggests, she rather eliminates slowly her negative inventory by tilting her quotes towards the buying side. By doing so the market-maker avoids directional bets and minimises her market impact at the end of the trading day.

7 Statistics of the PnL distributions

Comparing linear and exponential utility strategies

We performed 100,000 simulations with $\eta \in \{0, 0.0001, 0.001\}$, $\gamma = 1$ and $\mu \in \{0.98, 1.00, 1.02\}$. The rest of the parameters have the same values as for the previous simulations.

		linear ABM	linear MR	exp ABM	exp MR
PnL	**mean**	11.039	14.290	10.668	11.084
	std dev	1.013	13.678	0.356	0.520
	skewness	0.075	-0.550	0.026	-1.008
	kurtosis	5.721	4.692	3.043	6.176
	Jarque Bera	30947.9	16970.7	19.1	58949.6
	VaR 5%	9.430	-10.909	10.086	10.177
	VaR 1%	8.346	-25.726	9.842	9.465
Inv	**mean**	0.087	-333.297	0.004	-3.311
	std dev	33.258	418.200	7.672	15.227
	skewness	-0.005	0.622	0.003	0.007
	kurtosis	2.981	2.346	3.001	3.010
	Jarque Bera	1.8	8232.0	0.2	1.3
	Q(T) 90%	[-55,55]	[-847,463]	[-13,13]	[-28,22]

Table 1A. Statistics of the PnL distribution. $\mu = 0.98$, $\eta = 0$.

		linear ABM	linear MR	exp ABM	exp MR
PnL	**mean**	10.982	11.576	10.607	10.945
	std dev	0.412	1.541	0.347	0.444
	skewness	0.023	-2.164	0.038	-0.605
	kurtosis	3.016	11.418	3.007	4.460
	Jarque Bera	10.2	373263.9	24.7	14982.6
	VaR 5%	10.308	8.640	10.041	10.183
	VaR 1%	10.026	5.794	9.809	9.691
Inv	**mean**	-0.020	-1.739	0.008	-0.785
	std dev	5.025	8.428	4.574	5.696
	skewness	0.006	0.004	-0.001	-0.003
	kurtosis	3.024	2.995	2.965	3.000
	Jarque Bera	3.0	0.4	5.2	0.2
	Q(T) 90%	[-8,8]	[-16,12]	[-8,8]	[-10,9]

Table 1B. Statistics of the PnL distribution. $\mu = 0.98$, $\eta = 0.0001$.

		linear ABM	**linear MR**	**exp ABM**	**exp MR**
PnL	**mean**	10.435	10.494	10.000	10.234
	std dev	0.336	0.364	0.324	0.342
	skewness	0.023	-0.158	0.037	-0.034
	kurtosis	3.013	3.306	3.021	3.073
	Jarque Bera	9.5	806.5	24.3	41.2
	VaR 5%	9.886	9.889	9.470	9.672
	VaR 1%	9.659	9.594	9.252	9.426
Inv	**mean**	-0.001	-0.018	0.005	-0.020
	std dev	1.667	1.680	1.674	1.673
	skewness	-0.011	-0.014	-0.010	-0.001
	kurtosis	3.004	3.030	3.022	3.029
	Jarque Bera	2.2	7.1	3.6	3.5
	Q(T) 90%	[-3,3]	[-3,3]	[-3,3]	[-3,3]

Table 1C. Statistics of the PnL distribution. $\mu = 0.98$, $\eta = 0.001$.

		linear ABM	**linear MR**	**exp ABM**	**exp MR**
PnL	**mean**	11.030	13.457	10.670	11.039
	std dev	0.992	11.708	0.356	0.501
	skewness	0.014	-0.676	0.022	-0.864
	kurtosis	5.666	5.176	2.996	5.632
	Jarque Bera	29607.8	27355.7	8.4	41315.1
	VaR 5%	9.460	-8.180	10.086	10.167
	VaR 1%	8.368	-22.205	9.850	9.524
Inv	**mean**	-0.021	0.055	-0.013	0.031
	std dev	33.126	468.213	7.682	15.245
	skewness	-0.009	0.000	0.007	0.007
	kurtosis	2.990	1.849	3.024	2.988
	Jarque Bera	1.8	5523.2	3.2	1.5
	Q(T) 90%	[-55,55]	[-732,731]	[-13,13]	[-25,25]

Table 2A. Statistics of the PnL distribution. $\mu = 1.00$, $\eta = 0$.

		linear ABM	**linear MR**	**exp ABM**	**exp MR**
PnL	**mean**	10.982	11.453	10.606	10.913
	std dev	0.411	1.317	0.345	0.433
	skewness	0.029	-2.070	0.036	-0.554
	kurtosis	3.031	10.934	2.992	4.443
	Jarque Bera	17.7	333649.4	22.1	13802.7
	VaR 5%	10.312	8.956	10.043	10.178
	VaR 1%	10.037	6.601	9.813	9.699
Inv	**mean**	-0.006	-0.046	0.016	-0.018
	std dev	5.034	8.378	4.570	5.664
	skewness	-0.003	0.006	0.008	-0.001
	kurtosis	2.992	2.974	3.008	2.969
	Jarque Bera	0.5	3.4	1.5	4.0
	Q(T) 90%	[-8,8]	[-14,14]	[-7,8]	[-9,9]

Table 2B. Statistics of the PnL distribution. $\mu = 1.00$, $\eta = 0.0001$.

		linear ABM	**linear MR**	**exp ABM**	**exp MR**
PnL	**mean**	10.436	10.483	10.000	10.226
	std dev	0.335	0.356	0.325	0.338
	skewness	0.031	-0.097	0.031	-0.023
	kurtosis	3.001	3.162	2.988	3.069
	Jarque Bera	15.5	266.4	16.5	28.6
	VaR 5%	9.888	9.889	9.469	9.668
	VaR 1%	9.658	9.623	9.246	9.431
Inv	**mean**	-0.011	-0.009	0.005	0.006
	std dev	1.673	1.684	1.672	1.676
	skewness	0.006	-0.004	-0.001	0.002
	kurtosis	3.026	2.992	3.025	3.026
	Jarque Bera	3.6	0.5	2.7	3.0
	Q(T) 90%	[-3,3]	[-3,3]	[-3,3]	[-3,3]

Table 2C. Statistics of the PnL distribution. $\mu = 1.00$, $\eta = 0.001$.

		linear ABM	**linear MR**	**exp ABM**	**exp MR**
PnL	**mean**	11.038	14.230	10.670	11.081
	std dev	1.011	13.653	0.356	0.522
	skewness	0.015	-0.554	0.021	-0.987
	kurtosis	5.638	4.702	2.986	5.941
	Jarque Bera	29009.6	17181.5	8.0	52288.0
	VaR 5%	9.435	-10.866	10.084	10.168
	VaR 1%	8.332	-26.196	9.849	9.451
Inv	**mean**	0.045	335.118	-0.013	3.297
	std dev	33.253	417.302	7.641	15.295
	skewness	0.008	-0.626	-0.003	0.006
	kurtosis	3.004	2.358	3.024	2.992
	Jarque Bera	1.2	8241.5	2.5	0.9
	Q(T) 90%	[-55,55]	[-459,847]	[-13,13]	[-22,28]

Table 3A. Statistics of the PnL distribution. $\mu = 1.02$, $\eta = 0$.

		linear ABM	**linear MR**	**exp ABM**	**exp MR**
PnL	**mean**	10.982	11.581	10.605	10.945
	std dev	0.412	1.517	0.345	0.446
	skewness	0.031	-2.074	0.028	-0.647
	kurtosis	3.052	10.611	3.011	4.642
	Jarque Bera	27.2	313076.3	13.9	18207.8
	VaR 5%	10.308	8.659	10.039	10.182
	VaR 1%	10.026	5.977	9.807	9.662
Inv	**mean**	-0.003	1.712	0.005	0.795
	std dev	5.024	8.420	4.574	5.681
	skewness	0.001	-0.005	0.010	0.008
	kurtosis	3.006	2.976	2.993	2.980
	Jarque Bera	0.2	2.8	1.9	2.8
	Q(T) 90%	[-8,8]	[-12,16]	[-7,8]	[-9,10]

Table 3B. Statistics of the PnL distribution. $\mu = 1.02$, $\eta = 0.0001$.

		linear ABM	linear MR	exp ABM	exp MR
PnL	**mean**	10.436	10.495	10.000	10.234
	std dev	0.335	0.365	0.326	0.342
	skewness	0.038	-0.154	0.033	-0.061
	kurtosis	3.004	3.348	2.990	3.103
	Jarque Bera	24.0	901.2	18.5	106.9
	VaR 5%	9.890	9.889	9.468	9.669
	VaR 1%	9.666	9.598	9.252	9.414
Inv	**mean**	0.006	0.022	-0.006	0.018
	std dev	1.678	1.677	1.672	1.671
	skewness	0.002	-0.001	-0.018	-0.001
	kurtosis	3.001	3.030	3.027	3.023
	Jarque Bera	0.0	3.8	8.5	2.2
	Q(T) 90%	[-3,3]	[-3,3]	[-3,3]	[-3,3]

Table 3C. Statistics of the PnL distribution. $\mu = 1.02$, $\eta = 0.001$.

In Tables 1A-3C we see that for $\eta = 0$ the ABM case has an intrinsic inventory control: its mean is zero and the Jarque-Bera test cannot reject the normality hypothesis. However, the MR case has a very strong inventory risk: the market-maker ends with more than 300 shares on her pocket at the end of the day, in the direction she anticipated the market would trend. Moreover, the parameter η is very effective in controlling the inventory risk. Indeed, from $\eta = 0$ to $\eta = 0.0001$ and $\eta = 0.001$, in the case of a directional bet the inventory goes from ± 335 to ± 2 and 0, respectively.

It is important to remark that none of the the PnL distributions are not normal, but in all cases the inventory distributions can be considered as Gaussian because of their low Jarque-Bera scores.

The effect of η on the linear-utility strategy

From the previous section it was not clear the role of η on the PnL distribution. We tested here the hypothesis that η must have some indirect control on the PnL distribution. In order to prove our claim we performed 20,000 Monte-Carlo simulations for $\mu = 1.00$. The rest of the parameters are as before.

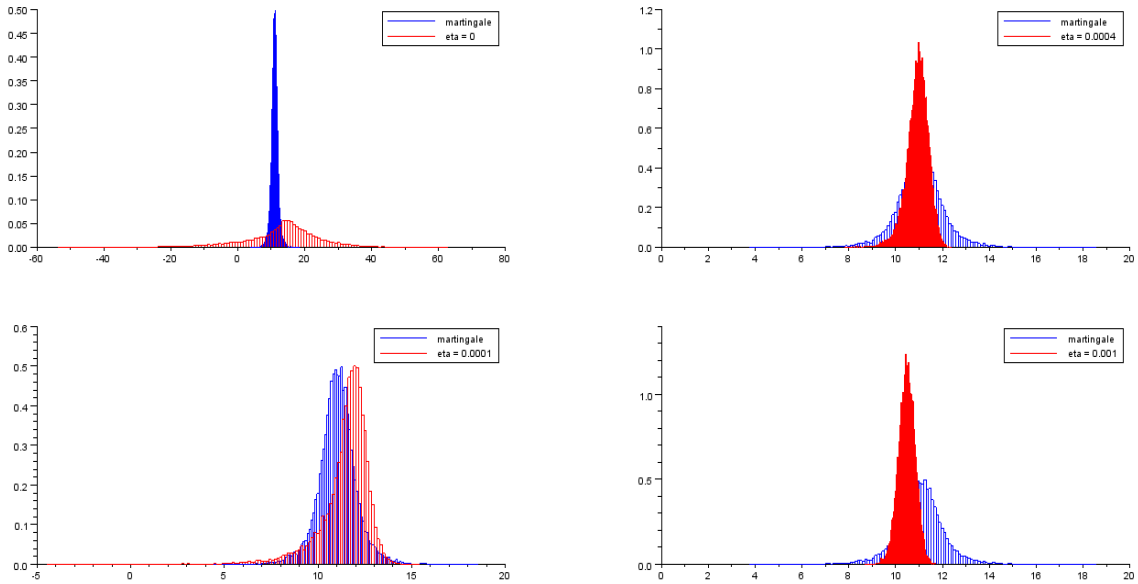

Figure 4. Histogram of PnL as function of η for linear utility function ($\mu = 1.00$). Blue: Linear martingale. Red: Linear MR. Upper-Left: $\eta = 0$, Lower-Left: $\eta = 0.0001$, Upper-Right: $\eta = 0.0004$, Lower-Right: $\eta = 0.001$.

	Martingale	$\eta = 0$	$\eta = 0.0001$	$\eta = 0.0004$	$\eta = 0.001$
mean	11.035	13.386	11.452	10.930	10.474
std dev	1.002	11.800	1.320	0.473	0.356
skewness	0.009	-0.645	-2.037	-0.810	-0.131
kurtosis	5.754	5.153	10.346	5.100	3.189
Jarque-Bera	6,322.0	5,250.2	58,797.2	5,863.1	86.6
VaR 5%	9.447	-8.343	9.001	10.115	9.882
VaR 1%	8.306	-22.297	6.435	9.502	9.599
Q(T) 90%	[-55,56]	[-730,732]	[-14,14]	[-4,4]	[-3,3]

Table 5. Statistics of the PnL distribution for linear MR as a function of η.

From Figure 4 and Table 5 we see that our guess was correct: as $\eta > 0$ increases, the PnL distribution has smaller mean, smaller standard deviation, skewness closer to zero and bigger 1% and 5% quantiles (the VaR). The case $\eta = 0$ seems to break this monotone behaviour in the skewness and kurtosis, but this is only due to the fact that the standard deviation is too big compared with any of the cases with $\eta > 0$. In fact, both the standard deviation and the VaR are gigantic when compared to the cases $\eta > 0$.

In summary, η controls directly the inventory but it also has indirect control on the PnL distribution.

The effect of γ on the exponential-utility strategy

We wanted to see if γ can control directly the PnL distribution, as it is expected from an exponential utility function due to its variance-reduction features. In order to assess the effect of γ separately from η, we performed 20,000 Monte-Carlo simulations for $\eta = 0$ and $\mu = 1.00$. The rest of the parameters are as before.

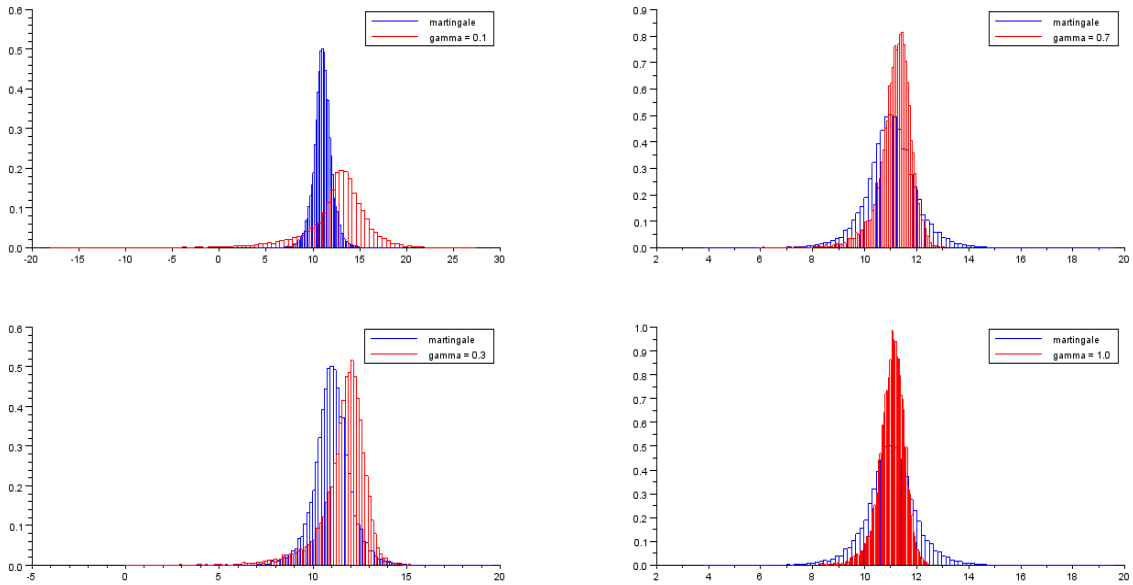

Figure 5. Histogram of the PnL as a function of γ for an exponential utility function ($\mu = 1.00$). Blue: Linear martingale. Red: Exponential MR. Upper-Left: $\gamma = 0.1$, Lower-Left: $\gamma = 0.3$, Upper-right: $\gamma = 0.7$, Lower-Right: $\gamma = 1.0$.

	Martingale	$\gamma = 0.1$	$\gamma = 0.3$	$\gamma = 0.7$	$\gamma = 1$
mean	11.023	12.609	11.581	11.193	11.039
std dev	0.977	3.862	1.230	0.611	0.493
skewness	0.055	-2.348	-1.881	-1.254	-0.808
kurtosis	5.411	13.242	10.143	6.939	5.185
Jarque-Bera	4,853.1	105,790.2	54,307.3	18,172.4	6,154.9
VaR 5%	9.421	6.225	9.273	10.044	10.186
VaR 1%	8.330	0.658	7.277	9.106	9.572
Q(T) 90%	[-55,55]	[-205,209]	[-73,73]	[-34,34]	[-25,25]

Table 6. Statistics of the PnL distribution for exponential MR as a function of γ ($\eta = 0$).

From Figure 5 and Table 6 we see that our claim was correct: as γ increases, the PnL distribution has smaller mean, smaller standard deviation, skewness closer to zero and bigger 1% and 5% quantiles (the VaR). However, if we compare Tables 5 and 6 we see that the linear MR with $\eta > 0$ seems to have

a better control on the standard deviation, skewness, kurtosis and VaR.

In summary, γ controls directly the PnL distribution and also has indirect control on the inventory, but the effect of η seems to be stronger in both risk factors.

Comparing $\eta = 0.0001$ and $\gamma = 0.3$

From Figures 4-5 we have that the linear MR with $\eta = 0.0001$ and the exponential MR with $\gamma = 0.3$ seem to have the same mode than the bechmark linear martingale. Here we compare these two distributions. We performed 20,000 Monte-Carlo simulations for $\mu = 1.00$ and kept the other parameters unchanged.

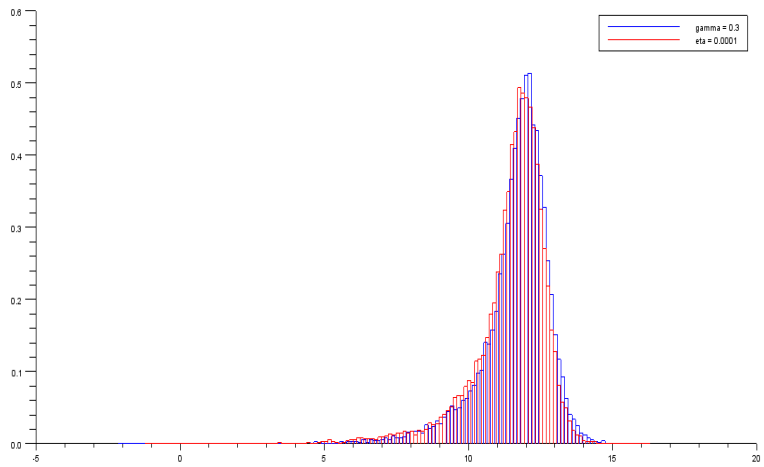

Figure 6. Comparing the PnL histograms for linear MR ($\eta = 0.0001$) and exponential MR ($\gamma = 0.3$). Blue: Linear MR. Red: Exponential MR.

	$\gamma = 0.3$	$\eta = 0.0001$	% change
mean	11.629	11.452	-0.0153
std dev	1.235	1.320	0.0691
skewness	-1.832	-2.037	0.1121
kurtosis	10.165	10.346	0.0178
Jarque-Bera	53967.9	58797.1	0.0895
VaR 5%	9.305	9.001	-0.0327
VaR 1%	7.281	6.435	-0.1163
Q(T) 90%	[-72,74]	[-14,14]	-0.808

Table 7. Statistical comparison between linear MR $\eta = 0.001$ and exponential $\gamma = 0.3$. The % change is the variation of the linear MR with respect to the exponential MR, i.e. "$\eta/\gamma - 1$".

As we can see in Table 7, when we compare linear MR with exponential MR it seems that exponential MR has better control on the PnL distribution: higher mean; smaller standard deviation, skewness and

kurtosis; and higher quantiles at 1% and 5% (VaR). However, the inventory of exponential MR is way too high compared with linear MR, around 5 times bigger.

8 Conclusions

On the market-making model

- In Theorem 4 we generalised the Avellaneda and Stoikov [1] approach for an exponential utility to any Markov process, provided its conditional expectation $\mathbb{E}_{t,s}[S(T)]$ is affine in s, e.g. an arithmetic Brownian motion with drift or an Ornstein-Uhlenbeck process. This allowed us to assess the effect that directional bets on the market mid-price would have on the PnL distribution of a high-frequency market-maker. Moreover, we also showed that the results of Lehalle *et al* [4] can be generalised to arithmetic Brownian motions with time-dependent drift and volatility, and that their approach cannot be extended further due to the choice of the ansatz.

- If instead of an exponential utility function we choose a linear utility function with inventory constraints, in Theorems 1-3 we showed that using the same stochastic-control approach we can find the optimal controls (i.e. the market-maker bid and ask quotes) for very general Markov processes, even with jumps (e.g. Lévy processes), provided the inventory-risk penalty $\pi(s)$ is chosen appropriately to ensure the finiteness of the value function $u(t, s, q, x)$. Moreover, the optimal controls in the linear case are independent of the volatility of the asset, which is very hard to estimate for high-frequency data, and as such the linear case is easier to calibrate with real data than the exponential case.

On the role of the parameters η and γ

- If η increases then the inventory risk decreases and eventually becomes negligeable. This was expected because the parameter η as added as a quadratic inventory penalty. However, as a nice side effect we have that η also reduces the variance on the PnL, in the sense that the variance and the kurtosis of the PnL distribution decrease as η increases. In other words, η controls perfectly the inventory risk, and by doing so it indirectly controls the PnL distribution.

- For γ we have the a similar effect. As γ increases the risks on the PnL distribution decrease, in particular the first four moments and the VaR. Moreover, as a side effect the inventory risk decreases as well. This implies that γ directly controls the PnL distribution and indirectly controls the inventory risk.

- This mirror-like, intertwined role of η and γ highlights a intimate relation between inventory risk and the risk on the PnL distribution: if one decreases via a control then necessarily the other has to do the same. This implies that a market-maker that chooses to reduce the risk on her PnL distribution necessarily reduces her inventory risk; conversely, a market-maker who reduces her inventory risk also reduces the risk on her PnL distribution. However, this risk reduction also implies a reduction on the average PnL.

- It seems that the control provided by η on the linear MR is better than the control provided by γ on the exponential MR. However, this claim has to be taken with a grain of salt because, for $\eta = 0.0001$ and $\gamma = 0.3$ (where qualitatively both distributions are comparable to the martingale

benchmark) the linear MR is less risky in terms of inventory whilst the exponential MR is less risky in terms of PnL distribution. That said, our numerical simulations show that an increase on η (from 0.0001 to 0.001) renders a more drastic reduction on inventory risk and stronger structural changes on PnL distribution than an increase on γ of the same order of magnitude (from 0.1 to 1).

On the directional bets and the risk profiles

There is a clear relation between risk and reward.

- When the market-maker makes a directional bet she improves her PnL up to 25% with respect to the martingale benchmark. However, by doing so she has to accept more risk, either on her PnL distribution (measured in terms of variance, skewness and kurtosis) or on her inventory.
- After a directional bet, the market maker can give up a percentage of her excess PnL in order to gain a direct control on her risks. If she chooses to control her PnL distribution directly (resp. inventory risk) then she gains some reduction on her inventory risk (resp. PnL distribution), but the latter cannot be controlled directly. In that spirit, the market-maker can choose to use an exponential MR strategy if she prices more dearly her risk on the PnL distribution and a linear MR strategy if her biggest concern is the inventory risk.
- In summary, directional bets enhance the PnL of the market-maker but add extra risk on her inventory and PnL distribution and both risks are positively correlated, so if she reduces one directly the other is indirectly reduced as well. Therefore, a market-maker in practice has to assess three factors, i.e. PnL increase, inventory risk and PnL distribution, in order to choose her optimal trading strategy because, in the current set-up at least, there is no strategy that is optimal in all three factors.

Further developments

- In our model there are no market orders, just limit orders. For the simulations we assumed that if $\delta_*^+ \leq 0$ (resp. $\delta_*^- \leq 0$) then the market-maker sends a selling (resp. buying) market order that is executed at the mid-price, i.e. we assumed that the market spread is zero and the impact of market orders on the PnL is half the market-maker's spread.
- For a detailed market-impact analysis we should add the market spread as another state variable and consider that the market-maker's limit orders affect both the market spread and the mid-price. For example, if she improves the best ask (resp. bid) then she reduces the market spread by one tick and pushes down (resp. up) the mid-price by half a tick.
- Another possibility is to incorporate market orders directly into the model, making a clear distinction between them and limit orders. However, the current framework does not seem to be easily extended for that purpose, which suggests that a new framework is needed. We are currently working on that direction.

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